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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

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S.J.L. van Eijndhoven

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Dept. of Mathematics & Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Abstract

By Fuglede's Theorem an operator B which commutes with a normal operator N also commutes with f(N) if f is a continuous function on the spectrum of N. In this paper we consider this theorem with "commuting" replaced by "almost commuting". We show that there are conditions for an operator topology τ such that $f(N)B - Bf(N)$ is τ-small as soon as $NB - BN$ is sufficiently τ-small and $\|B\| < K$ for some $K > 0$.

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Introduction

In the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space a normal operator N and an operator B commute if and only if N^* and B commute, this is the well-known theorem of Fuglede (2). Rosenblum (5) gave an elegant proof of this theorem, thus inspiring Moore (3) to the following extension.

For all $\varepsilon > 0$ *and* $K > 0$ *there exists* $\delta > 0$ *such that* $\|NB - BN\| < \delta$ *and* $\|B\| < K$ *imply* $\|N^*B - BN^*\| < \varepsilon$

Rogers proved that the norm topology in Moore's theorem may be replaced by the strong and weak operator topology. We shall prove that each operator topology that satisfies rather natural conditions may replace the norm topology. To prove this, we use techniques different from those of Moore and Rogers.

The main problem in all Fuglede-like theorems is how to relate $N^*B - BN^*$ to $NB - BN$. Or, equivalently, if we consider

$$H(\lambda) := e^{i\lambda N} B e^{-i\lambda N}, \quad (\lambda \in \mathbb{C})$$

and

$$G(\lambda) := e^{-i\lambda N^*} B e^{i\lambda N^*}, \quad (\lambda \in \mathbb{C})$$

how to relate $H'(0)$ to $G'(0)$.

To obtain such a relation, note first that $G'(0)$ is represented by the Cauchy type integral

$$G'(0) = \frac{1}{2\pi} \int_{|\mu|=r} \frac{G(\mu)}{\mu^2}\, d\mu$$

with integration along $|\mu| = r$ in the positive sense, and then that

$$G(\mu) = e^{-i(\mu N^* + \bar{\mu} N)} H(\bar{\mu})\, e^{i(\mu N^* + \bar{\mu} N)}.$$

Now, $H(\bar{\mu})$ can be obtained from H' by integration from 0 to $\bar{\mu}$ along a straight line segment. Hence

$$(*) \qquad G'(0) = \frac{1}{2\pi i} \int_{|\mu|=r} \frac{e^{-i(\mu N^* + \bar{\mu} N)}}{\mu^2} \left\{ \int_0^{\bar{\mu}} H'(\lambda)\, d\lambda + H(0) \right\} e^{i(\mu N^* + \bar{\mu} N)}\, d\mu.$$

Formula (*), together with the observation that the operators $e^{i(\mu N^* + \bar{\mu} N)}$ are unitary, is the central argument in the proof of Theorem 3.

The main difference between the approach of Moore and Rogers and our's is in the estimation of $H(\bar{\mu})$. They use the power series of the exponential function together with the identity

$$N^k B - B N^k = \sum_{j=0}^{k-1} N^j (NB - BN) N^{k-j-1},$$

while we express $H(\bar{\mu})$ as an integral which is then estimated in a simple way.

Results

The norm topology, the weak operator topology and the strong operator topology are algebraic topologies on $\mathcal{B}(H)$ in our terminology. Here is the definition.

Definition 1. A topology τ on $\mathcal{B}(H)$ is called algebraic if

(1.1) τ is coarser than the norm topology,

(1.2) $\mathcal{B}(H)$ with topology τ is a locally convex, topological vector space,

(1.3) the mappings $A \to BAC$ are τ-continuous for fixed $B, C \in \mathcal{B}(H)$.

The following lemma is an immediate consequence of the first two of these conditions. The proof depends on a simple compactness argument.

Lemma 2: Let τ be algebraic, and $\Omega \in \tau$ a convex open neighbourhood of 0. Let $T > 0$ and let $f: [0,T] \to \mathcal{B}(H)$ be norm continuous with $f(t) \in \Omega$ for all $t \in [0,T]$. Then

$$\int_0^T f(t)dt \in T\Omega .$$

Analogously to results of Moore and Rogers we now prove

Theorem 3. Let τ be algebraic, let $N \in \mathcal{B}(H)$ be normal and let $(B_\alpha) \subset \mathcal{B}(H)$ be a norm bounded net with $NB_\alpha - B_\alpha N \to 0$ in τ-sense. Then $N^*B_\alpha - B_\alpha N^* \to 0$ in τ-sense.

Proof. We may as well assume that $\|N\| = 1$. Let $\Omega \in \tau$ be a convex and circled open neighbourhood of 0, and let $K > 0$ with $\|B_\alpha\| < K$ for all α. Fix $r > 0$ such that $\{A \in \mathcal{B}(H) \,|\, \|A\| < \frac{K}{r}\} \subset \frac{1}{2}\Omega$. Put $S_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq r\}$ and $C_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| = r\}$. Define $U(\mu) := e^{-i(\mu N^* + \bar{\mu}N)}$, $(\mu \in \mathbb{C})$. Since $\|U(\mu)B_\alpha U(\mu)^*\| \leq \|B_\alpha\| < K$, we have for all α

$$\text{(i)} \qquad U(\mu)B_\alpha U(\mu)^* \in \frac{r}{2}\Omega , \quad (\mu \in \mathbb{C}).$$

Let $(\lambda,\mu) \in S_r \times C_r$. The mapping $A \to U(\mu)e^{i\lambda N}Ae^{-i\lambda N}U(\mu)^*$ is τ-continuous by (1.3). So there exists an open neighbourhood of 0, $\Omega_{\lambda,\mu}$, say, with

$$U(\mu)e^{i\lambda N}Ae^{-i\lambda N}U(\mu)^* \in \frac{1}{4}\Omega , \quad (A \in \Omega_{\lambda,\mu}).$$

For each A with $\|A\| \leq 1$, let F_A be the mapping on $S_r \times C_r$ that sends (λ,μ) into $U(\mu)e^{i\lambda N}Ae^{-i\lambda N}U(\mu)^*$. The mappings F_A are norm continuous on $S_r \times C_r$ and even uniformly equicontinuous. Since $S_r \times C_r$ is compact, there is a finite set $E := \{(\lambda_j,\mu_\ell) \,|\, j = 1,\dots k;\ \ell = 1,\dots m\}$ in $S_r \times C_r$ such that for each $(\lambda,\mu) \in S_r \times C_r$ there exists $(\lambda_j,\mu_\ell) \in E$ with

$$\|F_A(\lambda,\mu) - F_A(\lambda_j,\mu_\ell)\| < \frac{1}{8r} , \quad (\|A\| \leq 1).$$

Now take α_1 such that

$$NB_\alpha - B_\alpha N \in \bigcap_{(\lambda_j,\mu_\ell)} \Omega_{\lambda_j,\mu_\ell} \quad , \quad (\alpha \geq \alpha_1).$$

Then $F_{[N,B_\alpha]}(\lambda_j,\mu_\ell) \in \frac{1}{4}\,\Omega$ for every $(\lambda_j,\mu_\ell) \in E$ as soon as $\alpha \geq \alpha_1$ (with $[N,B_\alpha] = NB_\alpha - B_\alpha N$). Let $(\lambda,\mu) \in S_r \times C_r$ and $\alpha \geq \alpha_1$. Since $\|[N,B_\alpha]\| < 2K$, we can find $(\lambda_j,\mu_\ell) \in E$ with

$$\|F_{[N,B_\alpha]}(\lambda,\mu) - F_{[N,B_\alpha]}(\lambda_j,\mu_\ell)\| < \frac{K}{4r} \quad .$$

So

$$F_{[N,B_\alpha]}(\lambda,\mu) = \{F_{[N,B_\alpha]}(\lambda,\mu) - F_{[N,B_\alpha]}(\lambda_j,\mu_\ell)\} +$$

$$+ F_{[N,B_\alpha]}(\lambda_j,\mu_\ell) \in \frac{1}{4}\,\Omega + \frac{1}{4}\,\Omega = \frac{1}{2}\,\Omega.$$

Since α does not depend on the choice of $(\lambda,\mu) \in S_r \times C_r$, we have shown

$$\text{(ii)} \qquad U(\mu)e^{i\lambda N}(NB_\alpha - B_\alpha N)e^{-i\lambda N}U(\mu)^* \in \frac{1}{2}\,\Omega$$

for every $(\lambda,\mu) \in S_r \times C_r$ and $\alpha \geq \alpha_1$.
Let H_α denote the function H defined in the introduction, in which B is replaced by B_α. Then relation (*) gives

$$N^*B_\alpha - B_\alpha N^* = \frac{1}{2\pi}\int_{Cr} \frac{1}{\mu^2}\,U(\mu)\left(\int_0^{\bar{\mu}} H_\alpha'(\lambda)d\lambda + B_\alpha\right)U(\mu)^* d\mu \ .$$

Applying (i),(ii) and lemma 2 we obtain

$$\text{(iii)} \qquad \int_0^{\bar{\mu}} U(\mu)H_\alpha'(\lambda)U(\mu)^* d\lambda + U(\mu)B_\alpha U(\mu)^* \in r\,\Omega$$

for $\alpha \geq \alpha_1$ and $\mu \in C_r$.
Finally, applying (iii) and, again, lemma 2 we conclude that

$$N^*B_\alpha - B_\alpha N^* \in \frac{2\pi r}{2\pi}\,\frac{1}{r^2}\,r\,\Omega = \Omega \ , \ (\alpha \geq \alpha_1).$$

□

In the next theorem N^* is replaced by $f(N)$ with $f \in C(\sigma(N))$, i.e. f is a continuous complex function on the spectrum $\sigma(N)$ of N. Theorem 3 is a special case of Theorem 4; the former is essential as a preparation to the proof of the latter, though.

Theorem 4. Let τ, N and (B_α) satisfy the conditions of Theorem 3, and let $f \in C(\sigma(N))$. Then $f(N)B_\alpha - B_\alpha f(N) \to 0$ in τ-sense.

Proof. Let $\Omega \in \tau$ be a convex and circled neighbourhood of 0. We divide the proof into two steps.

Step one. Assume first that f is a polynomial p,

$$p(\lambda,\bar{\lambda}) = \sum_{i+j\le m} c_{ij}\lambda^i\bar{\lambda}^j ,$$

say. For each $i,j \in \mathbb{N}$ with $i + j \le m$, we have

$$N^iN^{*j}B_\alpha - B_\alpha N^iN^{*j} = N^i(N^{*j}B_\alpha - B_\alpha N^{*j}) + (N^iB_\alpha - BN^i)N^{*j}$$

and

$$N^{*j}B_\alpha - B_\alpha N^{*j} = \sum_{k=0}^{j-1} N^{*j-k-1}(N^*B_\alpha - B_\alpha N^*)N^{*k} ,$$

$$N^iB_\alpha - B_\alpha N^i = \sum_{\ell=0}^{i-1} N^{i-\ell-1}(NB_\alpha - B_\alpha N)N^\ell .$$

Consider the mappings

$$\Psi\colon A \to \sum_{i+j\le m} c_{ij}N^i(\sum_{k=0}^{j-1} N^{*j-k-1}A\, N^{*k})$$

and

$$\Phi\colon A \to \sum_{i+j\le m} c_{ij}(\sum_{\ell=0}^{i-1} N^{i-\ell-1}A\, N^\ell)N^{*j} .$$

Since τ is algebraic, Φ and Ψ are τ-continuous. Hence there exist convex and circled open neighbourhoods of 0, $\Omega_\Phi \in \tau$, $\Omega_\Psi \in \tau$, say, with

$$\Psi(\Omega_\Psi) \subset \frac{1}{2}\,\Omega \text{ and } \Phi(\Omega_\Phi) \subset \frac{1}{2}\,\Omega \,.$$

According to Theorem 3 there is an open neighbourhood $\tilde{\Omega}_\Psi$ of 0 such that $N^*B_\alpha - B_\alpha N^* \in \Omega_\Psi$ whenever $NB_\alpha - B_\alpha N \in \tilde{\Omega}_\Psi$. Now take α_1 such that $NB_\alpha - B_\alpha N \in \tilde{\Omega}_\Psi \cap \Omega_\Phi$ for all $\alpha \geq \alpha_1$. Then

$$p(N,N^*)B_\alpha - B_\alpha p(N,N^*) = \Psi(N^*B_\alpha - B_\alpha N^*) + \Phi(NB_\alpha - B_\alpha N) \in \Omega$$

as soon as $\alpha \geq \alpha_1$.

<u>Step two</u>. Let $f \in C(\sigma(N))$, and let $K > 0$ be a bound for $\|B_\alpha\|$. Fix $\rho > 0$ such that

$$\{A \in \mathcal{B}(H) \,|\, \|A\| < \rho\} \subset \frac{1}{3K}\,\Omega \,.$$

Since $\sigma(N)$ is compact, there is a polynomial $p(\lambda,\bar{\lambda})$ with $\sup_{\lambda\in\sigma(N)} |f(\lambda) - p(\lambda,\lambda)| < \rho$. Since $\|f(N) - p(N,N^*)\| < \rho$ and $\|B_\alpha\| < K$

(i) $$(f(n) - p(N,N^*))B_\alpha \in \frac{1}{3}\,\Omega \text{ and } B_\alpha(f(N) - p(N,N^*)) \in \frac{1}{3}\,\Omega.$$

According to the first part of the proof there exists α_1 such that

(ii) $$p(N,N^*)B_\alpha - B_\alpha\, p(N,N^*) \in \frac{1}{3}\,\Omega, \quad (\alpha \geq \alpha_1).$$

Combining (i) and (ii), and taking $\alpha \geq \alpha_1$ our proof is complete.

□

<u>Remark 1</u>. In Theorem 3 and Theorem 4 we may take the (ultra-)weak or (ultra-)strong operator topology of $\mathcal{B}(H)$. In each case it is necessary to require the net to be bounded. This may be shown by a construction

of Bastians and Harrison (cf. the proof of the last part of Theorem 3 in [1]).

Remark 2. Let $N_1, N_2 \in \mathcal{B}(H)$ be normal. Analogously to relation (*) in part I, we have

$$N_1^* B - B N_2^* = \frac{1}{2\pi i} \int_{|\mu|=r} \frac{e^{-i(\mu N_1^* + \bar{\mu} N_1)}}{\mu^2} \left\{ \int^{\mu} \tilde{H}'(\lambda)\, d\lambda + \tilde{H}(0) \right\} e^{i(\mu N_2^* + \bar{\mu} N_2)}\, d\mu$$

with

$$\tilde{H}(\lambda) = e^{i\lambda N_1} B e^{-i\lambda N_2}, \quad (\lambda \in \mathbb{C}),$$

and further more

$$N_1^i N_1^{*j} B - B N_2^i N_2^{*j} = N_1^i \left(\sum_{k=0}^{j-1} N_1^{*j-k-1} (N_1^* B - B N_2^*) N_2^{*k} \right) + \\ + \left(\sum_{\ell=0}^{i-1} N_1^{i-\ell-1} (N_1 B - B N_2) N_2^{\ell} \right) N_2^{*j} \quad .$$

Hence we may prove the following Putnam-like version of Theorem 4.

Theorem 5. Let τ be algebraic, let $N_1, N_2 \in \mathcal{B}(H)$ be normal, let $f \in C(\sigma(N_1) \cup \sigma(N_2))$, and let $(B_\alpha) \subset \mathcal{B}(H)$ be a normbounded net with $N_1 B_\alpha - B_\alpha N_2 \to 0$ in τ-sense. Then $f(N_1) B_\alpha - B_\alpha f(N_2) \to 0$ in τ-sense.

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