Long term structural dynamics of mechanical systems with local nonlinearities

R. H. B. Fey
TNO Building and Construction Research,
Centre for Mechanical Engineering,
P.O. Box 29, 2600 AA Delft,
The Netherlands

D. H. van Campen

A. de Kraker
Department of Mechanical Engineering,
Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven,
The Netherlands

Long Term Structural Dynamics of Mechanical Systems With Local Nonlinearities

This paper deals with the long term behavior of periodically excited mechanical systems consisting of linear components and local nonlinearities. The number of degrees of freedom of the linear components is reduced by applying a component mode synthesis technique. Lyapunov exponents are used to identify the character of the long term behavior of a nonlinear dynamic system, which may be periodic, quasi-periodic or chaotic. Periodic solutions are calculated efficiently by solving a two-point boundary value problem using finite differences. Floquet multipliers are calculated to determine the local stability of these solutions and to identify local bifurcation points. The methods presented are applied to a beam system supported by a one-sided linear spring, which reveals very rich, complex dynamic behavior.

1 Introduction

Mechanical systems consisting of linear components with many degrees of freedom and local nonlinearities are frequently met in engineering practice. Examples of such systems are: rotating mechanical systems with nonlinear bearings, mechanical systems with dry friction and backlash phenomena in certain connections, etc. From a spatial point of view, the local nonlinearities constitute only a small part of the mechanical system. However, their presence can have important consequences for overall dynamic behavior.

The subject of this paper is the long term behavior of mechanical systems with many degrees of freedom and local nonlinearities, excited by periodic external loads. The numerical determination of the long term behavior for a nonlinear model with many degrees of freedom in general may require much computational time and offer computational problems. In this paper the number of degrees of freedom of the linear components of a system with local nonlinearities is reduced by applying a component mode synthesis technique based on free-interface eigenmodes and residual flexibility modes (Section 2).

The long term behavior of a nonlinear dynamic system can have a periodic, quasi-periodic or chaotic character. In Subsection 3.1 periodic solutions are calculated by solving a two-point boundary value problem by applying a finite difference method. How the periodic solution is influenced by a change in a so-called design variable of the system is investigated by applying a path following technique. In Subsection 3.2 the local stability of a periodic solution is investigated using Floquet theory. On the branches of periodic solutions three types of local bifurcations can be found, namely the cyclic fold bifurcation, the flip (or period doubling) bifurcation and the Neimark (or secondary Hopf) bifurcation. The steady-state behavior is also investigated by means of standard numerical time integration. In this case the character of the long term behavior (periodic, quasi-periodic or chaotic) is identified by calculation of the Lyapunov exponents.

In Section 4 the methods outlined above are applied to a harmonically excited discretized beam system supported by a one-sided linear spring. The excitation frequency is taken as a design variable. Superharmonic and subharmonic resonances are evaluated and the bifurcations mentioned above are met frequently. In the system three routes to chaos are observed: a period doubling route (Feigenbaum, 1983), intermittency (Pomeau and Manneville, 1980) and a quasi-periodic–locked–chaotic route (Newhouse et al., 1978). All calculations in this paper were carried out using a development release of the finite element package DIANA (1994) (module STRDYN for nonlinear dynamic analysis).

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received July 1994. Associate Technical Editor: R. Ibrahim.

2 Reduction of the Number of Degrees of Freedom

The equations of motions of a linear elastic component are:

$$M\ddot{x} + B\dot{x} + Kx = f \tag{2.1}$$

where M, B and K are the mass matrix, damping matrix and stiffness matrix, respectively, all of size (n_x, n_x); $x = [x_B^t, x_I^t]^t$ is a n_x-column with degrees of freedom (dof), which is divided in a n_B-column x_B with loaded boundary dof (i.e., externally loaded dof and interface dof loaded by adjacent linear components or local nonlinearities), and a n_I-column x_I with unloaded internal dof. On empirical grounds it has been concluded in linear dynamics that the following Ritz-approximation of the component displacement field in general offers a large reduction of the number of dof and consequently of the CPU-time needed for analysis, whereas simultaneously the decrease in accuracy of the system response is only small, if the frequency spectrum of the n_x-column with loads $f = [f_B^t, 0_I^t]^t$ ranges from zero till some cut-off frequency $f_c = \omega_c/2\pi$ and if the assumption of proportional damping is justified:

$$x = Tp, \quad T = [\Phi_k \quad \Phi^B], \quad p = [p_k^t \quad p_B^t]^t \tag{2.2}$$

Here, the columns of the (n_x, n_k) matrix Φ_k with kept elastic eigenmodes are the mass normalized solutions ($\varphi_i^t M \varphi_i = 1$) of the undamped eigenproblem for $\omega_i \in (0, \omega_c]$ $(i = 1, \ldots, n_k)$:

$$(-\omega_i^2 M + K)\varphi_i = 0 \tag{2.3}$$

The (n_x, n_B) matrix Φ^B with residual flexibility modes is defined as follows:

$$\Phi^B = [K^{-1} - \Phi_k \Omega_{kk}^{-2} \Phi_k^t][I_{BB} \quad 0_{IB}^t]^t \tag{2.4}$$

where Ω_{kk} is a (n_k, n_k) diagonal matrix with the kept angular eigenfrequencies lower than or equal to ω_c. A residual flexibility mode is defined for each boundary dof and guarantees unaffected static load behavior of the reduced system model. The above holds for a statically determinate component; if the component can move as a rigid body, an alternative formulation can be applied (Craig, 1985).

Using transformation (2.2) the reduced component equations become:

$$T^tMT\ddot{p} + T^tBT\dot{p} + T^tKTp = T^tf \quad (2.5)$$

Again using (2.2) the dof p_B are replaced by the boundary dof x_B to permit simple coupling of the reduced component equations. Subsequently the reduced system model is assembled by demanding compatibility of interface dof and equilibrium of interface loads. The n_q-column with independent system dof q contains the modal dof $p_k^{(i)}$ of components i ($i = 1, \ldots, N_c$) and a column y, containing all boundary dof of the system. Local nonlinearities, which for simplicity are assumed to be only a function of y and $\dot{y}$, are added by means of the n_q-column with internal loads f_{nl} and external loads are collected in f_q. The equations of motion of the reduced system are given by:

$$M_q\ddot{q} + B_q\dot{q} + K_qq + f_{nl} = f_q \quad (2.6)$$

with:

$$q^t = [p_k^{(1)t}, \ldots, p_k^{(N_c)t}, y^t],$$
$$f_{nl}^t = [0_k^{(1)t}, \ldots, 0_k^{(N_c)t}, f_{ny}^t(\dot{y}, y)],$$
$$f_q^t = [0_k^{(1)t}, \ldots, 0_k^{(N_c)t}, f_y^t]$$

If $f_{nl} = f_q = 0$ and $B_q = 0$, eigenfrequencies of (2.6) below f_c Hz should be very accurate. In general, higher eigenfrequencies will be inaccurate. These inaccurate eigenfrequencies may cause superharmonic resonances in the nonlinear system in the low-frequency range. The accuracy of the results obtained with the reduced model can be checked by investigating the frequency spectrum of the external load minus the internal loads caused by the local nonlinearities, and by investigating the influence of the deleted (higher) eigenmodes on this frequency spectrum (de Kraker et al., 1989).

3 Steady-state Behavior

In our case the external load acting on the system is periodic with period $T_e = 1/f_e$:

$$f_q(t) = f_q(t + 1/f_e) = a_0 + \sum_{m=1}^{\infty} [a_m \cos m\phi(t) + b_m \sin m\phi(t)], \quad \phi(t) = 2\pi f_e t + \phi_e \quad (3.1)$$

The (bounded) steady-state behavior of the system, i.e., the attractor which is reached after the transient has damped out, might be periodic, quasi-periodic or chaotic. Several steady-states can coexist for a single set of Eqs. (2.6). Which of these steady-states will be reached, depends on the initial augmented state $s_a(t_0)$:

$$s_a(t) := [s^t(t), \phi(t)]^t, \quad s(t) := [q^t(t), \dot{q}^t(t)]^t$$

In case the solution is periodic with period T_p and the response is periodic with period T_r, we call the solution harmonic if $T_p = T_r = T_e$ and subharmonic of order r_T/e_T if $T_p = r_T T_r = e_T T_e$ ($r_T \in \mathbb{N}$, $e_T \in \mathbb{N}\setminus\{1\}$, $r_T < e_T$).

If the solution is quasi-periodic, the solution is a function of two or more periodic signals, which have incommensurate frequencies.

A chaotic solution is characterized by an extreme dependence on the initial state. Consider a state $s_1(t_1)$ on a chaotic solution and a state $s_2(t_1) = s(t_1) + \delta s(t_1)$, $\delta s(t_1)$ being an infinitesimally small perturbation. Now the two trajectories starting from both states will diverge for $t \geq t_1$, on average of time, but will eventually fill the same linear subspace, being the chaotic attractor.

3.1 Periodic Solutions. Periodic solutions are calculated by solving a two-point boundary value problem, which is defined by (2.6) supplemented with the boundary condition $s(t) = s(t + T_p)$. Approximations of periodic solutions can be obtained efficiently by using a finite difference technique. The time is discretized by n_t equidistant points $t_i = iT_p/n_t =: i\Delta t$ in one period T_p. The following approximations $\tilde{\dot{q}}_i$ and $\tilde{\ddot{q}}_i$ are used for the velocities $\dot{q}_i$ and accelerations $\ddot{q}_i$ respectively for $i = 0, \ldots, n_t - 1$ ($0(\Delta t^2)$ central difference scheme, Q_i is an abbreviation for quantity $Q(t_i)$):

$$\tilde{\dot{q}}_i = (\tilde{q}_{i+1} - \tilde{q}_{i-1})/(2\Delta t),$$
$$\tilde{\ddot{q}}_i = (\tilde{q}_{i+1} - 2\tilde{q}_i + \tilde{q}_{i-1})/(\Delta t^2) \quad (3.2)$$

Substitution of (3.2) in the two-point boundary value problem leads to a set of $n_q * n_t$ nonlinear algebraic equations:

$$h(z, r) = 0, \quad z = [\tilde{q}_0^t, \ldots, \tilde{q}_{n_t-1}^t]^t \quad (3.3)$$

where r represents a design variable, for example the excitation frequency. If r is given a value $r_{s,1}$, the discretized periodic solution $z_{s,1}$ can be solved from (3.3) using the iterative (damped) Newton process. Naturally, the solution, which will be found, depends on the initial estimate z_0. If d^3q/dt^3 and d^4q/dt^4 exist, the global discretization error of the $0(\Delta t^2)$ solution can be estimated and the solution can be improved to a $0(\Delta t^4)$ solution by applying a deferred correction technique (Pereyra, 1966).

By applying a path following (pf) technique the designer of a dynamic system is able to investigate how a periodic solution is influenced by a change in r. In essence the technique consists of a predictor-corrector mechanism. Starting from a known solution $z_{s,k}$, $r_{s,k}$ the prediction of pf-step k is chosen on the tangent to the solution branch at the point $z_{s,k}$, $r_{s,k}$ ($h_{(Q)} := \partial h/\partial Q$):

$$[z_{p,k}^t \quad r_{p,k}]^t = [z_{s,k}^t \quad r_{s,k}]^t + \sigma_{p,k}[(-h_{(z)}^{-1}h_{(r)})^t \quad 1]^t \quad (3.4)$$

where $\sigma_{p,k}$ is a well-chosen step size. Subsequently this prediction is corrected iteratively using the orthogonal trajectory method of Fried (1984) ($z_{c,k,1} = z_{p,k}$, $r_{c,k,1} = r_{p,k}$):

$$\begin{bmatrix} z_{c,k,m+1} \\ r_{c,k,m+1} \end{bmatrix} = \begin{bmatrix} z_{c,k,m} \\ r_{c,k,m} \end{bmatrix} - \begin{bmatrix} (h_{(z)} + h_{(r)}(h_{(z)}^{-1}h_{(r)})^t)^{-1}h \\ (h_{(z)}^{-1}h_{(r)})^t(h_{(z)} + h_{(r)}(h_{(z)}^{-1}h_{(r)})^t)^{-1}h \end{bmatrix} \quad (3.5)$$

In (3.5) the correction is orthogonal to the solution space of $h(z, r) = h(z_{c,k,m}, r_{c,k,m})$. The correction process is stopped if some convergence criterion is satisfied. More details about the path following method can be found in Fey (1992).

3.2 Local Stability, Local Bifurcations and Lyapunov Exponents. The local stability of a solution $q(t)$ is investigated by linearizing the equations of motion (2.6) around the solution and examining the evolution in time of an infinitesimally small perturbation. Using a first order formulation and neglecting higher order terms, substitution of the perturbed solution $q(t) + \delta q(t)$ in (2.6) gives:

$$\delta\dot{s} = A(t)\delta s,$$

$$A(t) = \begin{bmatrix} 0 & I \\ -M_q^{-1}(K_q + f_{nl(q)}(t)) & -M_q^{-1}(B_q + f_{nl(\dot{q})}(t)) \end{bmatrix} \quad (3.6)$$

with initial conditions $\delta s(t_0) = \delta s_0$. The general solution of (3.6) is:

$$\delta s(t) = \Theta(t, t_0)\delta s_0, \quad \dot{\Theta}(t) = A(t)\Theta(t),$$
$$\Theta(t_0, t_0) = I \quad (3.7)$$

Table 1 Situations just before and after bifurcation points. SP, UP: Stable, Unstable Periodic solution. SPD, UPD: Stable, Unstable Periodic solution with Double period. SQP, UQP: Stable, Unstable Quasi-Periodic solution. Cases *a* are called supercritical; cases *b* are called subcritical.

name:	cyclic fold or turning point	flip or period doubling	Neimark or secondary Hopf
$r = r_{bif}^-$:	SP and UP	a. SP b. UP	a. SP b. UP
$r = r_{bif}$:	$\mu_1 = 1$	$\mu_1 = -1$	$\mu_1 = \bar{\mu}_2, \lvert\mu_1\rvert = 1$
$r = r_{bif}^+$:	locally no periodic sol.	a. UP and SPD b. SP and UPD	a. UP and SQP b. SP and UQP

If the solution is periodic ($A(t) = A(t + T_p)$), Floquet theory (see for example Seydel, 1988) shows that the fundamental matrix $\Theta(t, t_0)$ satisfies:

$$\Theta(t + T_p, t_0) = \Theta(t, t_0)\Theta(t_0 + T_p, t_0) =: \Theta(t, t_0)\Theta_\mu \quad (3.8)$$

where Θ_μ is the so-called monodromy matrix, which is assumed to have a spectral decomposition $\Theta_\mu = \Psi_\mu \lceil \mu_i \rfloor \Psi_\mu^{-1}$. Using (3.8) it can easily be shown that ($t^* = t - t_0 - \kappa T_p$, $0 \le t^* < T_p$, $\kappa \in \mathbb{Z}$):

$$\Theta(t, t_0) = \Theta(t_0 + t^*, t_0)\Psi_\mu \lceil \mu_i \rfloor^\kappa \Psi_\mu^{-1} \quad (3.9)$$

So, if the solution is periodic, the long term behavior of $\delta s(t)$ is predestinated by the eigenvalues μ_i of $\Theta_\mu(|\mu_{i+1}| \le |\mu_i|)$, the so-called Floquet-multipliers. The following stability conditions result:

- (asymptotically) stable periodic solution, periodic attractor (in figures s), if: $|\mu_i| < 1, \forall i$
- marginally stable periodic solution, if: $|\mu_1| = 1$
- unstable periodic solution, periodic saddle or repellor (in figures u), if: $|\mu_1| > 1$

In case the periodic solution is marginally stable for $r = r_{bif}$, an infinitesimally small perturbation of the design variable r can change both the quantitative and the qualitative steady-state behavior drastically: the system is not structurally stable. For a more exact definition of structural stability, see Guckenheimer and Holmes (1983). The marginally stable periodic solution is called a dynamic bifurcation point and r_{bif} the bifurcation value. Because we are varying only one design variable at a time, we generically only meet one of the three types of (co-dimension 1) bifurcation points described in Table 1; these bifurcation points are identified by μ_1.

In case a solution is calculated by solving an initial value problem with an arbitrary initial augmented state $s_a(t_0)$, Lyapunov exponents λ_i can be used to identify its character and its stability. Let $\rho_1(t), \ldots, \rho_{2n_q}(t)$ ($|\rho_i| \ge |\rho_{i+1}|$) be the eigenvalues of $\Theta(t, t_0)$. Whenever the limit exists, Lyapunov exponent λ_i is defined by:

$$\lambda_i := \lim_{t\to\infty} \frac{1}{t} \log_2 |\rho_i(t)| \quad i = 1, \ldots, 2n_q \quad (3.10)$$

An algorithm for calculating Lyapunov exponents in a numerically stable manner can be found in Fey (1992). The solution is an attractor if contraction outweighs expansion ($\sum_{i=1}^{2n_q} \lambda_i < 0$). The character of an attractor is identified by the Lyapunov exponents λ_i. In our nonautonomous case $\lambda_i = (1/T_p) \log_2 |\mu_i| < 0$, $\forall i$ reveals a periodic attractor, $\lambda_1 = \ldots = \lambda_q = 0$ an attracting $(q + 1)$D-torus (or $(q + 1)$-dimensional quasi-periodic attractor) and $\lambda_1 > 0$ a chaotic attractor.

4 A Beam Supported by a One-sided Linear Spring

Consider a 2D pinned-pinned beam (length 3 m, mass density 7850 kg/m^3, modulus of elasticity 2.1 10^{11} N/m^2, area of cross-section 1.7593 10^{-4} m^2, second moment of area 1.7329 10^{-8} m^4), which halfway its length is excited by a periodic transversal force $f_y = 39.386 \cos(2\pi f_e t + \phi_e)$ and supported by a one-sided linear spring (spring force $f_{ny} = 0$ if $y \ge 0$, $f_{ny} = 39386y$ if $y < 0$) and a linear damper with constant $b = 233.22\xi$ Ns/m. The amplitude of the external load is nothing but a scaling factor in this system. The quotient of the stiffness of the one-sided linear spring and the stiffness of the beam is 6. Because the excitation force and the nonlinear internal force both act on the middle of the beam, it is sufficient to consider only half the system, which is discretized using 25 beam elements (pure bending). The four lowest eigenfrequencies of the system without support are 8.96 Hz, 80.6 Hz, 224 Hz and 439 Hz. Two reduced models have been made to investigate the long term behavior of the system: a single dof model and a four dof model, in which the displacement field of the beam is approximated by the first free-interface eigenmode and by the first three free-interface eigenmodes plus one residual flexibility mode, respectively. In the four dof model ($f_c = 300$ Hz) the fourth eigenfrequency is 538 Hz. The long term behavior of these two models is investigated taking f_e as the design variable.

4.1 Single Dof Model.

Figure 1 shows the amplitude-frequency plot of the single dof system for two values of the damping. Unless stated otherwise, $n_t = 600$. Harmonic resonance occurs near the first bilinear eigenfrequency $f_{b1} = 13$ Hz. For $\xi = 0.01$ branches of 1/2 subharmonic solutions are found in the frequency intervals 7.55–8.89 Hz and 20.64–38.50 Hz; at the boundaries of these intervals flip bifurcations are found. One closed branch with 1/3 subharmonic solutions is found in the interval 36.06–48.65 Hz; the boundaries of this interval are formed by cyclic fold bifurcation points.

Investigation of the stability of the branch with 1/2 subharmonics in the interval 7.55–8.89 Hz shows that the branch contains quite a number of stable and unstable regions: flip as well as cyclic fold bifurcations are met. In small frequency intervals also 1/4, 1/8 ($n_t = 800$) and 1/16 ($n_t = 1600$) subharmonic branches were calculated, see the inset of Fig. 1. Numerical integration (Runge-Kutta-Merson method, $s_0 = 0$, $\phi_e = 0$, required precision: 10 significant digits, integration time: $10000T_e$) for $f_e = 8.196$ Hz, $f_e = 8.193$ Hz, $f_e = 8.1905$ Hz, $f_e = 8.189$ Hz and $f_e = 8.185$ Hz showed 1/4, 1/8, 1/16, 1/32 subharmonic attractors and a chaotic attractor ($\lambda_1 \approx +0.842$, $\lambda_2 \approx -2.47$), respectively. The results obtained strongly suggest a Feigenbaum-route to chaos, i.e., an infinite cascade of period doublings. If the damping is increased to $\xi = 0.1$ the subharmonic and chaotic solutions disappear and the harmonic solution becomes stable.

The frequency response calculated is similar to frequency responses calculated by Shaw and Holmes (1983), Thompson and Stewart (1986), and Natsiavas (1990). Compared to these studies, here we find an additional branch with 1/2 subharmonics and an extra period doubling route to chaos before the first harmonic resonance peak.

The branch with 1/2 subharmonics in the interval 20.64–38.50 Hz is stable. At $f_e = 22$ Hz, three periodic solutions are found: one harmonic saddle and two 1/2 subharmonic attractors. The two 1/2 subharmonic attractors merge into one another, if one of them is shifted over T_e s. Figure 2 shows the Poincaré section ($\phi_e = 0$) of W^s and W^u, the stable and unstable manifolds of the unstable harmonic at $f_e = 22$ Hz. The Poincaré section is defined as the $2n_q$ dimensional state space, stroboscopically lighted at times $t = (\phi_e/2\pi f_e) + i/f_e$ (i integer). The unstable manifolds connect the harmonic saddle with the two 1/2 subharmonic attractors. The stable manifolds form the separatrices of the domains of attraction of the two 1/2 subharmonic attractors. In the calculation of the stable (unstable) manifolds of the unstable harmonic use is made of that eigenmode of Θ_μ, which corresponds with $|\mu_i| < 1$ ($|\mu_i| > 1$) (Parker and Chua, 1989, Van de Vorst et al., 1993).

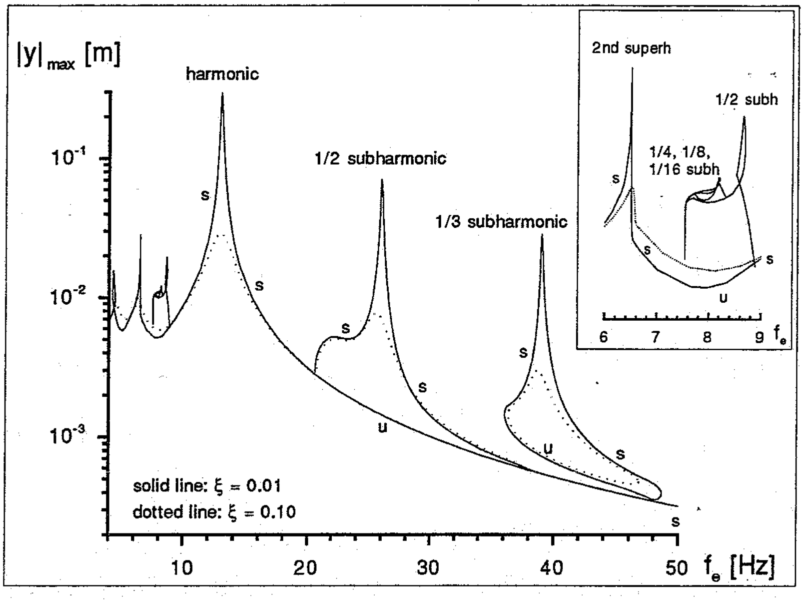

Fig. 1 Amplitude-frequency plot, 1 dof

4.2 Four Dof Model. Figure 3 shows the amplitude-frequency plot of the four dof system (n_t is 400 for the harmonic solutions, 800 for the 1/2 subharmonic solutions and 750 for the 1/3 subharmonic solutions). Globally, Fig. 3 is very similar to Fig. 1. A closer look, however, reveals a number of differences.

First, a large number of superharmonic resonances with moderate to small amplitudes are found in Fig. 3, which are caused by higher bilinear eigenfrequencies ($f_{b2} \approx 82$ Hz). Superharmonic resonances near $1/2\, f_{b2}$, $1/3\, f_{b2}$ and $1/4\, f_{b2}$ are clearly recognized.

Figure 4 shows for $\xi = 0.01$ as well as for $\xi = 0.05$ a small region near 32.5 Hz, where no periodic attractors are found using the finite difference method. For $\xi = 0.05$ numerical integration (variable order, variable step Adams' method, $s_0 = 0$, $\phi_e = -\pi/2$, required precision: 9 significant digits, integration time: $10000T_e$) is applied to investigate the steady-state behavior in this frequency range. Firstly, the stable 1/2 subharmonics for $f_e = 32.6$ Hz, $f_e = 32.58$ Hz and $f_e = 32.56$ Hz (just outside the region) calculated with the finite difference method were verified with numerical integration. All calculations resulted in two points in the Poincaré section, indicating a 1/2 subharmonic solution. The points found with the finite difference method coincided with those found with the numerical integration method. If f_e is further reduced, a cyclic fold bifurcation point is reached and the region without periodic attractors is entered. Figure 5 shows the time history of y resulting from numerical integration for $f_e = 32.55$ Hz in the time interval $t = 6000T_e - 6500T_e$. In a large part of this time interval the solution seems to be a 1/2 subharmonic, see inset 1. Then, suddenly, there appears a burst in the signal for a short period of time, see inset 2 (8 excitation periods), after which the signal recovers again. This type of chaotic behavior is called intermittency. As the chaotic region is entered further, the time intervals between two subsequent bursts become shorter. Eventually the intervals with almost periodic behavior will disappear.

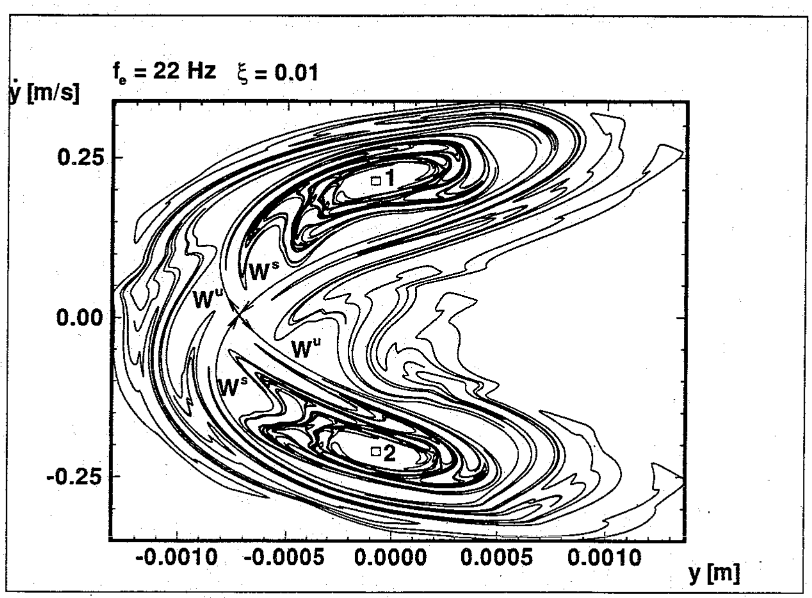

Fig. 2 Stable and unstable manifolds

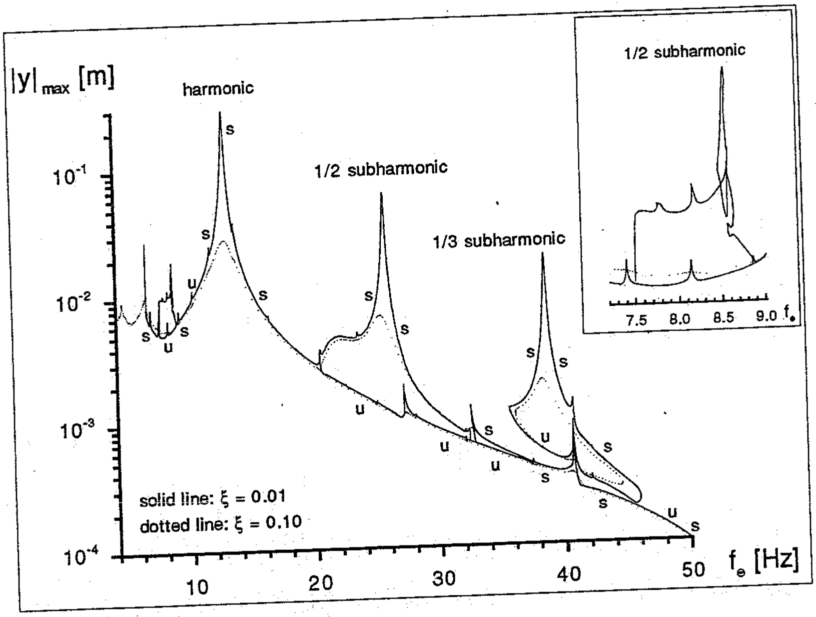

Fig. 3 Amplitude-frequency plot, 4 dof

For $f_e = 50$ Hz there is a large relative difference between the amplitudes of the single dof model (0.3 mm, Fig. 1) and the four dof model (0.1 mm, Fig. 3). This is caused by the anti-resonance near $f_e = 56$ Hz in the four-dof model, which of course does not exist in the single dof model. Before the anti-resonance the branch with harmonic periodic solutions becomes unstable via Neimark bifurcations in the interval 47.33–49.53 Hz for $\xi = 0.05$. In this interval a quasi-periodic → locked → chaotic route is observed, which is described below. Again numerical integration is used to investigate the steady-state behavior (Adams' method, same conditions as in the intermittency investigation). Figure 6 shows the Poincaré section for $f_e = 49.58$ Hz, $f_e = 49.50$ Hz, $f_e = 48.40$ Hz and $f_e = 48.15$ Hz ($\xi = 0.05$). For $f_e = 49.58$ Hz the solution is a stable harmonic resulting in one point in the Poincaré section. If f_e is reduced to 49.50 Hz, the Poincaré section shows a closed curve (i.e., a transection of a 2D torus). Shaw et al. (1989) reported quasi-periodic motion in a two dof system with a cubic stiffening spring for excitation frequencies $f_e \approx (f_1 + f_2)/2$ (f_1 and f_2 being the eigenfrequencies of the system without the nonlinear spring). In fact, this is also the frequency range under consideration here. A reduction of f_e to 49.05 Hz results in a subharmonic solution of order 1/22 (not visible). This phenomenon, in which the ratio of the forced frequency and the free frequency becomes rational, is called frequency-locking or mode-locking. In fact, in the frequency range 47.33–49.53 Hz a very large number of closed branches of subharmonic solutions (locked states), appearing in very small frequency intervals, can be found. For $f_e = 48.70$ Hz the attractor is quasi-periodic again, but the transection of the 2D torus starts to develop wrinkles (not visible). Wrinkles arise in the 2D torus if the onset of chaos is approached (Thompson and Stewart, 1986). A further reduction of f_e to 48.40 Hz again results in a locked state (subharmonic of order 1/10) and finally in a chaotic attractor for $f_e = 48.15$

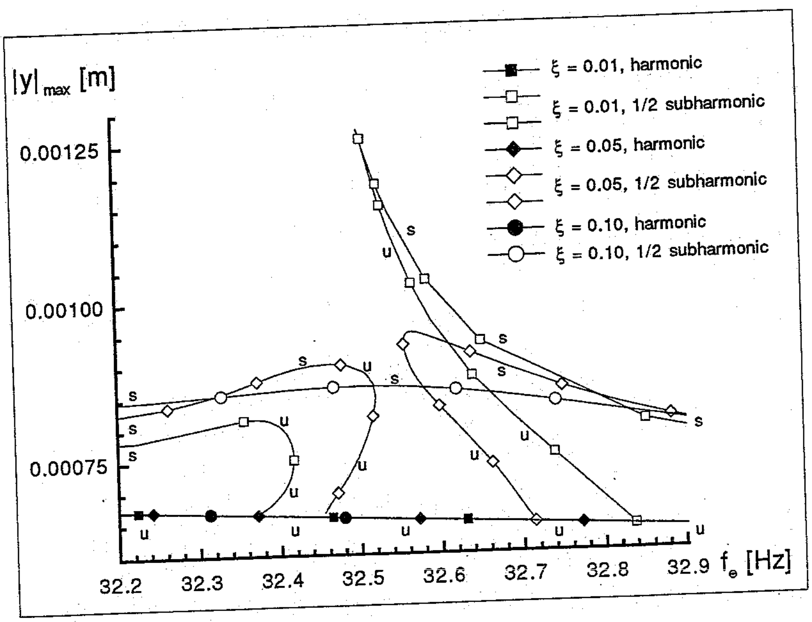

Fig. 4 Gap without periodic attractors

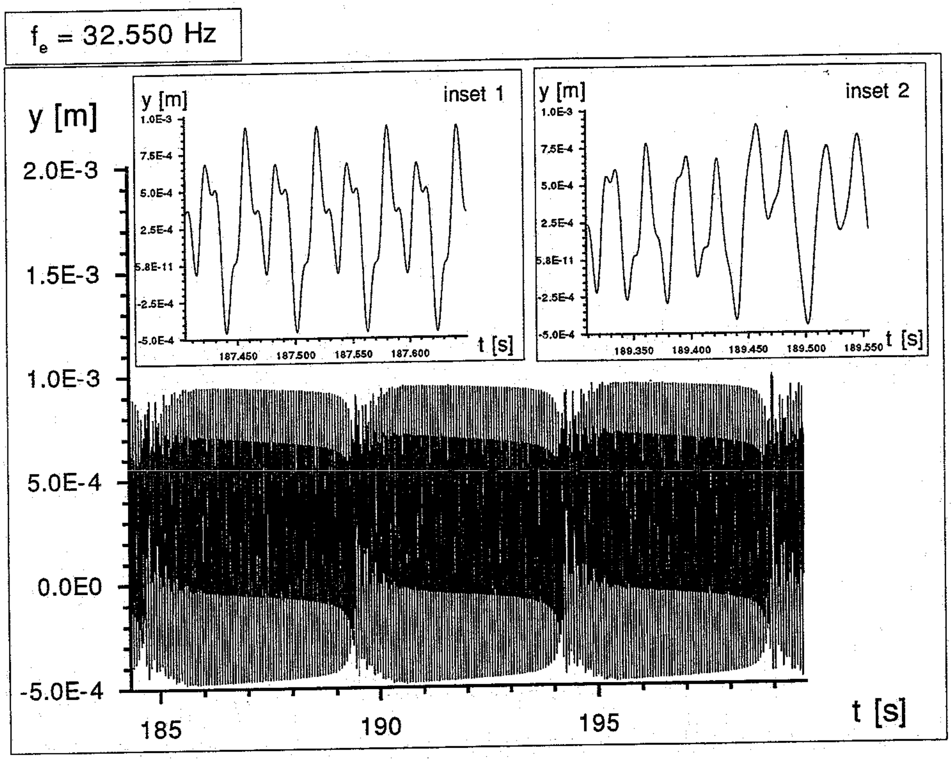

Fig. 5 Intermittency

Hz ($\lambda_1 \approx +3.65$, $\lambda_2 \approx -3.05$, $\lambda_3 \approx -3.97$, $\lambda_4 \approx -4.11$, $\lambda_5 \approx -5.01$, $\lambda_6 \approx -10.7$, $\lambda_7 \approx -10.9$, $\lambda_8 \approx -11.7$).

An eight dof model was used to verify the results of the four-dof model. The differences between the results obtained with the four dof model and the eight dof model appeared to be neglectable, whereas the CPU-time needed for the eight dof model was approximately seven times higher than the CPU-time needed for the four dof model.

5 Conclusions

By application of the finite difference method in combination with the path following method branches of periodic solutions

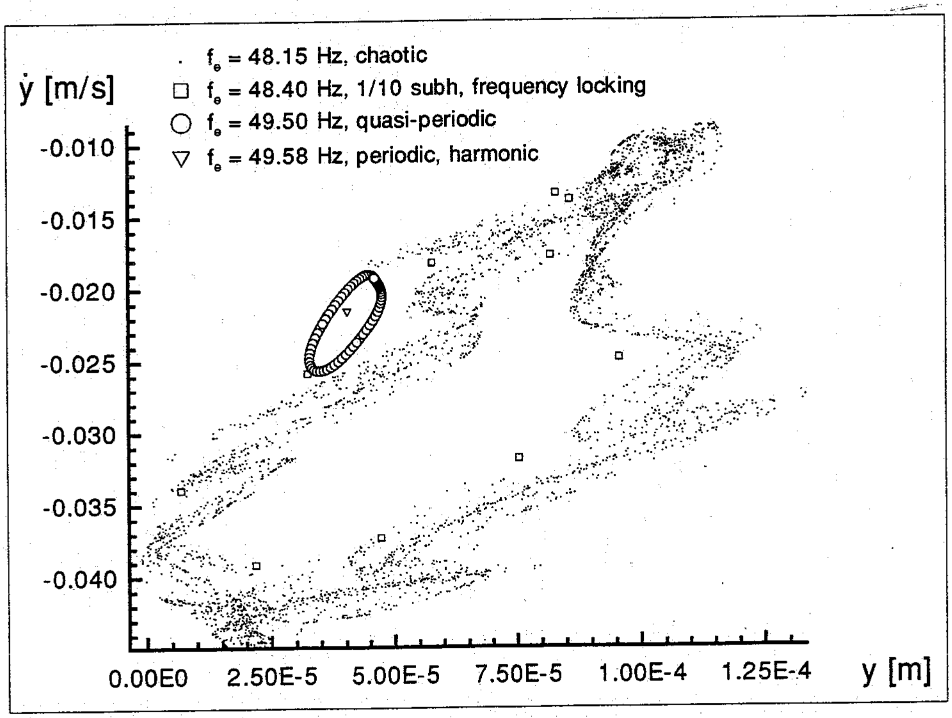

Fig. 6 Quasi-periodic-locked-chaotic route

can be followed for varying design variable. An important advantage of the finite difference method is the fact that stable as well as (very) unstable solutions can be determined easily. By combining these methods with the reduction method the steady-state behavior of complex dynamic systems with local nonlinearities can be analyzed very efficiently. CPU-time consuming numerical integration techniques have to be applied only in those regions of the design variable, where quasi-periodic or chaotic attractors are suspected, e.g., in regions where no stable periodic solutions of a two-point boundary value problem can be found.

The methods mentioned above were successfully applied in the investigation of the long term behavior of a beam system, supported by a one-sided linear spring. Super- and subharmonic resonances were calculated and cyclic fold, flip and Neimark bifurcations were found. In the single dof model, a period doubling route to chaos was observed. In the four dof model in addition an intermittency route and a quasi-periodic-locked-chaotic route were detected. Differences between the steady-state behavior of the four dof model and the eight dof model were neglectable, so one may conclude the four dof model to retain all the essential dynamic characteristics of the system in the frequency range under consideration (say 4–50 Hz). Moreover, a large amount of CPU-time is saved, if the four dof model is used instead of the eight dof model. It seems as if there exists a level of damping for which the single dof model works quite well over the frequency range of interest. Only for very small damping levels do the higher modes play a role, and even then only the fine features of the frequency response are changed. However, this is due to the fact that the eigenmodes of this system are weakly coupled. In general this will not hold for other systems.

References

Craig, Jr., R. R., 1985, "A Review of Time-Domain and Frequency-Domain Component Mode Synthesis Methods," *Combined Experimental/Analytical Modeling of Dynamic Structural Systems Using Substructure Synthesis*, D. R. Martinez and A. K. Miller, ed., pp. 1–31.

de Kraker, A., Fey, R. H. B., van Campen, D. H., and Langeveld C. J., 1989, "Some Aspects of the Analysis of Systems with Local Nonlinearities," *Nonlinear Dynamics in Engineering Systems*, W. Schiehlen, ed., IUTAM Springer-Verlag, pp. 165–172.

DIANA, 1994, *DIANA User's Manual 6.0 edition*, TNO Building and Construction Research, Delft, The Netherlands.

Feigenbaum, M. J., 1983, "Universal Behavior in Nonlinear Systems," *Physica*, Vol. 7D, pp. 16–39.

Fey, R. H. B., 1992, "Steady-State Behaviour of Reduced Dynamic Systems with Local Nonlinearities," Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands.

Fried, I., 1984, "Orthogonal Trajectory Accession to the Nonlinear Equilibrium Curve," *Computer Methods in Applied Mechanics and Engineering*, Vol. 47, pp. 283–297.

Guckenheimer, J., and Holmes, P., 1983, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields," *Applied Mathematical Sciences*, 42, Springer-Verlag.

Natsiavas, S., 1990, "On the Dynamics of Oscillators with Bi-Linear Damping and Stiffness," *Int. Journal of Non-Linear Mechanics*, Vol. 25, pp. 535–554.

Newhouse, S., Ruelle, D., and Takens, F., 1978, "Occurence of Strange Axiom-a Attractors Near Quasi-Periodic Flow on T^m, $m \leq 3$," *Commun. Math. Phys.*, Vol. 64, pp. 35–40.

Parker, T. S., and Chua, L. O., ed., 1989, "Practical Numerical Algorithms for Chaotic Systems," Springer-Verlag.

Pereyra, V., 1966, "On Improving an Approximate Solution of a Functional Equation by Deferred Corrections," *Numerische Mathematik*, Vol. 8, pp. 376–391.

Pomeau, Y., and Manneville, P., 1980, "Intermittent Transition to Turbulence in Dissipative Dynamical Systems," *Commun. Math. Phys.*, Vol. 74, pp. 189–197.

Seydel, R., 1988, *From Equilibrium to Chaos, Practical Bifurcation and Stability Analysis*, Elsevier Science Publishing Co., Inc.

Shaw, J., Shaw, S. W., and Haddow, A. G., 1989, "On the Response of the Nonlinear Vibration Absorber," *Int. Journal of Non-Linear Mechanics*, Vol. 24, pp. 281–293.

Shaw, S. W., and Holmes, P. J., 1983, "A Periodically Forced Piecewise Linear Oscillator," *Journal of Sound and Vibration*, Vol. 90, pp. 129–155.

Thompson, J. M. T., and Stewart, H. B., 1986, *Nonlinear Dynamics and Chaos*, John Wiley & Sons Ltd.

van de Vorst, E. L. B., Fey, R. H. B., van Campen, D. H., and de Kraker, A., 1993, "Manifolds of Nonlinear Single-DOF Systems," *Topics in Applied Mechanics, Integration of Theory & Applications in Applied Mechanics*, J. F. Dijksman and F. T. M. Nieuwstadt, ed., Kluwer Academic Publishers, pp. 293–303.