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FINITE SUBGROUPS OF $G_2(\mathbb{C})$

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0. Introduction.

The octaves or real Cayley numbers have been studied intensively since their appearance in the Literature (see [2] for an overview and further references). The automorphism group of the real octaves is the Lie group $G_2(\mathbb{R})$ as demonstrated in [13], [19] and elsewhere. However, no list of conjugacy classes of its finite subgroups, known to the authors, is present in the literature. These finite groups of octave automorphisms are classified in this paper.

Heavy use is made of the natural seven-dimensional representation of such a group. The exposition employs basic techniques of ordinary and modular character theory. The deep results of the classification of simple groups are not used. Many of the techniques of [22] are used but the main result is not employed. Our purpose is to provide an argument using results about the Lie group $G_2(\mathbb{R})$ and basic facts about representations of finite groups.

As any finite subgroup of the complex Lie group $G_2(\mathbb{C})$ is compact, it is contained in a maximal compact subgroup of $G_2(\mathbb{C})$ and hence is conjugate to a subgroup of $G_2(\mathbb{R})$ (cf.[20]). Thus the classification of finite automorphism groups of the real octaves coincides with the classification of the finite subgroups of $G_2(\mathbb{C})$.

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1. <u>Fundamental Notions and Notations.</u>

We let $\mathbb{O}$ be the real division algebra of the octaves. This algebra is nonassociative and has a basis $e_0 = 1, e_1, e_2, \ldots, e_7$ over $\mathbb{R}$ such that its multiplication is determined by the rules $e_i^2 = -1$ and $e_ie_j = e_k$ whenever (ijk) is one of the 3-cycles $(1+r, 2+r, 4+r)$ (i,j,k,r running through the integers mod 7 and all values taken in $\{1,2,\ldots,7\}$). For details concerning $\mathbb{O}$ the reader is referred to [16] or [19].

We need the anisotropic nondegenerate quadratic form Q on $\mathbb{O}$ given by

$$Q\left(\sum_{i=0}^{7} e_i\alpha_i\right) = \sum_{i=0} \alpha_i^2 (\alpha_i \in \mathbb{R}).$$

This form is multiplicative: $Q(xy) = Q(x)\,Q(y) (x,y \in \mathbb{O})$. The bilinear form $(.|.)$ corresponding to Q is given by $2(.|.) = Q(x+y) - Q(x) - Q(y)$. It is used to define the involutory anti-automorphism $x \to \tilde{x}$ on $\mathbb{O}$ in the following way: $\tilde{x} = (x|1)1-x$. We regard the field $\mathbb{R}$ as a subfield of $\mathbb{O}$ by means of the natural embedding $\mathbb{R} \to \mathbb{R}.1$.

Any automorphism of $\mathbb{O}$ preserves the bilinear form $(.|.)$ as well as the trilinear form f given by $f(x,y,z) = (xy|z) (x,y,z, \in \mathbb{O})$. As it must fix 1, such an automorphism may be viewed as an orthogonal transformation on $\mathbb{R}^{\perp}$ stabilizing f. On the other hand, it is easily derived from the nondegeneracy of Q that any orthogonal transformation g on $\mathbb{R}^{\perp}$ stabilizing f can be extended uniquely to an automorphism of $\mathbb{O}$ by $g(\alpha+x) = \alpha + g(x)$ $(\alpha \in \mathbb{R}, x \in \mathbb{R}^{\perp})$. Thus the automorphism group $\mathrm{Aut}(\mathbb{O})$ of $\mathbb{O}$ can be identified with the subgroup of the 7-dimensional linear orthogonal group $O(\mathbb{R}^{\perp})$ stabilizing a particular alternating trilinear form, namely $f|\mathbb{R}^{\perp}$.

There is another interpretation of $\mathrm{Aut}(\mathbb{O})$ that we shall frequently employ, namely the above mentioned isomorphism with $G_2(\mathbb{R})$. This leads for instance to the existence of a maximal torus T consisting of all automorphisms $t_{\theta,\eta} (\theta, \eta \in \mathbb{R})$ where $t_{\theta,\eta}$ has matrix

$$(1.1)\quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\eta & 0 & 0 & 0 & \sin\eta \\ 0 & -\sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta+\eta) & -\sin(\theta+\eta) & 0 \\ 0 & 0 & 0 & 0 & \sin(\theta+\eta) & \cos(\theta+\eta) & 0 \\ 0 & 0 & -\sin\eta & 0 & 0 & 0 & \cos\eta \end{pmatrix}$$

with respect to $e_1, \ldots, e_7$.

Moreover, the normalizer N of T inside $\mathrm{Aut}(\mathbb{O})$ is the semi-direct product of T and the dihedral group W of order 12 generated by the automorphisms

$$a = \delta^-_{\{1,4,6,7\}}(235)(476)$$

and

$$b = \delta^-_{\{2,3,4,5,6,7\}}(23)(47)$$

Here, the notation is the following: A permutation π on 7 letters stands for the linear transformation determined by $e_i \to e_{\pi(i)} (1 \leq i \leq 7)$, and δ^-_K for K a subset of $\{1,2,\ldots,7\}$ stands for the linear map sending e_i to $-e_i$ whenever $i \in K$ and fixing e_i if $i \notin K$.

Extending scalars to the complex numbers, we obtain the complexified algebra $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ for which extensions of Q, $(.|.)$, $\sim$, and f can be defined without difficulty. We shall not distinguish the notation for these operations and their extensions. Note that Q is still nondegenerate, but no longer anisotrophic. Defining complex conjugation $x \mapsto \overline{x}$ on $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ as usual by

$$\sum_{i=0}^{7} e_i\alpha_i \to \sum_{i=0}^{7} e_i\overline{\alpha}_i (\alpha_i \in \mathbb{C}),$$

we regain $\mathbb{O}$ from $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ as the set of fixed points with respect to complex conjugation.

Similarly to what we have seen for $\mathrm{Aut}(\mathbb{O})$, the group $\mathrm{Aut}(\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C})$ can be identified with both $G_2(\mathbb{C})$ and the group of transformations of $\mathbb{C}^{\perp}$, orthogonal with respect to Q and preserving f. Clearly, $\mathrm{Aut}(\mathbb{O})$ is the subgroup of $\mathrm{Aut}(\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C})$ consisting of all real transformations.

2. Some Properties of Finite Subgroups of $\mathrm{Aut}(\mathbb{O})$.

The above indicated interpretations of $\mathrm{Aut}(\mathbb{O})$ lead to a number of properties stated in the next three lemmas. The notation is as in the preceding section. Moreover C_m^ℓ for ℓ, $m \in \mathbb{N}$ denotes the direct product of ℓ copies of the cyclic group C_m of order m.

Lemma 1 (Borel-Serre). If G is a finite nilpotent subgroup of $\mathrm{Aut}(\mathbb{O})$, then

(i) G is conjugate to a subgroup of the normalizer N of T;

(ii) If $G \cong C_p^\ell$ for some prime p and a natural number ℓ, then either $\ell = 3$ and $p = 2$, or $\ell \leq 2$.

Proof. See [4] for the proof of (i) as well as for the proof of $\ell \leq 3$, and $\ell = 2$ if $p \neq 2, 3$ in case $G \cong C_p^\ell$ as in (ii). It remains to establish that $G \cong C_3^3$ does not occur inside $G_2(\mathbb{R})$. If such a subgroup G exists, then up to conjugacy we may assume that G is in N and G contains $a^2 = (253)(467)$. Now $t_{\theta,\eta}$ is centralized by a^2 if and only if $t_{\theta,\eta} = t_{\eta,-\theta-\eta} = t_{-\theta-\eta,\theta}$. Thus $t_{\theta,\eta}$ belongs to G whenever $t_{\theta,\eta}$ is a power of $t_{2\pi/3,2\pi/3}$. It follows that the number $|G/G\cap T|$ is a multiple of 3^2, which contradicts that $|G/G\cap T|$ divides $|N/T| = 2^2 \cdot 3$. This finishes the proof. □

Lemma 2. Let G be a finite subgroup of $\mathrm{Aut}(\mathbb{O})$ and let χ be the character of G on $\mathbb{R}^\perp$ (so $\chi(g) = \mathrm{trace}(g|_{\mathbb{R}^\perp})$ for $g \in G$). The following holds:

(i) If χ is irreducible, then $\nu(\chi) = 1$, where

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2);$$

(ii) $(\chi^{[3]}|1) \geq 1$, where $\chi^{[3]}$ is the alternating part of χ^3, i.e.

$$\chi^{[3]}(g) = \frac{\chi(g)^3 + 2\chi(g^3) - 3\chi(g^2)\chi(g)}{6} \quad (g \in G);$$

(iii) If p is a prime $\neq 2, 3$ dividing $|G|$ and S is a p-Sylow subgroup of G, then S is abelian, and conjugate to a subgroup of T;

(iv) If $g \in G$ is order m, then there are k, $\ell \in \mathbb{Z}$ such that

$$\chi(g) = 1 + 2\cos\frac{2\pi k}{m} + 2\cos\frac{2\pi \ell}{m} + 2\cos\frac{2\pi(k+\ell)}{m};$$

(v) If $g \in G$ is of order 2, then $\chi(g) = -1$;

(vi) If $g \in G$, then $\det(g) = 1$ (i.e. G is unimodular);

(vii) If χ is irreducible, then the centre $Z(G)$ of G is trivial.

Proof. (i) follows from the fact that the restriction to $\mathbb{R}^{\perp}$ is a real representation of G.

(ii) expresses the fact that G preserves the alternating trilinear form f restricted to $\mathbb{R}^{\perp}$.

(iii) is a direct consequence of Lemma 1 (i), as S is nilpotent.

(iv) it is well-known from the theory of Lie groups that any $g \in G$ (being semi-simple) is conjugate to an element of T (see for instance [4]).

(v), (vi) are obtained by application of (iv) to g.

(vii) as a consequence of the irreducibility of G, any element of $Z(G)$ has only one eigenvalue (disregarding multiplicities). By (iv), this eigenvalue must always be 1. □

Lemma 3. Let G, χ be as in Lemma 2. If $(\chi^{[3]}|1) = 1$, then the conjugacy class of G within $\mathrm{Aut}(\mathbb{D})$ coincides with the intersection of $\mathrm{Aut}(\mathbb{D})$ and the conjugacy class of G within the full linear group of $\mathbb{C}^{\perp}$.

Proof. Suppose G, H are subgroups of $\mathrm{Aut}(\mathbb{D})$ and the linear transformation t on $\mathbb{C}^{\perp}$ satisfies $tGt^{-1} = H$. Then a standard argument shows that we may assume that t is orthogonal and has determinant 1. Now $(xy|z)$ and $((tx)(ty)|tz)$ are both trilinear alternating forms on $\mathbb{C}^{\perp}$, so by hypothesis $(\chi^{[3]}|1) = 1$ the one is a scalar multiple, say α, of the other:

$$((tx)(ty)|tz) = \alpha(xy|z).$$

Extending t to $\mathbb{D}$ by prescribing $t1 = 1$, we get $(\alpha xy|z) = ((tx)(ty)|tz) = (t^{-1}((tx)(ty))|z)$, and by the nondegeneracy of Q,

$$\alpha t(xy) = (tx)(ty).$$

Substitution of $x = y = 1$ yields $\alpha = 1$ and $t \in \mathrm{Aut}(\mathbb{D})$. □

Whenever in the sequel the terms *(ir) reducibility* and *(im) primitivity* are used with respect to a subgroup G of $\mathrm{Aut}(\mathbb{D})$, they are meant to pertain to G viewed as a linear group on $\mathbb{C}^{\perp}$.

The study of subgroups G of $\mathrm{Aut}(\mathbb{O})$ is divided into three cases, according as G is reducible, imprimitive (and irreducible), primitive.

3. Reducible Groups.

We shall now outline the structure of a finite group G of automorphisms of $\mathbb{O}$ stabilizing in its action on $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ a nontrivial linear subspace $\mathbb{C}^{\perp}$ of dimension, say $m (1 \leq m < 7)$. If $m \geq 4$, there must be another linear subspace of $\mathbb{C}^{\perp}$ of dimension ≤ 3 left invariant by G, so without loss of generality we may assume $m \leq 3$.

We contend that $\mathbb{C}^{\perp}$ contains a G-invariant linear subspace of dimension 3. This is obviously true if all irreducible constituents of G have degree 1. Suppose that U is a linear subspace of dimension 2 on which G acts irreducibly. Then the $\mathbb{C}^{\perp}$-part of the subalgebra spanned by U is G-invariant. If $U = \overline{U}$ (i.e. U is real), then this subalgebra is 3-dimensional. As $U \cap \overline{U}$ is a G-invariant subspace of U, we are left with $U \cap \overline{U} = \{0\}$. Then $U + \overline{U}$ is 4-dimensional and G-invariant, so its orthoplement $(U + \overline{U} + \mathbb{C})^{\perp}$ is as required.

Let U be a 3-dimensional G-invariant linear subspace of $\mathbb{C}^{\perp}$. Assume, first of all, that $U + \mathbb{C}$ is a quaternion subalgebra of $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. From [19], it follows that G is a finite subgroup of the semidirect product of $\mathrm{Aut}(U)$ with the norm-1 subgroup $U_1(U)$ of the multiplicative group of U consisting of all elements $x \in U$ with $Q(x) = 1$. Here $\mathrm{Aut}(U)$ acts on $U_1(U)$ in its natural representation.

We next claim that if G does not stabilize a real quaternion subalgebra it must fix a 1-dimensional real subspace of $\mathbb{R}^{\perp}$.

Suppose not; then, as G stabilizes $U \cap \overline{U}$, the dimension of $U \cap \overline{U}$ cannot be 1. So if $U \cap \overline{U}$ is a nontrivial subspace of U, its dimension is 2. But then $U \cap \overline{U}$ generates a real quaternion subalgebra invariant under G. Hence either $U \cap \overline{U} = \{0\}$ or $U = \overline{U}$. Define V to be $U + \overline{U}$ in the first case and $U + \sum_{x,y \in U} xy\mathbb{C}$ in the second case. Then V is a

6-dimensional real vector space invariant under G and thus admits a 1-dimensional real orthoplement in $\mathbb{C}^{\perp}$. This establishes the claim.

Since $\mathrm{Aut}(\mathbb{O})$ is transitive on the points of $\mathbb{R}^{\perp}$, we may assume without harming the generality that G leaves invariant $e_1\mathbb{R}$, so that G is contained in the stabilizer of $\{\pm e_1\}$ within $\mathrm{Aut}(\mathbb{O})$. It is well known, however, that the stabilizer of e_1 in $\mathrm{Aut}(\mathbb{O})$ is isomorphic to $SU_3(\mathbb{C})$. It is easily derived that the stabilizer of $\{\pm e_1\}$ is isomorphic to $SU_3(\mathbb{C})$ extended (as a semi-direct product) by complex conjugation. The possible G_{e_1}, being isomorphic to subgroups of $SU_3(\mathbb{C})$, can be read from [1]. It is straightforward to determine all possible extensions of G_{e_1} of degree 2 within the semidirect product of $SU_3(\mathbb{C})$ and the group generated by complex conjugation. We shall not bother to write them down. The results of this section are summarized in the next theorem.

Theorem. Any finite subgroup G of $\mathrm{Aut}(\mathbb{O})$ that is reducible in $\mathbb{C}^{\perp}$ is isomorphic to a subgroup of either (i) the semidirect product of $U_1(\mathbb{H})$ by $\mathrm{Aut}(\mathbb{H})$, where $U_1(\mathbb{H})$ is the group of elements having norm 1 in the real quaternion division algebra $\mathbb{H}$; or (ii) the semidirect product of the special unitary group $SU_3(\mathbb{C})$ by the group of order 2 generated by complex conjugation.

4. Imprimitive Groups.

Suppose for the duration of this section that G is a finite irreducible imprimitive subgroup of $\mathrm{Aut}(\mathbb{O})$.

The blocks of a system of imprimitivity for G must have dimensions 1, so the stabilizer H of G of all the blocks in such a system is abelian.

By Lemma 1 this group H is up to conjugacy of G contained in the normalizer N of the torus T (notation of Section 2). As N stabilizes $\{\pm e_1\}$, the complement K of the 2-Sylow subgroup of H has a nonzero fixed space. But K is normal in G, so this space is G-invariant and thus equal to the whole space $\mathbb{R}^{\perp}$ by the irreducibility of G. Thus $K = 1$ and H is a 2-group. Inspection of the character of H on $e_1\mathbb{R}$ yields that $H \cong C_2^m$ for some $m \in \mathbb{N}$. Lemma 1 yields that $m \leq 3$, so that according to Clifford's theorem $m = 0$ or $m = 3$.

Suppose that $H \cong C_2^3$. If $g \in G$ would centralize H, then g would leave invariant all the blocks of the system of imprimitivity (because the characters of H on any two of these blocks differ) and g would belong to H. Therefore G/H is isomorphic to a subgroup of $\mathrm{Aut}(C_2^3) \cong PS\ell_2(7)$. On the other hand, if $H = 1$, then G is isomorphic to a transitive subgroup of the symmetric group $\mathrm{Sym}(7)$ on 7 letters. It is a well-known fact that $PS\ell_2(7)$ is the only subgroup of $\mathrm{Sym}(7)$ having an irreducible representation over $\mathbb{C}$ of degree 7. This implies that $G \cong PS\ell_2(7)$. The conclusion is stated in the following theorem.

Theorem. If G is a finite irreducible imprimitive subgroup of $\mathrm{Aut}(\mathbb{O})$, then either G is isomorphic to $PS\ell_2(7)$ or G has a normal subgroup H isomorphic to C_2^3 such that G/H is isomorphic to a subgroup of $PS\ell_2(7)$.

Examples. In fact both possibilities occur. H.S.M. Coxeter [9] exhibited a nonsplit extension of C_2^3 by $PS\ell_2(7)$ inside $\mathrm{Aut}(\mathbb{O})$. In the notation of section 1 the generators for such a group may be taken to be (1234567), $(124)(365)$, $\delta^-_{\{1,2,4,7\}}(12)(36)$.

The existence of an irreducible subgroup of $\mathrm{Aut}(\mathbb{O})$ isomorphic to $PS\ell_2(7)$ will follow from the result in the next section, where the group $PG\ell_2(7)$ containing $PS\ell_2(7)$ is proved to be isomorphic to an irreducible subgroup of $\mathrm{Aut}(\mathbb{O})$. Note that the group isomorphic to $PS\ell_2(7)$ must be irreducible as the index of $PS\ell_2(7)$ in $PG\ell_2(7)$ is 2. There is only one irreducible representation of $PS\ell_2(7)$ of degree 7. A straightforward computation using character values shows that $(\chi^{[3]}|1) = 1$ where χ is the character. Using lemma 3, it can be seen that any $PS\ell_2(7)$ subgroup of $\mathrm{Aut}(\mathbb{O})$ is contained in a unique $PG\ell_2(7)$ of $\mathrm{Aut}(\mathbb{O})$.

5. Primitive Groups.

Suppose in this section that G is a finite irreducible subgroup of $\mathrm{Aut}(\mathbb{O})$ which is also primitive. It is possible to utilize the result of [22] to obtain the possibilities for G. This result lists all finite quasi-primitive complex unimodular linear groups of degree 7 up to conjugacy. As described in Section 2, there are conditions imposed on G by being a subgroup of $\mathrm{Aut}(\mathbb{O})$ which are more stringent than those imposed by being a subgroup of $GL(7,\mathbb{C})$. Rather than use [22], we make use of these and the techniques of [22] to determine the possible groups G. These are described in the following theorem.

Theorem. There are five conjugacy classes of finite primitive subgroups of $\mathrm{Aut}(\mathbb{O})$. Such a subgroup is isomorphic to $PSL_2(13)$, $PSL_2(8)$, $PGL_2(7)$, $U_3(3)$, or $G_2(2)$.

Proof. Each of the five groups listed in the statement of the theorem has an irreducible character χ of degree 7 satisfying $(\chi|\overline{\chi}) = (\chi^{[3]}|1) = 1$, as is easily verified by use of the formula in lemma 2(ii). If χ is such a character, the corresponding representation is an embedding in $\mathrm{Aut}(\mathbb{O})$ (up to conjugacy) by the fact that $\mathrm{Aut}(\mathbb{O})$ represents the unique conjugacy class of subgroups of $O_7(\mathbb{R})$ that stabilize a trilinear alternating form on $\mathbb{R}^7$ and are irreducible in their natural representations. This result is a direct consequence of the classification of alternating trilinear forms on $\mathbf{R}^7$ by J. A. Schouten [17].

Embeddings of the five groups in $\mathrm{Aut}(\mathbb{O})$ can be given explicitly. For $S\ell_2(8)$ and $PS\ell_2(13)$ this is done at the end of this section. As to $PG\ell_2(7)$, $U_3(3)$ and $G_2(2)$, they are all three contained in $G_2(2)$, which is known to be the full group of automorphisms of the integral octaves [3]. In view of Lemma 3, it suffices to prove the second statement of the theorem.

Let G be finite primitive subgroup of $\mathrm{Aut}(\mathbb{O})$ and let χ be the character of its action on $\mathbb{R}^{\perp}$. As χ is irreducible, all parts of Lemma 2 apply. In particular, G is unimodular and has a trivial center. As G is primitive with trivial center, it can have no normal abelian subgroup as the eigenspaces would be permuted in blocks of imprimitivity. As in [22], the analysis proceeds by restricting the possible primes in the group order, $|G|$. A central tool will be the special restrictions placed on the character degrees of G by the fact that G will have a cyclic subgroup of prime order.

As follows from [7-2b and 2c], no prime larger than 7 could divide $|G|$ unless $G \cong PSL_2(13)$. This is one of the groups mentioned in the Theorem. Of course $7 \,|\, |G|$ as G has an irreducible representation of degree 7.

From now on we may assume that 2, 3, 5, 7 are the only possible primes in the order of G. By [7-8a], G has a simple commutator subgroup G^1. Notice that G^1 is nonabelian and that χ restricted to G^1, denoted $\chi|G^1$, is irreducible. Suppose the theorem has been established for simple groups. Then either G^1 is imprimitive and $G^1 \cong PS\ell_2(7)$ by Section 4, or $G^1 \cong PS\ell_2(8)$, $U_3(3)$. Reasoning as in [22, p. 233] readily yields that $G \cong G^1$, $PG\ell_2(7)$ or $G_2(2)$.

It remains to prove the theorem for simple G of order $2^a3^b5^c7^d$ (a,b,c,d nonnegative integers). We shall do so in 6 steps.

(1) We first show that $|G| = 2^a.3^b.5^c.7$ with $a \leq 7$, $b \leq 4$, $c \leq 2$. If $\pi \in G$ is of order 7, then $\chi(\pi) = 0$ and $C(\pi) = \langle\pi\rangle$, where $C(\pi)$ is the centralizer of π in G.

To begin, let P be a Sylow 7-subgroup of G. As G has a character of degree 7, 7 divides $|G|$ and so P is not trivial. By Lemma 2 (iii), P is abelian. If g is in P, $\frac{\chi(g)}{c(g)} \frac{|G|}{7}$ must be an algebraic integer where $c(g)$ is the order of the centralizer of g (see [10, Section 33]). As P is abelian, the powers of 7 dividing $c(g)$ and $|G|$ are identical and so $\frac{\chi(g)}{7} v$ must be an algebraic integer with $(v,7) = 1$. In turn, $\frac{\chi(g)}{7}$ must be an algebraic integer. The only possibility for this using Lemma 2 (iv) for nontrivial g is for $\chi(g) = 0$. This occurs for example when g has order 7, $k = 1$, and $\ell = 2$. This means $\chi|P$ vanishes except at the identity and $\chi(1) = 7$. However, as $\frac{1}{|P|} \sum_{g \in P} \chi(g)$ must be an integer [10,(2.9)], $|P|$ has order 7. This shows P is cyclic of order 7. Let π be a generator for P. We have shown $\chi(\pi) = 0$.

There are several applications of ordinary and modular character theory which will be utilized to analyze G. These will be described as they are used. The first application of modular theory is reduction mod p. Let p be any prime. It is possible to change the basis of $R^{\perp}$ to obtain a group for which the coefficients are all p-local integers in some algebraic extension of $\mathbb{Q}$ [10, Chapter 10]. If the coefficients are all reduced mod p we obtain a homomorphism of G to a subgroup of $GL_7(F)$ where F is a finite field of characteristic p. The kernel of this map is a normal p-group. As G is simple, this can only be the identity. Suppose τ is an element centralizing

π with τ of order p prime to 7. Reduce G mod p. The element π has distinct eigenvalues. As τ centralizes π, τ must be diagonal over an extension field of F containing the eigenvalues of π. But τ has order p and so τ is I. This means τ is the identity in G as well. This argument shows $C(\pi) = \langle\pi\rangle$ where $C(\pi)$ is the centralizer of π.

Let Q be a Sylow 5-group of G and suppose $5^3 \mid |Q|$. By reducing mod 5, using the primitivity, and considering the Jordan Normal Form of the reduced elements, it can be seen as in [7-3B] that elements of Q have order five or twenty-five. The form of (1.1) shows Q has rank two with an element $\sigma = t_{\theta,\eta}$ for which $\theta = 2\pi/25$. The basis can be changed so that the matrices of Q are all of the form $t_{\theta,\eta}$. As Q is not cyclic, there must be an element γ of the form $t_{0,\eta'}$. If γ has order 25, a power has $\eta' = 2\pi/25$. This element has 1 as an eigenvalue and all other eigenvalues are within $2\pi/6$ radians of this. This contradicts Blichfeldt's Theorem [1, p. 96] and shows $|Q| = 125$ with $Q \cong \langle\sigma\rangle \times \langle\gamma\rangle$. If η were 1 or a fifth root of 1, an element $\sigma\gamma^j$ for some j would be $t_{\theta,0}$ which would contradict Blichfeldt's Theorem. This means η is a primitive twenty-fifth root of 1. By replacing σ by $\sigma\gamma^k$ for suitable k, η can be chosen to be θ^{-j} for $j = 1, 2, 3$, or 4. This element contradicts Blichfeldt's Theorem as all eigenvalues are within $8\pi/25$ from 0 and $8\pi/25 \leq \pi/3$. We have shown that $|Q| \leq 5^2$, i.e. $c \leq 2$.

We now analyze the Sylow 3-group of G. Let R be a Sylow 3-group. It follows from Lemma 1 that R has an abelian subgroup A of index at most 3 which lies in T. Reducing mod 3 and using Jordan Canonical Form shows that elements of A have order at most 9. If $|A| = 3^4$, there would be an element contradicting Blichfeldt's Theorem [1, p. 96] with $\theta = 2\pi/9$, $\eta = 0$. This means $|A| \leq 3^3$ and $|R| \leq 3^4$. This means $b \leq 4$.

Let S be a Sylow 2-group of G and set $B = S \cap T$. The rank of B is at most 2 and elements of B have order at most 8 as in our discussions of Sylow groups for 3 and 5. If $|B| = 2^6$, there would be an element contradicting Blichfeldt's Theorem with $\theta = \pi/4$ and $\eta = 0$. This means $|B| \leq 2^5$ and so $|S| \leq 2^7$. In particular $a \leq 7$.

(2) Analysis of the principal 7-block $B_0(7)$.

The remaining analysis depends heavily on the theory of groups whose order is divisible by a prime to the first power [5]. This is the case for $p = 7$. Let $s = |N(\langle\pi\rangle)/\langle\pi\rangle|$ where $N\langle\pi\rangle$ is the normalizer of $\langle\pi\rangle$. The possible values of s are 1, 2, 3, 6. If $s = 1$, $N(\langle\pi\rangle) = \langle\pi\rangle$ and by Burnside's Theorem, G has a normal 7-complement K, see [12]. As $7 \nmid |K|$, Clifford's Theorem [10, Section 49] implies K is abelian and so is trivial. This shows $s = 2, 3$, or 6.

The irreducible characters of G whose degrees are prime to 7 constitute the principal 7-block $B_0(7)$. They can be described as follows [7, Section 8]. There are $s + 6/s$ such characters; $\chi_1, \chi_2, \ldots, \chi_s$ and $\chi_0^1, \ldots, \chi_0^t$ where $t = 6/s$. Assume χ_1 is the trivial character of degree one. The characters $\chi_1, \ldots, \chi_s$ have degrees congruent to $\pm 1 \bmod 7$; the characters $\chi_0^1, \ldots, \chi_0^t$ all have the same degree congruent to $\pm s \pmod 7$. If $\chi_i(1) \equiv \varepsilon_i \pmod 7$ and $\chi_0^1(1) \equiv \delta s \pmod 7$ where ε_i and δ are ± 1, the following equation holds for all elements σ in G of order prime to 7 for all $j = 1, 2, \ldots, t$:

$$\varepsilon_1\chi_1(\sigma) + \varepsilon_2\chi_2(\sigma) + \cdots + \varepsilon_s\chi_s(\sigma) + \delta\chi_0^j(\sigma) = 0. \tag{5.1}$$

If σ is the identity, this is called the *degree equation.*

The values of $\chi_i(\pi)$ are ε_i. The values of $\chi_0^j(\pi)$ are algebraic conjugates of $-\delta(\varepsilon + \varepsilon^{-1})$ if $t = 3$, $-\delta(\varepsilon + \varepsilon^2 + \varepsilon^4)$ if $t = 2$, or δ if $t = 1$ where $\varepsilon = \varepsilon^{2\pi i/7}$. Note $\sum_{j=1}^{t} \chi_0^j(\pi) = \delta$. It follows from (5.1) that $\chi_0^j(\sigma)$ is independent of j.

The character χ is real valued and can be realized as a representation over the reals by Lemma 2. This means $\chi\bar{\chi} = \chi^2 = \chi^{(2)} + \chi^{[2]}$ where $\chi^{(2)}$ is the symmetric part of χ^2 and $\chi^{[2]}$ is the antisymmetric part. As χ can be realized over the reals, $\chi^{(2)}$ contains χ_1 as a constituent with multiplicity one. As $\chi^{(2)}$ has degree 28, it contains some irreducible constituent, say χ_i, with $\varepsilon_i = -1$ or all χ_0^j and $\delta = 1$, [7, Section 8]. The possibilities for such $\chi_i(1)$ are 6, 20, 27. The possibilities for $\chi_0^j(1)$ are 3 or 10 if $s = 3$ and 2 or 9 if $s = 2$. Groups with representations of degree 2 and 3 have been determined [1]. Neither χ_i nor χ_0^j could have nontrivial kernel by the simplicity of G. The only possibility is $G \cong PSL_3(2)$.

This was handled in Section 4. There is a representation of degree 7 but it is imprimitive. It can be assumed, then, that if χ_0^j occurs as a constituent of $\chi^{(2)}$, the degree is 9 or 10.

(3) $|G|$ is not divisible by 25.

If Q has order 5^2, Q is either cyclic and generated by an element $\sigma = t_{\theta,\eta}$ with $\theta = 2\pi/25$ or is generated by elements σ and γ of the form

$$\sigma = t_{\theta,0} \tag{5.2}$$

$$\eta = t_{0,\theta} \quad \text{with} \quad \theta = 2\pi/5.$$

Suppose first that Q is cyclic of order 25 generated by $\sigma = t_{\theta,\eta}$ as above. If σ had a multiple root, the argument would have to be 0, θ, or η and σ itself would contradict Blichfeldt's Theorem. It follows that the eigenvalues of σ are all distinct. By reducing mod p for various primes dividing $|C(\sigma)|$ and using the primitivity, we see $C(\sigma) = \langle\sigma\rangle = Q$. We now use the results of Dade [11] describing the principal 5-block of G. If $e = |N(\langle\sigma\rangle)/C(\sigma)|$, this consists of $24/e$ exceptional characters of the same degree and e distinct characters with degree congruent to $\pm 1 \pmod{25}$. The sum of the exceptional character degrees is congruent to $\pm 1 \pmod{25}$. All characters of degree prime to 5 are in this block. In particular the trivial character, χ_1, of degree 1 is in this block as are the characters of degree 7. As 7 is not congruent to $\pm 1 \pmod{25}$, these must be the exceptional characters. As e must be 1, 2, or 4, the sum of the exceptional degrees is not congruent to $\pm 1 \pmod{25}$. (Here e is 1, 2, or 4 as $N(\langle\sigma\rangle)/C(\sigma)$ is isomorphic to a subgroup of the automorphism group of $\langle\sigma\rangle$ of order prime to 5). This shows Q is not cyclic of order 25.

Suppose that Q has order 25 and so is isomorphic to $\langle\sigma\rangle \times \langle\eta\rangle$ as in (5.2). Note $\chi(\sigma) = 3 + 4\cos\theta$. Evaluating $\chi^{(2)}(\sigma) = \frac{1}{2}((\chi(\sigma))^2 + \chi(\sigma^2)) = 7 + 6\cos\theta$. As was shown above, $\chi^{(2)}$ has χ_i or χ_0^j as a constituent. We consider the various possibilities for χ_i or χ_0^j.

If $\chi^{(2)} = \chi_1 + \chi_i$ with χ_i of degree 27, then $\chi_i(\sigma) = 6 + 6\cos\theta$. This contradicts Tuan's Theorem [21, Theorem C] which states that the character values, $\chi_i(\sigma)$, must be in the prime field when reduced (mod 7.). This is not the case as $2\cos\theta = \frac{-1+\sqrt{5}}{2}$ is in a quadratic extension of the prime field with seven elements as it is a root of $X^2 + X - 1$.

Suppose that $\chi^{(2)}$ has a constituent, χ_i, of degree 6. By [18], χ_i restricted to Q could not have a rational character. This means there are at least two characters of degree 6. If $s = 3$ the degree equation must be $1 + 11 = 6 + 6$. This is impossible as $11 \nmid |G|$. There are too many characters for s to be 2. If $s = 6$, restrict χ_i to $N(\langle\pi\rangle)$. Here $N(\langle\pi\rangle)$ is a Frobenius Group of order $7 \cdot 6$. The character of degree 6 of $N(\langle\pi\rangle)$ must agree with $\chi_i | N(\langle\pi\rangle)$. If τ is an involution in $N(\langle\pi\rangle)$, $\chi_i(\tau) = 0$. Let $\mathfrak{X}_i$ be the representation affording χ_i. Then $\mathfrak{X}_i(\tau)$ has determinant -1. This means G must have a subgroup G_1 of index 2, contradicting the simplicity of G.

Suppose now $s = 2$ and $\chi_0^1(1) = 9$ with $\sum_{j=1}^{3} \chi_0^j$ a constituent of $\chi^{(2)}$. Now $\chi^{(2)}(\sigma) = \sum_{j=1}^{3} \chi_0^j(\sigma) + 1 = 7 + 6\cos\theta$.

As $\chi_0^j(\sigma)$ is independent of j, $\chi_0^j(\sigma) = 2 + 2\cos\theta$. Now there are at least six algebraic conjugates of χ_0^1 contradicting (5.1).

We have handled all possibilities for the constituent χ_i except for $\chi_i(1) = 20$ and $\chi_0^1(1) = 10$. Suppose that G has a character of degree 10 or 20. As $|G| = 5^2 \cdot g_0$ where $5 \nmid g_0$, these characters are in a nonprincipal 5-block [6, Section 6]. As a consequence, there is some nontrivial element η of Q such that $C(\eta)/\langle\eta\rangle$ is a group with more than one Sylow 5-group [14]. However this is impossible by the restrictions imposed on $C(\eta)$ by (1.1) and the limited possibilities for the irreducible constituents of G restricted to $C(\eta)$. Indeed the irreducible constituents have degrees summing to 3, 2 and 2. If $C(\eta)$ had a quasisimple subgroup, it would be A_5 or $SL(2,5)$. It could not be $SL(2,5)$ as a Sylow 5-group would have an involution centralizing it. If it were A_5, it would have a 3-dimensional constituent and five linear ones. This contradicts Lemma 2, for any of the nontrivial elements. If $C(\eta)$ has no such subgroups, the fitting subgroup of $C(\eta)$ has an automorphism of order five which contradicts (1.1). This shows $5^2 \nmid |Q|$.

(4) G has no elements of order 5.

We now consider the case $|Q| = 5$. We will again use the theory of groups whose order is divisible by 5 to the first power described in [5]. We suppose at first that the Sylow 5-group is self-centralizing. This means there are no elements of order 10 or 15. The character χ of degree 7 is in the principal 5-block as its degree is prime to 5. As this degree is congruent to $2 (\mathrm{mod}\ 5)$, this block consists of the trivial character χ_1, two characters of degree 7, and a fourth character of degree 6. The degree equation is $1 + 6 = 7$. All other characters have degrees divisible by 5. We may assume then that G has a character of degree 6. Furthermore, it has rational character values. Denote this character χ_2. By restricting χ_2 to $N(\langle\pi\rangle)$ as above, it can be assumed that $s = 3$. Let R be a Sylow 3-group, and let $\chi_2|A = \sum_{i=1}^{6}\varphi_i$ where A is any abelian subgroup of R containing an element of order 9. As $\chi_2|R$ is rational, the φ_i are algebraic conjugates of φ_1. In particular, they all have the same kernel which is the kernel of $\chi_2|A$. This shows $|A| \leq 3^2$ and $|R| \leq 3^3$.

The degree equation is $1 \pm y_1 \pm y_2 = 6$. As one y_i must be odd, the possibilities are easily enumerated. The possible odd values are 5, 15, 45, 135. The values 5 and 135 are not congruent to ± 1 or $\pm 3 (\mathrm{mod}\ 7)$. The possible degree equations are $1 + 15 = 6 + 10$ and $1 + 50 = 6 + 45$. Only the first of these has all character degrees dividing $|G|$. The degrees obtained so far are 1, 6, 7, 7, 10, 15. All remaining degrees are divisible by 35. Now $\chi^{[2]}$ is a character of degree 21 of which χ is a constituent and χ_1 is not by Lemma 2. Thus $\chi^{[2]}$ has a constituent of degree 7 with multiplicity ≥ 2. This conflicts with the behavior of χ and $\chi^{[2]}$ on $N(\langle\pi\rangle)$.

We have shown that if $|Q| = 5$, there must be elements of order $5 \cdot 2$ or $5 \cdot 3$. A careful analysis of the possibilities for elements of order 10 or 15 using [7] as in [22, p. 210] for a representation of degree 7 shows the elements of order 2 or 3 must have eigenspaces of dimension 5. However, this contradicts Lemma 1. We have now shown $|Q| = 1$.

(5) If $s = 2, 3$ then $G \cong S\ell_2(8), U_3(3)$ respectively.

We now consider the possible degree equations. As the degrees are of the form $3^\alpha \cdot 2^\beta$ and one degree is 1, there must be another degree of the form 3^α

and another of the form 2^β. Suppose $s = 2$. The equation must be $1 \pm 2^\beta \pm 3^\alpha = 0$. The only solution is $1 + 8 = 9$. Suppose $s = 3$. The degree equation is of the form $1 \pm 2^\alpha \pm 3^\beta \pm 2^\gamma 3^\delta = 0$. The only solutions are $1 + 8 = 6 + 3$ and $1 + 32 = 27 + 6$. Recall $\alpha \leq 4$ and $\beta \leq 7$ as degrees divide $|G|$. We have already discussed the case $1 + 8 = 6 + 3$.

We describe a technique known as block separation which in some situations enables the group order to be determined by the degree equation. It makes use of a theorem of Brauer and Tuan [8, Lemmas 2 and 3] which applies to G. Let B be a given q-block for $q = 2$ or 3. The theorem states that

$$\sum_{y \in B_0(7) \cap B} y(1)y(\pi) \equiv 0 \pmod{q^d}$$

where a Sylow q-group of G has order q^d. Furthermore, a character of degree q^α for $\alpha \geq 1$ cannot be in the principal q-block. Applying these results for the degree equation $1 + 8 = 9$ implies $|G| = 7 \cdot 3^2 \cdot 2^3$. Applying it to $1 + 32 = 27 + 6$ implies $|G| = 7 \cdot 3^3 \cdot 2^5$. (The power 2^5 occurs here as a character degree has order 2^5 and so $2^5 \mid |G|$ and because none of 1, 1-27, 1-6, 1-6-27 is divisible by a power of 2 higher than 2^5. These are the possibilities for $\sum_{y \in B_0(7) \cap B_0(2)} y(1)y(\pi)$.) We have shown that if $s = 2, 3$, G is a group of order $7 \cdot 9 \cdot 8$ or $7 \cdot 3^3 \cdot 2^5$. There are unique simple groups of these orders [15], which give $PSL_2(8)$ and $U_3(3)$.

(6) $s \neq 6$.

Suppose $|G| = 7 \cdot 3^b \cdot 2^a$, $s = 6$.

The degree equation has exactly one character of degree 1 and so has a character of degree 3^α and one of degree 2^β. As all such character degrees are congruent to $\pm 1 \pmod 7$, these degrees are 27 and 8 or 64. Recall from (1) that $a \leq 7$ and $b \leq 4$. By Sylow's Theorem $(3^{b-1})(2^{a-1}) \equiv 1 \pmod 7$. As there is a character of degree 27, $b = 3$ or 4. This means $b = 3$ and $a = 3$ or 6. The degree equation is now $1 + 2^\beta + \cdots = 27 + \cdots$. If there are no more characters of degree 27, there must be a character of degree $2 \cdot 3^\gamma$ as can be seen by reducing $\pmod 4$. The only possibility congruent to $\pm 1 \pmod 7$ is 6. However a character of degree 6 implies a subgroup of index 2 which has been ruled out. Consequently there must be three characters of degree 27. The degree equation is $1 + 2^\alpha + \cdots = 27 + 27 + 27 +$. The possible degrees less than $\sqrt{|G|}$ are 8, 27, 36, 48 and 64. If $2^\alpha = 64$, the only solutions are

$$1 + 64 + 64 = 27 + 27 + 27 + 48$$

$$1 + 64 + 8 + 8 = 27 + 27 + 27.$$

If $2^\alpha = 8$, the only possible degree equations with degrees less than $\sqrt{|G|}$ are

$$1 + 8 + 36 + 36 = 27 + 27 + 27$$

$$1 + 8 + 8 + 64 = 27 + 27 + 27.$$

The case $1 + 8 + 8 + 64 = 27 + 27 + 27$ is impossible by 2-block separation as $1,\ 1 - 27,\ 1 - 54,\ 1 - 81$ are not divisible by 64. The case $1 + 8 + 36 + 36 = 27 + 27 + 27$ is impossible by 2-block separation as the order of $|G|$ must be $7 \cdot 27 \cdot 64$ as the sum of the squares of the degrees of characters in $B_0(7)$ is greater than $7 \cdot 27 \cdot 8$. The only remaining case is $1 + 64 + 64 = 27 + 27 + 27 + 48$. Here $|G| = 7 \cdot 27 \cdot 64$. To eliminate this note that the sum of squares of the degrees of characters in $B_0(7)$ is greater than $|G|$.

This contradiction completes the final case. □

To finish, we present $S\ell_2(8)$ and $PS\ell_2(13)$ as groups generated by explicitly given automorphisms of $\mathbb{O}$. Generators for $S\ell_2(8)$ are (in the notation of Section 1)

$$\delta^-_{\{1,2,4,6\}}(1437526)$$

b, $t_{\pi,0}$, $t_{0,\pi}$ and the transformation whose matrix with respect to $e_1, \ldots, e_7$ is

$$\frac{1}{4}\begin{pmatrix} \sigma^2+\sigma-2 & 1-\sigma^2 & \sigma^2-2 & 1+\sigma & 1 & 3-\sigma-\sigma^2 & \sigma \\ 1-\sigma^2 & -\sigma & 2-\sigma-\sigma^2 & 3-\sigma-\sigma^2 & \sigma^2-2 & 1 & 1+\sigma \\ \sigma^2-2 & 2-\sigma-\sigma^2 & -1 & -\sigma & \sigma^2+\sigma-3 & -1-\sigma & 1-\sigma^2 \\ 1+\sigma & 3-\sigma-\sigma^2 & -\sigma & 2-\sigma^2 & 1-\sigma^2 & \sigma^2+\sigma-2 & -1 \\ 1 & \sigma^2-2 & \sigma^2+\sigma-3 & 1-\sigma^2 & -1-\sigma & -\sigma & \sigma^2+\sigma-2 \\ 3-\sigma-\sigma^2 & 1 & -1-\sigma & \sigma^2+\sigma-2 & -\sigma & 1-\sigma^2 & 2-\sigma^2 \\ \sigma & 1+\sigma & 1-\sigma^2 & -1 & \sigma^2+\sigma-2 & 2-\sigma^2 & \sigma^2+\sigma-3 \end{pmatrix}$$

where $\sigma = 2\cos\frac{2\pi}{9}$.

The generators for $PS\ell_2(13)$ are

$$\delta^-_{\{1,4,6,7\}}(235)(476),$$

$t_{\frac{2\pi}{13},\frac{6\pi}{13}}$ and the linear transformation whose matrix with respect to $e_1, \ldots, e_7$ is

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & -2 & -2 \\ 0 & c_0 & c_1 & 0 & c_2 & 0 & 0 \\ 0 & c_1 & c_2 & 0 & c_0 & 0 & 0 \\ -2 & 0 & 0 & u & 0 & v & w \\ 0 & c_2 & c_0 & 0 & c_1 & 0 & 0 \\ -2 & 0 & 0 & v & 0 & w & u \\ -2 & 0 & 0 & w & 0 & u & v \end{pmatrix}$$

where

$$c_r = \frac{-7+\sqrt{13}+4\cos(\frac{2\pi 4^r}{13}) + (13-\sqrt{13})\cos^2(\frac{2\pi 4^r}{13})}{2} \qquad (r = 0,1,2)$$

and

$$u = (c_0 + 2c_2 - 2c_1)/\sqrt{13},$$

$$v = (c_2 + 2c_1 - 2c_0)/\sqrt{13},$$

$$w = (c_1 + 2c_0 - 2c_2)/\sqrt{13}.$$

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