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G.J.K. Regterschot, K. van Donselaar and W.G.M.M. Rutten

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Graduate School of Industrial Engineering and Management Science
Eindhoven University of Technology
P.O. Box 513, Paviljoen F11
NL-5600 MB Eindhoven
The Netherlands
Phone: +31.40.473828

A FUNDAMENTAL IDENTITY IN THE LOST SALES INVENTORY PROBLEM

Abstract

A fundamental identity has been derived for the determination of the service level in a periodic review inventory model with a positive lead time and without backordering. The event of a stockout in the lead time, which is written as a union of events, in which each event represents a stockout during a specific period in the lead time, can be written as an intersection of events in which each event represents the total demand exceeding the scheduled receipts during (a part of) the lead time.

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1. INTRODUCTION

In inventory control theory it is often assumed that the demand which is not satisfied from stock immediately, will be backordered. For models with this assumption together with several other assumptions (see e.g. [Heyman & Sobel, 1990, chapter 12]) a simple policy has been proven to be optimal: order every period and thus raise the inventory position up to a critical level S. The inventory position used here is defined as the totals of the inventory on hand and the scheduled receipts minus the backorders and other commitments to customers.

The model complementary to the backorder model above is 'the lost sales model'. In this model demand is lost if there is no inventory available. For this type of model the simple policy, which is based on the inventory position only, is no longer optimal (see [Karlin & Scarf, 1958] and [Morton, 1969]). Morton [1971] and Nahmias [1979] derived approximations for the optimal order quantity function. In their best approximations the reorder decision depends not only on the inventory position, but rather on a state vector of dimension **leadtime+1**, containing the inventory on hand as well as the scheduled receipts.

The simplest way to deal with the lost sales model in practice is to use the backorder model (see e.g. [Silver & Peterson, 1985]) as an approximation. However as soon as the service level, which is the probability that demand can be met from stock, should be low, this approximation is not very precise (see [Rutten *et al.*, 1992]).

In order to see why a low service level can be benificial, consider the following situation, which was encountered in food-industry. For a particular raw material A an alternative slightly better raw material B is available for a slightly higher price. There is a lot of inventory for raw material B, since B is part of the standard recipe of many endproducts. In such a situation it may be more efficient to require a low service level for raw material A and to supply the customers of raw material A with raw material B if raw material A is out of stock. The advantage here is a consequence of the law of variance for (nearly) independent demands: it is cheaper to have a common safety stock (raw material B) for (part of) the demand of A and the demand of B rather than to have large safety stocks for both raw materials A and B separately. If the difference in price between A and B is low and if the coefficient of variation of the demand for the product which uses A is relatively high, it will be benificial to use a low safety stock norm for raw material A.

Since in practice a low service level may well be advantageous, in this article a fundamental identity for the lost sales model will be derived which holds for all service levels: both high and low. From Rutten *et al.* ([1992]) it is clear that the stockout event in a lost sales model can be written as the union of events X_i, so the stockout probability is given by $1-\alpha = Pr\{\bigcup X_i\}$. Unfortunately this expression is difficult to handle. Therefore in this article the stockout event will be rewritten into an expression which is easier to handle. In this expression the stockout event will be written as the intersection of events Y_i, so the stockout event is given by $1-\alpha = Pr\{\bigcap Y_i\}$.

Before doing this we introduce some notation and give a short model description. For more information on the model we refer to Rutten *et al.* [1992].

2. MODEL AND NOTATION

Introduce the notation $x^+ = \max(0, x)$, so x^+ is the nonnegative part of x.
We use the following variables:

k lead time in periods, $k \in \mathbb{N}$
Q_{t-k} quantity ordered at the start of period t-k
(which, by definition, will arrive at the start of period t)
ξ_t demand during period t
I_t virtual inventory at the start of period t, before order arrival,
the real inventory on hand is I_t^+ (see Remark at the end of this paper)
α desired service level, probability that demand in a period does not exceed the available inventory at the start of that period, i.e.

$$\alpha = Pr\{\ \xi_t \le I_t^+ + Q_{t-k}\}$$

The reorder cycle in the periodic review inventory model is as follows:

1. starting inventory on hand in period t (equals the resulting inventory on hand in period t-1) is I_t^+
2. the reorder quantity Q_t is determined and the reordered quantity Q_{t-k} is received
3. the inventory on hand before satisfying demand during period t equals $I_t^+ + Q_{t-k}$
4. the demand ξ_t during period t is met as long as inventory is available.

In our inventory policy, backordering is not allowed. Moreover, the order-up-to level is dynamic; every period t the order-up-to level is chosen such that after reordering the expected service level in period t+k equals α. In other words, the reorder quantity Q_t is taken as large as is required to assure that the probability of a stockout occurence in period t+k is 1-α. For lead time k this gives the following probability:

$$Pr\{\ \xi_{t+k} > I_{t+k}^+ + Q_t\} = 1 - \alpha.$$

The difficulty with the above probability is to determine the value of I_{t+k}^+ since this variable is not known and has to be expressed in known variables.

3. THE FUNDAMENTAL IDENTITY

Since for the lost sales model with lead time k we have the relation

$$I_{t+k+1} = I^+_{t+k} + Q_t - \xi_{t+k}, \tag{1}$$

we consider the stockout event in period t+k

$$X = \{I_{t+k+1} < 0\} = \{\xi_{t+k} > I^+_{t+k} + Q_t\}. \tag{2}$$

For the service level α we need to have $\Pr(X) = 1 - \alpha$.

We can distinguish the situations $\xi_{t+k} > I^+_{t+k} + Q_t$ $(I_{t+k+1} < 0)$ and $\xi_{t+k} \le I^+_{t+k} + Q_t$ $(I_{t+k+1} \ge 0)$. Only in the first case there is a stockout.

Now there exist two possible situations:

a) for some i, i=0,...,k we have $I_{t+k-i}<0$ (there is a stockout in period t+k-i-1)
b) for all i, i=0,...,k we have $I_{t+k-i}\ge 0$ (there is never a stockout in periods t-1,..,t+k-1)

For case a) define $n = \min\{i, i=0,1,...,k \mid I_{t+k-i} < 0\}$ then $I_{t+k-n}<0$ and $I_{t+k-i}\ge 0$ for i=0,1,...,n-1, i.e. the last stockout before period t+k occurs in period t+k-n-1.
Since $I^+_{t+k-n} = 0$ it easily follows for n=0,1,...,k,

$$I_{t+k} = I^+_{t+k} = \sum_{i=1}^{n} Q_{t-i} - \sum_{i=1}^{n} \xi_{t+k-i}. \tag{3a}$$

For case b) we have $I_{t+k} = I^+_{t+k} = I^+_t + \sum_{i=1}^{k} Q_{t-i} - \sum_{i=1}^{k} \xi_{t+k-i}$. (3b)

Note that for n=k the expression for case a) coincides with the special case of b) where $I_t < 0$ $(I^+_t = 0)$. We combine these two situations in b).

For convenience of notation we define for i=0,1,..,k-1,

$$X_i = \{\xi_{t+k} > I^+_{t+k} + Q_t;\ I_{t+k-i} < 0,\ I_{t+k-j} \ge 0,\ j=0,1,...,i-1\} \text{ and } Y_i = \{\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j}\}. \tag{4}$$

Further define $X_k = \{\xi_{t+k} > I^+_{t+k} + Q_t;\ I_{t+k-j} \ge 0,\ j=0,1,...,k-1\}$ and $Y_k = \{\sum_{j=0}^{k} \xi_{t+k-j} > I^+_t + \sum_{j=0}^{k} Q_{t-j}\}$.

Note that X_i, i=0,1,...,k-1, represents the event that a stockout occurs in period t+k and the last stockout before period t+k occurred in period t+k-i-1. This enables us to write X as follows, $X = \bigcup_{i=0}^{k} X_i$.

Next introduce the events R_i, i=0,1,..,k-1,

$$R_i = \{\xi_{t+k-i-1} > I^+_{t+k-i-1} + Q_{t-i-1}\} = \{I_{t+k-i} < 0\},\ i=0,1,..,k-1,$$

the event of having a stockout in period t+k-i-1.

The following lemma states the first part of the identity for the stockout event in period t+k.

Lemma
For the stockout event in period t+k we have the following identity

$$\{I_{t+k+1}<0\} = \bigcup_{i=0}^{k-1} \{\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j};\ I_{t+k-i}<0,\ I_{t+k-j}\geq 0,\ j=0,1,\ldots,i-1\}$$
$$\cup\ \{\sum_{j=0}^{k} \xi_{t+k-j} > I_t^* + \sum_{j=0}^{k} Q_{t-j},\ I_{t+k-j}\geq 0,\ j=0,1,\ldots,k-1\}$$

Proof
First we restate the Lemma as follows,

$$X = \bigcup_{i=0}^{k} X_i = \bigcup_{i=0}^{k-1} (Y_i \cap (\bigcap_{j=0}^{i-1} \bar{R}_j) \cap R_i) \cup (Y_k \cap (\bigcap_{j=0}^{k-1} \bar{R}_j)) \tag{5}$$

For i=0,1,..,k-1, if $\bigcap_{j=0}^{i-1} \bar{R}_j \cap R_i$ holds, we have $I^*_{t+k-j} = I_{t+k-j} = I_{t+k-j-1} + Q_{t-j-1} - \xi_{t+k-j-1},\ j=0,\ldots,i-1$

and $I_{t+k-i}=0$, so it follows from repeated substitution in (3a) that $\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j}$. Thus

the events $\{\xi_{t+k} > I^*_{t+k} + Q_t\}$ and $\{\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j}\}$ are identical, i.e. $X_i = Y_i$.

Similarly if $\bigcap_{j=0}^{k-1} \bar{R}_j$ holds we get from equation (3b) $\sum_{j=0}^{k} \xi_{t+k-j} > I_t^* + \sum_{j=0}^{k} Q_{t-j}$, so the events

$\{\xi_{t+k} > I^*_{t+k} + Q_t\}$ and $\{\sum_{j=0}^{k} \xi_{t+k-j} > I_t^* + \sum_{j=0}^{k} Q_{t-j}\}$ are identical, i.e. $X_k = Y_k$.

So we have $X_i \cap (\bigcap_{j=0}^{i-1} \bar{R}_j \cap R_i) = Y_i \cap (\bigcap_{j=0}^{i-1} \bar{R}_j \cap R_i)$, i=0,1,..,k-1 and $X_k \cap (\bigcap_{j=0}^{k-1} \bar{R}_j) = Y_k \cap (\bigcap_{j=0}^{k-1} \bar{R}_j)$.
Further we have by definition

$$\begin{aligned}
X_0 &= X_0 \cap R_0 \\
X_1 &= X_1 \cap \bar{R}_0 \cap R_1 \\
X_2 &= X_2 \cap \bar{R}_0 \cap \bar{R}_1 \cap R_2 \\
&\ldots \\
X_i &= X_i \cap (\bigcap_{j=0}^{i-1} \bar{R}_j) \cap R_i \\
&\ldots \\
X_k &= X_k \cap (\bigcap_{j=0}^{k-1} \bar{R}_j))
\end{aligned} \tag{6}$$

Now $X = \bigcup_{i=0}^{k} X_i = \bigcup_{i=0}^{k-1} (Y_i \cap (\bigcap_{j=0}^{i-1} \bar{R}_j) \cap R_i) \cup (Y_k \cap (\bigcap_{j=0}^{k-1} \bar{R}_j))$ follows, which proves the lemma.
Note that the events X_i in this lemma are mutually disjoint. ⊠

Now we can prove the fundamental identity in which the union of the disjoint events X_i can be written as the intersection of the (non-disjoint) events Y_i.

Theorem
For the event of a stockout in period t+k we have the identity

$$
\begin{aligned}
\{I_{t+k+1} < 0\} &= \{\xi_{t+k} > I^{+}_{t+k} + Q_t\} \\
&= \bigcup_{i=0}^{k-1} \{\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j};\ I_{t+k-i} < 0,\ I_{t+k-j} \geq 0,\ j = 0,1,\ldots,i-1\} \\
&\quad \bigcup \{\sum_{j=0}^{k} \xi_{t+k-j} > I^{+}_{t} + \sum_{j=0}^{k} Q_{t-j},\ I_{t+k-j} \geq 0,\ j = 0,1,\ldots,k-1\} \\
&= \bigcap_{i=0}^{k-1} \{\sum_{j=0}^{i} \xi_{t+k-j} > \sum_{j=0}^{i} Q_{t-j}\} \cap \{\sum_{j=0}^{k} \xi_{t+k-j} > I^{+}_{t} + \sum_{j=0}^{k} Q_{t-j}\}
\end{aligned}
$$

Proof
The second equality in this Theorem is identical to the Lemma.
First we prove the following equality

$$\bigcup_{j=i}^{k} X_j = (\bigcap_{j=i}^{k} Y_j) \cap (\bigcap_{j=0}^{i-1} \bar{R}_j). \tag{7}$$

For i=k (7) follows directly from the definitions of X_k, Y_k and R_j, j=0,...,k-1, and (3b).

If (7) holds for i, i.e. $\bigcup_{j=i}^{k} X_j = (\bigcap_{j=i}^{k} Y_j) \cap (\bigcap_{j=0}^{i-1} \bar{R}_j)$

then it also holds for i-1:

$$
\begin{aligned}
\bigcup_{j=i-1}^{k} X_j &= \bigcup_{j=i}^{k} X_j \cup X_{i-1} \\
&= [(\bigcap_{j=i}^{k} Y_j) \cap (\bigcap_{j=0}^{i-1} \bar{R}_j)] \cup [Y_{i-1} \cap (\bigcap_{j=0}^{i-2} \bar{R}_j) \cap R_{i-1}] \\
&= (\bigcap_{j=0}^{i-2} \bar{R}_j) \cap [[\bigcap_{j=i}^{k} Y_j \cap \bar{R}_{i-1}] \cup [Y_{i-1} \cap R_{i-1}]] \\
&= (\bigcap_{j=0}^{i-2} \bar{R}_j) \cap [(\bigcap_{j=i-1}^{k} Y_j) \cup (\bigcap_{j=i}^{k} Y_j \cap \bar{Y}_{i-1} \cap \bar{R}_{i-1}) \cup (\overline{\bigcap_{j=i}^{k} Y_j} \cap Y_{i-1} \cap R_{i-1})] \\
&= (\bigcap_{j=0}^{i-2} \bar{R}_j) \cap (\bigcap_{j=i-1}^{k} Y_j),
\end{aligned}
$$

since it is easily seen that $\bigcap_{j=i}^{k} Y_j \cap \bar{Y}_{i-1} \cap \bar{R}_{i-1} = \emptyset$ and $\overline{\bigcap_{j=i}^{k} Y_j} \cap Y_{i-1} \cap R_{i-1} = \emptyset$.

The result $\bigcup_{j=0}^{k} X_j = (\bigcap_{j=0}^{-1} \bar{R}_j) \cap (\bigcap_{j=0}^{k} Y_j)$ follows from applying a reverse induction argument

for i=0, using the convention that $\bigcap_{j=0}^{-1} \bar{R}_j = \Omega$, where Ω denotes the universal set.

This yields $X = X_0 \cup \ldots \cup X_k = Y_0 \cap \ldots \cap Y_k$ which proves the Theorem. ⊠

Remark
Note that in our definition we distinguished between the virtual inventory I_t and the real inventory I_t^*. In many literature the real inventory is denoted by I_t and we have the relation $I_{t+k+1} = [I_{t+k} + Q_t - \xi_{t+k}]^+$. In this paper we used an alternative notation since the event $I_{t+k}^* + Q_t - \xi_{t+k} \leq 0$ implies a stockout in the (<)-case and no stockout in the (=)-case. To avoid difficulties in our proofs we have chosen for the definitions above.

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