Shift-invariant operators

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 94-14
August 1994
Shift-invariant operators

by

S.J.L. van Eijndhoven
J.M. Soethoudt

Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507

Shift–invariant operators

S.J.L. van Eijndhoven & J.M. Soethoudt
Department of Mathematics and Computing Science
University of Technology, Eindhoven
PO Box 513
5600 MB Eindhoven
The Netherlands

18-11-1993

Abstract

Let $n \in \mathbb{N}$, $C(\mathbb{R}, \mathbb{C}^n)$ denotes the space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}^n$. In this paper a complete characterization of the closed translation–invariant operators from $C(\mathbb{R}, \mathbb{C}^q)$ into $C(\mathbb{R}, \mathbb{C}^p)$ is derived. It yields an explicit description of the duals of $C^k(\mathbb{R}, \mathbb{C}^n)$ and $C^\infty(\mathbb{R}, \mathbb{C}^n)$. An application can be found in the field of fundamental system theory.

Keywords: convolution operators, shift–invariance, Fourier transform.

1 Introduction

For $n \in \mathbb{N}$ let $C(\mathbb{R}, \mathbb{C}^n)$ denote the space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}^n$. Endowed with the seminorms

$$q_m(f) = \sup_{t \in [-m,m]} |f(t)|_n, (m \in \mathbb{N}, f \in C(\mathbb{R})) \tag{1}$$

$C(\mathbb{R}, \mathbb{C}^n)$ is a Fréchet space. Here $|.|_n$ denotes the Euclidean norm in $\mathbb{C}^n$. We write $C(\mathbb{R})$ instead of $C(\mathbb{R}, \mathbb{C})$. Besides we consider the space $C^k(\mathbb{R}, \mathbb{C}^n)$ of all k-times continuously differentiable functions from $\mathbb{R}$ into $\mathbb{C}^n$ and the space $C^\infty(\mathbb{R}, \mathbb{C}^n)$,

$$C^\infty(\mathbb{R}, \mathbb{C}^n) = \bigcap_{k=0}^{\infty} C^k(\mathbb{R}, \mathbb{C}^n) \tag{2}$$

Starting from a well known representation of the dual of $C(\mathbb{R})$, namely the function-space $ba_c(\mathbb{R})$, we introduce a one-to-one correspondence between $ba_c(\mathbb{R})$ and the class of all translation-invariant operators on $C(\mathbb{R})$. Thus a convolution structure is established in $ba_c(\mathbb{R})$ and $ba_c(\mathbb{R})$ turns into a convolution ring without zero divisors. Also the Fourier transform and a (weak) differentiablity-structure is introduced on $ba_c(\mathbb{R})$. The elements of the matrix ring $\mathcal{M}^{p\times q}(ba_c(\mathbb{R}))$ are in one-one correspondence with continuous linear mappings from $C(\mathbb{R}, \mathbb{C}^q)$ into $C(\mathbb{R}, \mathbb{C}^p)$ which are translation invariant.

Next we establish a similar structure for $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$, which is the testspace for the distributions of compact support, $\mathcal{E}'(\mathbb{R})$ (cf. [3]). Then $\mathcal{E}'(\mathbb{R})$ is linked to the vector space $ba_c(\mathbb{R}) \oplus \mathcal{P}$ where $\mathcal{P}$ is the space of all complex polynomials. This correspondence turns out to be non-isomorphic. The vector space $ba_c(\mathbb{R}) \oplus \mathcal{P}$ is related to the class of all shift-invariant operators on $\mathcal{E}(\mathbb{R})$ and a natural convolution structure on $ba_c(\mathbb{R}) \oplus \mathcal{P}$ is imposed. If we denote the matrices over the dual space $\mathcal{E}'(\mathbb{R})$ by $\mathcal{M}^{p\times q}(\mathcal{E}'(\mathbb{R}))$ then $\mathcal{M}^{p\times q}(\mathcal{E}'(\mathbb{R}))$ can be linked to the class of all translation-invariant operators from $C^\infty(\mathbb{R}, \mathbb{C}^n)$ into $C^\infty(\mathbb{R}, \mathbb{C}^p)$.
The final step is the characterization of all translation-invariant closed linear mappings from $C(\mathbb{R}, \mathbb{C}^n)$ into $C(\mathbb{R}, \mathbb{C}^p)$, having $C^\infty(\mathbb{R}, \mathbb{C}^n)$ in their domain.

2 The convolution ring $ba_c(\mathbb{R})$

We recall that $C(\mathbb{R})$ denotes the space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$ endowed with the seminorms

$$q_m(f) = \sup_{t\in[-m,m]} |f(t)|, (m \in \mathbb{N}, f \in C(\mathbb{R}))$$

Thus $C(\mathbb{R})$ is a Fréchet space, a complete metrizable locally convex space.

Definition 2.1 A function $\mu : \mathbb{R} \to \mathbb{R}$ is of *bounded variation* if there exists a $K > 0$ such that for all $n \in \mathbb{N}, \{t_0, ..., t_n\} \subset \mathbb{R}$

$$(t_{i-1} < t_i) \ \wedge \ \sum_{i=1}^{n} |\mu(t_i) - \mu(t_{i-1})| < K$$

The infimum of all K satisfying this inequality is denoted by *var(μ)*. By $ba(\mathbb{R})$ we denote the vector space of all right-continuous functions $\mu : \mathbb{R} \to \mathbb{C}$ which are of bounded variation.

Definition 2.2 By $ba_c(\mathbb{R})$ we denote the subspace of $ba(\mathbb{R})$ consisting of all μ with the property that there exists a $T > 0$ such that

$$\begin{cases} \mu(t) = 0 & t \leq -T \\ \mu(t) = \mu(T) & t \geq T \end{cases}$$

There is the following characterization of the dual of $C(\mathbb{R})$, cf. [2], theorem 6.19.

Lemma 2.3 *A linear functional $\mathcal{L}$ on $C(\mathbb{R})$ is continuous if and only if there exists exactly one $\mu \in ba_c(\mathbb{R})$ such that*

$$\forall_{f \in C(\mathbb{R})} \ [\ \mathcal{L}(f) = < f, \mu > := \int_{\mathbb{R}} f d\mu\ ]$$

(One should interpret this integral as a Riemann-Stieltjes integral)

Definition 2.4 The translation operators σ^t, $t \in \mathbb{R}$ on $C(\mathbb{R})$ are defined by

$$(\sigma^t f)(\tau) = f(t + \tau),\ \ \tau \in \mathbb{R},\ \ f \in C(\mathbb{R})$$

Lemma 2.5 *The set $\{\sigma^t \ | t \in \mathbb{R}\}$ is a one parameter c_0-group on $C(\mathbb{R})$, i.e.*

$$\begin{aligned} \forall_{t,\tau \in \mathbb{R}} &\ :\ \ \sigma^{t+\tau} = \sigma^t \sigma^\tau,\ \sigma^0 = I \\ \forall_{f \in C(\mathbb{R})} &\ :\ \ \lim_{t \to 0} \sigma^t f = f \end{aligned}$$

To each $\mu \in ba_c(\mathbb{R})$ we associate the convolution operator C_μ from $C(\mathbb{R})$ into $C(\mathbb{R})$ as follows

$$(C_\mu f)(t) = < \sigma^t f, \mu >,\ \ f \in C(\mathbb{R}),\ t \in \mathbb{R}$$

Since $\{\sigma^t \ | t \in \mathbb{R}\}$ is a c_0-group on $C(\mathbb{R})$, $C_\mu f \in C(\mathbb{R})$; indeed

$$|(C_\mu f)(t) - (C_\mu f)(\tau)| \leq \text{var}(\mu) \cdot q_a(\sigma^t f - \sigma^\tau f)$$

for $a > 0$ sufficiently large. Moreover C_μ is continuous, since

$$\begin{aligned} q_m(C_\mu f) &= \sup_{t \in [-m,m]} | < \sigma^t f, \mu > | \\ &\leq (\sup_{t \in [-m,m]} q_a(\sigma^t f)) \cdot \text{var}(\mu) \\ &= q_{a+m}(f) \cdot \text{var}(\mu) \end{aligned}$$

Convolution operators are characterized by the following property

Lemma 2.6 *A continuous linear operator K from $C(\mathbb{R})$ into $C(\mathbb{R})$ is a convolution operator, i.e. $K = C_\mu$ for some unique $\mu \in ba_c(\mathbb{R})$ if and only if $K\sigma^t = \sigma^t K$ for all $t \in \mathbb{R}$.*

Proof: Sufficiency is clear. So we prove necessity. Let $K : C(\mathbb{R}) \to C(\mathbb{R})$ be a continuous linear mapping with $K\sigma^t = \sigma^t K$ for all $t \in \mathbb{R}$. Then $f \mapsto (Kf)(0)$ is a continuous linear functional, so $(Kf)(0) =< f, \mu >$ for precisely one $\mu \in ba_c(\mathbb{R})$. And for all $t \in \mathbb{R}$

$$(Kf)(t) = (K(\sigma^t f))(0) =< \sigma^t f, \mu >= (C_\mu f)(t)$$

□

It is not hard to check that the mapping $\mu \mapsto C_\mu$ defined on $ba_c(\mathbb{R})$ is linear and injective.

Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$. Then $C_{\mu_1} \circ C_{\mu_2}$ is a continuous linear mapping on $C(\mathbb{R})$, which commutes with all $\sigma^t (t \in \mathbb{R})$, and hence there exists a unique $\mu \in ba_c(\mathbb{R})$ such that $C_{\mu_1} \circ C_{\mu_2} = C_\mu$. This leads to the following definition:

Definition 2.7 Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$. Then $\mu := \mu_1 * \mu_2 \in ba_c(\mathbb{R})$ denotes the unique $\mu \in ba_c(\mathbb{R})$ satisfying

$$C_{\mu_1} \circ C_{\mu_2} = C_\mu = C_{\mu_1 * \mu_2}$$

Next we introduce the Fourier transform on $ba_c(\mathbb{R})$. For $\mu \in ba_c(\mathbb{R})$, the analytic function $\mathcal{F}_\mu$ on $\mathbb{C}$ is defined by

$$\mathcal{F}_\mu(\omega) =< e_\omega, \mu >, \ \omega \in \mathbb{C}$$

where $e_\omega \in C(\mathbb{R})$ is defined by

$$e_\omega(t) = e^{-i\omega t}, \ t \in \mathbb{R}$$

The function $\mathcal{F}_\mu$ is called the Fourier transform of μ.

Lemma 2.8 *The Fourier transform $\mathcal{F}$ on $ba_c(\mathbb{R})$ is linear and injective.*

Proof: Linearity is evident from the definition. To establish injectivity we note that

$$\mathcal{F}_\mu \equiv 0 \Leftrightarrow \left(\frac{d}{d\omega}\right)^n \mathcal{F}_\mu \equiv 0$$

And so $< p, \mu >= 0$ for all polynomials p. This yields $\mu = 0$. □

Lemma 2.9 *Let $\mu \in ba_c(\mathbb{R})$. Then $\mathcal{F}_\mu$ is of exponential type 1 and bounded on the real axis.*

Proof: For $a > 0$ sufficiently large,

$$\begin{aligned} |\mathcal{F}_\mu(\omega)| &\leq \text{var}(\mu) \cdot p_a(e_\omega) \\ &= \text{var}(\mu) \cdot e^{a|Im(\omega)|} \end{aligned}$$

□

For all $\mu \in ba_c(\mathbb{R})$, the function e_ω is an eigenfunction of C_μ,

$$(C_\mu e_\omega)(t) = < \sigma^t e_\omega, \mu > = < e_\omega, \mu > e_\omega(t)$$

so that

$$C_\mu e_\omega = \mathcal{F}_\mu(\omega) e_\omega$$

Lemma 2.10 *For all $\mu_1, \mu_2 \in ba_c(\mathbb{R})$: $\mathcal{F}_{\mu_1 * \mu_2} = \mathcal{F}_{\mu_1} \cdot \mathcal{F}_{\mu_2}$ and so $\mu_1 * \mu_2 = \mu_2 * \mu_1$ and $C_{\mu_1} C_{\mu_2} = C_{\mu_2} C_{\mu_1}$.*

Proof: For all $\omega \in \mathbb{C}$,

$$C_{\mu_1 * \mu_2}\, e_\omega = C_{\mu_1} C_{\mu_2} e_\omega = \mathcal{F}_{\mu_1}(\omega) \mathcal{F}_{\mu_2}(\omega) e_\omega$$

and

$$C_{\mu_1 * \mu_2}\, e_\omega = \mathcal{F}_{\mu_1 * \mu_2}(\omega) e_\omega$$

□

Theorem 2.11 *$(ba_c(\mathbb{R}), +, *)$ is a commutative ring with no zero divisors and an identity. The mapping $\mu \mapsto C_\mu$, $\mu \in ba_c(\mathbb{R})$ is a representation of this ring in the algebra of continuous linear mappings from $C(\mathbb{R})$ into $C(\mathbb{R})$. The mapping $\mu \mapsto \mathcal{F}_\mu$ is a representation of this ring in the algebra of analytic functions of exponential type 1.*

Proof: We only prove that there are no zero divisors. Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$ and $\mu_1 * \mu_2 = 0$. Then for all $\omega \in \mathbb{C}$

$$0 = (\mu_1 * \mu_2)(e_\omega) = \mu_1(e_\omega)\mu_2(e_\omega) = \mathcal{F}_{\mu_1}(\omega)\mathcal{F}_{\mu_2}(\omega)$$

Since $\mathcal{F}_{\mu_1}, \mathcal{F}_{\mu_2}$ are analytic functions at least one of them must be zero. Injectivity of the Fourier transform on the class $ba_c(\mathbb{R})$ implies $\mu_1 = 0$ or $\mu_2 = 0$. Let H be the Heaviside-function, then $< f, H > = f(0)$, $f \in C(\mathbb{R})$ and so $C_H = I$, the identity operator. □

3 Algebraic properties of convolution operators

As a consequence of theorem 2.11 we obtain the following corollaries:

Corollary 3.1 *Let $K : C(\mathbb{R}) \to C(\mathbb{R})$ be continuous and injective such that $K\sigma^t = \sigma^t K$, $t \in \mathbb{R}$. Then there exist $a \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that $K = \lambda\sigma^a$. Consequently, K is invertible.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ with $K = C_\mu$. Then $\mathcal{F}_\mu$ is an analytic function of exponential type 1, bounded on the real axis, with no zeroes. Therefor there are $a \in \mathbb{R}$, $b \in \mathbb{C}$ such that

$$\mathcal{F}_\mu(\omega) = e^{ia\omega+b}$$

Hence $C_\mu e_\omega = e^{ia\omega+b} = e^b\sigma^a e_\omega$. Since $\text{span}\{e_\omega|\omega \in \mathbb{C}\}$ is dense in $C(\mathbb{R})$, $C_\mu = e^b\sigma^a$. □

Corollary 3.2 *For all nonzero $\mu \in ba_c(\mathbb{R})$* Range(C_μ) *is dense in $C(\mathbb{R})$.*

Proof: Let $\mu \in ba_c(\mathbb{R}), \mu \neq 0$. If $\nu \in ba_c(\mathbb{R})$ and $\nu|_{\text{Range}(C_\mu)} = 0$, then $C_\nu C_\mu f = 0$ for all $f \in C(\mathbb{R})$. It follows that $\nu * \mu = 0$, whence $\nu = 0$. □

As we have seen the set of convolution operators $\{C_\mu|\mu \in ba_c(\mathbb{R})\}$ equals the commutant in $\mathcal{L}(C(\mathbb{R}))$ of the set $\{\sigma^t|t \in \mathbb{R}\}$. Another relation between the two sets is presented in the next theorem.

Theorem 3.3 *Every convolution operator C_μ on $C(\mathbb{R})$ is the strong (i.e. pointwise) limit of a sequence in the linear* span$< \{\sigma^t|t \in \mathbb{R}\} >$. *I.e. for all $f \in C(\mathbb{R})$ there exists a sequence $(G_n)_{n\in\mathbb{N}}$ in $< \{\sigma^t|t \in \mathbb{R}\} >$ such that $G_n f \to C_\mu f$ if $n \to \infty$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and suppose $\text{supp}(\mu) \subseteq [-N, N]$.
For $n \in \mathbb{N}$ define

$$t_{j,n} = -N + \frac{j}{n},\ j = 0, 1, ..., 2Nn$$

Then $t_{j+1,n} - t_{j,n} = \frac{1}{n}$.
Put

$$a_{j,n} := \mu(t_{j+1,n}) - \mu(t_{j,n}) = \int_{t_{j,n}}^{t_{j+1,n}} d\mu(\tau),\ j = 0, 1, ..., 2Nn - 1$$

and define $G_n \in \text{span} < \{\sigma^t | t \in \mathbb{R}\} >$ by

$$G_n = \sum_{j=0}^{2Nn-1} a_{j,n}\sigma^{t_{j,n}}$$

Now let $f \in C(\mathbb{R})$ and $m \in \mathbf{N}$. Then there exists for given $\varepsilon > 0$, $n_\varepsilon \in \mathbf{N}$ such that for $n > n_\varepsilon$

$$|f(s) - f(\sigma)| < \frac{\varepsilon}{\text{var}(\mu)}$$

whenever $s, \sigma \in [-N - m, N + m]$ with $|s - \sigma| < \frac{1}{n}$.
Let $\varepsilon > 0$ and choose n_ε as indicated. Then

$$\begin{aligned} q_m(G_n f - C_\mu f) &= \max_{t \in [-m,m]} |\sum_{j=0}^{2Nn-1} \int_{t_{j,n}}^{t_{j+1,n}} [f(t_{j,n} + t) - f(\tau + t)] d\mu(\tau)| \\ &< \sum_{j=0}^{2Nn-1} \frac{\varepsilon}{\text{var}(\mu)} \int_{t_{j,n}}^{t_{j+1,n}} |d\mu(\tau)| \leq \varepsilon \end{aligned}$$

□

The above theorem can be applied in establishing some results for *closed* translation invariant subspaces and *closed* translation invariant mappings. First a definition.

Definition 3.4 Let K with domain $\mathcal{D}(K) \subseteq C(\mathbb{R})$ be a linear mapping from $\mathcal{D}(K)$ into $C(\mathbb{R})$. Then K is said to be *a closed translation invariant operator in* $C(\mathbb{R})$ if the graph of K is closed in $C(\mathbb{R}) \times C(\mathbb{R})$ and if $\sigma^t(\mathcal{D}(K)) = \mathcal{D}(K), t \in \mathbb{R}$ with $K\sigma^t f = \sigma^t K f, f \in \mathcal{D}(K)$.

Lemma 3.5 *Let $\mathcal{M}$ be a closed translation-invariant subspace of $C(\mathbb{R})$. Then $C_\mu(\mathcal{M}) \subseteq \mathcal{M}$ for all $\mu \in ba_c(\mathbb{R})$.*

Proof: Since $\sigma^t \mathcal{M} = \mathcal{M}$ for all $t \in \mathbb{R}$, and for all $G \in \text{span}\{\sigma^t | \ t \in \mathbb{R}\}$ $G(\mathcal{M}) \subseteq \mathcal{M}$. Let $\mu \in ba_c(\mathbb{R})$ and $(G_n)_{n \in \mathbf{N}}$ be a sequence in $\text{span}\{\sigma^t | \ t \in \mathbb{R}\}$ such that $G_n \to C_\mu$ strongly. Then $C_\mu f = \lim_{n \to \infty} G_n f \in \mathcal{M}$. □

Lemma 3.6 *Let K be a closed translation-invariant operator in $C(\mathbb{R})$ with domain $\mathcal{D}(K)$. Then for all $\mu \in ba_c(\mathbb{R})$, $C_\mu(\mathcal{D}(K)) \subseteq \mathcal{D}(K)$ and $C_\mu K f = K C_\mu f$ for all $f \in \mathcal{D}(K)$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and $(G_n)_{n\in\mathbb{N}}$ be a sequence in span$< \{\sigma^t | t \in \mathbb{R}\} >$ such that for all $h \in C(\mathbb{R})$, $G_n h \to C_\mu h$. Then by definition 3.4 we have, if $f \in \mathcal{D}(K)$ then for all n, $G_n f \in \mathcal{D}(K)$ and $KG_n f = G_n K f$, $f \in \mathcal{D}(K)$. For $f \in \mathcal{D}(K)$

$$\begin{cases} G_n f & \to C_\mu f \ \ (n \to \infty) \\ KG_n f & \to C_\mu K f \ \ (n \to \infty) \end{cases} \quad \text{both in } C(\mathbb{R})\text{-sense}$$

and so, since K is a closed operator it follows that $C_\mu f \in \mathcal{D}(K)$ and $KC_\mu f = C_\mu K f$. □

For $n \in \mathbb{N}$ by $C(\mathbb{R}, \mathbb{C}^n)$ we denote the space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}^n$. So each $f \in C(\mathbb{R}, \mathbb{C}^q)$ is $f = (f_1, ..., f_n)$, where $f_j \in C(\mathbb{R})$, $j = 1, ..., n$. It is natural to endow $C(\mathbb{R}, \mathbb{C}^n)$ with the seminorms (cf. the introduction)

$$q_{n,m}(f) = \sup_{t\in[-m,m]} |f(t)|_n, (m \in \mathbb{N}, f \in C(\mathbb{R}))$$

Definition 3.7 Let V be a function space in which the shift-operators σ^t, $t \in \mathbb{R}$ are well defined. If $f = (f_1, ..., f_n) \in V^n$ then we define

$$\sigma_n^t f := (\sigma^t f_1, ..., \sigma^t f_n)$$

For each M in the matrix ring $\mathcal{L}^{p\times n}(ba_c(\mathbb{R}))$,

$$M = (\mu_{ij})_{i=1,j=1}^{p,n}$$

we define the linear mapping C_M from $C(\mathbb{R}, \mathbb{C}^n)$ into $C(\mathbb{R}, \mathbb{C}^p)$ by

$$C_M f = (\sum_{j=1}^{n} C_{\mu_{1j}} f_j, ..., \sum_{j=1}^{n} C_{\mu_{pj}} f_j)$$

It is not hard to check that C_M is a continuous linear mapping from $C(\mathbb{R}, \mathbb{C}^n)$ into $C(\mathbb{R}, \mathbb{C}^p)$ which satisfies

$$\sigma_p^t C_M = C_M \sigma_n^t$$

The following characterization is natural in comparison with lemma 2.6.

Theorem 3.8 *Let K from $C(\mathbb{R}, \mathbb{C}^n)$ into $C(\mathbb{R}, \mathbb{C}^p)$ be a continuous linear operator. Then $K\sigma_n^t = \sigma_p^t K$ for all $t \in \mathbb{R}$ if and only if $K = C_M$ for some $M \in \mathcal{L}^{p\times n}(ba_c(\mathbb{R}))$.*

Before we give the proof we need an auxiliary result.

Lemma 3.9 *Let $F : C(\mathbb{R}, \mathbb{C}^n) \to \mathbb{C}^p$ be a continuous linear mapping. Then there is $M \in \mathcal{L}^{p\times n}(ba_c(\mathbb{R}))$, $M = (\mu_{ij})_{i=1,j=1}^{p,n}$ such that*

$$Ff = (\sum_{j=1}^{n} < f_j, \mu_{1j} >, ..., \sum_{j=1}^{n} < f_j, \mu_{pj} >) = (C_M f)(0)$$

Proof: Let $\beta_1, ..., \beta_p$ denote the standard base of $\mathbb{C}^p$. Then there are linear functions F_{ij} on $C(\mathbb{R})$ such that

$$Ff = \sum_{i=1}^{p} \sum_{j=1}^{n} F_{ij}(f_j)\beta_i$$

Since F is continuous, the F_{ij} are continuous. So $F_{ij}(f) = (f, \mu_{ij})$ for certain $\mu_{ij} \in ba_c(\mathbb{R})$. □

Now we apply this lemma in the proof of theorem 3.8.

Proof: Let $K : C(\mathbb{R}, \mathbb{C}^q) \to C(\mathbb{R}, \mathbb{C}^p)$ satisfy the requirements. Then $F : C(\mathbb{R}, \mathbb{C}^q) \to \mathbb{C}^p$ defined by $Ff = (Kf)(0)$ is continuous and linear. So that $(Kf)(0) = (C_M f)(0)$ with $M = (\mu_{ij})_{i=1,j=1}^{p,n}$. □

It will be clear to the reader that the matrix ring $\mathcal{L}^{p\times n}(ba_c(\mathbb{R}))$, which is non-commutative, can be treated much similar to the ring $ba_c(\mathbb{R})$. We mention only a few results in this direction:

1. By $ba_c(\mathbb{R}, \mathbb{C}^{p\times n})$ we denote all right-continuous matrix-valued functions M from $\mathbb{R}$ into $\mathbb{C}^{p\times n}$ for which there exists a $K > 0$ such that for all $-\infty < t_0 < t_1 < ... < t_n < \infty$

$$\sum_{j=1}^{n} |M(t_j) - M(t_{j-1})|_{p\times n} \leq K$$

and

$$\begin{cases} M(t) = 0 & t \leq -T \\ M(t) = M(T) & t \geq T \end{cases}$$

Then $\mathcal{L}^{p\times n}(ba_c(\mathbb{R}))$ and $ba_c(\mathbb{R}, \mathbb{C}^{p\times n})$ can be identified.

2. The space $ba_c(\mathbb{R}, \mathbb{C}^{1\times n})$ represents the dual of $C(\mathbb{R}, \mathbb{C}^n)$.

3. Define the Fourier transform $\mathcal{F}_M$ of $M \in ba_c(\mathbb{R}, \mathbb{C}^{p\times n})$ componentwise. Then for $v \in \mathbb{C}^n$ and $\omega \in \mathbb{C}$

$$M(e_\omega \otimes v) = \mathcal{F}_M(\omega)v$$

4. The Fouriertransform $\mathcal{F}_M$ is an analytic matrix valued function which is of exponential type 1 and bounded on the real axis. In fact there are $A > 0$ and $a > 0$ such that

$$|(\mathcal{F}_M)(\omega)|_{p\times n} \leq \ A \cdot e^{a|Im(\omega)|}$$

4 Smoothing properties of convolution operators

For convenience we set $D := \frac{d}{dt}$. For each $k \in \mathbb{N}$ the space $C^k(\mathbb{R})$ consists of all k-times continuously differentiable functions from $\mathbb{R}$ into $\mathbb{R}$. The Fréchet-topology on $C^k(\mathbb{R})$ is brought about by the seminorms

$$q_m^{(k)}(f) = \sum_{j=0}^{k} q_m(D^j f)$$

It is clear that for each polynomial p with degree $p = d$, $p(D)$ is a continuous linear mapping from $C^k(\mathbb{R})$ into $C^{k-d}(\mathbb{R})$ for $k \geq d$. For convenience we introduce the Volterra-integral operators $I_j(\lambda)$, $j = 0, 1, \ldots, \lambda \in \mathbb{R}$

$$(I_j(\lambda)f)(t) := \int_0^t e^{\lambda(t-\tau)} \frac{(t-\tau)^{j-1}}{(j-1)!} f(\tau)d\tau$$

where $t \in \mathbb{R}$ anf $f \in C(\mathbb{R})$. We observe that

$$I_j(\lambda) = e^{\lambda t} I_j(0) e^{-\lambda t}$$

and

$$I_{j_1}(\lambda) I_{j_2}(\lambda) = I_{j_1+j_2}(\lambda)$$

Straightforward estimations show that $I_j(\lambda)$ maps $C^k(\mathbb{R})$ into $C^{k+j}(\mathbb{R})$. Moreover $(D-\lambda)^j I_j(\lambda) = I$ the identity mapping. We summarize as follows.

Lemma 4.1 *Let p be a monic polynomial with $p(\lambda) = \prod_{j=1}^{s}(\lambda - \lambda_j)^{j_r}$. Define*

$$S = p(D) \quad d = \sum_{r=1}^{s} j_r \quad \text{and} \quad J = \prod_{r=1}^{s} I_{j_r}(\lambda_r)$$

Then S maps $C^{k+d}(\mathbb{R})$ into $C^k(\mathbb{R})$, J maps $C^k(\mathbb{R})$ into $C^{k+d}(\mathbb{R})$ and $SJ = I$.

Remark 4.2 Since $JSJS = JS$, $P = I - JS$ is a projection mapping in $C^{k+d}(\mathbb{R})$ onto $\ker(S)$, along Range(J), and $C^{k+d}(\mathbb{R}) = \ker(S) \oplus \text{Range}(J)$. We observe that $\ker(S)$ is finite dimensional and Range(J) is closed. □

Lemma 4.3 *Let p be a polynomial of degree d. Then the linear mapping $p(D)$ from $C(\mathbb{R})$ into $C(\mathbb{R})$ with domain $C^d(\mathbb{R})$ is closed.*

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C^d(\mathbb{R})$ such that

$$\begin{cases} u_n & \to u \ (n \to \infty) \\ p(D)u_n & \to v \ (n \to \infty) \end{cases} \quad \text{both in } C(\mathbb{R})\text{-sense}$$

We have to prove $u \in C^d(\mathbb{R})$ and $p(D)u = v$. Let J be as indicated and $Q = Jp(D)$. Then $Qu_n \to Qu$ and $Qu_n \to Jv$ so that $Qu = Jv \in C^d(\mathbb{R})$, whence

$$u = Pu + Qu \in \ker(S) + C^d(\mathbb{R}) \subseteq C^d(\mathbb{R})$$

since $\ker(S) \subseteq C^\infty(\mathbb{R})$. Moreover

$$p(D)u = p(D)Qu = p(D)Jv = v$$

□

Corollary 4.4 *Let p be a polynomial of degree d. Then for all $f \in C^d(\mathbb{R})$ $C_\mu p(D)f = p(D)C_\mu f$.*

Proof: By lemma 3.6. □

Now we are going from smoothness properties of the element $\mu \in ba_c(\mathbb{R})$ to smoothing properties of the corresponding C_μ. For that we introduce the notion of weakly differentiability for the elements of $ba_c(\mathbb{R})$.

Definition 4.5 Let $k \in \mathbb{N}$. A function $\mu \in ba_c(\mathbb{R})$ is said to be *k-times weakly differentiable* if

$$(i\omega)^k \mathcal{F}_\mu(\omega) = \mathcal{F}_\nu(\omega)$$

for some $\nu \in ba_c(\mathbb{R})$.

The weakly differentiability of elements of $ba_c(\mathbb{R})$ arises in the following way.

Lemma 4.6 *Let* $\nu \in ba_c(\mathbb{R})$ *with* $< p_j, \nu >= 0,\ j = 0, 1, ..., k-1$ *where* $p_j(t) = t^j$. *Then*

$$\nu_{[k]} : t \mapsto \int_{-\infty}^{t} \frac{(t-\tau)^{k-1}}{(k-1)!} d\nu(\tau)$$

belongs to $ba_c(\mathbb{R})$ *and is k-times weakly differentiable. We have*

$$\mathcal{F}_{\nu_{[k]}}(\omega) = (i\omega)^k \mathcal{F}_\nu(\omega)$$

Proof: The condition on ν ensures that $\nu_{[k]}(t) = 0$ for t sufficiently large. Since $\nu_{[k]} \in C^{k-1}(\mathbb{R})$ we get $\nu_{[k]} \in ba_c(\mathbb{R})$. Further a straightforward computation of the Fourier transform $\mathcal{F}_{\nu_{[k]}}$ gives the desired result. □

Corollary 4.7 *Let* $\mu \in ba_c(\mathbb{R})$ *be k-times weakly differentiable with k-th derivative* ν. *Then*

$$\mu(t) = \int_{-\infty}^{t} \frac{(t-\tau)^{k-1}}{(k-1)!} d\nu(\tau)$$

and so $\mu \in ba_c(\mathbb{R}) \cap C^{k-1}(\mathbb{R})$.

Theorem 4.8 *The following statements are equivalent for* $\mu \in ba_c(\mathbb{R})$

1. μ *is k-times weakly differentiable*
2. C_μ *maps* $C(\mathbb{R})$ *into* $C^k(\mathbb{R})$ *continuously*
3. *For each polynomial of degree k there exists* $\rho \in ba_c(\mathbb{R})$ *such that*

$$\forall f \in C^k(\mathbb{R}) :\ C_\mu p(D) f = C_\rho f$$

Proof:
(1⇒2): Then for all $f \in \text{span}\{e_\omega | \omega \in \mathbb{C}\}$ $D^k C_\mu f = C_\nu f$. Let $f \in C(\mathbb{R})$

and $(f_n)_{n\in\mathbb{N}}$ be a sequence in span$\{e_\omega | \omega \in \mathbb{C}\}$ such that $f_n \to f$. Then $D^k C_\mu f_n = C_\nu f_n \to C_\nu f$ and

$$\begin{cases} C_\mu f_n & \to C_\mu f \quad (n \to \infty) \\ D^k C_\mu f_n & \to C_\nu f \quad (n \to \infty) \end{cases}$$

Since D^k is a closed operator from $C(\mathbb{R})$ into $C(\mathbb{R})$ we have $C_\mu f \in \mathcal{D}(D^k) = C^k(\mathbb{R})$ and $D^k C_\mu f = C_\nu f$.
(2⇒3): Since $p(D)C_\mu$ is a continuous linear mapping from $C(\mathbb{R})$ into $C(\mathbb{R})$ with $\sigma^t p(D)C_\mu = p(D)C_\mu \sigma^t$ we have $p(D)C_\mu = C_\rho$ for some $\rho \in ba_c(\mathbb{R})$.
(3⇒1): Take $p(\lambda) = \lambda^k$. Then there exists $\nu \in ba_c(\mathbb{R})$ such that $D^k C_\mu e_\omega = C_\nu e_\omega$ and so $(i\omega)^k(\mathcal{F}_\mu)(\omega) = \mathcal{F}_\nu(\omega)$. □

5 Characterizations of convolution operators on $C^k(\mathbb{R})$.

We start by introducing two characterizations of the dual of $C^k(\mathbb{R})$ where $k \in \mathbb{N}$ is fixed. The first one is based on the Riemann's remainder formula.

Lemma 5.1 *A linear functional $\mathcal{L}$ on $C^k(\mathbb{R})$ is continuous if and only if for each $a \in \mathbb{R}$ there exists a polynomial p_a of degree $\leq k-1$ and a $\mu_a \in ba_c(\mathbb{R})$ such that*

$$\mathcal{L}(f) = (p_a(D)f)(a) + < D^k f, \mu_a >$$

Proof: The sufficiency part is clear. We prove only necessity. So let $\mathcal{L} : C^k(\mathbb{R}) \to \mathbb{C}$ be linear and continuous. For each $f \in C^k(\mathbb{R})$ we have

$$f = \sum_{j=0}^{k-1} \frac{(D^j f)(a)}{j!} \sigma^{-a} p_j + \sigma^{-a} I_k \sigma^a D^k f$$

So

$$\mathcal{L}(f) = \sum_{j=0}^{k-1} \frac{(D^j f)(a)}{j!} \mathcal{L}(\sigma^{-a} p_j) + (\mathcal{L} \circ \sigma^{-a} I_k \sigma^a)(D^k f)$$

Put $p_a(\lambda) = \sum_{j=0}^{k-1} \mathcal{L}(\sigma^{-a} p_j) \frac{\lambda^j}{j!}$ and observe that $\mathcal{L} \circ \sigma^{-a} I_k \sigma^a$ is a continuous linear functional on $C(\mathbb{R})$ so that

$$(\mathcal{L} \circ \sigma^{-a} I_k \sigma^a)(D^k f) = < D^k f, \mu_a >$$

for some $\mu_a \in ba_c(\mathbb{R})$. □

Remark 5.2

$$[\forall f \in C^k(I\!R) : (p_a(D)f)(a) + < D^k f, \mu_a >= 0] \Leftrightarrow [p_a = 0 \wedge \mu_a = 0]$$

□

This result shows the uniqueness of the representation of an element in $C^k(I\!R)'$

Now fix $\mathcal{L} \in C^k(I\!R)'$. Then there exists $m > 0$ and $C > 0$ such that

$$|\mathcal{L}(f)| \leq C q_m^k(f)$$

so that

$$|\mathcal{L}(e_\omega)| \leq C(\sum_{j=0}^{k} |\omega|^j) e^{m|Im(\omega)|} \leq \tilde{C}(1 + |\omega|^k) e^{m|Im(\omega)|} \tag{3}$$

Define the function $\hat{\mathcal{L}}$ by

$$\hat{\mathcal{L}}(\omega) = \mathcal{L}(e_\omega),\ \omega \in \mathbb{C}$$

Then from the above it follows that for each $a \in I\!R$, there exist $p_a \in \mathcal{P}$, $\mu_a \in ba_c(I\!R)$ such that

$$\hat{\mathcal{L}}(\omega) = p_a(-i\omega)e^{ia\omega} + (-i\omega)^k(\mathcal{F}_{\mu_a})(\omega)$$

and so $\hat{\mathcal{L}}$ is analytic, of exponential type 1 and polynomially bounded on the real axis. Now suppose $\hat{\mathcal{L}}$ has a finite number of zeroes, say l. Then we obtain

$$\hat{\mathcal{L}}(\omega) = p(-i\omega)e^{ic\omega}$$

for p a polynomial of degree l and $c \in I\!R$. Whence taking $a = c$ in lemma 5.1 yields

$$(-i\omega)^k \mathcal{F}_{\mu_c}(\omega) = [p(-i\omega) - p_c(-i\omega)]e^{ic\omega}$$

Since $\mathcal{F}_{\mu_c}$ has no zeroes or countably many it follows that

$$p(-i\omega) = (-i\omega)^k + p_c(-i\omega)$$

and

$$\mathcal{F}_{\mu_c}(\omega) = e^{ic\omega}$$

whence $l \leq k$ and $\mathcal{L}(f) = (p(D)f)(c)$. We come to the following conclusion.

Lemma 5.3 *Let $\mathcal{L} \in C^k(\mathbb{R})'$. Then the analytic function $\omega \mapsto \mathcal{L}(e_\omega)$, $\omega \in$ $\mathbb{C}$ has a finite number l of zeroes with $l \leq k$, counting with multiplicity, in which case $\mathcal{L}(f) = (p(D)f)(c)$ for some $p \in \mathcal{P}$ with degree(p)$= l$ or countably many zeroes.*

Lemma 5.4 *Let $K : C^k(\mathbb{R}) \to C(\mathbb{R})$ be continuous. Then $K\sigma^t = \sigma^t K$ for all $t \in \mathbb{R}$ if and only if there exists $\mathcal{L} \in C^k(\mathbb{R})'$ such that $\mathcal{L}(f) = (Kf)(t)$, $t \in \mathbb{R}$.*

Proof: To prove '$\Rightarrow$' use the fact that $(Kf)(0)$ is a continuous linear functional and the shift invariance of K. Conversely, to prove '$\Leftarrow$' we can use the explicit form for $\mathcal{L}$ as given in lemma 5.1. □

On the basis of the above auxiliary result we get a second characterization of the dual of $C^k(\mathbb{R})$.

Theorem 5.5 *Each continuous linear functional $\mathcal{L}$ on $C^k(\mathbb{R})$ is of the form*

$$\mathcal{L}(f) = < p(D)f, \mu >$$

for some $\mu \in ba_c(\mathbb{R})$ and polynomial p of degree $\leq k$.

Proof: Define $K : C^k(\mathbb{R}) \to C(\mathbb{R})$ by $(Kf)(t) = \mathcal{L}(\sigma^t f)$. Then $\ker(K)$ is a closed translation-invariant subspace of $C^k(\mathbb{R})$. We have

$$e_\omega \in \ker(K) \Leftrightarrow \hat{\mathcal{L}}(\omega) = 0$$

If $\ker(K)$ is finite dimensional we can apply lemma 5.3. If not, we can select $\omega_1, ..., \omega_k$ mutually different (!!) such that $\omega_j \in \ker \hat{\mathcal{L}}$ whence $e_{\omega_j} \in \ker(K)$. Put

$$p(\lambda) = \prod_{j=1}^{k} (\lambda - \omega_j)$$

According to remark 4.2 there exists $J : C(\mathbb{R}) \to C^k(\mathbb{R})$ is such that $p(D)J = I$. Defining the projection $P := \mathbf{I} - Jp(D)$ we have

$$\ker(K) = P(\ker(K)) + (\mathbf{I} - P)(\ker(K))$$

and $KP = 0$. We show that KJ is a shift-invariant operator.

$$\begin{aligned}
\sigma^t J - J\sigma^t &= \sigma^t JSJ - J\sigma^t SJ \\
&= (\sigma^t JS - JS\sigma^t)J \\
&= (\sigma^t(\mathbf{I} - P) - (\mathbf{I} - P)\sigma^t)J \\
&= (P\sigma^t - \sigma^t P)J
\end{aligned}$$

Herewith

$$\begin{aligned} K(\sigma^t J - J\sigma^t) &= \sigma^t KJ - KJ\sigma^t \\ &= K(P\sigma^t - \sigma^t P)J \\ &= 0 \end{aligned}$$

Now $KJ = C_\mu$ and $K = p(D)KJ = p(D)C_\mu$. □

We mention the following interesting results:

Corollary 5.6 *Let $K : C^k(\mathbb{R}) \to C(\mathbb{R})$ be a translation-invariant continuous linear mapping. Then $K = p(D)C_\mu$ for some polynomial p and $\mu \in ba_c(\mathbb{R})$.*

Corollary 5.7 *Let $K : C^k(\mathbb{R}) \to C^k(\mathbb{R})$ be a translation-invariant continuous linear mapping. Then $K = C_\nu$ for some $\nu \in ba_c(\mathbb{R})$.*

Proof: $K = p(D)C_\mu$ for some polynomial p and $\mu \in ba_c(\mathbb{R})$ and so with J such that $p(D)J = I$, we have $KJ = C_\mu : C(\mathbb{R}) \to C^k(\mathbb{R})$. I.e. $\mu \in ba_c^{(k)}(\mathbb{R})$ and $p(D)C_\mu = C_\nu$. □

6 The convolution ring $\mathcal{E}'(\mathbb{R})$

By $\mathcal{E}(\mathbb{R})$ we denote the space of all infinitely differentiable functions on $\mathbb{R}$ endowed with the intersection topology induced by the spaces $C^n(\mathbb{R})$, i.e.

$$\mathcal{E}(\mathbb{R}) = \bigcap_{n=0}^{\infty} C^n(\mathbb{R})$$

So a seminorm on $\mathcal{E}(\mathbb{R})$ is continuous if and only if it extends continuously to a seminorm on $C^n(\mathbb{R})$ for some $n \in \mathbb{N}$. As a natural consequence we have

Lemma 6.1 *Each continuous linear functional F on $\mathcal{E}(\mathbb{R})$ is the restriction to $\mathcal{E}(\mathbb{R})$ of a continuous linear functional $\tilde{F}$ on $C^n(\mathbb{R})$ for some $n \in \mathbb{N}$ dependent on F.*

Using the characterization of continuous linear functionals on $C^n(\mathbb{R})$ as presented in theorem 5.5 we get the following result:

Theorem 6.2 *1. Let $\mu \in ba_c(\mathbb{R})$ and let p denote a polynomial. Then the linear functional*

$$f \mapsto < p(D)f, \mu >, \quad f \in \mathcal{E}(\mathbb{R})$$

is continuous on $\mathcal{E}(\mathbb{R})$.

2. Let F be a continuous linear functional on $\mathcal{E}(\mathbb{R})$. Then there exists $\mu \in ba_c(\mathbb{R})$ and a polynomial p such that

$$F(g) =< p(D)g, \mu >$$

for all $g \in \mathcal{E}(\mathbb{R})$.

For convenience we denote the supposed correspondence between $\mathcal{E}'(\mathbb{R})$ and $ba_c(\mathbb{R}) \oplus P$ by $F = [\mu; p]$. But be aware, this correspondence is not linear and not one-one.

Now let $F = [\mu; p]$. Then

$$F(\sigma^t g) = (C_\mu p(D)g)(t), \; t \in \mathbb{R}, g \in \mathcal{E}(\mathbb{R})$$

and so $[\mu; p]$ is linked with the translation invariant operator $p(D)C_\mu$ which maps $\mathcal{E}(\mathbb{R})$ into $\mathcal{E}(\mathbb{R})$ continuously. If, conversely, $K : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is a continuous linear mapping with $\sigma^t K = K\sigma^t$ for all $t \in \mathbb{R}$, then with $\mu \in ba_c(\mathbb{R})$ and $p \in P$ such that

$$[\mu; p](g) = (Kg)(0), \; g \in \mathcal{E}(\mathbb{R})$$

we get

$$K = p(D)C_\mu$$

It yields the natural convolution structure in $\mathcal{E}'(\mathbb{R})$, without the use of distribution theory.

Definition 6.3 *Let $F_1 = [\mu_1; p_1]$ and $F_2 = [\mu_2; p_2]$. Then*

$$F_1 * F_2 := [\mu_1 * \mu_2; p_1 p_2]$$

This is a natural definition, since for all $t \in \mathbb{R}$ and $g \in \mathcal{E}(\mathbb{R})$

$$(F_1 * F_2)(\sigma^t g) = (\underbrace{p_1(D)p_2(D)}_{=(p_1p_2)(D)} C_{\mu_1 * \mu_2} g)(t)$$

With the above convolution, $\mathcal{E}'(\mathbb{R})$ is a convolution ring with identity and without zero divisors. In connection we recall the Paley-Wiener result:

For an analytic function ψ there exists $F \in \mathcal{E}'(\mathbb{R})$ such that $\psi(\omega) = F(e_\omega)$, $\omega \in \mathbf{C}$, if and only if there are $A > 0, B > 0$ and $N \in \mathbf{N}$ such that

$$|\psi(\omega)| \leq A(1+|\omega|)^N e^{B|Im(\omega)|} \ \omega \in \mathbf{C}$$

The Paley-Wiener characterization is nicely in line with our results, since

$$[\mu; p](e_\omega) = p(i\omega)(\mathcal{F}_\mu)(\omega)$$

(cf. [1], theorem 10.2.2).

For $\mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ we see from theorem 4.8, that C_μ is a continuous linear mapping from $C(\mathbb{R})$ into $\mathcal{E}(\mathbb{R})$. For completeness, note that $\mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ if and only if there exists $\varphi \in \mathcal{D}(\mathbb{R})$ such that

$$\mu(t) = \int_{-\infty}^{t} \varphi(\tau)d\tau, \ t \in \mathbb{R}$$

If $\varphi \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(t)dt = 1$, then $\varphi_n(t) = n\varphi(nt)$ satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n(t)f(t)dt = f(0), \ f \in C(\mathbb{R})$$

Put $\mu_n(t) = \int_{-\infty}^{t} \varphi_n(\tau)d\tau$. Then $\mu_n \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and the sequence $(\mu_n)_{n\in\mathbf{N}}$ is called an *approximate identity*, since $C_{\mu_n}f \to f$ for all $f \in C(\mathbb{R})$. With the aid of this concept a number of results can be proved.

Theorem 6.4 *Let $\mathcal{M}$ be a closed subspace of $C(\mathbb{R})$ such that $\sigma^t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \in \mathbb{R}$. Then $\mathcal{M} \cap C^\infty(\mathbb{R})$ is dense in $\mathcal{M}$.*

Proof: Let $(\mu_n)_{n\in\mathbf{N}}$ be an approximate identity in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then $C_{\mu_n}(\mathcal{M}) \subseteq \mathcal{M}$ by lemma 3.5 and $C_{\mu_n}(\mathcal{M}) \subseteq C^\infty(\mathbb{R})$ by the preceding remark. Hence $C_{\mu_n}(\mathcal{M}) \subseteq [\mathcal{M} \cap C^\infty(\mathbb{R})]$. Since $C_{\mu_n}f \to f$, $\mathcal{M} \cap C^\infty(\mathbb{R})$ is dense in $\mathcal{M}$. □

Theorem 6.5 *Let K be a closed translation-invariant operator from $\mathcal{D}(K) \subseteq C(\mathbb{R})$ into $C(\mathbb{R})$, such that $C^\infty(\mathbb{R}) \subset \mathcal{D}(K)$. Then $K = p(D)C_\mu$ for some $p \in P$ and $\mu \in ba_c(\mathbb{R})$.*

Proof: The restriction $K|_{C^\infty(\mathbb{R})} : C^\infty(\mathbb{R}) \to C(\mathbb{R})$ is continuous and translation invariant and so $K|_{C^\infty(\mathbb{R})} = (p(D)C_\mu)|_{C^\infty(\mathbb{R})}$. Now let $(\mu_n)_{n\in\mathbb{N}}$ be an approximate identity in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Let $f \in \mathcal{D}(K)$. Then $C_{\mu_n} f \in C^\infty(\mathbb{R}) \subseteq \mathcal{D}(K)$ and $C_{\mu_n} f \to f$ as $n \to \infty$. So

$$KC_{\mu_n} f = C_{\mu_n} Kf \to Kf$$

and

$$KC_{\mu_n} f = p(D)C_\mu C_{\mu_n} f$$

It follows that $f \in \mathcal{D}(p(D)C_\mu)$ and

$$p(D)C_\mu f = Kf$$

Similarly for $f \in \mathcal{D}(p(D)C_\mu)$ we get

$$f \in \mathcal{D}(K) \text{ and } Kf = p(D)C_\mu f$$

Note that for p a polynomial of degree n

$$\mathcal{D}(p(D)C_\mu) = \{f \in C(\mathbb{R}) |\ C_\mu f \in C^n(\mathbb{R})\}$$

□

Next we want to characterize closed linear mappings from $C(\mathbb{R}, \mathbb{C}^n)$ into $C(\mathbb{R}, \mathbb{C}^p)$, with $\mathcal{E}(\mathbb{R}, \mathbb{R}^n)$ in their domain.

Definition 6.6 By $\mathcal{P}\ltimes\mathcal{M}$ we define a class of linear operators from $\mathcal{E}(\mathbb{R}, \mathbb{R}^n)$ into $\mathbb{C}^p$ which can be represented as a $p \times n$-matrix operator with entries of the form $p_{ij}\mu_{ij}$. An element of that class will be denoted by $p \ltimes \mu$ where $\mathbf{p} \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\mu \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$.

Definition 6.7 By $\mathcal{P} \ltimes C_\mathcal{M}$ we define a class of linear operators from $\mathcal{E}(\mathbb{R}, \mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}, \mathbb{R}^p)$ which can be represented as a matrix operator with entries of the form $p_{ij}C_{\mu_{ij}}$. An element of that class will be denoted by $\mathbf{p}\ltimes C_\mu$ where $\mathbf{p} \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\mu \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$.

Remark 6.8 Note that we do not claim that in either one of the two classes of operators the elements can be written as a product of two matrices with on one side all the polynomial elements and on the other side the elements in $ba_c(\mathbb{R})$. □

An almost repetition of the proof of lemma 3.9 yields

Lemma 6.9 *Let $\mathcal{L} : \mathcal{E}(\mathbb{R}, \mathbf{C}^n) \to \mathbf{C}^p$ be a continuous linear mapping. Then $\mathcal{L} = \mathbf{p} \ltimes \boldsymbol{\mu}$ for some $\mathbf{p} \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\boldsymbol{\mu} \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$.*

We apply this lemma in the following theorem.

Theorem 6.10 *Let K from $\mathcal{E}(\mathbb{R}, \mathbf{C}^n)$ into $\mathcal{E}(\mathbb{R}, \mathbf{C}^p)$ be a continuous linear mapping satisfying $K\sigma_n^t = \sigma_p^t K$ for all $t \in \mathbb{R}$. Then there are $\mathbf{p} \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\boldsymbol{\mu} \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$ such that*

$$Kf = \mathbf{p} \ltimes C_{\boldsymbol{\mu}} \tag{4}$$

Proof: The linear mapping $f \mapsto (\mathcal{L}f)(0)$ from $\mathcal{E}(\mathbb{R}, \mathbf{C}^n)$ into $\mathbf{C}^p$ is continuous. Hence

$$(\mathcal{L}f)(0) = \mathbf{p} \ltimes \boldsymbol{\mu}$$

for suitable $\mathbf{p} \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\boldsymbol{\mu} \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$. □

The following theorem is a generalization of theorem 6.5.

Theorem 6.11 *Let K be a closed linear mapping from $\mathcal{D}(K) \subseteq C(\mathbb{R}, \mathbf{C}^n)$ into $C(\mathbb{R}, \mathbf{C}^p)$ with $\mathcal{E}(\mathbb{R}, \mathbf{C}^n) \subseteq \mathcal{D}(K)$. Then*

$$K = p \ltimes C_\mu$$

for certain $p \in \mathcal{M}^{p\times n}(\mathcal{P})$ and $\mu \in \mathcal{M}^{p\times n}(ba_c(\mathbb{R}))$.

Proof: We observe that $KC_{\mu I_n} = C_{\mu I_p}K$ for all $\mu \in ba_c(\mathbb{R})$, since K is closed and $\text{span}\{\sigma_r^t | t \in \mathbb{R}\}$ is strongly dense in the set $\{C_{\mu I_r} | \mu \in ba_c(\mathbb{R})\}$, $r = p, n$. With this in mind, it is clear that the statement can be proved with similar arguments as in the proof of theorem 6.5. □

The results in this paper can be applied in the field of system theory. Their relevance will be shown in a future paper where we treat the problem of finding descriptions of systems that are shift-invariant subspaces.

References

[1] **Friedlander,F.G.** Introduction to the theory of distributions, Cambridge University Press (1982).

[2] **Rudin,W.** Real and Complex Analysis, 3rd edition, McGraw-Hill (1986).

[3] **Schwartz,L.** Théorie des distributions, Vols. 1 and 2. Hermann et Cie, Paris (1950-1951).