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Abstract.

Starting with a Hilbert space $L_2(\mathbb{R},\mu)$ we introduce the dense subspace $R(L_2(\mathbb{R},\mu))$ where R is a positive self-adjoint Hilbert-Schmidt operator on $L_2(\mathbb{R},\mu)$. For the space $R(L_2(\mathbb{R},\mu))$ a measure theoretical Sobolev lemma is proved. The results for the spaces of type $R(L_2(\mathbb{R},\mu))$ are applied to nuclear analyticity spaces $S_{X,A} = \bigcup_{t>0} e^{-tA}(X)$ where e^{-tA} is a Hilbert-Schmidt operator on the Hilbert space X for each $t > 0$. We solve the so-called generalized eigenvalue problem for a general self-adjoint operator T in X.

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Introduction

Let $L_2(\mathbb{R},\mu)$ denote the Hilbert space of equivalence classes of square integrable functions on $\mathbb{R}$ with respect to some Borel measure μ. In this paper we only consider finite nonnegative Borel measures. The elements of $L_2(\mathbb{R},\mu)$ will be denoted by $[\cdot]$.

Consider the orthonormal basis $([\varphi_k])_{k\in\mathbb{N}}$ in $L_2(\mathbb{R},\mu)$. Then every $[f] \in L_2(\mathbb{R},\mu)$ can be written as

$$(0.1) \qquad [f] = \sum_{k=1}^{\infty} ([f],[\varphi_k])[\varphi_k]$$

where $(\cdot,\cdot)$ denotes the inner product of $L_2(\mathbb{R},\mu)$. The series (0.1) converges in L_2-sense, i.e.

$$(0.2) \qquad \int_{\mathbb{R}} |\hat{f} - \sum_{k=1}^{N} ([f],[\varphi_k])\hat{\varphi}_k|^2 d\mu \to 0 \quad \text{as } N \to \infty$$

for all $\hat{f} \in [f]$ and all $\hat{\varphi}_k \in [\varphi_k]$, $k \in \mathbb{N}$. However, in general, not very much can be said about the possible convergence of the series (0.1).

For a positive self-adjoint Hilbert-Schmidt operator R on $L_2(\mathbb{R},\mu)$, the dense subspace $D(R^{-1})$ of $L_2(\mathbb{R},\mu)$ is defined by

$$(0.3) \qquad [f] \in D(R^{-1}) \Leftrightarrow \sum_{k=1}^{\infty} \rho_k^{-2} \, |([f],[\varphi_k])|^2 < \infty$$

where $\rho_k > 0$, $k \in \mathbb{N}$, are the eigenvalues of R and $[\varphi_k]$ its eigenvectors.

In [EG_{II}] we have shown that for any choice of representants $\tilde{\varphi}_k \in [\varphi_k]$, $k \in \mathbb{N}$, there exists a null set $\tilde{N}_\mu$ such that for all $[f] \in D(R^{-1})$ the series

$$(0.4) \qquad \sum_{k=1}^{\infty} ([f],[\varphi_k])\, \tilde{\varphi}_k$$

converges pointwise outside the set $\widetilde{N}_\mu$. In the present paper we make the canonical choice

$$(0.5) \qquad \widetilde{\varphi}_k(x) = \lim_{h\downarrow 0} \mu([x-h,x+h])^{-1} \int_{x-h}^{x+h} \widetilde{\varphi}_k \, d\mu \, .$$

It will lead to a measure theoretical version of Sobolev's lemma.

The first sections of this paper contain the measure theoretical results which we need to solve the so-called generalized eigenvalue problem for self-adjoint operators.

In order to get a theory of generalized eigenfunctions we need a theory of generalized functions, of course. Here we employ De Graaf's theory [G]. This theory is based on the triplet

$$(0.6) \qquad S_{X,A} \subset X \subset T_{X,A}$$

where A is a nonnegative self-adjoint operator in a Hilbert space X. The space $S_{X,A}$ is called an analyticity space and $T_{X,A}$ a trajectory space; they are each other's strong duals. We give a short summary of this theory in the preliminaries.

Here we look at nuclear analyticity spaces $S_{X,A}$. We shall prove that to any self-adjoint operator T in the Hilbert space X there can be associated a total set of generalized functions in $T_{X,A}$ which together establish a so-called Dirac basis. (Cf. [EG_{II}] for the terminology.) If T is also a continuous linear mapping from $S_{X,A}$ into itself, then each element of this Dirac basis is a generalized eigenfunction of T. In addition it follows that to almost each point with multiplicity m in the spectrum there corresponds at least m non-trivial independent generalized eigenfunctions. In order to obtain this result we employ the commutative multiplicity theory for self-adjoint operators. (Cf. [Br] for this theory.)

Preliminaries

In a Hilbert space X consider the evolution equation

$$\text{(p.1)} \qquad \frac{du}{dt} = -A\,u \quad , \quad t > 0$$

where A is a nonnegative unbounded self-adjoint operator. A solution F of (p.1) is called a trajectory if F satisfies

$$\text{(p.2.i)} \qquad \forall_{t>0} : F(t) \in X$$

$$\text{(p.2.ii)} \qquad \forall_{t>0}\forall_{\tau>0} : e^{-\tau A} F(t) = F(t+\tau)\,.$$

We remark that $\lim_{t\downarrow 0} F(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $T_{X,A}$. The space $T_{X,A}$ is considered as a space of generalized functions in [G]. The Hilbert space X is embedded in $T_{X,A}$ by means of emb : $X \hookrightarrow T_{X,A}$,

$$\text{(p.3)} \qquad \mathrm{emb}(w) : t \mapsto e^{-tA} w \quad , \quad w \in X\,.$$

The analyticity space $S_{X,A}$ is defined as the dense linear subspace of X consisting of smooth elements of the form $e^{-\tau A} w$ where $w \in X$ and $\tau > 0$. So $S_{X,A} = \bigcup_{t>0} e^{-tA}(X) = \bigcup_{n\in\mathbb{N}} e^{-\frac{1}{n}A}(X)$. We note that for each $f \in S_{X,A}$ there exists $\tau > 0$ with $e^{\tau A} f \in S_{X,A}$ and, also, that for each $F \in T_{X,A}$ and for all $t > 0$ we have $F(t) \in S_{X,A}$. The space $S_{X,A}$ is the test function space in [G].

In $T_{X,A}$ the topology can be described by the seminorms

(p.4) $\quad F \mapsto \|F(t)\|_X \quad , \quad F \in T_{X,A}$,

where $t > 0$. The space $T_{X,A}$ is a Frechet space. In $S_{X,A}$ we take the inductive limit topology. This inductive limit is not strict. A set of seminorms is produced in [G] which generates the inductive limit topology. The pairing $<\cdot,\cdot>$ between $S_{X,A}$ and $T_{X,A}$ is defined by

(p.5) $\quad <g,F> := (e^{\tau A} g, F(\tau))_X \quad , \quad g \in S_{X,A}, F \in T_{X,A}$.

Here $(\cdot,\cdot)$ denotes the inner product of X. Definition (p.5) makes sense for $\tau > 0$ sufficiently small. Due to the trajectory property it does not depend on the choice of τ. The spaces $S_{X,A}$ and $T_{X,A}$ are reflexive in the given topologies.

The space $S_{X,A}$ is nuclear if and only if A generates a semigroup of Hilbert-Schmidt operators on X. In this case A has an orthonormal basis of eigenvectors v_k, $k \in \mathbb{N}$, with eigenvalues λ_k. In addition, for all $t > 0$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges. It can be shown that $f \in S_{X,A}$ if and only if there exists $\tau > 0$ such that

(p.6) $\quad (f,v_k) = O(e^{-\lambda_k \tau})$

and $F \in T_{X,A}$ if and only if

(p.7) $\quad <v_k,F> = O(e^{\lambda_k t})$

for all $t > 0$. For examples of these spaces, see [G], [EG_I], [EGP].

1. A measure theoretical Sobolev lemma

Let μ denote a finite nonnegative Borel measure on $\mathbb{R}$. Let $([\varphi_k])_{k\in\mathbb{N}}$ be an orthonormal basis in $L_2(\mathbb{R},\mu)$ and let $(\rho_k)_{k\in\mathbb{N}}$ be an ℓ_2-sequence with $\rho_k > 0$, $k \in \mathbb{N}$. Let R denote the Hilbert-Schmidt operator on $L_2(\mathbb{R},\mu)$ which satisfies $R[\varphi_k] = \rho_k[\varphi_k]$, $k \in \mathbb{N}$. Then we define $D(R^{-1}) \subset L_2(\mathbb{R},\mu)$ by

$$[f] \in D(R^{-1}) :\Leftrightarrow \sum_{k\in\mathbb{N}} \rho_k^{-2} \, |([f],[\varphi_k])|^2 < \infty .$$

Here $(\cdot,\cdot)$ denotes the inner product of $L_2(\mathbb{R},\mu)$. The unbounded inverse R^{-1} with domain $D(R^{-1})$ is defined by

$$R^{-1}[f] = \sum_{k\in\mathbb{N}} \rho_k^{-1} \, ([f],[\varphi_k]) \, [\varphi_k] .$$

R^{-1} is a self-adjoint operator in $L_2(\mathbb{R},\mu)$. The sesquilinear form $(\cdot,\cdot)_\rho$,

$$([f],[g])_\rho = (R^{-1}[f],R^{-1}[g])$$

is an inner product in $D(R^{-1})$ and thus $D(R^{-1})$ becomes a Hilbert space. We note that the sequence $([f_n])_{n\in\mathbb{N}}$ converges to $[f]$ in $D(R^{-1})$ if and only if $(R^{-1}[f_n])_{n\in\mathbb{N}}$ converges to $R^{-1}[f]$ in $L_2(\mathbb{R},\mu)$.

Here we shall prove that in each class $[f] \in D(R^{-1})$ there can be chosen a canonical representant. This canonical choice takes out the continuous representant of each member of $D(R^{-1})$ if such a representant should exist. To this end, we first define the support of a measure.

(1.1) Definition.

The support of μ, denoted by $\mathrm{supp}(\mu)$, is defined by

$$\mathrm{supp}(\mu) := \{x \in \mathbb{R} \mid \forall_{h>0} : \mu([x-h,x+h]) > 0\} .$$

It is not hard to prove that $\mathrm{supp}(\mu)$ is the complement of the largest open set $\mathcal{O}$ for which $\mu(\mathcal{O}) = 0$. So the complement of $\mathrm{supp}(\mu)$ is a null set with respect to μ. (Cf. [E], p. 11.)

In the sequel the closed interval $[x-h,x+h]$ is denoted by $Q_h(x)$. Consider the following theorem.

(1.2) Theorem

Let $[w] \in L_1(\mathbb{R},\mu)$ and let $\hat{w} \in [w]$. Then there exists a null set $N([w])$ such that the limit

$$\tilde{w}(x) = \lim_{h \downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \hat{w}\, d\mu$$

exists for all $x \in \mathrm{supp}(\mu) \backslash N([w])$. The function $x \to \tilde{w}(x)$ can be extended to an everywhere defined representant of $[w]$ by taking $\tilde{w}(x) = 0$ for $x \in N([w]) \cup \mathrm{supp}(\mu)^*$. The representant w is independent of the choice of $\hat{w} \in [w]$.

Proof. Cf. [WZ], Theorem 10.49. $\square$

Since μ is a finite measure it follows that $L_2(\mathbb{R},\mu) \subset L_1(\mathbb{R},\mu)$. So by the previous theorem there exist null sets $N_{k,\mu}$ such that

$$(1.3) \qquad \tilde{\varphi}_k(x) = \lim_{h \downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \hat{\varphi}_k\, d\mu \quad , \quad x \in \mathrm{supp}(\mu) \backslash N_{k,\mu}$$

exists. If we define $\tilde{\varphi}_k(x) = 0$ for $x \in \text{supp}(\mu)^* \cup N_{k,\mu}$, then $\tilde{\varphi}_k$ is an everywhere defined representant of the class $[\varphi_k]$. The definition of $\tilde{\varphi}_k$ does not depend on the choice of $\hat{\varphi}_k \in [\varphi_k]$.

In order to prove our measure theoretical version of Sobolev's lemma we shall extend the null set $\bigcup_{k\in\mathbb{N}} N_{k,\mu}$. It is clear that the functions $|\tilde{\varphi}_k|^2$, $k \in \mathbb{N}$, and $\sum_{k\in\mathbb{N}} \rho_k^2 |\tilde{\varphi}_k|^2$ are integrable. So by Theorem (1.2) there exists a null set $\tilde{N}_\mu \supseteq \left(\bigcup_{k\in\mathbb{N}} N_{k,\mu}\right)$ with the property that for all $x \in \text{supp}(\mu)\backslash\tilde{N}_\mu$,

$$(1.4) \qquad |\tilde{\varphi}_k(x)|^2 = \lim_{h\downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_k|^2 \, d\mu$$

and

$$(1.5) \qquad \sum_{k=1}^{\infty} \rho_k^2 |\tilde{\varphi}_k(x)|^2 = \lim_{h\downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k\in\mathbb{N}} \rho_k^2 |\tilde{\varphi}_k|^2\right) d\mu \, .$$

For convenience we take $\tilde{\varphi}_k(x) = 0$ for $x \in \text{supp}(\mu)^* \cup \tilde{N}_\mu$. By (1.5) the following definition makes sense.

(1.6) <u>Definition</u>

We define $[\tilde{e}_x] \in D(R^{-1})$ by

$$[\tilde{e}_x] = \sum_{k=1}^{\infty} \rho_k^2 \, \overline{\tilde{\varphi}_k(x)} \, [\varphi_k] \, .$$

Note that $[\tilde{e}_x] = 0$ for $x \in \text{supp}(\mu)^* \cup \tilde{N}_\mu$.

The following lemma is fundamental for this paper.

(1.7) Lemma.

For $h > 0$ and $x \in \mathrm{supp}(\mu)\backslash\widetilde{N}_\mu$ we write

$$[e_x\{h\}] = \sum_{k=1}^{\infty} \rho_k^2 \left(\mu(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\widetilde{\varphi}_k(x)}\, d\mu\right) [\varphi_k] .$$

Then $[\widetilde{e}_x]$ satisfies

$$[\widetilde{e}_x] = \lim_{h\downarrow 0} [e_x\{h\}]$$

where the limit is taken in the norm topology of $D(R^{-1})$.

Proof. Let $x \in \mathrm{supp}(\mu)\backslash\widetilde{N}_\mu$ and let $\varepsilon > 0$. Then we first fix $k_0 \in \mathbb{N}$ so large that

$$(*) \qquad \sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\widetilde{\varphi}_k(x)|^2 < \varepsilon^2 .$$

Next, by the relations (1.3), (1.4) and (1.5) there exists $h_0 > 0$ so small that for all h, $0 < h < h_0$

$$(**) \qquad |\widetilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \widetilde{\varphi}_k \, d\mu| < \varepsilon \quad , \quad k = 1,\ldots,k_0$$

and, also,

$$(***) \qquad \sum_{k=k_0+1}^{\infty} \rho_k^2 \, \mu(Q_h(x))^{-1} \int_{Q_h(x)} |\widetilde{\varphi}_k|^2 \, d\mu < 2\varepsilon^2 .$$

Thus we obtain

$$\| [\widetilde{e}_x] - [e_x\{h\}] \|^2 =$$

$$= \left(\sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty}\right) \rho_k^2 \, |\widetilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \widetilde{\varphi}_k \, d\mu|^2 .$$

Now we have the following inequalities for $0 < h < h_0$. By (**)

$$\sum_{k=1}^{k_0} \rho_k^2 \, |\tilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu|^2 < \varepsilon^2 \sum_{k=1}^{\infty} \rho_k^2$$

and by (*) and (***)

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\tilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu|^2 \leq$$

$$\leq 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\tilde{\varphi}_k(x)|^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu|^2 <$$

$$< 2\varepsilon^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \, \mu(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_k|^2 \, d\mu < 6\varepsilon^2 .$$

It leads to the result

$$\| [\tilde{e}_x] - [\tilde{e}_x\{h\}] \|_\rho^2 < \varepsilon^2 \left(6 + \sum_{k=1}^{\infty} \rho_k^2\right) .$$

Since $\varepsilon > 0$ was taken arbitrarily, the proof is complete. □

The previous lemma enables us to prove the following major theorem.

(1.8) Theorem (Measure theoretical Sobolev lemma).

For every element $[f] \in D(R^{-1})$ there can be chosen as representant $\tilde{f} \in [f]$ such that the following properties hold

(i) $\tilde{f} = \sum_{k=1}^{\infty} ([f],[\varphi_k]) \, \tilde{\varphi}_k$ where the series converges pointwise on $\mathbb{R}$.

(ii) The point evaluation $\delta_x : [f] \mapsto \tilde{f}(x)$ is a continuous linear functional on the Hilbert space $D(R^{-1})$ for all $x \in \mathbb{R}$. Its Riesz representant in $D(R^{-1})$ is $[\tilde{e}_x]$. So each sequence, convergent in the Hilbert space norm of $D(R^{-1})$ is pointwise convergent.

(iii) If $\sum_{k=1}^{\infty} \rho_k^2 [|\varphi_k|^2] \in L_\infty(\mathbb{R},\mu)$, then there exists a null set $\tilde{M}_\mu$ such that the convergence in (i) is uniform on $\mathbb{R}\backslash\tilde{M}_\mu$.

(iv) Let $x \in \operatorname{supp}(\mu)\backslash\tilde{N}_\mu$. Then

$$\tilde{f}(x) = \lim_{h\downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \hat{f}\, d\mu$$

where $\hat{f}$ is an arbitrary member of $[f]$.

Proof.

Let $[f] \in D(R^{-1})$ and put $\tilde{f} = \sum_{k=1}^{\infty} ([f],[\varphi_k])\, \tilde{\varphi}_k$.

(i) $$([f],[\tilde{e}_x])_\rho = \sum_{k=1}^{\infty} ([f],[\varphi_k])\, \tilde{\varphi}_k(x) \quad , \quad x \in \mathbb{R}.$$

Thus the assertion follows.

(ii) Since $\tilde{f}(x) = ([f],[\tilde{e}_x])_\rho$ it follows that the linear functional $[f] \mapsto \tilde{f}(x)$ is continuous.

(iii) The function $\sum_{k=1}^{\infty} \rho_k^2 |\tilde{\varphi}_k|^2$ is essentially bounded if and only if there exists a null set $\tilde{M}_\mu$ such that

$$S := \sup_{x\in\mathbb{R}\backslash\tilde{M}_\mu} \left(\sum_{k=1}^{\infty} \rho_k^2 |\tilde{\varphi}_k(x)|^2\right)^{\frac{1}{2}} < \infty .$$

Thus we obtain for $x \in \mathbb{R}\backslash\widetilde{M}_\mu$ and all $K \in \mathbb{N}$

$$\left|\sum_{k=K}^{\infty} ([f],[\varphi_k])\,\widetilde{\varphi}_k(x)\right| \le S\left(\sum_{k=K}^{\infty} \rho_k^{-2}\,|([f],[\varphi_k])|^2\right)^{\frac{1}{2}} .$$

In addition we note that $D(R^{-1}) \subset L_\infty(\mathbb{R},\mu)$.

(iv) Let $x \in \operatorname{supp}(\mu)\backslash\widetilde{N}_\mu$. Then we have by Lemma (1.7)

$$\widetilde{f}(x) = ([f],[\widetilde{e}_x])_\rho = ([f],\lim_{h\downarrow 0}[e_x\{h\}])_\rho =$$

$$= \lim_{h\downarrow 0}([f],[e_x\{h\}])_\rho =$$

$$= \lim_{h\downarrow 0}\left(\sum_{k=1}^{\infty} ([f],[\varphi_k])\,\mu(Q_h(x))^{-1}\int_{Q_h(x)} \widetilde{\varphi}_k\,d\mu\right).$$

Because of the inequality

$$\left(\sum_{k=1}^{\infty}\int_{Q_h(x)}\right) |([f],[\varphi_k])\,\widetilde{\varphi}_k|\,d\mu \le$$

$$\le \frac{1}{2}\mu(Q_h(x))\sum_{k=1}^{\infty}\rho_k^{-2}\,|([f],[\varphi_k])|^2 + \frac{1}{2}\sum_{k=1}^{\infty}\rho_k^2\int_{Q_h(x)}|\widetilde{\varphi}_k|^2\,d\mu$$

and because of the Fubini-Tonelli theorem, summation and integration can be interchanged. It yields the result

$$\widetilde{f}(x) = \lim_{h\downarrow 0}\mu(Q_h(x))^{-1}\int_{Q_h(x)}\left(\sum_{k=1}^{\infty}([f],[\varphi_k])\,\widetilde{\varphi}_k\right)d\mu$$

$$= \lim_{h\downarrow 0}\mu(Q_h(x))^{-1}\int_{Q_h(x)}\widetilde{f}\,d\mu .$$

A posteriori it follows that the limit does not depend on the choice of $\widetilde{f}$. $\square$

The following lemma will be used later.

(1.8) Lemma.

The set $\Gamma_0 = \bigcap_{k=1}^{\infty} \tilde{\varphi}_k^{\leftarrow}(0)$ is a null set with respect to μ.

Proof. Observe first that Γ_0 is a Borel set. Let χ_{Γ_0} be the characteristic function of the set Γ_0. Then for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}} \tilde{\varphi}_k \cdot \chi_{\Gamma_0} \, d\mu = \int_{\Gamma_0} \tilde{\varphi}_k \, d\mu = 0 .$$

So $[\chi_{\Gamma_0}] = [0]$, i.e. Γ_0 is a null set. □

2. δ-functions in trajectory spaces

Let μ_j, $j \in \mathbb{N}$, denote finite nonnegative Borel measures on the Borel sets in $\mathbb{R}$ and let Y denote the Hilbert space $\bigoplus_{j=1}^{\infty} L_2(\mathbb{R},\mu_j)$. We recall that for $f,g \in Y$, $f = ([f_1],[f_2],\ldots)$, $g = ([g_1],[g_2],\ldots)$

$$(f,g)_Y = \sum_{j=1}^{\infty} ([f_j],[g_j])_{L_2(\mathbb{R},\mu_j)} .$$

In this section we consider a nuclear analyticity space $S_{Y,B}$ and its corresponding trajectory space $T_{Y,B}$. So we assume that B has a discrete spectrum $\{\lambda_k \mid k \in \mathbb{N}\}$ and an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of eigenvectors such that $B\varphi_k = \lambda_k \varphi_k$, $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty$ for all $t > 0$. For convenience we take $0 \le \lambda_1 \le \lambda_2 \le \ldots$. See the preliminaries.

Let φ_k have components $[\varphi_{k,j}] \in L_2(\mathbb{R},\mu_j)$. Let $t > 0$. Then by assumption the series

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \, \| [\varphi_{k,j}] \|^2_{L_2(\mathbb{R},\mu_j)} \le \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty .$$

So for each fixed $j \in \mathbb{N}$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t} [|\varphi_{k,j}|^2]$ represents a member of $L_1(\mathbb{R},\mu_j)$. As in Section 1 it follows that there are representants $\tilde{\varphi}_{k,j} \in [\varphi_{k,j}]$ and a null set $\tilde{N}_{\mu_j}(t)$ with the following properties

$$(2.1.i)\qquad \tilde{\varphi}_{k,j}(x) = \lim_{h\downarrow 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_{k,j}\, d\mu_j$$

$$(2.1.ii)\qquad |\tilde{\varphi}_{k,j}(x)|^2 = \lim_{h\downarrow 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_{k,j}|^2\, d\mu_j$$

$$(2.1.iii)\qquad \sum_{k=1}^{\infty} e^{-2\lambda_k t} |\tilde{\varphi}_{k,j}(x)|^2 = \lim_{h\downarrow 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k=1}^{\infty} e^{-2\lambda_k t} |\tilde{\varphi}_{k,j}|^2\right) d\mu_j$$

where we take $x \in \operatorname{supp}(\mu_j)\backslash\tilde{N}_{\mu_j}(\frac{1}{n})$.

Now put $\tilde{N}_{\mu_j}(\mathcal{B}) = \bigcup_{n\in\mathbb{N}} \tilde{N}_{\mu_j}(\frac{1}{n})$ and for convenience take $\tilde{\varphi}_{k,j}(x) = 0$ for $x \in \operatorname{supp}(\mu_j)^* \cup \tilde{N}_{\mu_j}(\mathcal{B})$. Then similar to Lemma (1.7) we get

(2.2) Lemma.

Let $j \in \mathbb{N}$ and let $x \in \mathbb{R}$. Put

$$E_x^{(j)}\{h\} = \sum_{k=1}^{\infty} \left(\mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\tilde{\varphi}_{k,j}}\, d\mu_j\right) \varphi_k$$

$$\tilde{E}_x^{(j)} \quad : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \overline{\tilde{\varphi}_{k,j}(x)}\, \varphi_k$$

Then the mapping $\tilde{E}_x^{(j)} \in T_{Y,\mathcal{B}}$, and for $x \in \operatorname{supp}(\mu_j)\backslash\tilde{N}_{\mu_j}(\mathcal{B})$

$$\tilde{E}_x^{(j)} = \lim_{h\downarrow 0} \tilde{E}_x^{(j)}\{h\}$$

where the limit has to be taken in the strong topology of $T_{Y,\mathcal{B}}$.

Proof. Let $t > 0$. Then $\sum_{k\in\mathbb{N}} e^{-2\lambda_k t} |\tilde{\varphi}_{k,j}(x)|^2 \le \sum_{k\in\mathbb{N}} e^{-\frac{2}{n}\lambda_k} |\tilde{\varphi}_{k,j}(x)|^2$ for all $n \in \mathbb{N}$ with $0 < \frac{1}{n} < t$. Hence it follows that $\tilde{E}_x^{(j)}(t) \in Y$. Furthermore, it is not hard to see that the properties 2.1(i) - (iii) imply

$$\|\tilde{E}_x^{(j)}(\tfrac{1}{n}) - e^{-\frac{1}{n}B}(E_x^{(j)}\{h\})\|_Y \to 0 \qquad \text{as } h \downarrow 0$$

for all $n \in \mathbb{N}$ exactly as in Lemma (1.7). Now for $n \in \mathbb{N}$ with $0 < \frac{1}{n} \le t$

$$\|\tilde{E}_x^{(j)}(t) - e^{-tB}(E_x^{(j)}\{h\})\|_Y \le$$

$$\le \|e^{-(t-\frac{1}{n})B}\| \, \|\tilde{E}_x^{(j)}(\tfrac{1}{n}) - e^{-\frac{1}{n}B}(E_x^{(j)}\{h\})\|_Y . \qquad \square$$

We note that the vector $E_x^{(j)}\{h\}$ corresponds to the characteristic function of the set $Q_h(x)$ in the direct summand $L_2(\mathbb{R},\mu_j)$.

(2.3) Theorem.

Let $j \in \mathbb{N}$. Then for any $f \in S_{Y,B}$ there can be chosen a representant $\tilde{f}_j \in [f_j]$ with the following properties

(i) $\tilde{f}_j = \sum_{k=1}^{\infty} (f,\varphi_k)\, \tilde{\varphi}_{k,j}$ where the series converges pointwise on $\mathbb{R}$.

(ii) The point evaluation $\delta_x^{(j)} : f \mapsto \tilde{f}_j(x)$ is a continuous linear functional on $S_{Y,B}$. Furthermore, $\delta_x^{(j)}(f) = \langle f,\tilde{E}_x^{(j)}\rangle$.

(iii) For all $x \in \operatorname{supp}(\mu_j)\setminus\tilde{N}_{\mu_j}(B)$,

$$\tilde{f}_j(x) = \lim_{h\downarrow 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{f}_j \, d\mu .$$

The proof of the above theorem is similar to the proof of Theorem (1.8). Cf. the preliminaries for the definition of $\langle\cdot\,,\cdot\rangle$.

The set $\{E_x^{(j)} \mid x \in \mathbb{R},\ j \in \mathbb{N}\}$ is a concrete example of a Dirac basis. (For the terminology we refer to our paper [EG_{II}].) To see this, let M denote the disjoint union $\bigcup_{j=1}^{\infty} \mathbb{R}_j$ where each $\mathbb{R}_j$ is a copy of $\mathbb{R}$. Points in M will be denoted by (x,j). A set $B \subset M$ is called measurable if $B = \bigcup_{j=1}^{\infty} B_j$ where each B_j is a Borel set in $\mathbb{R}$. The σ-finite measure $\mu = \bigoplus_{j=1}^{\infty} \mu_j$ on M is defined by

$$\mu(B) = \sum_{j=1}^{\infty} \mu_j(B_j)$$

for all measurable sets $B = \bigcup_{j=1}^{\infty} B_j$ in M. Put $\tilde{E} : M \to T_{Y,B} : (x,j) \to \tilde{E}_x^{(j)}$. Then $(M,\mu,\tilde{E},T_{Y,B})$ is a Dirac basis in $T_{Y,B}$. (See [EG_{II}], Definition (2.1).)

It now follows from [EG_{II}] that $f \in S_{Y,B}$ can be expanded with respect to this Dirac basis.

$$(2.4) \qquad f = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} \langle f,\tilde{E}_x^{(j)}\rangle\ \tilde{E}_x^{(j)}\, d\mu_j(x)\ .$$

By this we mean

$$(2.4') \qquad f = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} \langle e^{\tau B} f,\tilde{E}_x^{(j)}\rangle\ \tilde{E}_x^{(j)}(\tau)\, d\mu_j(x)\ ,$$

where $\tau > 0$ has to be taken so small that $e^{\tau B} f \in S_{Y,B}$. Relation (2.4') does not depend on the choice of $\tau > 0$.

Furthermore, for $F \in T_{Y,B}$ we obtain

$$F(t) = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} \langle F(t-\tau),\tilde{E}_x^{(j)}\rangle\ \tilde{E}_x^{(j)}(\tau)\, d\mu_j(x)$$

with $t > \tau > 0$.

In [EG_{II}] we have written

$$|F\rangle = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} \langle \tilde{E}_x^{(j)} \,|\, F\rangle \, |\tilde{E}_x^{(j)}\rangle \, d\mu_j(x)$$

in the spirit of Dirac ([Di], p. 64).

Let Q_j denote multiplication by the identity function in $L_2(\mathbb{R},\mu_j)$. Then the operator $\mathrm{diag}(Q_\ell)$ defined by

$$\mathrm{diag}(Q_\ell)(f) = (Q_1[f_1], Q_2[f_2], \ldots)$$

with domain $\bigoplus_{\ell=1}^{\infty} D(Q_\ell)$ is self-adjoint in Y. For the operator $\mathrm{diag}(Q_\ell)$ we have the following result.

(2.5) Theorem.

Let $j \in \mathbb{N}$ and let $x \in \mathrm{supp}(\mu_j)\backslash\tilde{N}_{\mu_j}(B)$. Then

$$\lim_{h\downarrow 0} \mathrm{diag}(Q_\ell)(E_x^{(j)}\{h\}) = x\tilde{E}_x^{(j)}$$

where the limit is taken in the strong topology of $T_{Y,B}$.

Proof. We note first that the null set $\tilde{N}_{\mu_j}(B)$ has been taken such that

$$\sum_{k=1}^{\infty} e^{-\frac{2}{n}\lambda_k} |\tilde{\varphi}_{k,j}(x)|^2 = \lim_{h\downarrow 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k=1}^{\infty} e^{-\frac{2}{n}\lambda_k} |\tilde{\varphi}_{k,j}|^2\right) d\mu_j$$

for all $n \in \mathbb{N}$. Now let $t > 0$. Then

$$\lim_{h\downarrow 0} e^{-tB} (\mathrm{diag}(Q_\ell) - xI)\, \tilde{E}_x^{(j)}\{h\} =$$

$$= \lim_{h\downarrow 0} \left(\sum e^{-\lambda_k t} (\mu_j(Q_h(x))^{-1} \int_{Q_h(x)} (y-x)\, \overline{\tilde{\varphi}_{k,j}(y)}\, d\mu_j(y)\right) \varphi_k .$$

This expression can be treated as follows

$$\sum_{k=1}^{\infty} e^{-2\lambda_k t} \left|\mu_j(Q_h(x))^{-1} \int_{Q_h(x)} (y-x)\, \overline{\tilde{\varphi}_{k,j}(y)}\, d\mu_j(y)\right|^2 \leq$$

$$\leq \left(\sum_{k=1}^{\infty} e^{-2\lambda_k t} \left\{\mu_j(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_k(y)|^2\, d\mu_j(y)\right\}\right) \cdot$$

$$\cdot \left(\mu_j(Q_h(x))^{-1} \int_{Q_h(x)} |y-x|^2\, d\mu_j(y)\right) \leq$$

$$\leq h^2 \left(1 + \sum_{k=1}^{n} e^{-\frac{2}{n}\lambda_k} |\tilde{\varphi}_k(x)|^2\right)$$

for sufficiently small $h > 0$ and $n \in \mathbb{N}$ with $0 < \frac{1}{n} \leq t$. □

(2.6) Corollary.

Suppose $\operatorname{diag}(Q_\ell)$ can be extended to a continuous linear mapping on $\mathcal{T}_{Y,\mathcal{B}}$. Then $\operatorname{diag}(Q_\ell)\, \tilde{E}_x^{(j)} = x\tilde{E}_x^{(j)}$ for all $j \in \mathbb{N}$ and all $x \in \operatorname{supp}(\mu_j)\backslash\tilde{N}_{\mu_j}(\mathcal{B})$.

Finally we prove that almost all $\tilde{E}_x^{(j)}$ are non-trivial.

(2.7) Lemma.

The set $\{x \mid \tilde{E}_x^{(j)} = 0\}$ is a null set with respect to μ_j for each $j \in \mathbb{N}$.

Proof. Let $j \in \mathbb{N}$. We note that $\{x \mid \tilde{E}_x^{(j)} = 0\} = \bigcap_{k\in\mathbb{N}} \overset{\leftarrow}{\varphi}_{k,j}(0)$. As in the proof of Lemma (1.9), it follows that the latter set is a null set with respect to μ_j. □

3. Commutative multiplicity theory

The commutative multiplicity theory enables us to set up a theory which ensures that the notion 'multiplicity of an eigenvalue' also makes sense for generalized eigenvalues. We shall summarize the version of multiplicity theory given by Reed and Simon in [RS]. This theory is also very well described by Nelson in [Ne], ch. VI and by Brown in [Br].

(3.1) Definition.

The Borel measure ν is absolutely continuous with respect to the Borel measure μ, notation $\nu \ll \mu$, if for every Borel set $\mathcal{B}$ with $\mu(\mathcal{B}) = 0$ also $\nu(\mathcal{B}) = 0$.

The Borel measure ν and μ are equivalent, $\nu \sim \mu$ if $\nu \ll \mu$ and $\mu \ll \nu$.

It is clear that $\nu \sim \mu$ implies $\mathrm{supp}(\nu) = \mathrm{supp}(\mu)$. So it makes sense to write $\mathrm{supp}(\langle\nu\rangle)$ meaning the support of each $\nu \in \langle\nu\rangle$.

(3.2) Definition.

The equivalence classes $\langle\nu\rangle$ and $\langle\mu\rangle$ are called disjoint if

$$\nu(\mathrm{supp}(\langle\nu\rangle) \cap \mathrm{supp}(\langle\mu\rangle)) = \mu(\mathrm{supp}(\langle\nu\rangle) \cap \mathrm{supp}(\langle\mu\rangle)) = 0 .$$

To get a listing of the eigenvalues of a matrix it is natural to list all eigenvalues of multiplicity one, two, etc. We need a way of saying that an operator is of uniform multiplicity one, two, etc. Therefore we introduce

(3.3) Definition.

A self-adjoint operator T is said to be of uniform multiplicity m, $1 \leq m \leq \infty$ if T is unitarily equivalent to multiplication by the identity function in $L_2(\mathbb{R},\mu) \oplus \ldots \oplus L_2(\mathbb{R},\mu)$ where there are m terms in the sum and where μ is a finite nonnegative Borel measure.

This definition makes sense. If T is also unitarily equivalent to multiplication by the identity function on $L_2(\mathbb{R},\nu) \oplus L_2(\mathbb{R},\nu) \oplus \ldots \oplus L_2(\mathbb{R},\nu)$ then $m = n$ and $\mu \sim \nu$, [Br].

(3.4) Theorem.

Let T be a self-adjoint operator in a Hilbert space X. Then there exists a decomposition $X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus \ldots$ such that

(i) T acts invariantly in each X_m.

(ii) $T \restriction X_m$ has uniform multiplicity m.

(iii) The measure classes $<\mu_m>$ associated with the spectral representation of $T \restriction X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2, \ldots$ (some of which may be zero) and the measure classes $<\mu_\infty>, <\mu_1>, \ldots$ are uniquely determined by (i), (ii) and (iii).

4. Generalized eigenfunctions

Let T be a self-adjoint operator in a Hilbert space X. In the previous section we have seen that there exists a unitary operator U which sends X into the countable direct sum Y

$$(4.1) \qquad Y = \left(\bigoplus_{m=1}^{\infty} \bigoplus_{j=1}^{m} L_2(\mathbb{R},\mu_m) \right) \oplus \left(\bigoplus_{j=1}^{\infty} L_2(\mathbb{R},\mu_\infty) \right)$$

where some of the finite nonnegative measures μ_m can be identically zero. In addition, the self-adjoint operator UTU^* acts invariantly in each of the summands of (4.1); UTU^* restricted to $\bigoplus_{j=1}^{m} L_2(\mathbb{R},\mu_m)$ equals m-times multiplication by the identity function.

Let A be a nonnegative self-adjoint operator in X with a discrete spectrum $\{\lambda_k \mid k \in \mathbb{N}\}$. Then there exists an orthonormal $(v_k)_{k\in\mathbb{N}}$ in X such that $Av_k = \lambda_k v_k$. Oncemore we assume that $\sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty$ for all $t > 0$. So the space $S_{X,A}$ is supposed to be nuclear.

Put $B = UAU^*$ and $\varphi_k = Uv_k$, $k \in \mathbb{N}$. Then it is not hard to see that $B\varphi_k = \lambda_k \varphi_k$, and further that $U(S_{X,A}) = S_{Y,B}$, $U(T_{X,A}) = T_{Y,B}$. We denote the components of the elements $f \in Y$ by $[f_j^{(m)}]$ where $m \in \mathbb{N} \cup \{\infty\}$ and $1 \leq j < m+1$. Following Section 2 there are representants $\tilde{\varphi}_{k,j}^{(m)} \in [\varphi_{k,j}^{(m)}]$ such that

$$(4.2) \qquad \tilde{G}_x^{(m,j)} : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \tilde{\varphi}_{k,j}^{(m)}(x)\, v_k$$

is an element of $T_{X,A}$, where $m \in \mathbb{N} \cup \{\infty\}$ and where $1 \leq j < m+1$. For $h > 0$ we put

$$(4.3) \qquad G_x^{(m,j)}\{h\} = \sum_{k=1}^{\infty} \left(\mu_m(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_{k,j}^{(m)}\, d\mu_m\right) v_k \,.$$

Then as in Section 2 it can be seen that

$$G_x^{(m,j)}\{h\} \in D(T) \quad , \quad h > 0$$

and

$$T(G_x^{(m,j)}\{h\}) = \sum_{k=1}^{\infty} \left((\mu_m(Q_h(x))^{-1} \int_{Q_h(x)} y\, \tilde{\varphi}_{k,j}^{(m)}(y)\, d\mu_m(y)\right) v_k \,.$$

Following Lemma (2.2), Lemma (2.7) and Theorem (2.5) we have

(4.4) Theorem.

Let $m \in \mathbb{N} \cup \{\infty\}$ and let $1 \le j < m+1$. Then there exists a null set $\tilde{N}_j^{(m)}(\mathcal{B})$ with respect to $<\mu_m>$ such that for all $x \in \text{supp}(<\mu_m>)\backslash\tilde{N}_j^{(m)}(\mathcal{B})$

(i) $\qquad \lim\limits_{h\downarrow 0} G_x^{(m,j)}\{h\} = \tilde{G}_x^{(m,j)}$.

(ii) $\qquad \tilde{G}_x^{(m,j)} \neq 0$.

(iii) $\qquad \lim\limits_{h\downarrow 0} T G_x^{(m,j)}\{h\} = x\,\tilde{G}_x^{(m,j)}$.

The limits are taken in the strong topology of $T_{X,A}$.

(4.5) Theorem.

Let T in addition be a continuous linear mapping on $S_{X,A}$. Let m be a number in the multiplicity sequence of T. Then there exists a null set $\tilde{N}^{(m)}(\mathcal{B})$ with respect to $<\mu_m>$ such that for all $x \in \text{supp}(<\mu_m>)\backslash\tilde{N}^{(m)}(\mathcal{B})$ there are m independent generalized eigenvectors in $T_{X,A}$.

Proof. Since T is symmetric and continuous on $S_{X,A}$, the linear mapping T can be continuously extended to $T_{X,A}$, cf. [G], Ch. IV.

Following the previous theorem there exist null sets $\tilde{N}_j^{(m)}(\mathcal{B})$ such that for all $x \in \text{supp}(\mu_m)\backslash\tilde{N}_j^{(m)}(\mathcal{B})$, $1 \le j < m+1$

$$\lim_{h\downarrow 0} T G_x^{(m,j)}\{h\} = x\, G_x^{(m,j)} .$$

Thus we find with

$$\overline{T} \lim_{h \downarrow 0} G_x^{(m,j)}\{h\} = \lim_{h \downarrow 0} T\, G_x^{(m,j)}\{h\}$$

that

$$\overline{T}\, \widetilde{G}_x^{(m,j)} = x\, \widetilde{G}_x^{(m,j)} \quad , \quad 1 \leq j < m+1 .$$

With $\widetilde{N}^{(m)}(B) = \bigcup_{j=1}^{m} \widetilde{N}_j^{(m)}(B)$ the proof is complete. □

It follows from Section 2 that the set $\{\widetilde{G}_x^{(m,j)} \mid m \in \mathbb{N} \cup \{\infty\},\ 1 \leq j < m+1,$ $x \in \mathrm{supp}(\mu_m) \backslash \widetilde{N}^{(m)}(B)\}$ produces a Dirac basis in $T_{X,A}$. If T happens to be continuous on $S_{X,A}$, this Dirac basis consists of generalized eigenfunctions of T.

Recapitulated: Let $T_{X,A}$ be a nuclear trajectory space. Then to any self-adjoint operator T in X there corresponds a Dirac basis in a canonical way. Moreover, if T can be extended to a closed operator in $T_{X,A}$ then this Dirac basis consists of generalized eigenvectors of T. This is the case e.g. if T has a continuous extension to $T_{X,A}$.

Finally we note that we have also investigated the case of a finite number of commuting self-adjoint operators. Our investigations have led to results similar to the results of the present paper. They can be found in [E].

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