Designing columns for maximal buckling load

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DESIGNING COLUMNS FOR MAXIMAL BUCKLING LOAD

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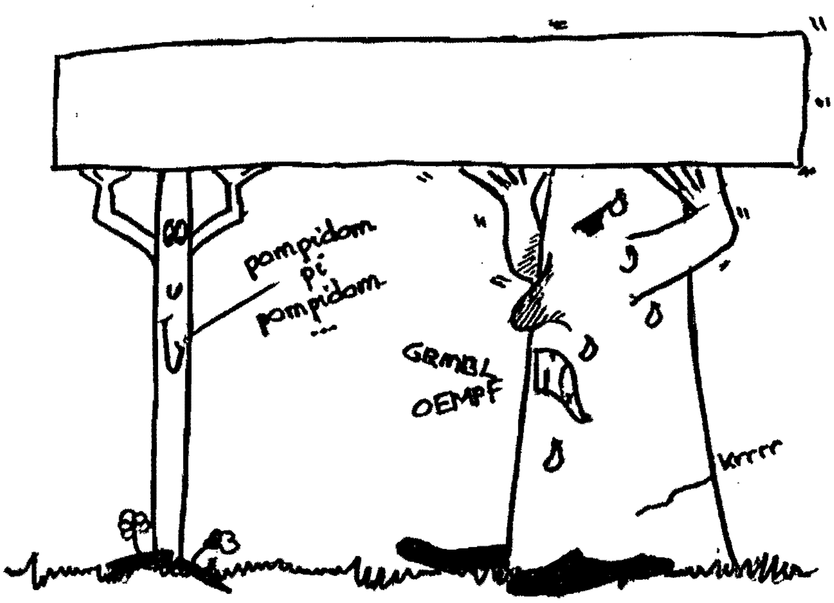

Acknowledgements

Thanks are due to Jos Brands for his valuable comments and suggestions, and for spending at least one weekend and – we suspect – some days of his vacations to help us solving column design problems. He also pointed out some quite serious mistakes, which of course have been corrected in the final version of this report.
Fons van de Ven must be thanked for his suggestions concerning mechanics, and for providing some useful literature.

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Introduction

The research described in this report is based upon the paper "The Shape of the Ideal Column" by S.J.Cox, which appeared in The Mathematical Intellingencer, Vol.14, in 1992.

The author of the paper claims to have found a shape for columns that deserves the adjective "optimal" or "ideal". The background for this claim is a – quite mathematical – reasoning involving functional analysis and other types of difficult calculations.
From a practical point of view, however, it is questionable if the column shape Cox presents is as good as claimed. How many constructors would approve of a column of which, at certain points, the cross section equals zero ? And would you like to live in a building supported by that kind of columns ?

We would not, and this (negative) desire motivated our quest for column shapes wearth to be called "optimal" and "ideal" in practice.

1 Buckling of a Column with Constant Cross-Section Surface

1.1 The Deflection of a Column with Constant Cross-Section Surface

In this paragraph, equations are derived describing the deflection of a column's middle line if – apart from a vertical force – a horizontal force is applied to it. The column is assumed to have a constant cross-section surface A along its length; it is hinged on both ends.

Figure 1 displays the situation we are interested in. The upper side of the column is loaded with a vertical force P; an equally large reaction force in the opposite direction results on the lower side. Both the force and the reaction force act along the middle line of the column. The column's middle line is subject to a horizontal force q.

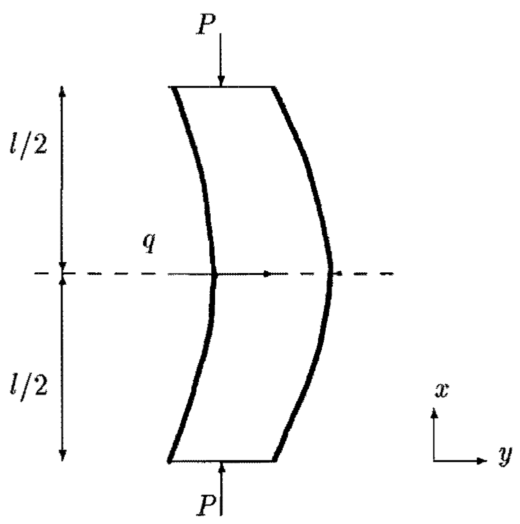

Figure 1: deflection of the column's middle line, and splitting the column in two parts to determine the moment M

The description of the column's deflection $y(x)$ is based upon the constitutive equation (see [2], pp.4-5)

$$E.I.\frac{d^2y}{dx^2} = -M, \tag{1}$$

combined with the hinged-hinged boundary conditions

$$y(0) = y(l) = \frac{d^2y}{dx^2}(0) = \frac{d^2y}{dx^2}(l) = 0. \tag{2}$$

E is the elasticity modulus of the column, I its inertia moment, and M denotes the column moment. The curve $y(x)$ describes the position of its middle line ([1]).

In the case of a column with constant cross-section surface, E only depends upon the material the column is made of, I is constant, and it is possible to derive an explicit formula for M !
To do so, we proceed as in [6], pp.2-5. Perform the thought experiment of splitting the column into two parts, as shown by the dotted line in Figure 1. We then find for the moment along the upper part of the column

$$M_U(x) = P.y + \frac{q}{2}.x,$$

and similarly for the lower part

$$M_L(x) = P.y + \frac{q}{2}.(l - x).$$

Combining these results with the constitutive equation (1) yields two differential equations: one for the lower part of the column, and one for the upper part.

$$\begin{aligned} L: \quad \frac{d^2y_L}{dx^2}.E.I &= -P.y_L - \frac{q}{2}.(l - x) \\ U: \quad \frac{d^2y_U}{dx^2}.E.I &= -P.y_U - \frac{q}{2}.x. \end{aligned}$$

Setting

$$k^2 = \frac{P}{E.I},$$

they can be rewritten as

$$\begin{aligned} L: \quad \frac{d^2y_L}{dx^2} &= -k^2.y_L - \frac{q}{2.E.I}.(l - x) \\ U: \quad \frac{d^2y_U}{dx^2} &= -k^2.y_U - \frac{q}{2.E.I}.x. \end{aligned}$$

The general solutions to these differential equations can be found by the method of variation of a constant. They look like

$$\begin{aligned} L: \quad y_L(x) &= A.\cos(kx) + B.\sin(kx) - \frac{q}{2.P}.(l-x) \\ U: \quad y_U(x) &= C.\cos(kx) + D.\sin(kx) - \frac{q}{2.P}.x; \end{aligned}$$

A, B, C, and D are constants to be determined from the hinged-hinged boundary conditions on $y(x)$. Instead of also using the conditions on $\frac{d^2y}{dx^2}$, we notice that $y(x)$ should be continuous and even differentiable in $x = l/2$ – as long as the column does not actually collapse.
The constants in the expressions for $y_L(x)$ and $y_U(x)$ can thus be found using

$$\begin{aligned} y_L(l) = y_U(0) &= 0 \\ y_L(\frac{l}{2}) &= y_U(\frac{l}{2}) \\ \frac{dy_L}{dx}(\frac{l}{2}) &= \frac{dy_U}{dx}(\frac{l}{2}). \end{aligned}$$

Some calculations, and substitution of the results into the expressions for $y_L(x)$ and $y_U(x)$ yield a description of the column's deflection.

$$\begin{aligned} L: \quad y_L(x) &= \frac{q}{2.P.k}.\frac{1}{\cos(kl/2)}.\sin(k(l-x)) - \frac{q}{2.P}.(l-x) \\ U: \quad y_U(x) &= \frac{q}{2.P.k}.\frac{1}{\cos(kl/2)}.\sin(kx) - \frac{q}{2.P}.x. \end{aligned}$$

1.2 The Buckling Load of a Column with Constant Cross-Section Surface

The buckling load of a column equals the vertical load at which even a small horizontal force causes the column deflection to be very large. In this paragraph, an expression is derived for the buckling load of a column with constant cross-section surface A, which in literature is called the Euler buckling-load. Details can be found in [6], pp.2-5.

From symmetry considerations – or from the expressions for $y_L(x)$ and $y_U(x)$, combined with hard labour – it appears that the maximal deflection of the column occurs at $x = l/2$. We have

$$y(\frac{l}{2}) = \frac{q}{2.P.k}.\frac{\sin(kl/2)}{\cos(kl/2)} - \frac{q}{2.P}.\frac{l}{2}$$

$$
\begin{aligned}
&= \frac{q}{2.P.k}.(\tan(kl/2) - (kl/2)) \\
&\to \infty \quad (k.\frac{l}{2} \uparrow \frac{\pi}{2}).
\end{aligned}
$$

Using $k^2 = P/E.I$, we conclude that the column deflection tends to infinity if the vertical load P tends to $E.I.\pi^2/l^2$.

Thus, the buckling load of a hinged-hinged column with constant cross-section surface equals

$$
P_{\text{buck}} = E.I.\frac{\pi^2}{l^2} \tag{3}
$$

1.3 The Maximal Buckling Load of a Column with Constant Cross-Section Surface, Volume V and Length l

If volume and length of the column under consideration are given to be V and l respectively, the (maximal) buckling load can be expressed as a function of these parameters. In fact, if the surface of the column cross-section equals A, its inertia moment is found from

$$
I = \kappa.A^2,
$$

where κ is a constant which only depends on the shape of the column's cross-section. For a circular cross-section $\kappa = 1/(4.\pi)$; an equilateral triangular cross-section has $\kappa = 1/(6.\sqrt{3})$ (see [4]).
The column volume satisfies

$$
\begin{aligned}
V = \int_0^l A dx &= l.A \\
A &= V/l.
\end{aligned}
$$

Substituting these results into the expression for the buckling load derived in the previous paragraph yields

$$
P_{\text{buck, max}} = \kappa.E.\frac{V^2}{l^4}.\pi^2. \tag{4}
$$

It is quite clear that the buckling load should be proportional to the elasticity modulus. Furthermore, if more material is used – if the volume is increased – the column gets stronger, while increasing the length has a negative influence on the size of the buckling load.
A less trivial result we may deduce from (4) is that columns with an equilateral triangular cross-section are stronger by a factor $6.\sqrt{3}/(4.\pi) = 1.209$ than columns of which the cross-section is circular ([4]) !

In the sequel, columns with a constant cross-section surface are referred to as prismatic columns.

2 General Considerations about Buckling of Columns

In the case of a prismatic column, the inertia moment I is constant, and the moment M can be determined explicitly. This is not the case in general, and it seems useful to derive expressions from which the buckling load can be determined for – in principle – all columns of which the cross-section shape does not change along the length. We assume the column to be homogeneous, and hinged on both ends.
The following derivation is taken from [2], pp.4-5.

2.1 A Differential Equation for the Column Deflection

Figure 2 displays the situation we consider. A column with cross-section $A(x)$ is loaded on the upper end with a force P, acting along the middle line of the column. This force causes an equally large reaction force to occur on the lower end. A force $q(x)$ is applied horizontally. If the column assumes its original straight state after removal of the lateral force $q(x)$, it is said to be in a stable equilibrium; if it retains its bent form, it can be considered to be in unstable equilibrium.

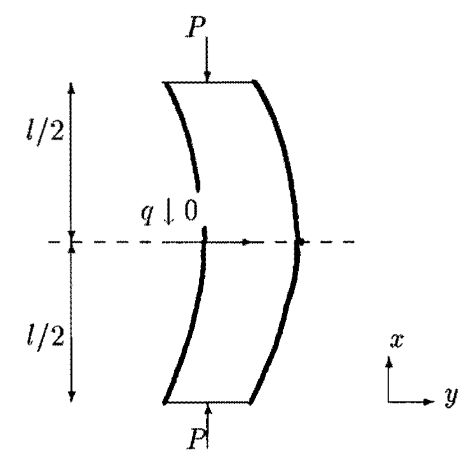

Figure 2: the column's unstable equilibrium deflection caused by an infinitesimal lateral load q

Consider now the equilibrium of the column shown in Figure 2 under an infinitesimal lateral load which places it in a slightly bent configuration. The external bending moment resulting from the axial load P and displacement $y(x)$ is

$$M = P.y.$$

The internal bending resistance follows from conventional beam theory:

$$M = \frac{E.I}{R}.$$

As before, E is its elasticity modulus, and I the inertia moment. R is the reciproque value of the column's curvature, which is given by

$$\frac{1}{R} = -\frac{d^2y/dx^2}{(1+(dy/dx)^2)^{3/2}}.$$

If the deflection $y(x)$ is assumed to be sufficiently small, the second-order term in the latter expression can be neglected, resulting in the approximation

$$\frac{1}{R} = -\frac{d^2y}{dx^2}.$$

Combining the above results, we find the following differential equation for the column's unstable equilibrium deflection y:

$$E.I.\frac{d^2y}{dx^2} + P.y = 0.$$

We only consider columns with hinged ends, resulting in boundary conditions

$$y(0) = \frac{d^2y}{dx^2}(0) = y(l) = \frac{d^2y}{dx^2}(l) = 0.$$

2.2 The Link between Buckling Load and Eigenvalues

When discussing the buckling load of a prismatic column, we identified this load as the one at which even a small horizontal force q applied on the column causes the deflection to be extremely large. In practice this means that applying a small force q makes the column collapse.
Thus, we are interested in solutions $y(x)$ of the differential equation derived in the previous sections, for which $y(l/2)$ tends to infinity if P approaches a certain maximal value.

Now consider

$$E.I.\frac{d^2y}{dx^2} + \lambda.y = 0.$$

Depending on the eigenvalue λ, this differential equation may have different eigenfunctions. Suppose we do not load the column at all. It is clear that in that case the solution to the latter differential equation is the trivial function $y(x) = 0$. Clearly, $y(l/2)$ does not tend to infinity in this case !
Now we let the vertical force P applied on the column increase slowly. Nothing much will happen, until the smallest eigenvalue λ is reached, in which case $y(x)$ is the first eigenfunction of the differential equation. This eigenfunction will be expressed – among others – in terms of P, and for P approaching a certain maximal value, $y(l/2)$ may tend to infinity.

This explains – at least intuitively – that the buckling load of the column can be identified with the smallest eigenvalue λ of the aforementioned differential equation ([1], p.18).

Thus, the buckling load of a homogeneous, hinged-hinged column can be found as the smallest eigenvalue of the equation

$$E.I.\frac{d^2y}{dx^2} + \lambda.y = 0 \tag{5}$$

$$y(0) = y(l) = 0; \tag{6}$$

E denotes the column's elasticity modulus, and I its inertia moment, satisfying

$$I(x) = \kappa.A^2(x). \tag{7}$$

$A(x)$ denotes the cross-section surface of the column at height x, and κ depends on the shape of the cross-section. This shape does not change along the column length.

3 Buckling of Some Columns Varying Cross-Section Surface

3.1 "Linear Surface" Columns

A "linear surface" column is a column of which the cross-section surface increases linearly with x:

$$A(x) = \frac{a}{\sqrt{\kappa}}.(1 + \gamma.\frac{x}{l}),$$

where l denotes the column length. An example of such a column, compared to a cilinderical one, is shown in Figure 3. The parameter γ is a measure for how much the cross-section varies along the column; it appears to be convenient to insert the constant κ in the expression for $A(x)$.
Notice that "linear surface" columns are not conical ! The surface varies linearly with x, and so the diameter of the circular column cross-section behaves like $\sqrt{x}$.

3.1.1 The Buckling Load of a "Linear Surface" Column

The buckling load of a "linear surface" column is found as the smallest eigenvalue λ of the differential equation (5), with hinged-hinged boundary conditions (6), after insertion of

$$I(x) = \kappa.A^2(x) = a^2.(1 + \gamma.\frac{x}{l})^2.$$

The resulting equation can be solved after substitution of

$$z = \ln(1 + \gamma.\frac{x}{l}),$$

resulting in the linear differential equation with constant terms

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + \left(\frac{l}{a.\gamma}\right)^2.\frac{\lambda}{E}.y = 0.$$

It is easily verified ([6], p.135) that the general solution of the latter differential equation looks like

$$\begin{aligned} y &= \sqrt{e^z}.\{A.\sin(\beta z) + B.\cos(\beta z)\} \\ &= \sqrt{1 + \frac{\gamma}{l}.x}.\left\{A.\sin\left(\beta.\ln(1 + \frac{\gamma}{l}.x)\right) + B.\cos\left(\beta.\ln(1 + \frac{\gamma}{l}.x)\right)\right\}, \end{aligned}$$

where

$$\beta = \sqrt{(\frac{l}{a.\gamma})^2.\frac{\lambda}{E} - \frac{1}{4}}.$$

Form the hinged-hinged boundary condition $y(0) = 0$ we find $B = 0$. Substituting this result into the requirement $y(l) = 0$ yields

$$\begin{aligned} \sin(\beta.\ln(1+\gamma)) &= 0 \\ \beta.\ln(1+\gamma) &= m.\pi, \quad m \in \mathbf{Z}. \end{aligned}$$

For the first eigenfunction, corresponding to the smallest eigenvalue λ – and therefore to the column's buckling load – we find

$$\begin{aligned} \beta.\ln(1+\gamma) &= \pi \\ \lambda &= \left(\frac{a.\gamma}{l}\right)^2.\left\{\frac{1}{4} + \frac{\pi^2}{\ln^2(1+\gamma)}\right\}.E. \end{aligned}$$

Thus, the buckling load of a hinged-hinged "linear surface" column equals

$$P_{\text{buck}} = \left(\frac{a.\gamma}{l}\right)^2.\left\{\frac{1}{4} + \frac{\pi^2}{\ln^2(1+\gamma)}\right\}.E. \tag{8}$$

3.1.2 The Maximal Buckling Load of a "Linear Surface" Column with Volume V and Length l

The shape of the "linear surface" column depends on γ, being a measure for how much the column cross-section varies with x, and so does the buckling load.
The question arises for which value of γ the buckling load of the corresponding "linear surface" column is maximal, given the column's volume V and its length l.

We have

$$\begin{aligned} V &= \int_0^l A(x)dx = \frac{a}{\sqrt{\kappa}}.\int_0^l (1 + \frac{\gamma}{l}.x)dx = \frac{a.l}{2.\sqrt{x\kappa}}.(2+\gamma), \\ a &= \sqrt{\kappa}.\frac{V}{l}.\frac{2}{2+\gamma}. \end{aligned}$$

Substitution into (8) yields

$$P_{\text{buck}} = \kappa.E.\frac{V^2}{l^4}.\left(\frac{2.\gamma}{2+\gamma}\right)^2.\left\{\frac{1}{4} + \frac{\pi^2}{\ln^2(1+\gamma)}\right\}. \tag{9}$$

In Figure 4, the buckling loads of hinged-hinged prismatic and "linear surface" columns are compared: we plotted $P_{\text{buck max}}$ as a function of γ for both cases – although the buckling load of a prismatic column does not depend on γ.
The "linear surface" column appears to be strongest for $\gamma = 0$, yielding a buckling load

$$P_{\text{buck max}} = \kappa.E.\frac{V^2}{l^4}.\pi^2; \tag{10}$$

this may be verified analytically from (9). Notice, however, that the "linear surface" column corresponding to $\gamma = 0$ is actually prismatic !

Thus, homogeneous, hinged-hinged prismatic columns are always stronger than their "linear surface" colleagues if the buckling load is the strength criterion.

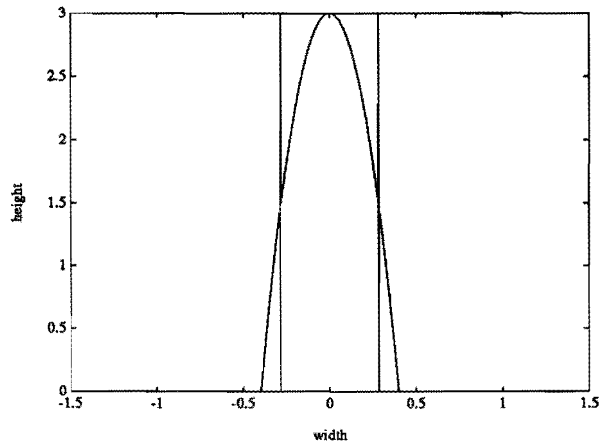

Figure 3: a "linear surface" column compared to a prismatic one[1]

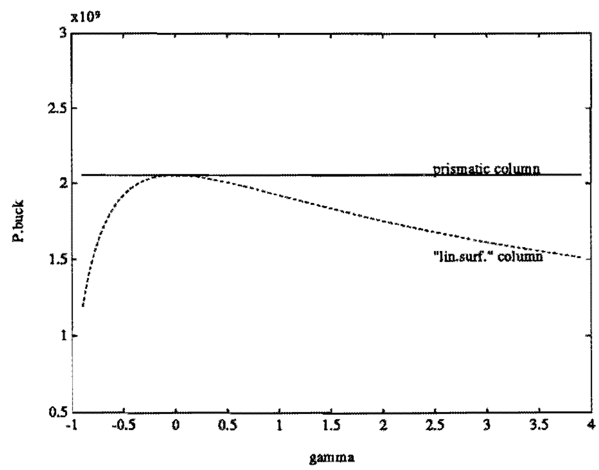

Figure 4: the hinged-hinged buckling load of a "linear surface" column, compared to that of a prismatic one[1]

[1]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\ N/m^2$.

3.2 Conical Columns

The columns used, for instance, in Greek temples do not have a constant cross-section surface, but they are tapered. We now investigate into the buckling load of these conical columns, for which the cross-section area satisfies

$$A(x) = \frac{a^2}{\sqrt{\kappa}}.(1+\gamma.\frac{x}{l})^2.$$

As before, l denotes the column length, and the parameter γ is a measure for "how conical" the column looks like. The closer γ is to zero, the more prismatic the column is. The constant κ is again included in $A(x)$.
Figure 5 shows a conical column, compared to a prismatic one.

3.2.1 The Buckling Load of a Conical Column

The buckling load of a conical column equals the smallest eigenvalue λ of (5) with boundary conditions (6), where

$$I(x) = \kappa.A^2(x) = a^4.(1+\gamma.\frac{x}{l})^4.$$

Combining (5) and the expression for $I(x)$, we find

$$\begin{aligned}(\alpha+\beta.x)^4.\frac{d^2y}{dx^2}+y &= 0\\ \alpha &= \left(\frac{E}{\lambda}\right)^{1/4}.a\\ \beta &= \left(\frac{E}{\lambda}\right)^{1/4}.\frac{\gamma}{l}.a.\end{aligned}$$

This differential equation can easily be solved after substitution of ([3], pp.401-403, p.497)

$$\begin{aligned}\eta(\xi) &= y(x)\\ \xi &= \alpha+\beta.x.\end{aligned}$$

We find

$$\beta^2.\xi^4.\frac{d^2\eta}{d\xi}+\eta = 0,$$

of which the general solution looks like

$$\begin{aligned}\eta &= A.\xi.\cos\left(\frac{1}{\beta.\xi}\right) + B.\xi.\sin\left(\frac{1}{\beta.\xi}\right)\\ y(x) &= A.(\alpha+\beta.x).\cos\left(\frac{1}{\beta.(\alpha+\beta.x)}\right) + B.(\alpha+\beta.x).\sin\left(\frac{1}{\beta.(\alpha+\beta.x)}\right).\end{aligned}$$

Form the hinged-hinged boundary conditions $y(0) = 0$ and $y(l) = 0$ we can derive respectively

$$\frac{A}{B} = -\tan\left(\frac{1}{\alpha.\beta}\right), \qquad \frac{A}{B} = -\tan\left(\frac{1}{\alpha.(\alpha+\beta.l)}\right),$$

leading to the conclusion that

$$\tan\left(\frac{1}{\alpha.\beta}\right) = \tan\left(\frac{1}{\alpha.(\alpha+\beta.l)}\right).$$

As we are interested in the first eigenfunction of the original differential equation – i.e. the one corresponding to the smallest eigenvalue –, this implies that

$$\frac{1}{\alpha.\beta} = \frac{1}{\alpha.(\alpha+\beta.l)} + \pi.$$

Inserting the definitions of α and β, this results in an expression for the smallest eigenvalue λ, and thus for the buckling load of a hinged-hinged, conical column.

$$P_{\text{buck}} = E.a^4.\frac{\pi^2}{l^2}.(1+\gamma)^2. \tag{11}$$

3.2.2 The Maximal Buckling Load of a Conical Column with Volume V and Length l

The constant a appearing in (11) can be expressed as a function of the column's length l, and volume V. This will make it possible to compare the buckling load of conical columns with that of prismatic and "linear surface" ones.

The column's volume satisfies

$$\begin{aligned}V &= \int_0^l A(x)dx = \frac{a^2}{\sqrt{\kappa}}.\int_0^l (1+\frac{\gamma}{l}.x)^2 dx = \frac{a^2}{\sqrt{\kappa}}.l.\left(1+\gamma+\frac{1}{3}.\gamma^2\right)\\ a^2 &= \sqrt{\kappa}.\frac{V}{l}.\frac{1}{1+\gamma+\gamma^2/3},\end{aligned}$$

from which

$$P_{\text{buck}} = \kappa.E.\frac{V^2}{l^4}.\pi^2.\left(\frac{1+\gamma}{1+\gamma+\gamma^2/3}\right)^2. \tag{12}$$

The buckling load of conical columns as a function of γ is plotted in Figure 6. It is compared to the buckling load of a prismatic column.
It appears that the conical column with the largest buckling load is again prismatic – which may be derived formally by considering P_{buck} as a function of γ, and determining its extrema. The corresponding buckling load is

$$P_{\text{buck max}} = \kappa.E.\frac{V^2}{l^4}.\pi^2; \tag{13}$$

Thus, from all column types considered yet, the simplest one is the best as far as buckling loads are concerned !
Still, a simple, intuitive argument shows that it must be possible to do better than that. Indeed, the prismatic column appeared to buckle in the middle. One may therefore expect the column to get stronger if some material is removed on both ends, and added in the middle !
This observation motivates an investigation into the buckling load of a column type which we will call parabolic in the sequel.

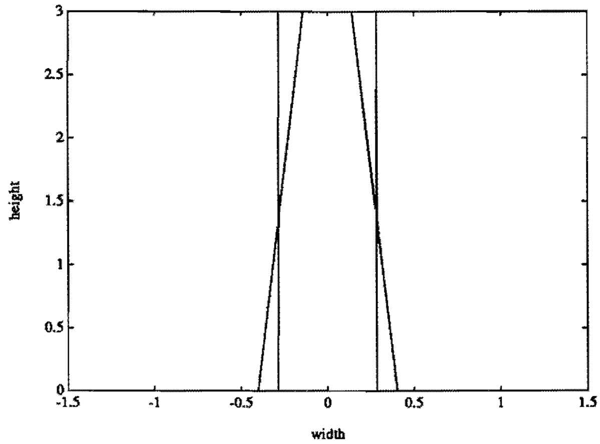

Figure 5: a conical column compared to a prismatic one[1]

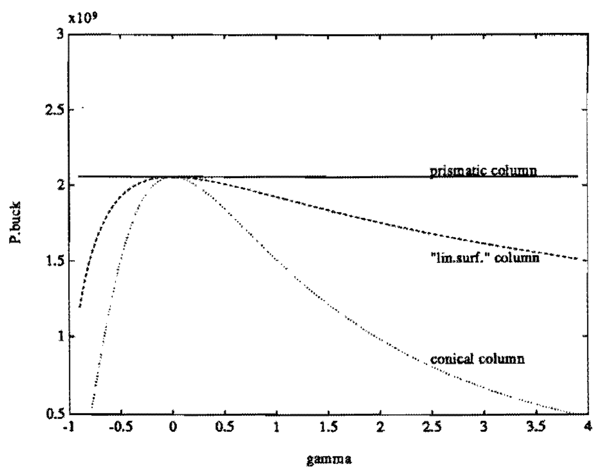

Figure 6: the hinged-hinged buckling load of a conical column, compared to that of prismatic and "linear surface" columns[1]

[1]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\;N/m^2$.

3.3 Parabolic Columns

Parabolic columns are columns that look like in Figure 7: the cross-section surface is a parabolic function of x, and

$$A(x) = (a.x^2 + b.x + c).$$

Notice that κ does not appear in the expression for $A(x)$ this time !
A parabolic column may be constructed from a prismatic one by removing some material on both ends, and adding it in the middle. Thus, it may be expected that strongest parabolic column with respect to buckling load is stronger than a prismatic one of the same volume and length.

3.3.1 The Buckling Load of a Parabolic Column

To find the buckling load of a parabolic column, we again have to determine the smallest eigenvalue λ of (5) with boundary conditions (6), but now with

$$I = \kappa.A^2(x) = \kappa.(a.x^2 + b.x + c)^2.$$

The substitutions leading to a solution of this equation are similar to those used in the conical column case – which could be expected, as in both cases the inertia moment I is a fourth degree polynomial in x. Writing ([3], nr.497)

$$\begin{aligned} y &= \sqrt{a.x^2 + b.x + c}.\eta(x) \\ \xi &= \int \frac{1}{a.x^2 + b.x + c} dx \end{aligned}$$

we find

$$\kappa.E.\sqrt{a.x^2 + b.x + c}.\left\{ \frac{d^2\eta}{d\xi^2} + (a.c - \frac{1}{4}b^2).\eta \right\} = -\lambda.\sqrt{a.x^2 + b.x + c}.\eta.$$

It is quite clear that the surface of the column cross-section should not vanish on $(0, l)$. We may therefore divide the latter equation by $\sqrt{a.x^2 + b.x + c}$, which eventually yields

$$\frac{d^2\eta}{d\xi^2} = -\left((a.c - \frac{1}{4}b^2) + \frac{\lambda}{\kappa.E} \right).\eta =: -k^2.\eta.$$

The general solution of this differential equation is easily found to be

$$\begin{aligned}
\eta(\xi) &= A.\cos(k.\xi) + B.\sin(k.\xi) \\
y(x) &= \sqrt{a.x^2 + b.x + c}. \\
&\left\{A.\cos\left(k.\int_0^x \frac{1}{a.\xi^2 + b.\xi + c}d\xi\right) + B.\sin\left(k.\int_0^x \frac{1}{a.\xi^2 + b.\xi + c}d\xi\right)\right\}.
\end{aligned}$$

The hinged-hinged boundary condition $u(0) = 0$ yields $A = 0$. Substituting this result into the requirement $u(l) = 0$ gives

$$\begin{aligned}
\sin\left(k.\int_0^l \frac{1}{a.\xi^2 + b.\xi + c}d\xi\right) &= 0 \\
k.\int_0^l \frac{1}{a.\xi^2 + b.\xi + c}d\xi &= m.\pi, \qquad m \in \mathbf{Z}.
\end{aligned}$$

We are interested in the differential equation's smallest eigenvalue, and thus in the first eigenfunction, corresponding to $m = 1$. Setting

$$h = \int_0^l \frac{1}{a.\xi^2 + b.\xi + c}d\xi = \int_0^l A^{-1}(x)dx,$$

we find

$$\begin{aligned}
k^2 &= \frac{\pi^2}{h^2} \\
(a.c - \frac{1}{4}b^2) + \frac{\lambda}{\kappa.E} &= \frac{\pi^2}{h^2} \\
\lambda &= \kappa.E.\frac{\pi^2}{h^2} - \kappa.E.(a.c - \frac{1}{4}b^2).
\end{aligned}$$

The buckling load of a hinged-hinged parabolic column thus equals

$$P_{\text{buck}} = \kappa.E.\frac{\pi^2}{h^2} - \kappa.E.(a.c - \frac{1}{4}b^2) \tag{14}$$

$$h = \int_0^l A^{-1}(x)dx. \tag{15}$$

3.3.2 The Maximal Buckling Load of a Parabolic Column with Volume V and Length l

If the volume V of the column, and its length l, are given, the constants a, b, and c in (14) can be eliminated. To do so, we assume symmetry – which is clear from the

intuitive idea of how we constructed the parabolic column. The material deleted in both ends is added in the middle, and thus the resulting column is symmetric with respect to $x = l/2$.
The mathematical consequence of this symmetry is that the derivative of $A(x)$ is zero for $x = l/2$, so $2.a.x + b = 0$ for $x = l/2$, or $b = -a.l$. Thus,

$$A(x) = a.x^2 - a.l.x + c$$

As noticed before, the column cross-section does not vanish on $(0, l)$. This means that the zeroes of $A(x)$ are not allowed to be elements of this interval. These zeroes occur at

$$x = \frac{l}{2} \pm \frac{\sqrt{a^2.l^2 - 4.a.c}}{2.a},$$

leading to the condition

$$\begin{aligned} \frac{\sqrt{a^2.l^2 - 4.a.c}}{2.a} &> \frac{l}{2} \\ -4.a.c &> 0. \end{aligned}$$

As $A(0) = c \geq 0$, this implies that $a < 0$. Setting $a' = -a$, we eventually find

$$A(x) = a'.(-x^2 + l.x + \frac{c}{a'}).$$

We proceed by deriving an explicit formula for h. Some calculations, and the use of standard integrals yield

$$\begin{aligned} h &= \int_0^l A^{-1}(x)dx = \int_0^l \frac{1}{-a'.x^2 + a'.l.x + c}dx \\ &= \frac{1}{\sqrt{a'^2 l^2 + 4.a'.c}}.\ln\left(\frac{a'.l + \sqrt{a'^2 l^2 + 4.a'.c}}{a'.l - \sqrt{a'^2 l^2 + 4.a'.c}}\right)^2. \end{aligned}$$

To simplify the formula's, we introduce a parameter γ. As in the cases of "linear surface" and conical columns, it indicates "how parabolic" the considered column actually is. We set

$$\gamma = \left(\frac{l}{2}\right)^2.\frac{a}{c},$$

which yields after some calculations

$$\frac{1}{h^2} = \left(\frac{4.c}{l}\right)^2.\gamma.(\gamma+1).\frac{1}{\ln^2\left(1 + 2.\gamma + 2.\sqrt{\gamma.(\gamma+1)}\right)^2}.$$

The factor $a.c - \frac{1}{4}b^2$, occurring in (14) can also be rewritten in terms of γ:

$$a.c - \frac{1}{4}b^2 = -c^2.\left(\frac{2}{l}\right)^2.\gamma.(1+\gamma).$$

We can now express P_{buck} in terms of γ and c only:

$$P_{\text{buck}} = \kappa.E.\left(\frac{2.c}{l}\right)^2.\gamma.(\gamma+1).\left\{\frac{4.\pi^2}{\ln^2\left(1+2.\gamma+2.\sqrt{\gamma.(\gamma+1)}\right)^2}+1\right\}.$$

Using the given column volume V, and its length l, we can finally eliminate c form this formula as well. In fact

$$\begin{aligned} V &= \int_0^l A(x)dx = \int_0^l (-a'.x^2 + a'.l.x + c)dx \\ &= c.l.(\frac{2}{3}.\gamma + 1), \\ c &= \frac{V}{l}.\frac{1}{2/3.\gamma+1} \end{aligned}$$

The buckling load of a hinged-hinged parabolic column as a function of its volume and length – and of the parameter γ – thus looks like ([1], p.20)

$$P_{\text{buck}} = \kappa.E.\frac{V^2}{l^4}.\frac{\gamma.(\gamma+1)}{(2/3.\gamma+1)^2}.\left\{\frac{4.\pi^2}{\ln^2\left(1+2.\gamma+2.\sqrt{\gamma.(\gamma+1)}\right)^2}+1\right\}. \qquad (16)$$

We determined the maximum of P_{buck} as a function of γ graphically; it appears that for $\gamma = 0.39$, the buckling load of the parabolic column is largest. The column corresponding to this value of γ is plotted in Figure 7.
Figure 8 shows the buckling load P_{buck} of parabolic columns as a function of γ, compared to the buckling load of a prismatic column.

We conclude that, for given volume V and length l, it is possible to construct a parabolic column that is stronger than a prismatic one of the same length and volume. The best one can do is to choose $\gamma = 0.39$, in which case the parabolic column is 1.65 % stronger than the prismatic one !

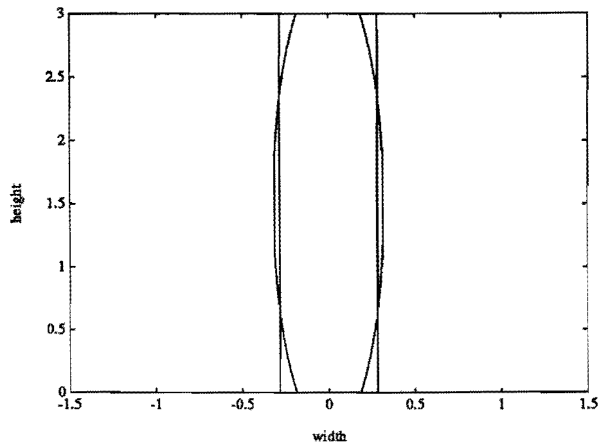

Figure 7: a parabolic column compared to a prismatic one; for the parabolic column, $\gamma = 0.39^1$

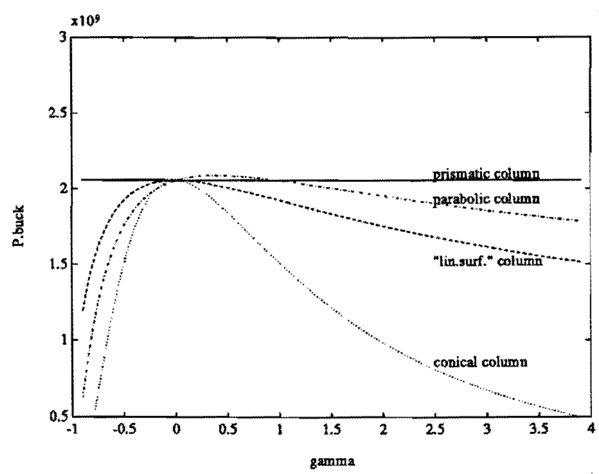

Figure 8: the hinged-hinged buckling load of a parabolic column, compared to that of prismatic, "linear surface", and conical columns[1]

[1]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\,N/m^2$.

4 Determining the Column Shape for Maximal Buckling Load

In the previous sections, we derived expressions for the buckling load of columns of different shapes: prismatic columns, "linear surface" columns, conical and parabolic columns were considered. The results obtained for the buckling load of parabolic columns show that it is possible to construct columns which are stronger than prismatic ones by deleting some material on both ends, and adding it in the middle of the column – i.e. by making columns in the shape of a cigar.
This section is devoted to finding the optimal cigar-shape as far as the buckling load is concerned.

4.1 A Condition on the Shape of a Column with Volume V, Length l, and Maximal Buckling Load

As mentioned in (5) and (6), the buckling load of a column with hinged-hinged boundary conditions can be determined as the smallest eigenvalue λ of the differential equation

$$\begin{aligned} E.I(x).\frac{d^2y}{dx^2} + \lambda.y &= 0 \\ y(0) = y(l) &= 0. \end{aligned}$$

The requirement that the column must have a certain volume V and length l can be expressed as

$$\int_0^l A(x)dx = V,$$

where $A(x)$ equals the surface of the column cross-section at height x; $A(x)$ describes the column's shape.

We now derive a necessary condition on $A(x)$ in terms of u for the column's buckling load to be maximal, given its volume V and its length l. We use the so-called method of variations, as proposed in [1], pp.21-23, and in [4], pp.22-23.

Suppose we have found the shape $\hat{A}(x)$ of the column which maximises the buckling load given V and l. Removing material somewhere from the column, and adding it elsewhere, then decreases the buckling load. Thus, any disturbance $A_0(x)$ of the optimal shape $\hat{A}(x)$, satisfying

$$\int_0^l A_0(x)dx = 0,$$

causes the column's buckling load to get smaller. It is intuitively clear that the more the optimal shape $\hat{A}(x)$ is disturbed, the smaller the resulting buckling load will be. Furthermore, it seems clear that this resulting buckling load is a continuous – and even differentiable – function of the disturbance.

More formally, one may notice that for each choice of the column shape $A(x)$, there is a buckling load. This means that there is some function $f : A \to \lambda(A)$ mapping shapes $A(x)$ to the corresponding buckling loads – being smallest eigenvalues of (5). The assumption that we found an optimal shape $\hat{A}(x)$ then implies that the function f must have a maximum among the shapes $A(x)$ corresponding to columns of volume V and length l !
The observation that the buckling load of the disturbed column is a continuous and differentiable function of the disturbance says that $g : t \to \lambda(\hat{A} + t.A_0)$ is differentiable for each disturbance $A_0(x)$ satisfying $\int_0^l A_0(x)dx = 0$.
Finally, as $\hat{A}(x)$ is the shape which maximises the column's buckling load, we must have

$$\frac{d}{dt}\lambda(\hat{A} + t.A_0)\bigg|_{t=0} = 0,$$

meaning that each change made to the column's shape is a change for the worse !

Now (5) can be rewritten as

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{\lambda}{\kappa.E}.A^{-2}.y &= 0, \\ y(0) = y(l) &= 0. \end{aligned}$$

Setting

$$A(x) = \hat{A}(x) + t.A_0(x), \qquad \text{where} \qquad \int_0^l A_0(x)dx = 0,$$

and determining the derivative with respect to t yields

$$\begin{aligned} & \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) + \frac{1}{\kappa.E}.\frac{d}{dt}\left(\lambda(\hat{A} + t.A_0)\right).A^{-2}.y \\ + & \frac{\lambda}{\kappa.E}.(-2).A^{-3}.\frac{d}{dt}\left(\hat{A} + t.A_0\right).y + \frac{\lambda}{\kappa.E}.A^{-2}.\frac{d}{dt}(y)\bigg|_{t=0} = 0. \end{aligned}$$

Using $\frac{d}{dt}\lambda(\hat{A} + t.A_0)\Big|_{t=0} = 0$, we find

$$\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) + \frac{\lambda}{\kappa.E}.\hat{A}^{-2}\frac{d}{dt}(y) = 2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.A_0.y.$$

This may not seem very helpful, until one notices that the right-hand side of this equation satisfies

$$\begin{aligned}\int_0^l (2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.A_0.y).ydx &= \int_0^l (\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) + \frac{\lambda}{\kappa.E}.\hat{A}^{-2}\frac{d}{dt}(y)).ydx \\ &= \int_0^l \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right).ydx - \int_0^l \frac{d^2y}{dx^2}.\frac{d}{dt}(y)dx \\ &= \frac{d}{dt}(\frac{dy}{dx}).y\Big|_0^l - \int_0^l \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dy}{dx}dx \\ &\quad - \frac{dy}{dx}.\frac{d}{dt}(y)\Big|_0^l + \int_0^l \frac{dy}{dx}.\frac{d}{dt}\left(\frac{dy}{dx}\right)dx.\end{aligned}$$

Both integrals on the right-hand add up to zero, and the remaining terms equal zero as a consequence of the hinged-hinged boundary conditions (6): $y(0) = y(l) = 0$, independent of t, and thus also $\frac{d}{dt}(y)(0) = \frac{d}{dt}(y)(l) = 0$. We eventually find

$$\int_0^l (2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.A_0.y).ydx = 0.$$

The latter equation holds for each disturbance $A_0(x)$ of the optimal shape $\hat{A}(x)$, as long as $\int_0^l A_0(x)dx = 0$; but this implies

$$\begin{aligned}\hat{A}^{-3}.y^2 &= \text{constant} =: c^2 \\ y &= c.\hat{A}^{3/2}. \end{aligned} \tag{17}$$

4.2 A Differential Equation for the Shape of a Column with Volume V, Length l, and Maximal Buckling Load

The condition (17) on the optimal column shape $\hat{A}(x)$ with respect to the buckling load can be substituted into (5). Some calculations then yield

$$\hat{A}.\frac{d^2\hat{A}}{dx^2} + \frac{1}{2}\left(\frac{d\hat{A}}{dx}\right)^2 + \frac{2}{3}.\frac{\lambda}{\kappa.E} = 0.$$

The hinged-hinged boundary conditions (6), combined with $y = c.\hat{A}^{3/2}$, give

$$\hat{A}(0) = \hat{A}(l) = 0.$$

The above differential equation for $\hat{A}$ can be solved after substitution of ([3], p.581)

$$\frac{d\hat{A}}{dx} = p(\hat{A}), \quad v(\hat{A}) = p^2(\hat{A}), \quad \alpha = \frac{2}{3}.\frac{\lambda}{\kappa.E},$$

resulting in the Bernoulli-type equation

$$\begin{aligned} \frac{1}{2}.\hat{A}.\frac{dv}{d\hat{A}} + \frac{1}{2}.v + \alpha &= 0 \\ \frac{dv}{d\hat{A}} &= -\frac{v}{\hat{A}} - 2.\frac{\alpha}{\hat{A}}. \end{aligned}$$

The solution of the latter differential equation for v is found by variation of a constant; the result is

$$\begin{aligned} v &= -2.\alpha + \frac{\beta}{\hat{A}} = p^2 \\ p &= \pm\sqrt{-2.\alpha + \frac{\beta}{\hat{A}}} = \frac{d\hat{A}}{dx} \\ \frac{d\hat{A}}{dx} &= \pm\sqrt{2.\alpha}.\sqrt{\frac{\beta/2.\alpha - \hat{A}}{\hat{A}}}, \end{aligned}$$

where β is an integration constant.

We thus have found a differential equation for the shape $\hat{A}(x)$ of the strongest column with volume V and length l with respect to the buckling load. Recalling the hinged-hinged boundary conditions on $\hat{A}$, we find

$$\begin{aligned} \frac{d\hat{A}}{dx} &= \pm\sqrt{2.\alpha}.\sqrt{\frac{\beta/2.\alpha - \hat{A}}{\hat{A}}}, \\ \hat{A}(0) = \hat{A}(l) &= 0. \end{aligned}$$

4.3 The Shape of a Column with Volume V, Length l, and Maximal Buckling Load

We now solve the differential equation for $\hat{A}(x)$ we derived in the preceeding paragraphs.

We assume the optimal shape of the column with respect to buckling load to be symmetrical – as we did when determining the best parabolic column. Symmetry implies that

$$\left.\frac{d\hat{A}}{dx}\right|_{x=l/2} = 0$$
$$\beta = 2.\alpha.\hat{A}(l/2),$$

and so

$$\frac{d\hat{A}}{dx} = \pm\sqrt{2.\alpha}.\sqrt{\frac{\hat{A}(l/2)-\hat{A}}{\hat{A}}}.$$

It appears that the solution of this differential equation with the hinged-hinged boundary conditions on $\hat{A}(x)$ is a cycloid, i.e. a curve with parameterisation

$$\begin{aligned} x(t) &= \frac{\hat{A}(l/2)}{2.\sqrt{2.\alpha}}.(\phi(t)-\sin\phi(t))+\xi \\ \hat{A}(t) &= \frac{\hat{A}(l/2)}{2}.(1-\cos\phi(t)), \qquad 0 \le t \le 2.\pi; \end{aligned}$$

we refer to [5], pp.382-388 for details. Substitution of $\alpha = \frac{2}{3}.\frac{\lambda}{\kappa.E}$ yields

$$\begin{aligned} x(t) &= \frac{1}{2}.\sqrt{\frac{3.\kappa.E}{4.\lambda}}.\hat{A}(l/2).(\phi(t)-\sin\phi(t))+\xi \\ \hat{A}(t) &= \frac{1}{2}.\hat{A}(l/2).(1-\cos\phi(t)), \qquad 0 \le t \le 2.\pi. \end{aligned}$$

As the column is symmetrical, $\phi(t) = 2.\pi - \phi(2.\pi - t)$, and we can see from the expression for $\hat{A}(t)$ that it is maximal at $\phi(t) = \pi$. But from symmetry of $\phi(t)$ it follows that this maximum occurs at $t = \pi$, and so $\phi(\pi) = \pi$.
We can therefore express the length l of the column as

$$\begin{aligned} l &= 2.(x(\pi)-x(0)) \\ &= \sqrt{\frac{3.\kappa.E}{4.\lambda}}.\hat{A}(l/2).(\phi(\pi)-\sin\phi(\pi)-\phi(0)+sin\phi(0)) \\ &= \sqrt{\frac{3.\kappa.E}{4.\lambda}}.\hat{A}(l/2).(\pi-\phi(0)+\sin\phi(0)). \end{aligned}$$

For the column's volume we find

$$\begin{aligned} V &= \int_0^l \hat{A}(x)dx = 2.\int_0^\pi \hat{A}(x(t)).\frac{dx}{dt}dt \\ &= \frac{1}{2}.\sqrt{\frac{3.\kappa.E}{4.\lambda}}.\hat{A}^2(l/2).\int_{\phi(0)}^\pi (1-\cos\phi)^2 d\phi \end{aligned}$$

$$= \frac{1}{2}.\sqrt{\frac{3.\kappa.E}{4.\lambda}}.\hat{A}^2(l/2).\left(\frac{3}{2}.(\pi - \phi(0)) + 2.\sin\phi(0) + \frac{1}{4}.\sin 2.\phi(0)\right).$$

Elimination of $\hat{A}(l/2)$ from the two latter results yields

$$\hat{A}(l/2) = 2.\frac{V}{l}.\frac{\pi - \phi(0) + \sin\phi(0)}{\frac{3}{2}.(\pi - \phi(0)) + 2.\sin\phi(0) + \frac{1}{4}.\sin 2.\phi(0)}.$$

We substitute this expression into the one for l, and rewrite the result to obtain

$$\lambda = \frac{4}{3}.\kappa.E.\frac{V^2}{l^4}.\frac{(\pi - \phi(0) + \sin\phi(0))^4}{\left(\pi - \phi(0) + \frac{4}{3}.\sin\phi(0) - \frac{1}{6}.\sin 2.\phi(0)\right)^2},$$

which would be a useful formula for the column's buckling load if we knew what to fill in for $\phi(0)$...

From the hinged-hinged boundary conditions we have

$$\hat{A}(0) = \frac{1}{2}.\hat{A}(l/2).(1 - \cos\phi(0)) = 0$$
$$\phi(0) = 0.$$

As $x(0) = 0$, this implies that the constant ξ in the parameterisation of x equals zero. Furthermore, as $\phi(0) = 0$, $\phi(\pi) = \pi$, and $\phi(2.\pi) = 2.\pi$, we might as well choose $\phi(t) = t$ for $0 \leq t \leq 2.\pi$.

We eventually find the following results ([1], p.22). The buckling load of a hinged-hinged column of volume V and length l is upper bounded by

$$P_{\text{buck max}} = \frac{4}{3}.\kappa.E.\frac{V^2}{l^4}.\pi^2. \tag{18}$$

The maximal cross-section surface of the column actually reaching this upper bound equals

$$\hat{A}(l/2) = \frac{4}{3}.\frac{V}{l},$$

and the parameter representation of its shape $\hat{A}$ looks like

$$x(t) = \frac{3}{4}.\frac{l}{\pi}.\left(\frac{2}{3}.(t - \sin t)\right) \tag{19}$$

$$\hat{A}(t) = \frac{V}{l}.\left(\frac{2}{3}.(1 - \cos t)\right), \quad 0 \leq t \leq 2.\pi. \tag{20}$$

The cycloid-shaped column is compared to a prismatic one in Figure 9; Figure 10 displays its buckling load compared to that over the other column types already considered.

Notice that the buckling load of the cycloid-shaped column is larger by a factor 4/3 than the buckling load of a prismatic column !

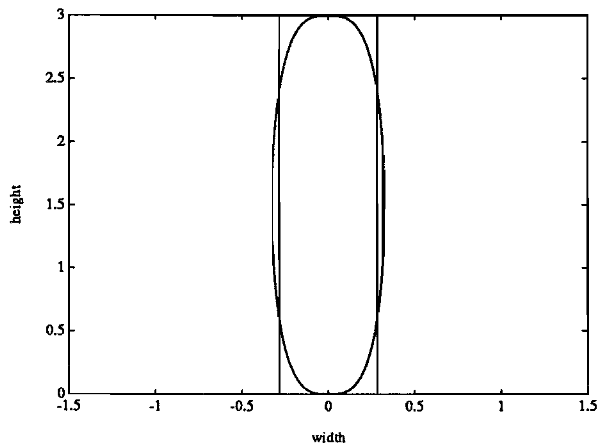

Figure 9: a cycloid-shaped column compared to a prismatic one[1]

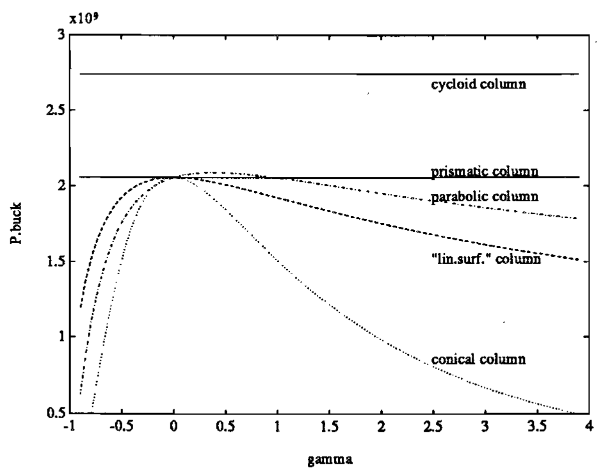

Figure 10: the hinged-hinged buckling load of a cycloid column, compared to that of prismatic, "linear surface", conical, and parabolic columns[1]

[1]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\ N/m^2$.

5 Numerical Computation of Buckling Loads

The analytically derived results in the previous sections are ideal in the mathematical sense. No attention was payed to constructability of the shapes. In practical situations, however, this is vital. No constructor needs a column that is very hard or expensive to build.
The problem might be solved by approximating the cycloid column with simpler, constructable shapes, while still gaining strength. Numerical results on strength are fairly easy to obtain.
A discretized version of the differential equation for the deflection of has been implemented. After verification of some results from the previous sections the program was adapted to solve (approximately) the maximum buckling load problem for a column consisting of three cylindrical parts.

5.1 Computing the Buckling Load of a Column

In Section 2.2 we found a relation between the maximum buckling load of a column and the following *Sturm-Liouville problem*:

$$\begin{aligned} E.I.y'' + \lambda.y &= 0 \\ y(0) = y(l) &= 0, \end{aligned}$$

where $I = \kappa.A^2(x)$. $A(x)$ is the cross-section surface of the column at height x. For any non-singular positve density function $d(x) := \frac{1}{E.I(x)}$ one can prove that all eigenvalues except 0 are positive . The smallest non-zero eigenvalue λ_1 is equal to the buckling load of a homogeneous, hinged-hinged column with cross-section $A(x)$. For the numerical computations the parameters V and l are both taken to be 1. The density is taken $d(x) = \frac{1}{a^2(x)}$.

The main ingredients of the numerical algorithm are finite differences and the power method for eigenvalues. First of all we dicretize the differential equation using a grid of m equidistant points x_i in the interval $(0, 1)$ and central differences for the second order derivative. The approximating problem then becomes:

Find $\mathbf{u} = (u_1, \ldots, u_m)^T$ and μ such that

$$\mathbf{Au} = \mu\,\mathbf{Du},$$

where the matrix $\mathbf{A}$ contains the differences and the boundary values and $\mathbf{D}$ is a diagonal matrix with elements $d_{ii} = d(x_i)$. The smallest eigenvalue μ_1 is an approximation for λ_1.

This problem is equivalent to

$$\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\mathbf{v} = \mathbf{v},$$

where $\mathbf{D}^{-1/2} = (\mathbf{D}^{1/2})^{-1}$, $(\mathbf{D}^{1/2})_{ij} := \sqrt{d_{ij}}$ and $\mathbf{v} = \mathbf{D}^{-1/2}\mathbf{u}$.

To determine the smallest eigenvalue μ_1 of the discretized Sturm-Liouville problem we use the power method to approximate the largest eigenvalue ρ_m of $\mathbf{D}^{1/2}\mathbf{A}^{-1}\mathbf{D}^{1/2}$. We have $1/\rho_m = \mu_1$.

The implemented algorithm uses a grid of $m = 34$ points (the maximum number the compiler could handle). The absolute relative error of the computed smallest eigenvalue is in this case always less than or equal to $\frac{1}{12}\pi^2\frac{1}{(m+1)^2} \approx 7.10^{-4}$. We can check this upperbound for the prismatic column, where $\lambda_1 = \pi^2$. It holds.

5.2 The Maximal Buckling Load of a Column which Consists of Three Cylindrical Parts

Based on the observations that the problem is symmetrical and that strength is gained by shifting mass from the ends to the middle, we attacked the optimization problem in the following way. First we loop over all the grid points in the interval $(0, \frac{1}{2})$. At each grid point x_i we vary the radius r_1 of the column such that

$$r(x) = \begin{cases} r_1 & \text{if } 0 \leq x \leq x_i \\ 1 & \text{if } x_i < x < 1 - x_i \\ r_1 & \text{if } 1 - x_i \leq x \leq 1 \end{cases}$$

The radii are then normalized such that the volume of the column is equal to 1. Then the buckling load for that shape is computed. For each x_i the optimal configuration is stored.

For this grid of 34 points and step size 1/50 we found the following results.

i	xi	Pbuck	r1/r2	r1
1	0.029	10.6039	0.52	0.299692
2	0.057	11.0984	0.60	0.351617
3	0.086	11.3911	0.66	0.391801
4	0.114	11.5254	0.70	0.420182
5	0.143	11.5419	0.74	0.447416
6	0.171	11.4709	0.76	0.463672
7	0.200	11.3501	0.80	0.487841
8	0.229	11.1936	0.82	0.501728
9	0.257	11.0157	0.84	0.514463
10	0.286	10.8346	0.84	0.519640
11	0.314	10.6527	0.86	0.530563
12	0.343	10.4715	0.88	0.540010
13	0.371	10.3038	0.88	0.544175
14	0.400	10.1471	0.90	0.551404
15	0.429	10.0054	0.92	0.557015
16	0.457	9.8900	0.96	0.562144
17	0.486	9.8630	1.00	0.564190

The maximum buckling load for this type of column is attained for $x_i = .143$ and a ratio of the radii of .74. The prismatic column appears for $i = 17$, where it is no longer advantageous to shift mass from top and and bottom to the middle (at least in this approximation). The value $P_{Buck} = 9.8630$ is an approximation for π^2. This type of column has an optimal buckling load that is a factor 1.17 larger than that of the prismatic column. Compared to the cycloid shaped column we're halfway there. Figure 11 displays both the optimal cycloid-shaped column, and its collegue consisting of three cilindrical parts. The dimensions of the latter one have been re-scaled in order to correspond with those of the cycloid-shaped column.

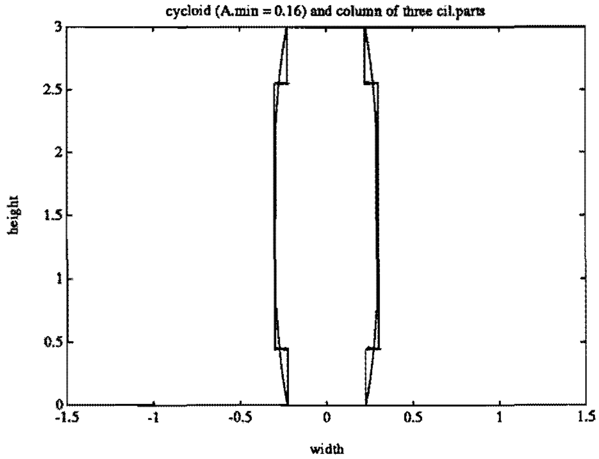

Figure 11: a column consisting of three cilindrical parts, compared to the optimal cycloid-shaped one.[1]

[1]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\ N/m^2$.

6 Did we Consider a Realistic Problem ? Material Strength, Carry-able Moments, and Reinforcements

In the previous sections, we considered a problem which is – according to some people, at least – quite appealing from a mathematical point of view. Interesting differential equations are involved, large formula's occur, and the result is a beautiful cycloid-shaped column.

The question arises, however, if it is interesting in practice to have information about the shape of a homogeneous, hinged-hinged column with volume V and length l, of which the buckling load is maximal.
We decided to ask some experts. The two engineers, working at the Construction Design Department of the Faculty of Civil Engineering at the Eindhoven University of Technology did not laugh aloud; but the winked, nudged and said no more...

According to them, for concrete columns, the buckling load is hardly ever a parameter which plays a role in the design. Columns usually carry ceilings, which tend to move as a consequence of wind or heavy people doing their morning gymnastics. They cause moments and tractive forces to occur on both ends. It is far more important to know how a column reacts upon them !
Often, steel reinforcements are put inside concrete columns to deal with the mentioned tractions. The assumption that the column is homogeneous is therefore not quite realistic.
Last but not least at all, everyone who has ever built a column must notice something strange about the cycloid-shaped one designed in the previous sections: its cross-section surface is zero on both ends ! This causes problems if the column is used to carry a load – which is quite likely to be the case. In fact, imagine a wooden toothpick that has been used to support a heavy book. The sharp extremities are almost surely damaged. Similarly, a large vertical load applied on one or both ends of the cycloid-shaped column will simply cause the extremities to break down !

In the next sections, we first redesign the cycloid-shaped column such that it can support a vertical load P. Then, we study the influence of the steel reinforcements, and investigate into the effect of moments and tractive forces applied on the column ends.
Hopefully, a more realistic column will result...

7 Taking Into Account the Material Strength

7.1 Tensions in the Column ! A Restriction on the Column Shape

If you sit down on a sugar cube, it may not be a nice feeling, but nothing will happen to the cube. If an elephant does the same, the sugar cube probably gives up and becomes a small, disappointed heap of sugar.
The same is true for columns. If a column has to carry a load which is in some sense too large, the material it is made of falls to pieces.

It is intuitively clear that this effect has something to do with the column's cross-section – which explains why larger columns are used when larger loads have to be carried. The smaller this cross-section is, the larger the load per surface unit that has to be carried, and thus the larger the tensions in the material. Indeed, the occurring tensions are defined as the load, or force, per surface unit.
For a homogeneous column of shape $A(x)$, loaded with a vertical force P, the material tensions $\sigma(x)$ at height x satisfy

$$\sigma(x) = \frac{P}{A(x)}.$$

Notice that the tensions are defined to be positive if P is a compressive force; for a tractive force a minus-sign occurs in the right-hand side of this equation.

If a column is made of a material which breaks down if tensions exceed a critical value $\sigma_{\max}$, we thus find a requirement on the column's shape $A(x)$:

$$\begin{aligned} |\sigma(x)| &\leq \sigma_{\max} \quad \text{on } [0,l] \\ A(x) &\geq \frac{P}{\sigma_{\max}} \quad \text{on } [0,l]. \\ A_{\min} &\geq \frac{P}{\sigma_{\max}}. \end{aligned} \tag{21}$$

A nasty situation occurs. Starting from a prismatic column, we decided to delete some material on both ends which could be added in the middle to increase the buckling load. This yielded a parabolic column at first, and a cycloid-shaped one at last.

Now, the cycloid-shaped column is seen to be of no use at all, unless $P = 0$. If P is larger than zero – but still very small –, it may suffice to remove some material from the middle, and add it on both ends to satisfy the condition on A_{min}; but the larger P gets, the more material has to be added to the column ends, and at a certain moment we are bound to find back the prismatic column we started from in the beginning !

Thus, a trade-off has to be made. If P is small, we can invest relatively much material to make the column look like the cycloid-shaped one, thus increasing the buckling load; but for large P, most of the material has to be used to satisfy the minimal surface condition, and the column will only have a slight resemblance with a cigar.

7.2 Reconstructing the Cycloid-Shaped Column to Carry a Load P

We redesign the cycloid-shaped column as to satisfy the minimal surface condition (21).

As for the case treated in Section 4.1 where no restrictions are made on A_{min}, we can derive that

$$\int_0^l (2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.A_0.y).ydx = 0.$$

In the case under consideration, this equation does not hold for all disturbances $A_0(x)$, thought, but only for those satisfying $\int_0^l A_0(x)dx = 0$, and $A(x) = \hat{A}(x) + t.A_0(x) \geq \frac{P}{\sigma_{\text{max}}}$ on $[0, l]$.

From this we can see that if $P = \hat{A}(x).\sigma_{\text{max}}$ for all x, no disturbances are allowed. Indeed, in that case the load is so large that no material can be used to increase the buckling load, and the column is prismatic. If $P = 0$, the situation treated in Section 4.1 occurs, and we can again derive

$$y = c.\hat{A}^{3/2}.$$

But what happens if P lies somewhere between these extremal values ?

In that case there is an $x_0 \in [0, l]$ such that

$$P < \hat{A}(x_0).\sigma_{\text{max}}.$$

As $\hat{A}(x)$ is a continuous function of x, this implies that it must exceed P/σ_{max} in a neighbourhood of x_0. Choose two x-values in this neighbourhood, say x_1 and x_2, and assume $a := \hat{A}(x1) \leq \hat{A}(x_2)$. Define the disturbance $A_0(x)$ by

$$A_0(x) = a.(\delta(x - x_2) - \delta(x - x_1)),$$

where $\delta(x)$ is the Dirac δ-function. Then

$$\int_0^l (2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.A_0.y).ydx = a.\left(2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.y^2\right)(x_2) - a.\left(2.\frac{\lambda}{\kappa.E}.\hat{A}^{-3}.y^2\right)(x_1) = 0,$$

from which

$$\hat{A}^{-3}(x_1).y^2(x_1) = \hat{A}^{-3}(x_2).y^2(x_2) =: c^2.$$

With a similar reasoning we deduce that $\hat{A}^{-3}(x).y^2(x) = c^2$ for all $x \in [0, l]$ satisfying $\hat{A}(x) > \frac{P}{\sigma_{\max}}$.

If there is enough material to make the column's surface larger than its minimal value $\frac{P}{\sigma_{\max}}$, this material must be used in the middle of the column in order to increase the buckling load. Thus, if there is material available to make a better column than a prismatic one, its shape will satisfy $A(x) = A_{\min}$ for $0 \leq x \leq a$ and $l - a \leq x \leq l$ for some a, and it will be cigar-like with $A(x) > A_{\min}$ for all x satisfying $a < x < l - a$. If there is even more material available, the whole column will look like a cigar, i.e. $a = 0$. In that case, the column cross-section equals its minimal value only for $x = 0$ and $x = l$.

7.2.1 Completely Cycloid-Shaped Columns: a Case we can Solve !

If there is enough material available to make a completely cycloid-shaped column, we have – in the notation introduced above – $a = 0$. In this case, we may conclude that if restriction (21) is made on the column's cross-section, the optimal shape $\hat{A}(x)$ with respect to buckling load satisfies

$$y = c.\hat{A}^{3/2} \quad \text{on } (0, l).$$

Notice that this equation may not be valid at the column ends $x = 0$ and $x = l$, and therefore does not violate the hinged-hinged boundary conditions (6).

As in Section 4.2, the latter restriction on u can be inserted into the original differential equation, which can then be solved after a number of substitutions. This yields

$$\begin{aligned} \frac{d\hat{A}}{dx} &= \pm\sqrt{2.\alpha}.\sqrt{\frac{\beta/2.\alpha - \hat{A}}{\hat{A}}}, \\ \hat{A}(0) = \hat{A}(l) &= \frac{P}{\sigma_{\max}} =: A_{\min} \end{aligned}$$

This differential equation can be solved in exactly the same way as shown in Section 4.3, with one exception: instead of finding that $\phi(0) = 0$, the hinged-hinged boundary conditions (6) now result in

$$\begin{aligned} \hat{A}(0) &= \frac{1}{2}.\hat{A}(l/2).(1-\cos\phi(0)) &= A_{\min} \\ && \cos\phi(0) = 1-\frac{2.A_{\min}}{\hat{A}(l/2)}. \end{aligned}$$

Instead of choosing $\phi(t) = t$, we set $\phi(t)$ to a linear function on $[0, 2.\pi]$, equal to $\phi(0)$ for $t = 0$, π for $t = \pi$, and $2.\pi - \phi(0)$ for $t = 2.\pi$. Figure 12a displays some of these functions $\phi(t)$ for various values of $A_{\min}$, while Figure 12b shows the relation between the column's minimal surface, and $\phi(0)$.
A consequence of the fact that $\phi(0) \neq 0$ is that the integration constant ξ appearing in the parameterisation (19) of $x(t)$ is not equal to zero. Requiring $x(0) = 0$, we find

$$\xi = -\frac{l}{2}.\frac{\phi(0)-\sin\phi(0)}{\pi-\phi(0)+\sin\phi(0)}.$$

Combining the results concerning the column of volume V, length l and minimal surface $A_{\min}$ which maximises the buckling load, we find for a parameterisation of its shape $\hat{A}$:

$$x(t) = \frac{l}{2}.\frac{1}{\pi-\phi(0)+\sin\phi(0)}.(\phi(t)-\sin\phi(t)-\phi(0)+\sin\phi(0)) \tag{22}$$

$$\hat{A}(t) = \frac{2}{3}.\frac{V}{l}.\frac{\pi-\phi(0)+\sin\phi(0)}{\pi-\phi(0)+\frac{4}{3}.\sin\phi(0)-\frac{1}{6}.\sin 2.\phi(0)}.(1-\cos\phi(t)), \tag{23}$$

$$0 \leq t \leq 2.\pi.$$

The value of $\phi(0)$ can be found from

$$1-\cos\phi(0)-\frac{3}{2}.l.\frac{A_{\min}}{V}.\frac{\pi-\phi(0)+\frac{4}{3}.\sin\phi(0)-\frac{1}{6}.\sin 2.\phi(0)}{\pi-\phi(0)+\sin\phi(0)} = 0. \tag{24}$$

The buckling load of this column, which looks like a truncated cycloid – see Figure 13 –, equals

$$P_{\text{buck max}} = \frac{4}{3}.\kappa.E.\frac{V^2}{l^4}.\frac{(\pi-\phi(0)+\sin\phi(0))^4}{\left(\pi-\phi(0)+\frac{4}{3}.\sin\phi(0)-\frac{1}{6}.\sin 2.\phi(0)\right)^2}. \tag{25}$$

Figure 12c illustrates how the buckling load depends of $\phi(0)$; and Figure 12d shows that – as we discussed in the beginning of this section – the column's buckling load decreases if the minimal surface $A_{\min}$ increases, i.e. if more material has to be used to make sure the column is able to carry the load P.

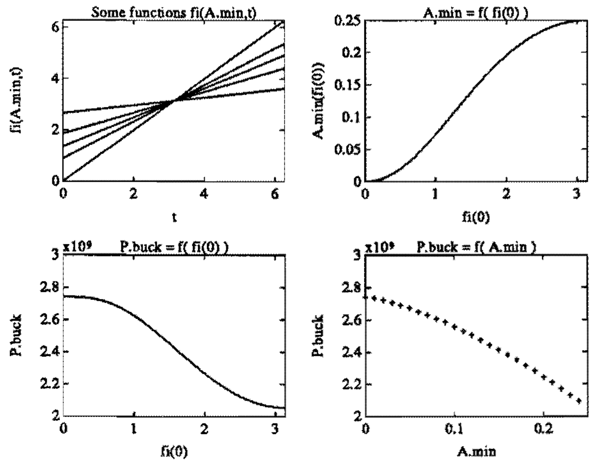

Figure 12:
a. Some functions $\phi(t)$ for various A_{min}
b. The relation between the column's minimal surface A_{min}, and $\phi(0)$
c. The maximal buckling load of a column of volume V and length l as a function of $\phi(0)$
d. The maximal buckling load of a column of volume V and length l as a function of its minimal surface A_{min}[2]

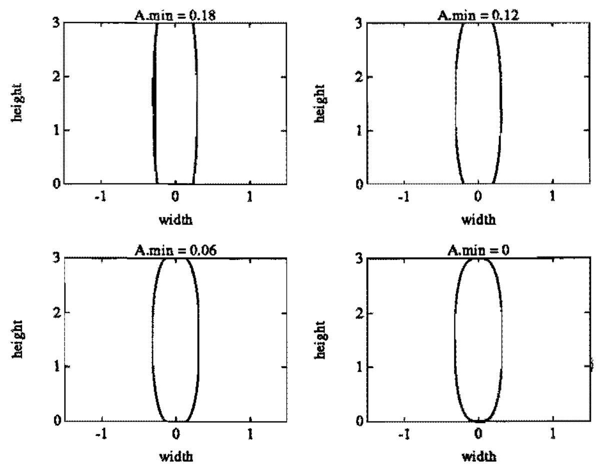

Figure 13: the hinged-hinged buckling load of a truncated cycloid column for various values of A_{min}[2]

[2]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\ N/m^2$ and $\sigma\text{max} = 2,5.10^7\ N/m^2$

If the column is required to carry a load P, the best we can do is to choose A_{min} such that

$$A_{\text{min}} = \frac{P}{\sigma_{\text{max}}}.$$

Furthermore, we must make sure that the buckling load of the designed column is not smaller than P, and if we want to minimise the amount of material used, we can suffice with choosing

$$\begin{aligned} P_{\text{buck max}} &= P = A_{\text{min}}.\sigma_{\text{max}} \\ \sigma_{\text{max}} &= \frac{P_{\text{buck max}}}{A_{\text{min}}}. \end{aligned}$$

As an example, suppose the column has to be made of concrete, for which a typical value of σ_{max} might be $\sigma_{\text{max}} = 2,5.10^7\,N/m^2$. If the column has to carry a load $1.10^6 N$ – which is quite realistic –, the size of the minimal column cross-section can be found from Figure 14. It suffices to draw a horizontal line at $\sigma = \sigma_{\text{max}}$, and to find the abscise of the intersection of this line with the plotted graph: $A_{\text{min}} = 0.16m^2$. The truncated cycloid-shaped column satisfying this minimal cross-section surface constraint is displayed in Figure 15. The corresponding buckling load can be found from Figure 12c. It is 22 % more than the buckling load of a prismatic column of the same length and volume.

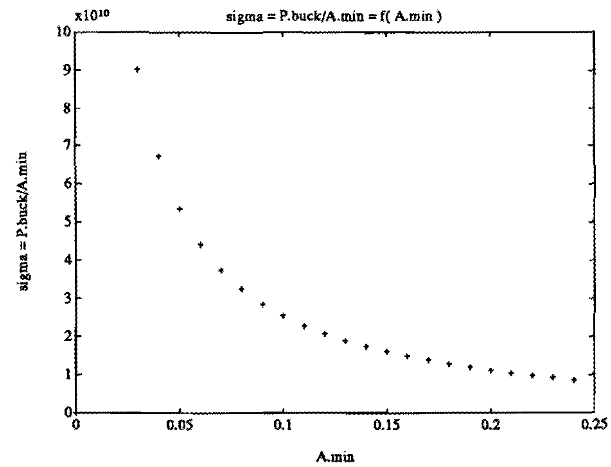

Figure 14: the material stresses σ the truncated cycloid column can support as a function of its minimal surface A_{min}[2]

[2]We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\,N/m^2$ and $\sigma_{\text{max}} = 2,5.10^7\,N/m^2$

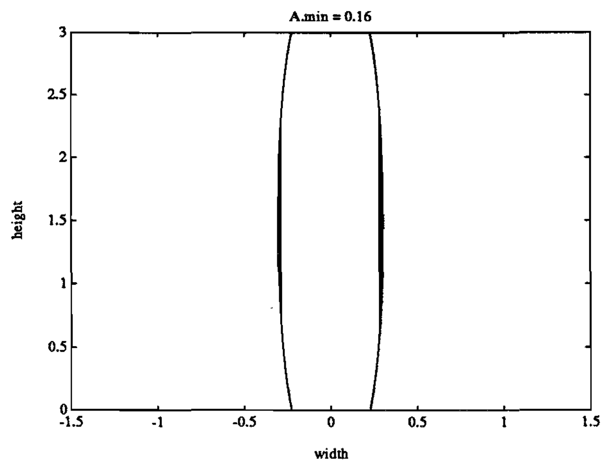

Figure 15: the optimal truncated cycloid column ![3]

7.2.2 Partly Cycloid-Shaped Columns: a More Complicated Case

When the volume we can use for a column is only a little bit more than $l.A_{\min}$, an obvious thing to do seems to be using the excess material to make only part of the column cycloid-shaped. Indeed, the buckling point lies in the middle of the column – the whole problem is symmetrical, in fact. It looks more effective to use all the extra material to reinforce that point, instead of a distributing it over the whole column.

Therefore, we now consider columns as in Figure 10, with

$$A(x) = A_{\min}\ ,\ x \in [0\ ,\ a] \cup [l - a\ ,\ l].$$

We can choose $a \in (0\ ,\ l/2)$.

In paragraph 7.2 we showed that $\hat{A}^{-3}u^2 = c^2$ holds for the optimal area $\hat{A}$ for all $x \in [0\ ,\ l]$ satisfying $\hat{A}(x) > A_{\min}$; and if $\hat{A}^{-3}u^2 = c^2$ the optimal shape becomes a cycloid. For the column depicted in Figure 10, this implies that it is cycloid-shaped for $x \in (a\ ,\ l - a)$. It can easily be shown that the formulas describing the cross-section area become:

$$\cos\phi(0) \quad = \quad 1 - \frac{3}{2}.\frac{A_{\min}.(l - 2.a)}{V - 2.a.A_{\min}}.\frac{\pi - \phi(0) + \frac{4}{3}.\sin\phi(0) - \frac{1}{6}.\sin 2.\phi(0)}{\pi - \phi(0) + \sin\phi(0)} \qquad (26)$$

[3] We have chosen the following parameters. The column length $l = 3\,m$, its volume being $V = 0.75\,m^3$. It is assumed to be made of concrete with an elasticity modulus $E = 3.10^{10}\ N/m^2$ and $\sigma\max = 2,5.10^7\ N/m^2$. The resulting value for $A_{\min}$ is $0.16\,m^2$.

$$x(t) = \frac{l-2.a}{2}.\frac{\phi(t)-\sin\phi(t)-\phi(0)+\sin\phi(0)}{\pi-\phi(0)+\sin\phi(0)} \tag{27}$$

$$\hat{A}(t) = \frac{2}{3}.\frac{V-2.a.A_{\text{min}}}{l-2.a}.\frac{(\pi-\phi(0)+\sin\phi(0))(1-\cos\phi(t))}{\pi-\phi(0)+\frac{4}{3}.\sin\phi(0)-\frac{1}{6}.\sin 2.\phi(0)}, \tag{28}$$

$$0 \le t \le 2.\pi.$$

Alas ! Formula 7.2.1 for the maximum buckling load can not be used because it only holds when the total column is a cycloid. In order to find a proper formula for P_{max} we need a formula for y but we did not manage to find one. It is possible, however, to find an optimal value for a numerically using the formula's derived in this section.

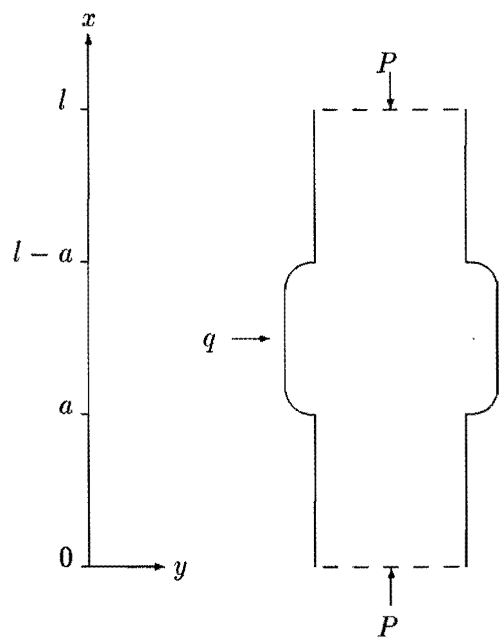

Figure 16: a column with the extra volume centred around the mid-point

8 Can Reinforcements be Neglected?

8.1 Introduction

So far we only considered homogeneous columns. But in reality most columns are made of concrete with reinforcements. Therefore the question arises: are we dealing with a realistic model? In Paragraph 8.2 we describe why reinforcements are used. In Paragraph 8.3 we describe a simple method to detect how big the error is when we neglect these reinforcements and restrict ourselves to the design of homogeneous columns.

8.2 Why are Reinforcements Used ?

When a column is made it is meant to carry something, so it has to deal with pressure forces. But in reality a column is a part of a bigger construction with all kinds of forces working on it. Therefore it also has to deal with tractive forces.
When one has to decide what material is to be used for a column, he or she might choose concrete because it can deal with a lot of pressure forces, it is relatively cheap and it has a long life time. Steel, for instance, can deal with more pressure forces but it is also more expensive and it has a shorter life time because of oxidation problems. But concrete has one disadvantage: it can not handle tractive forces. Therefore columns in buildings are made of concrete to deal with pressure forces and some reinforcements to handle tractive forces, mostly made of steel because it has the same expansion coëfficiënt as concrete, it firms very good to concrete and it can deal with a lot of tractive force. Moreover the steel will not oxidate because the concrete protects it when it is put all around the steel.

8.3 What Happens if we Forget About the Reinforcements ? The Exclusion Error

We assume that there is enough reinforcement to deal with tractive forces (2% of the cross-section area will do). If the minimal cross-section area is big enough we can concentrate on the buckling load, so we are dealing with equations (5) and (7):

$$\kappa\ E.A^2(x)\,.\frac{d^2y}{dx^2} + \lambda.y = 0.$$

Including reinforcement will change $E.A^2$ into $\tilde{E}.\tilde{A}^2$. What we want to know is: what is the (relative) error, err, introduced by modelling the column as a homogeneous one, or: how large is

$$err = \frac{|E.A^2 - \tilde{E}.\tilde{A}^2|}{\tilde{E}.\tilde{A}^2}. \tag{29}$$

Suppose that the reinforcement has the same elasticity modulus E as the material the rest of the column is made of. Then we can write: $E.A^2 = E_b.(A_b + A_r)^2 = (\sqrt{E_b}.A_b + \sqrt{E_b}.A_r)^2$; the subscript b refers to the basic material the column is made of and subscript r to the reinforcements.
Changing the material the reinforcements are made of will only influence the E in the second term:

$$\tilde{E}.\tilde{A}^2 = (\sqrt{E_b}.A_b + \sqrt{E_r}.A_r)^2 = E_b.(A_b + \sqrt{\frac{E_r}{E_b}}.A_r)^2.$$

Thus, (29) becomes:

$$err = \left| \left(\frac{A_b + A_r}{A_b + \sqrt{\frac{E_r}{E_b}}.A_r} \right)^2 - 1 \right|. \tag{30}$$

To get a first order approximation of this relative error, we substitute realistic values (see [7], p.55,57):

$$\begin{aligned} E_b &= 3 \cdot 10^{10} N/m^2 \\ E_r &= 2 \cdot 10^{11} N/m^2 \\ A_s &= 0.25 m^2 \\ A_r &= 0.005 m^2 \text{ (2 \% of } A_b \text{)}, \end{aligned}$$

and we find

$$err = 6\% \tag{31}$$

An error of 6 % seems to be quite reasonable. Therefore we conclude that modelling a column as a homogeneous one is reasonable with respect to buckling forces.
Notice that excluding reinforcements would increase the cross-section of the resulting optimal column; we are on the safe side of the optimum.

9 A Couple of Moments, Please !

9.1 A Differential Equation for the Column Deflection as a Consequence of Moments Acting on the Ends

In Section 1.1, we derived a differential equation describing the column deflection as a consequence of a horizontal force $q(x)$ acting on the column's middle line:

$$E.I.\frac{d^2y}{dx^2} + P.y = q.$$

E is the elasticity modulus of the material the column is made of; I is the inertia moment; and P is the vertical force with which the column is loaded.
If the column is hinged on both ends, the boundary conditions to this differential equation are

$$y(0) = \frac{d^2y}{dx^2}(0) = y(l) = \frac{d^2y}{dx^2}(l) = 0,$$

l denoting the column length.

Suppose we take

$$q(x) = \frac{M}{x_q}.(\delta(x - x_q) + \delta(x - l + x_q)) \tag{32}$$

i.e. we apply a horizontal force of size M/x_q at $x = x_q$ and at $x = l - x_q$. Taking limits for x_q tending to zero, the force $q(x)$ can be interpreted as a moment of size M about $x = 0$ and $x = l$ ([6]) !

Notice that in the case of buckling, we were interested in small horizontal forces q; it appeared to be practical to study the homogeneous differential equation obtained by taking limits for q tending to zero. Now, considering moments, we have to let x_q tend to zero in (32), so that the size of q tends to infinity. This explains why the homogeneous equation is not quite useful here; we will have to deal with the more difficult, non-homogeneous one now !

Inserting the expression (32) into the deflection differential equation, we find ([6], pp.6-7)

$$E.I.\frac{d^2y}{dx^2} + \lambda.y = M \tag{33}$$

$$y(0) = y(l) = 0, \tag{34}$$

where q is given by (32). We recall that

$$I(x) = \kappa.A^2(x),$$

A(x) being the cross-section surface, and κ depending on the shape of this surface.

The latter differential equation can also be derived using the method of Section 1.1 . It suffices to notice that applying a moment M in $x = 0$ causes a reaction force F to occur at $x = l$, and vice versa. The resulting moment at height x of the column equals

$$(l - x).F + x.F = l.F.$$

From the fact that the moment in $x = 0$ equals M, we find

$$l.F = M.$$

9.2 Trying to Derive a Condition on the Shape of a Column with Volume V, Length l, and Maximal Buckling Load, Able to Carry a Load P, and to Resist to Moments M

In this section, we try to derive a condition on $A(x)$ in terms of u for the column's buckling load to be maximal, given its volume V and its length l. The column must be able to carry a load P – leading to a lower bound $A_{\min}$ on the minimal surface –, and must resist to moments M acting on both ends. We proceed as in Sections 4.1 and 7.2, using the method of variations.
It must be noticed that the following results are not quite well derived; we advise the reader to be very careful...

Equations (33) and (34) can be rewritten as

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{\lambda}{\kappa.E}.A^{-2}.y &= M.A^{-2}, \\ y(0) = y(l) &= 0. \end{aligned}$$

Setting

$$A(x) = \hat{A}(x) + t.A_0(x), \quad \text{where} \quad \int_0^l A_0(x)dx = 0,$$

we consider the influence of a disturbance of the optimal column shape $\hat{A}(x)$. As the column must be able to carry a load P, we must require $A(x) \geq \frac{P}{\sigma_{\max}}$ on $[0, l]$.

Determining the derivative of the above non-homogeneous differential equation with respect to t, and using $\frac{d}{dt}\lambda(\hat{A} + t.A_0)\Big|_{t=0} = 0$, we find

$$\int_0^l \frac{2}{\hat{A}^3}.\left(\frac{\lambda}{\kappa.E}.y - M\right).y.A_0dx = \int_0^l (\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) + \frac{\lambda}{\kappa.E}.\hat{A}^{-2}\frac{d}{dt}(y)).ydx$$

$$
\begin{aligned}
&= \int_0^l \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right).ydx + \int_0^l \left(\frac{M}{\hat{A}^2} - \frac{d^2y}{dx^2}.\right).\frac{d}{dt}(y)dx \\
&= \int_0^l \frac{M}{\hat{A}^2}.\frac{d}{dt}(y)dx \;=\; \int_0^l \frac{M}{\hat{A}^2}.\frac{dy}{dA}.\frac{dA}{dt}dx \\
&= \int_0^l \frac{q}{\hat{A}^2}.\frac{dy}{dA}.A_0dx,
\end{aligned}
$$

$$
\int_0^l \left\{\frac{2}{\hat{A}^3}.\left(\frac{\lambda}{\kappa.E}.y - M\right).y - \frac{M}{\hat{A}^2}.\frac{dy}{dA}\right\}.A_0dx = 0.
$$

We again wish to warn the reader: the introduction of dy/dA, for instance, can only be correct if the mapping assigning to each A a y is bijective on the part of the column under consideration. Things might go wrong here !

By a similar reasoning as in Section 4.1, we conclude that for all $x \in (0, l)$,

$$
\begin{aligned}
\frac{2}{\hat{A}^3}.\left(\frac{\lambda}{\kappa.E}.y - M\right).y - \frac{M}{\hat{A}^2}.\frac{dy}{dA} &= \text{constant} =: c^2 \\
\frac{2}{\hat{A}^3}.\frac{\lambda}{\kappa.E}.y^2 - c^2 &= \frac{M}{\hat{A}^3}.\left(2.y + \hat{A}.\frac{dy}{dA}\right).
\end{aligned}
$$

It is a relief that for $M = 0$, the same expression results as in Section 4.1; but this does not guarantee the latter differential equation for y as a function of A to be correct. It is therefore maybe even good luck that we did not manage to solve it, so that no wrong conclusions have been made concerning the column design...

10 Gravity

We investigated into the effect of gravity on the optimal column shape. In fact, if the column cross-section surface is $A(x)$, and if its density is constant and equal to ρ, applying a vertical force P on the upper end of the column results in a total vertical load at height x of

$$P(x) = P + \int_x^l g.\rho.A(x)\,dx. \tag{35}$$

The differential equation to consider therefore becomes more complicated.

We did not manage to find any directly applicable results; however, we wish to thank Jos Brands, who invested lots of time into the problem. He transformed the analytical problem of solving a complicated differential equation into a numerical one by deriving three equations from which $A(x)$ could be calculated iteratively.

What we have done is substituting realistic values in equation (10) to get an estimate for the error made by excluding the influence of gravity. The data used are (see [7],p.233):

$$\begin{aligned} P_{max} &= 4.5 \cdot 10^5\ N \\ g &= 9.8\ m/s^2 \\ \rho &= 2.4 \cdot 10^3\ kg/m^3 \\ V &= 0.75\ m^3 \end{aligned}$$

The corresponding relative error is:

$$err = \frac{g.\rho.V}{P_{max} + g.\rho.V},$$

yielding $err = 4\%$ after substitution. Therefore it is not really necessary to include the gravity term in calculations to be applied in practice.

List of Symbols Used[1]

Symbol		*Description*	*Unit*
$A(x)$	:	Column's cross-section area	m^2
$A_{\min}$	:	Minimal cross-section area for allowable material tensions	m^2
E	:	Elasticity modulus of the used material	N/m^2
g	:	Gravity acceleration	m/s^2
$I(x)$	:	Square area moment (moment of inertia)	m^4
M	:	Moment	Nm
l	:	Length of the column	m
P	:	Vertical load	N
P_{buck}	:	Smallest vertical load that makes the column buckle	N
$P_{buck\ max}$	:	Maximum of all P_{buck}'s in a class of columns	N
q	:	Horizontal load	N
u	:	Second moment of deflection (y'')	$1/m$
V	:	Volume of the column	m^3
x	:	Vertical coördinate of the column's axis	m
y	:	Horizontal coördinate of the column's axis	m
κ	:	Constant only dependent on the shape of the column's cross-section shape	$--$
$\sigma(x)$	:	Material tension	N/m^2
σ_{max}	:	Maximum tension of the used material	N/m^2

[1] Local variables not included

References

[1] Cox, S.J.,
"The Shape of the Ideal Column",
The Mathematical Intelligencer,
Vol.14, No.1, 1992,
pp.16-24

[2] Gerard. G.,
Introduction to Structural Stability Theory,
McGraw-Hill Book Company,
New-York – Toronto – London, 1962

[3] Kamke, E.,
Differentialgleichungen: Lösungsmethoden und Lösungen I,
Akademische Verlagsgesellschaft Geest & Partig,
Leipzig, 1959

[4] Keller, J.B.,
"The shape of the strongest column",
Arch. for Rat. Mechanics and Anal.,
Vol.5, 1960,
pp.275-285

[5] Serret, J.A., Scheffers, G.,
Lehrbuck der Differential- und Integralrechnung I,
B.G. Teubner,
Leipzig – Berlin, 1908

[6] Timoshenko, S.,
Theory of Elastic Stability,
McGraw-Hill Book Company,
New-York – London, 1936

[7] Vis, W.C., Sagel, R.,
Constructief ontwerpen in beton,
Vereniging Nederlandse Cementindustrie VNC,
's-Hertogenbosch, 1992