De berekening van $\int\int_\Delta x%5Ep y%5Eq \mathrm{d}x\mathrm{d}y$

WE-71-~~28~~39

DE BEREKENING VAN $\iint\limits_{\Delta} x^p y^q dxdy$

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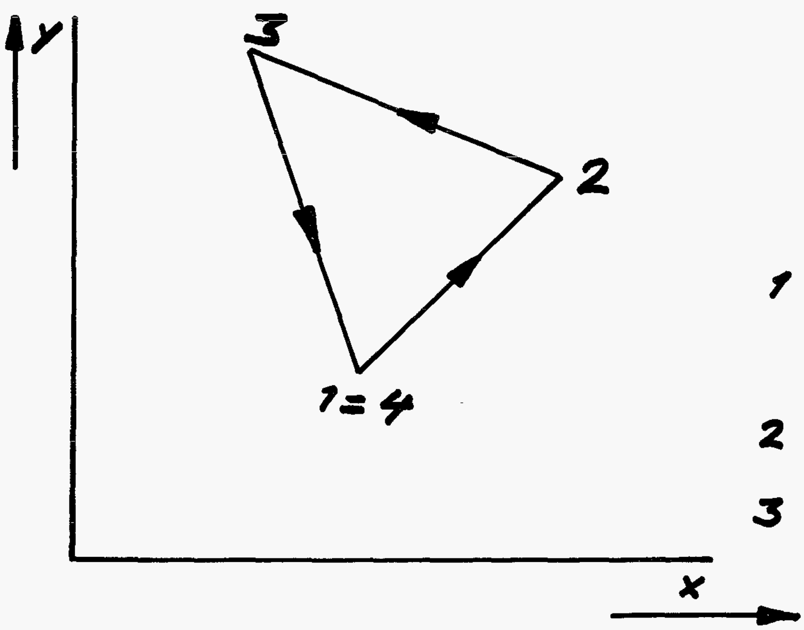
~~WE-71-28~~

De berekening van $\iint_{\Delta} x^p y^q \, dx\,dy$

Voor een beperkt aantal combinaties van p en q zullen we eenvoudig-programmeerbare relaties afleiden voor:

$$\chi_q^p = \iint_{\Delta} x^p y^q \, dx\,dy$$

waarbij het integratiegebied wordt gevormd door een driehoek.
Deze driehoek ligt vast door middel van de coördinaten van de hoekpunten. De volgorde van nummering is aangegeven in onderstaande figuur:



1 : (x_1, y_1)
of (x_4, y_4)
2 : (x_2, y_2)
3 : (x_3, y_3)

Voor positieve waarden van p en q, de waarde nul niet uitgezonderd, geldt:

$$\chi_q^p = \frac{2\,p!\,q!}{(p+q+2)!}\sum_{\ell=1}^{3} A_\ell \sum_{i=0}^{p}\sum_{j=0}^{q} \binom{i+j}{i}\binom{p+q-i-j}{p-i} x_\ell^{p-i}\, x_{\ell+1}^{i}\, y_\ell^{q-j}\, y_{\ell+1}^{j}$$

met $A_\ell = \frac{1}{2}(x_\ell\, y_{\ell+1} - x_{\ell+1}\, y_\ell)$

Wanneer we definiëren:

$$\alpha(p,q) = \frac{2\,p!\,q!}{(p+q+2)!}$$

$$\beta(p,q,i,j) = \binom{i+j}{i}\binom{p+q-i-j}{p-i}$$

geldt:

$$\chi_q^p = \alpha(p,q)\sum_{\ell=1}^{3} A_\ell \sum_{i=0}^{p}\sum_{j=0}^{q}\beta(p,q,i,j)\, x_\ell^{p-i}\, x_{\ell+1}^{i}\, y_\ell^{q-j}\, y_{\ell+1}^{j}$$

Voor slechts die waarden van p en q, die voldoen aan de onderstaande relaties, zal de formule voor χ_q^p verder worden uitgewerkt.

$$p \geq 0$$

$$q \geq 0$$

$$p+q \leq 4$$

$\boxed{\alpha}$

$q \backslash p$	0	1	2	3	4
0	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$
1	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{60}$	
2	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{90}$		
3	$\frac{1}{10}$	$\frac{1}{60}$			
4	$\frac{1}{15}$				

$\boxed{\beta}$

<table>
<tr><th>$q \backslash p$</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>0</td>
<td>$\begin{array}{c|c} j\backslash i & 0 \\ \hline 0 & 1 \end{array}$</td>
<td>$\begin{array}{c|cc} j\backslash i & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}$</td>
<td>$\begin{array}{c|ccc} j\backslash i & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 1 \end{array}$</td>
<td>$\begin{array}{c|cccc} j\backslash i & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 1 \end{array}$</td>
<td>$\begin{array}{c|ccccc} j\backslash i & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 \end{array}$</td></tr>
<tr><td>1</td>
<td>$\begin{array}{c|c} j\backslash i & 0 \\ \hline 0 & 1 \\ 1 & 1 \end{array}$</td>
<td>$\begin{array}{c|cc} j\backslash i & 0 & 1 \\ \hline 0 & 2 & 1 \\ 1 & 1 & 2 \end{array}$</td>
<td>$\begin{array}{c|ccc} j\backslash i & 0 & 1 & 2 \\ \hline 0 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{array}$</td>
<td>$\begin{array}{c|cccc} j\backslash i & 0 & 1 & 2 & 3 \\ \hline 0 & 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 4 \end{array}$</td>
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<td>$\begin{array}{c|c} j\backslash i & 0 \\ \hline 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{array}$</td>
<td>$\begin{array}{c|cc} j\backslash i & 0 & 1 \\ \hline 0 & 3 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{array}$</td>
<td>$\begin{array}{c|ccc} j\backslash i & 0 & 1 & 2 \\ \hline 0 & 6 & 3 & 1 \\ 1 & 3 & 4 & 3 \\ 2 & 1 & 3 & 6 \end{array}$</td>
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<td>$\begin{array}{c|c} j\backslash i & 0 \\ \hline 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{array}$</td>
<td>$\begin{array}{c|cc} j\backslash i & 0 & 1 \\ \hline 0 & 4 & 1 \\ 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 4 \end{array}$</td>
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<tr><td></td>
<td>$\begin{array}{c|c} j\backslash i & 0 \\ \hline 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{array}$</td>
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Bovenstaande tabellen geven de waarden van α en β in afhankelijkheid van de bepalende variabelen.

$$\chi_0^0 = \sum_{\ell=1}^{3} A_\ell$$

$$\chi_1^0 = \frac{1}{3} \sum_{\ell=1}^{3} A_\ell \left[y_\ell + y_{\ell+1} \right]$$

$$\chi_2^0 = \frac{1}{6} \sum_{\ell=1}^{3} A_\ell \left[y_\ell^2 + y_\ell y_{\ell+1} + y_{\ell+1}^2 \right]$$

$$\chi_3^0 = \frac{1}{10} \sum_{\ell=1}^{3} A_\ell \left[y_\ell^3 + y_\ell^2 y_{\ell+1} + y_\ell y_{\ell+1}^2 + y_{\ell+1}^3 \right]$$

$$\chi_4^0 = \frac{1}{15} \sum_{\ell=1}^{3} A_\ell \left[y_\ell^4 + y_\ell^3 y_{\ell+1} + y_\ell^2 y_{\ell+1}^2 + y_\ell y_{\ell+1}^3 + y_{\ell+1}^4 \right]$$

$$\chi_0^1 = \frac{1}{3} \sum_{\ell=1}^{3} A_\ell \left[x_\ell + x_{\ell+1} \right]$$

$$\chi_1^1 = \frac{1}{12} \sum_{\ell=1}^{3} A_\ell \left[x_\ell y_{\ell+1} + 2 x_\ell y_\ell + 2 x_{\ell+1} y_{\ell+1} + x_{\ell+1} y_\ell \right]$$

$$\chi_2^1 = \frac{1}{30} \sum_{\ell=1}^{3} A_\ell \Big[3 x_\ell y_\ell^2 + x_{\ell+1} y_\ell^2 + 2 x_\ell y_\ell y_{\ell+1} + 2 x_{\ell+1} y_\ell y_{\ell+1} + \\ + x_\ell y_{\ell+1}^2 + 3 x_{\ell+1} y_{\ell+1}^2 \Big]$$

$$\chi_3^1 = \frac{1}{60} \sum_{\ell=1}^{3} A_\ell \Big[4 x_\ell y_\ell^3 + x_{\ell+1} y_\ell^3 + 3 x_\ell y_\ell^2 y_{\ell+1} + 2 x_{\ell+1} y_\ell^2 y_{\ell+1} + \\ + 2 x_\ell y_\ell y_{\ell+1}^2 + 3 x_{\ell+1} y_\ell y_{\ell+1}^2 + x_\ell y_{\ell+1}^3 + 4 x_{\ell+1} y_{\ell+1}^3 \Big]$$

$$\chi_0^2 = \frac{1}{6} \sum_{\ell=1}^{3} A_\ell \left[x_\ell^2 + x_\ell x_{\ell+1} + x_{\ell+1}^2 \right]$$

$$\chi_1^2 = \frac{1}{30} \sum_{\ell=1}^{3} A_\ell \Big[3 y_\ell x_\ell^2 + y_{\ell+1} x_\ell^2 + 2 y_\ell x_\ell x_{\ell+1} + 2 y_{\ell+1} x_\ell x_{\ell+1} + \\ + y_\ell x_{\ell+1}^2 + 3 y_{\ell+1} x_{\ell+1}^2 \Big]$$

$$\chi_2^2 = \frac{1}{90} \sum_{l=1}^{3} A_l \Big[6x_l^2 y_l^2 + 3x_l^2 y_l y_{l+1} + x_l^2 y_{l+1}^2 + 3x_l x_{l+1} y_l^2 + 4x_l x_{l+1} y_l y_{l+1} + 3x_l x_{l+1} y_{l+1}^2 + x_{l+1}^2 y_l^2 + 3x_{l+1}^2 y_l y_{l+1} + 6x_{l+1}^2 y_{l+1}^2 \Big]$$

$$\chi_0^3 = \frac{1}{10} \sum_{l=1}^{3} A_l \Big[x_l^3 + x_l^2 x_{l+1} + x_l x_{l+1}^2 + x_{l+1}^3 \Big]$$

$$\chi_1^3 = \frac{1}{60} \sum_{l=1}^{3} A_l \Big[4y_l x_l^3 + y_{l+1} x_l^3 + 3y_l x_l^2 x_{l+1} + 2y_{l+1} x_l^2 x_{l+1} + 2y_l x_l x_{l+1}^2 + 3y_{l+1} x_l x_{l+1}^2 + y_l x_{l+1}^3 + 4y_{l+1} x_{l+1}^3 \Big]$$

$$\chi_0^4 = \frac{1}{15} \sum_{l=1}^{3} A_l \Big[x_l^4 + x_l^3 x_{l+1} + x_l^2 x_{l+1}^2 + x_l x_{l+1}^3 + x_{l+1}^4 \Big]$$

Door middel van de volgende afkortingen kunnen de gegeven formules enigszins vereenvoudigd worden.

$a = x_l + x_{l+1}$ $\qquad aa = y_l + y_{l+1}$

$b = x_l^2 + x_{l+1}^2$ $\qquad bb = y_l^2 + y_{l+1}^2$

$c = x_l x_{l+1}$ $\qquad cc = y_l y_{l+1}$

$d = x_l y_l + x_{l+1} y_{l+1}$

De resultaten zijn neergeschreven in een direct programmeerbare vorm

$$\chi_0^0 = \sum_{\ell=1}^{3} A_\ell$$

$$\chi_1^0 = \frac{1}{3}\sum_{\ell=1}^{3} A_\ell\left[aa\right]$$

$$\chi_2^0 = \frac{1}{6}\sum_{\ell=1}^{3} A_\ell\left[bb+cc\right]$$

$$\chi_3^0 = \frac{1}{10}\sum_{\ell=1}^{3} A_\ell\left[aa*bb\right]$$

$$\chi_4^0 = \frac{1}{15}\sum_{\ell=1}^{3} A_\ell\left[bb*(bb+cc)-cc*cc\right]$$

$$\chi_0^1 = \frac{1}{3}\sum_{\ell=1}^{3} A_\ell\left[a\right]$$

$$\chi_1^1 = \frac{1}{12}\sum_{\ell=1}^{3} A_\ell\left[a*aa+d\right]$$

$$\chi_2^1 = \frac{1}{30}\sum_{\ell=1}^{3} A_\ell\left[2*aa*d+a*bb\right]$$

$$\chi_3^1 = \frac{1}{60}\sum_{\ell=1}^{3} A_\ell\left[a*aa*(bb-2*cc)+d*(3*bb+4*cc)\right]$$

$$\chi_0^2 = \frac{1}{6}\sum_{\ell=1}^{3} A_\ell\left[b+c\right]$$

$$\chi_1^2 = \frac{1}{30}\sum_{\ell=1}^{3} A_\ell\left[2*a*d+aa*b\right]$$

$$\chi_2^2 = \frac{1}{90}\sum_{\ell=1}^{3} A_\ell\left[b*(bb+3*cc)+c*(3*bb-6*cc)+5*d*d\right]$$

$$\chi_0^3 = \frac{1}{10}\sum_{\ell=1}^{3} A_\ell\left[a*b\right]$$

$$\chi_1^3 = \frac{1}{60}\sum_{\ell=1}^{3} A_\ell\left[a*aa*(b-2*c)+d*(3*b+4*c)\right]$$

$$\chi_0^4 = \frac{1}{15}\sum_{\ell=1}^{3} A_\ell\left[b*(b+c)-c*c\right]$$