

# Coordination of two robot manipulators based on position measurements only

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**Coordination of two robot  
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# Coordination of two robot manipulators based on position measurements only

A. Rodriguez-Angeles\*, H. Nijmeijer †

## Abstract

In this note we propose a controller that solves the problem of coordination of two (or more) robots, under a master-slave scheme, in the case when only position measurements are available. The controller consists of a feedback control law, and two nonlinear observers. It is shown that the controller yields ultimate uniformly boundedness of the closed loop errors, and a relation between this bound and the gains on the controller is established. Simulations results on two two-link robot systems show the predicted convergence performance.

## 1 Introduction

Synchronization, coordination, and cooperation are intimately linked subjects, and sometimes they are used as synonymous to describe the same kind of behavior, mainly in mechanical systems. Nowadays, there are several papers related with synchronization of rotating bodies and electrical-mechanical systems, see for instance (Blekman *et al.* 1995), (Huijberts *et al.* 2000), and communication systems (Pecora and Carroll 1990). Rotating mechanical structures form a very important and special class of systems that, with or without the interaction through some coupling, exhibit synchronized motion. On the other hand, for mechanical systems synchronization is of great importance as soon as two machines have to cooperate. Typically robot coordination, and cooperation of manipulators, see (Brunt 1998), (Liu *et al.* 1997), (Liu *et al.* 1999), form important illustrations of the same goal, where it is desired that two or more mechanical systems, either identical or different, are asked to work in synchrony.

In robot coordination the basic problem is to ascertain synchronous motion of two (or more) robotic systems. This is obviously a control problem, where at least for one of the robots a suitable feedback controller has to be designed, such that this robot (slave) follows the other robot (master). This problem is further complicated by the fact that frequently only position measurements of both master and slave robots are available. This partial access to the state of the system has been the reason to develop model-based observers, which are integrated in the feedback control loop.

In practice, robot manipulators are equipped with high precision position sensors, such as encoders. On the other hand the velocity measurements are obtained by means of tachometers, which are often contaminated by noise, or moreover, velocity sensing equipment is frequently omitted due to the savings in cost, volume, and weight that can be obtained. For these reasons, a number of model-based robot control methods have been proposed

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(Nicosia and Tomei 1990), (Canudas *et al.* 1992). In these methods a velocity observer is integrated in the control loop, although exact knowledge of the nonlinear robot dynamics is assumed, which in practice is generally not available. To overcome this drawback, robust tracking controllers only based on position measurements have been proposed (Canudas and Fixot 1991), (Berghuis and Nijmeijer 1994), (Wong Lee and Khalil 1997). However, all the aforementioned papers deal with the tracking control problem, and not with the robot coordination problem.

In this paper we present a novel approach for the coordination of two robot manipulators, assuming only position measurements of both robots. This approach is based on the design of two nonlinear observers and a state-feedback controller. The general setup to be considered is as follows.

Consider two fully actuated robot manipulators with  $n$  joints each, such that one of these robots (master) is driven by an input torque  $\tau_m(\cdot)$ , that ensures convergence of the joint variables  $q_m, \dot{q}_m \in \mathbb{R}^n$  to a desired trajectory  $q_d, \dot{q}_d \in \mathbb{R}^n$ . However, the input torque  $\tau_m$  is unknown, at least for the controller design of the second robot (slave), as well as the joint velocity and acceleration variables  $\dot{q}_m, \ddot{q}_m$ . Under these assumptions, the goal is to design a control law  $\tau_s(\cdot)$  for the slave robot, such that its joint variables  $q_s, \dot{q}_s \in \mathbb{R}^n$  synchronize with the variables  $q_m, \dot{q}_m$  of the master robot. Also we assume that the joint velocities and accelerations  $\dot{q}_s, \ddot{q}_s$  are not available; therefore from this fact and the assumption that  $\dot{q}_m, \ddot{q}_m$  are not available, the control law  $\tau_s$ , that is to be designed, can only depend on position measurements of both robots, i.e.  $q_m, q_s$ , and estimated values of the joint velocities and accelerations  $\dot{q}_m, \ddot{q}_m, \dot{q}_s, \ddot{q}_s$ . Notice that the goal is to follow the trajectories of the master robot  $q_m, \dot{q}_m$ , and not the desired trajectories  $q_d, \dot{q}_d$ ; therefore knowledge of  $q_d, \dot{q}_d$  is not necessary to design the control law  $\tau_s$  for the slave robot.

This paper is organized as follows. In Section 2 the dynamic model of the robot and some of its properties are presented. The feedback control law and the observers for slave and master velocities are proposed in Section 3. In Section 4 the convergence properties of the closed loop system are examined. In Section 5 a simulation study shows the predicted convergence performance. Sections 6 and 7 present some remarks and general conclusions. Throughout this paper standard notation is used, in particular, vector norms are Euclidean, and for matrices the induced norm  $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$  is employed, with  $\lambda_{\max}(\cdot)$  the maximum eigenvalue. Moreover, for any positive definite matrix  $A$  we denote by  $A_m$  and  $A_M$  its minimum and maximum eigenvalue respectively.

## 2 Dynamic model of the robot manipulators

Consider a pair of rigid robots, each one with the same number of joints, i.e.  $q_i \in \mathbb{R}^n$ , where  $i = m, s$  identifies the master ( $m$ ) and slave ( $s$ ) robot respectively, and all the joints are rotational, actuated and, without loss of generality, frictionless. This does not mean, however, that they are identical in their parameters (masses, inertias, etc.).

For each of the robots, the kinetic energy is given by  $T_i(q_i, \dot{q}_i) = \frac{1}{2} \dot{q}_i^T M_i(q_i) \dot{q}_i$ ,  $i = m, s$ , with  $M_i(q_i) \in \mathbb{R}^{n \times n}$  the symmetric, positive-definite inertia matrix, and the potential energy is denoted by  $U_i(q_i)$ . Hence, applying the Euler-Lagrange formalism (Spong and Vidyasagar 1989) the dynamic model of the robot is given by

$$M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + g_i(q_i) = \tau_i \quad i = m, s \quad (1)$$

where  $g_i(q_i) = \frac{\partial}{\partial q_i} U_i(q_i) \in \mathbb{R}^n$  denotes the gravity forces,  $C_i(q_i, \dot{q}_i) \dot{q}_i \in \mathbb{R}^n$  represents the Coriolis and centrifugal forces, and  $\tau_i$  denotes the  $[n \times 1]$  vector of input torques.

In the subsequent sections we use the following properties.

- If the matrix  $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n \times n}$  is defined using the Christoffel symbols (Spong and Vidyasagar 1989), then the matrix  $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$  is skew symmetric, i.e.

$$x^T \left( \dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i) \right) x = 0 \quad \text{for all } x \in \mathbb{R}^n \quad (2)$$

- In addition, for the previous choice of the matrix  $C_i(q_i, \dot{q}_i)$ , the Coriolis and centrifugal term  $C_i(q_i, \dot{q}_i)$  can be written as

$$C_i(q_i, \dot{q}_i) = \begin{bmatrix} \dot{q}_i^T C_{i1}(q_i) \\ \vdots \\ \dot{q}_i^T C_{in}(q_i) \end{bmatrix} \quad (3)$$

where  $C_{ij}(q_i) \in \mathbb{R}^{n \times n}$   $j = 1, \dots, n$  are symmetric matrices (Craig 1988). It follows that

$$\begin{aligned} C_i(q_i, x)y &= C_i(q_i, y)x \\ C_i(q_i, z + \alpha x)y &= C_i(q_i, z)y + \alpha C_i(q_i, x)y \end{aligned} \quad (4)$$

for any scalar  $\alpha$  and for all  $q_i, x, y, z \in \mathbb{R}^n$ .

- The matrices  $M_i(q_i)$ ,  $C_i(q_i, \dot{q}_i)$  are bounded with respect to  $q_i$ , (Lewis *et al.* 1993), so

$$0 < M_{i,m} \leq \|M_i(q_i)\| \leq M_{i,M} \quad \text{for all } q_i \in \mathbb{R}^n \quad (5)$$

$$\|C_i(q_i, x)\| \leq C_{i,M} \|x\| \quad \text{for all } q_i, x \in \mathbb{R}^n. \quad (6)$$

### 3 Feedback controller

As stated in Section 1, it is assumed that there is no access to  $(\dot{q}_m, \ddot{q}_m)$  and  $(\dot{q}_s, \ddot{q}_s)$ , but only joint positions  $q_m$  and  $q_s$  can be measured. Therefore, the controller  $\tau_s$  can only depend on positions measurements  $(q_m, q_s)$  and estimated values for the velocities  $(\hat{q}_m, \hat{q}_s)$  and accelerations  $(\hat{\ddot{q}}_m, \hat{\ddot{q}}_s)$ .

#### 3.1 Feedback control law

If the variables  $(q_m, q_s)$  and  $(\hat{q}_m, \hat{q}_s, \hat{\ddot{q}}_m, \hat{\ddot{q}}_s)$  were available and all the parameters of the slave robot were known, then the control law  $\tau_s$  can be considered of the form proposed by Paden and Panja (1988)

$$\tau_s = M_s(q_s)\hat{\ddot{q}}_m + C_s(q_s, \hat{q}_s)\hat{q}_m + g_s(q_s) - K_d\dot{e}_s - K_p e_s \quad (7)$$

where the tracking errors  $e_s, \dot{e}_s \in \mathbb{R}^n$  are defined by

$$e_s := q_s - q_m, \quad \dot{e}_s := \dot{q}_s - \dot{q}_m, \quad (8)$$

$M_s(q_s)$ ,  $C_s(q_s, \hat{q}_s)$ ,  $g_s(q_s)$  are defined as in equation (1), and  $K_p, K_d \in \mathbb{R}^{n \times n}$  are positive definite gain matrices.

With the control law proposed by Paden and Panja (1988) in mind, and under the assumptions that the estimated values are available, and the nonlinearities and parameters of the slave robot are known, we propose the controller  $\tau_s$  for the slave robot as

$$\tau_s = M_s(q_s)\hat{\ddot{q}}_m + C_s(q_s, \hat{q}_s)\hat{q}_m + g_s(q_s) - K_d\hat{\dot{e}}_s - K_p e_s \quad (9)$$

where  $\hat{q}_s, \hat{\dot{e}}_s, \hat{q}_m, \hat{\ddot{q}}_m \in \mathbb{R}^n$  represent the estimates of  $\dot{q}_s, \dot{e}_s, \dot{q}_m$  and  $\ddot{q}_m$  respectively.

### 3.2 An observer for the tracking errors $(e_s, \dot{e}_s)$

We denote estimated values for the tracking errors  $e_s, \dot{e}_s$  (8) by  $\hat{e}_s, \hat{\dot{e}}_s$ ; these estimated values are obtained by the nonlinear Luenberger observer

$$\begin{aligned}\frac{d}{dt}\hat{e}_s &= \hat{\dot{e}}_s + \Lambda_1 \hat{e} \\ \frac{d}{dt}\hat{\dot{e}}_s &= M_s(q_s)^{-1} \left[ -C_s(q_s, \hat{q}_s) \hat{e}_s - K_d \hat{\dot{e}}_s - K_p \hat{e}_s \right] + \Lambda_2 \hat{e}\end{aligned}\quad (10)$$

where the estimation position and velocity tracking errors  $\hat{e}, \hat{\dot{e}}$  are defined by

$$\hat{e} := e_s - \hat{e}_s, \quad \hat{\dot{e}} := \dot{e}_s - \hat{\dot{e}}_s, \quad (11)$$

and  $\Lambda_1, \Lambda_2 \in \mathbb{R}^{n \times n}$  are positive definite gain matrices.

### 3.3 An observer for the slave joint variables $(q_s, \dot{q}_s)$

Lets  $\hat{q}_s, \hat{\dot{q}}_s$  denote estimated values for  $q_s, \dot{q}_s$ , to compute these estimated values, we propose the nonlinear observer

$$\begin{aligned}\frac{d}{dt}\hat{q}_s &= \hat{\dot{q}}_s + L_{p1} \hat{e}_\eta \\ \frac{d}{dt}\hat{\dot{q}}_s &= M_s(q_s)^{-1} \left[ -C_s(q_s, \hat{q}_s) \hat{e}_s - K_d \hat{\dot{e}}_s - K_p e_s \right] + L_{p2} \hat{e}_\eta\end{aligned}\quad (12)$$

where the estimation position and velocity errors  $\hat{e}_\eta$  and  $\hat{\dot{e}}_\eta$  are defined by

$$\hat{e}_\eta := q_s - \hat{q}_s \quad \hat{\dot{e}}_\eta := \dot{q}_s - \hat{\dot{q}}_s, \quad (13)$$

and  $L_{p1}, L_{p2} \in \mathbb{R}^{n \times n}$  are positive definite gain matrices.

### 3.4 Estimated values for $\dot{q}_m, \ddot{q}_m$

As stated, the master robot variables  $\dot{q}_m, \ddot{q}_m$  are not available, therefore estimated values for  $\dot{q}_m, \ddot{q}_m$  are used in  $\tau_s$  (9). From (8) and the definition of the estimated variables  $\hat{e}_s, \hat{\dot{e}}_s, \hat{q}_s, \hat{\dot{q}}_s$ , we can consider that estimated values for  $q_m, \dot{q}_m, \ddot{q}_m$  are given by

$$\begin{aligned}\hat{q}_m &= \hat{q}_s - \hat{e}_s \\ \hat{\dot{q}}_m &= \hat{\dot{q}}_s - \hat{\dot{e}}_s \\ \hat{\ddot{q}}_m &= \frac{d}{dt}\hat{\dot{q}}_s - \frac{d}{dt}\hat{\dot{e}}_s.\end{aligned}\quad (14)$$

## 4 Ultimate boundedness of the closed loop system

In the closed loop system formed by the slave robot (1), the control law (9), and both observers (10) and (12), the closed loop errors are the tracking errors  $(e_s, \dot{e}_s)$ , the estimation tracking errors  $(\hat{e}, \hat{\dot{e}})$ , and the estimation position and velocity errors  $(\hat{e}_\eta, \hat{\dot{e}}_\eta)$ , which are defined by (8), (11), and (13).

To simplify the stability analysis, we make the following assumptions on the positive definite gain matrices  $K_p, K_d, L_{p1}, L_{p2}, \Lambda_1, \Lambda_2$ .

**Assumption 1** *The gain matrices  $\Lambda_1, \Lambda_2$  and  $L_{p1}, L_{p2}$  satisfy*

$$\Lambda_1 = L_{p1}, \quad \Lambda_2 = L_{p2}. \quad (15)$$

**Assumption 2** The gains  $K_p, K_d, L_{p1}, L_{p2}$  are symmetric matrices.

In addition, the following assumption is required.

**Assumption 3** The signals  $\dot{q}_m(t)$  and  $\ddot{q}_m(t)$  are bounded by  $V_M$  and  $A_M$ , i.e.

$$V_M = \sup_t \|\dot{q}_m(t)\|, \quad (16)$$

$$A_M = \sup_t \|\ddot{q}_m(t)\|. \quad (17)$$

In practice, it is often not difficult to obtain on the basis of the desired motion  $q_d(t), \dot{q}_d(t)$  and  $\ddot{q}_d(t)$  of the master robot bounds on  $\dot{q}_m(t)$  and  $\ddot{q}_m(t)$ , although due to friction effects, tracking errors, etc., the actual motion of the master robot may differ from its desired motion.

Our main result can be formulated as follows.

**Theorem 4** Consider the master and slave robots, which are described by (1), and the slave robot in closed loop with the control law (9), and both observers (10), (12). Given scalar parameters  $\varepsilon_o, \lambda_o, \mu_o, \gamma_o$ , such that

$$\lambda_o > 0, \quad \mu_o > 0, \quad \gamma_o > 0, \quad \varepsilon_o > \max\{0, \varepsilon_{q6}\}, \quad (18)$$

and if the gain matrices  $K_d, K_p, L_{p1}, L_{p2}$  are chosen such that

$$\begin{aligned} L_{p2,m} &> \max\{\mu_o^2, \gamma_o^2, L_{p2q4}, L_{p2q5}, L_{p2q6}\}, \\ L_{p1,m} &> \max\{\mu_o, \gamma_o, L_{p1q5}\}, \\ K_{p,m} &> \max\{K_{pq2}, K_{pq3}\}, \\ K_{d,m} &> \max\{K_{dq1}, K_{dq3}, K_{dq5}, K_{dq6}\}, \end{aligned} \quad (19)$$

then, the errors  $\dot{e}_s, e_s, \hat{e}, \dot{e}, \hat{e}_\eta, \dot{e}_\eta$  in the closed loop system are uniformly ultimately bounded. Moreover, this bound can be made small, by a proper choose of  $K_{p,m}$  and  $L_{p1,M}$ . The scalars  $\varepsilon_{q6}, L_{p2q4}, L_{p2q5}, L_{p2q6}, L_{p1q5}, K_{pq2}, K_{pq3}, K_{dq1}, K_{dq3}, K_{dq5}, K_{dq6}$  are given in Appendix A.

*Proof:* The proof of the theorem is divided into two steps. First the formulation of the closed loop error dynamics is given in Subsection 4.1, and then the stability analysis is presented in Subsection 4.2.

#### 4.1 Closed loop error dynamics

To simplify the closed loop error dynamics two coordinate transformations are introduced.

**Lemma 5** Consider the tracking errors  $(e_s, \dot{e}_s)$ , the estimation tracking errors  $(\hat{e}, \dot{\hat{e}})$  and the estimation position and velocity errors  $(e_\eta, \dot{e}_\eta)$ , which are defined by (8), (11), and (13).

Introduce the coordinate transformation defined by

$$\tilde{q} := \hat{e} - \dot{e}_\eta \quad (20)$$

and

$$\begin{aligned} \bar{q} &:= e_s - \tilde{q} \\ \dot{\bar{q}} &:= \dot{e}_s - \dot{\tilde{q}} + L_{p1}\tilde{q} \\ \ddot{\bar{q}} &:= \ddot{e}_s - \ddot{\tilde{q}} + L_{p1}\dot{\tilde{q}} - L_{p1}L_{p1}\tilde{q}. \end{aligned} \quad (21)$$

Define the vectors  $x, y \in \mathbb{R}^{6n}$  as

$$x^T := \begin{bmatrix} \dot{e}_s^T & e_s^T & \hat{e}^T & \hat{e}^T & \hat{e}_\eta^T & \hat{e}_\eta^T \end{bmatrix}, \quad (22)$$

$$y^T := \begin{bmatrix} \dot{\tilde{q}}^T & \tilde{q}^T & \dot{\tilde{q}}^T & \tilde{q}^T & \hat{e}_\eta^T & \hat{e}_\eta^T \end{bmatrix}, \quad (23)$$

then  $x$  and  $y$  are related by

$$x = Ty \quad (24)$$

where

$$T = \begin{bmatrix} I & 0 & I & -L_{p1} & 0 & 0 \\ 0 & I & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \quad (25)$$

*Proof:* The proof follows from the definition of the coordinate transformations.  $\square$

In the new set of error coordinates, the closed loop error dynamics can be formulated as follows.

**Lemma 6** Consider the closed loop system formed by the slave robot (1), the control law (9), and both observers (10), (12). Then, in the variables  $(\hat{e}_\eta, \hat{e}_\eta)$ ,  $(\tilde{q}, \tilde{q})$ , and  $(\dot{\tilde{q}}, \tilde{q})$ , defined by (13), (20), and (21), the closed loop error dynamics are given by

$$\begin{aligned} M_s(q_s) \ddot{\tilde{q}} + C_s(q_s, \dot{q}_s) \dot{\tilde{q}} + K_d \dot{\tilde{q}} + K_p \tilde{q} = & -C_s(q_s, \dot{q}_s) \hat{e}_\eta + C_s(q_s, \hat{e}_\eta) \dot{\tilde{q}} + \\ & -C_s(q_s, \hat{e}_\eta) L_{p1} \tilde{q} + C_s(q_s, \dot{q}_s) L_{p1} \tilde{q} + \\ & + K_d \hat{e}_\eta + K_d L_{p1} \tilde{q} - K_p \tilde{q} + \\ & + M(q_s) \left( L_{p1} \dot{\tilde{q}} - L_{p1} L_{p1} \tilde{q} \right), \end{aligned} \quad (26)$$

$$\frac{d}{dt} \tilde{q} = \dot{\tilde{q}} - L_{p1} \tilde{q} \quad (27)$$

$$\frac{d}{dt} \dot{\tilde{q}} = -M_s(q_s)^{-1} K_p (\tilde{q} + \hat{e}_\eta) - L_{p2} \tilde{q} - \ddot{q}_m, \quad (28)$$

$$\frac{d}{dt} \hat{e}_\eta = \hat{e}_\eta - L_{p1} \hat{e}_\eta \quad (29)$$

$$\frac{d}{dt} \hat{e}_\eta = M_s(q_s)^{-1} \{-K_p (\tilde{q} + \hat{e}_\eta) - 2C_s(q_s, \dot{q}_s) \hat{e}_\eta + C_s(q_s, \hat{e}_\eta) \hat{e}_\eta\} - L_{p2} (\tilde{q} + \hat{e}_\eta). \quad (30)$$

*Proof:* See Appendix B.  $\square$

## 4.2 Stability of the closed loop error dynamics

First we introduce a result that supports the stability analysis in the following subsections. This result is a modified version of a theorem by Chen and Leitmann (1987), (see also (Berghuis and Nijmeijer 1994)), which states that a system is uniformly ultimately bounded if it has a Lyapunov function whose time-derivative is negative definite in an annulus of a certain width around the origin.



**Lemma 7** (Berghuis and Nijmeijer 1994) Consider the function  $g(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$

$$g(y) = \alpha_0 - \alpha_1 y + \alpha_2 y^2, \quad y \in \mathbb{R}^+ \quad (31)$$

where  $\alpha_i > 0$ ,  $i = 0, 1, 2$ . Then  $g(y) < 0$  if  $y_1 < y < y_2$ , where

$$y_1 = \frac{\alpha_1 - \sqrt{(\alpha_1^2 - 4\alpha_2\alpha_0)}}{2\alpha_2}, \quad y_2 = \frac{\alpha_1 + \sqrt{(\alpha_1^2 - 4\alpha_2\alpha_0)}}{2\alpha_2}. \quad (32)$$

**Proposition 8** Let  $x(t) \in \mathbb{R}^n$  be the solution of the differential equation

$$\dot{x}(t) = f(x(t), t)$$

with  $f(x(t), t)$  Lipschitz and initial condition  $x(t_0) = x_0$ , and assume there exists a function  $V(x(t), t)$  that satisfies

$$P_m \|x(t)\|^2 \leq V(x(t), t) \leq P_M \|x(t)\|^2, \quad (33)$$

$$\dot{V}(x(t), t) \leq \|x(t)\| \cdot g(\|x(t)\|) < 0 \quad \text{for all } y_1 < \|x(t)\| < y_2 \quad (34)$$

with  $P_m$  and  $P_M$  positive constants,  $g(\cdot)$  as in (31), and  $y_1, y_2$  as in (32). Define  $\delta := \sqrt{P_m^{-1}P_M}$ . If  $y_2 > \delta y_1$ , then  $x(t)$  is locally uniformly ultimately bounded, that is, given  $d_m = \delta y_1$ , there exists  $d \in (d_m, y_2)$  such that

$$\|x_0\| \leq r \Rightarrow \|x(t)\| \leq d \quad \text{for all } t \geq t_0 + T(d, r),$$

where

$$T(d, r) = \begin{cases} 0 & r \leq R \\ \frac{P_M r^2 - P_m R^2}{-\alpha_0 R + \alpha_1 R^2 - \alpha_2 R^3} & R < r < \delta^{-1} y_2 \end{cases}$$

and  $R = \delta^{-1} d$ .

Consider the vector  $y \in \mathbb{R}^{6n}$  defined by (23), and take as a candidate Lyapunov function

$$V(y) = \frac{1}{2} y^T P(y) y \quad (35)$$

where  $P(y) = P(y)^T$  is given by

$$P(y) = \begin{bmatrix} \varepsilon_o \begin{bmatrix} M(q_s) & \lambda_o M(q_s) \\ \lambda_o M(q_s) & K_p + \lambda_o K_d \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} I & \mu(\tilde{q})I \\ \mu(\tilde{q})I & L_{p2} \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} I & \gamma(\hat{e}_\eta)I \\ \gamma(\hat{e}_\eta)I & L_{p2} \end{bmatrix} \end{bmatrix}, \quad (36)$$

$\varepsilon_o, \lambda_o \in \mathbb{R}$  are positive constants to be determined, and  $\mu(\tilde{q}), \gamma(\hat{e}_\eta)$  are defined by

$$\mu(\tilde{q}) := \frac{-\mu_o}{1 + \|\tilde{q}\|}, \quad \gamma(\hat{e}_\eta) := \frac{-\gamma_o}{1 + \|\hat{e}_\eta\|} \quad (37)$$

with  $\mu_o, \gamma_o \in \mathbb{R}$  positive constants to be determined,  $\mu(\tilde{q}), \gamma(\hat{e}_\eta)$  are bounded, such that

$$-\mu_o \leq \mu(\tilde{q}) < 0, \quad \text{and} \quad -\gamma_o \leq \gamma(\hat{e}_\eta) < 0. \quad (38)$$

Sufficient conditions for positive definiteness of  $P(y)$  are

$$K_{d,m} > \lambda_o M_{s,M}, \quad L_{p2,m} > \max \{ \mu_o^2, \gamma_o^2 \} \quad (39)$$

Therefore, conditions (18-19), with the boundedness from above of  $\mu(\tilde{q})$ ,  $\gamma(\hat{e}_\eta)$ , imply that there exist constants  $P_m$  and  $P_M$  such that

$$\frac{1}{2} P_m \|y\|^2 \leq V(y) \leq \frac{1}{2} P_M \|y\|^2. \quad (40)$$

Along the error dynamics (26-30), and under Assumption 2, the time derivative of (35) becomes

$$\dot{V}(y) = -y^T Q(y)y + \beta(y, \dot{q}_s, \ddot{q}_m) \quad (41)$$

where

$$\begin{aligned} \beta(y, \dot{q}_s, \ddot{q}_m) = & \varepsilon_o \dot{\tilde{q}}^T C_s(q_s, \hat{e}_\eta) \dot{\tilde{q}} + \hat{e}_\eta^T M_s(q_s)^{-1} \left( C_s(q_s, \hat{e}_\eta) - 2C_s(q_s, \dot{q}_s) \right) \hat{e}_\eta + \\ & + \varepsilon_o \lambda_o \dot{\tilde{q}}^T \dot{M}(q_s) \tilde{q} + \varepsilon_o \lambda_o \tilde{q}^T \left( C_s(q_s, \hat{e}_\eta) - C_s(q_s, \dot{q}_s) \right) \dot{\tilde{q}} + \\ & - \varepsilon_o \dot{\tilde{q}}^T \left( C_s(q_s, \hat{e}_\eta) - C_s(q_s, \dot{q}_s) \right) L_{p1} \tilde{q} - \varepsilon_o \dot{\tilde{q}}^T C_s(q_s, \dot{q}_s) \hat{e}_\eta + \\ & - \varepsilon_o \lambda_o \tilde{q}^T \left( C_s(q_s, \hat{e}_\eta) - C_s(q_s, \dot{q}_s) \right) L_{p1} \tilde{q} - \varepsilon_o \lambda_o \tilde{q}^T C_s(q_s, \dot{q}_s) \hat{e}_\eta + \\ & + \dot{\mu} \dot{\tilde{q}}^T \tilde{q} + \dot{\gamma} \hat{e}_\eta^T \hat{e}_\eta + \hat{e}_\eta^T M_s(q_s)^{-1} \left( C_s(q_s, \hat{e}_\eta) - 2C_s(q_s, \dot{q}_s) \right) \hat{e}_\eta + \\ & - \dot{\tilde{q}}^T \ddot{q}_m - \mu \tilde{q}^T \ddot{q}_m \end{aligned} \quad (42)$$

and  $Q(y) = Q(y)^T$  is given by

$$Q(y) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix} \quad (43)$$

with the block matrices

$$\begin{aligned} Q_{11} &= \varepsilon_o \begin{bmatrix} K_d - \lambda_o M_s(q_s) & 0 \\ 0 & \lambda_o K_p \end{bmatrix} \\ Q_{12} &= \frac{\varepsilon_o}{2} \begin{bmatrix} -M_s(q_s) L_{p1} & K_p - (K_d - M_s(q_s) L_{p1}) L_{p1} \\ -\lambda_o M_s(q_s) L_{p1} & \lambda_o (K_p - (K_d - M_s(q_s) L_{p1}) L_{p1}) \end{bmatrix} \\ Q_{13} &= \frac{\varepsilon_o}{2} \begin{bmatrix} -K_d & 0 \\ -\lambda_o K_d & 0 \end{bmatrix} \\ Q_{22} &= \begin{bmatrix} -\mu I & \frac{1}{2} (M_s(q_s)^{-1} K_p + \mu L_{p1}) \\ \frac{1}{2} (M_s(q_s)^{-1} K_p + \mu L_{p1})^T & \mu M_s(q_s)^{-1} K_p + L_{p2} (\mu I + L_{p1}) \end{bmatrix} \\ Q_{23} &= \begin{bmatrix} 0 & \frac{1}{2} M_s(q_s)^{-1} K_p \\ \frac{1}{2} (M_s(q_s)^{-1} K_p + L_{p2}) & \frac{1}{2} ((\mu + \gamma) M_s(q_s)^{-1} K_p + \gamma L_{p2}) \end{bmatrix} \\ Q_{33} &= \begin{bmatrix} -\gamma I & \frac{1}{2} (M_s(q_s)^{-1} K_p + \gamma L_{p1}) \\ \frac{1}{2} (M_s(q_s)^{-1} K_p + \gamma L_{p1})^T & \gamma M_s(q_s)^{-1} K_p + L_{p2} (\gamma I + L_{p1}) \end{bmatrix}. \end{aligned}$$

To conclude stability of the variable  $y$  defined by (23), we require positive definiteness of  $Q(y)$  and boundedness of the term  $\beta(y, \dot{q}_s, \ddot{q}_m)$  along the closed loop error dynamics, these two requirements are developed in the following subsections.

#### 4.2.1 Boundedness of $\beta(y, \dot{q}_s, \ddot{q}_m)$

First, from the definition of  $\mu(\tilde{q})$ ,  $\gamma(\hat{e}_\eta)$  (37), it follows that

$$\dot{\mu} \tilde{q}^T \tilde{q} = -\mu \left( \frac{\tilde{q}^T \dot{\tilde{q}}}{1 + \|\tilde{q}\|} \right) \tilde{q}^T \tilde{q} \leq -\mu \|\dot{\tilde{q}}\|^2 \quad (44)$$

$$\dot{\gamma} \hat{e}_\eta^T \hat{e}_\eta = -\gamma \left( \frac{\hat{e}_\eta^T \dot{\hat{e}}_\eta}{1 + \|\hat{e}_\eta\|} \right) \hat{e}_\eta^T \hat{e}_\eta \leq -\gamma \|\dot{\hat{e}}_\eta\|^2. \quad (45)$$

Then by boundedness of  $\mu(\tilde{q})$ ,  $\gamma(\hat{e}_\eta)$  (38) we obtain that

$$\dot{\mu} \tilde{q}^T \tilde{q} \leq \mu_o \|\dot{\tilde{q}}\|^2, \quad \text{and} \quad \dot{\gamma} \hat{e}_\eta^T \hat{e}_\eta \leq \gamma_o \|\dot{\hat{e}}_\eta\|^2. \quad (46)$$

On the other hand, the definition of the tracking errors (8) implies that

$$\dot{q}_s = \dot{e}_s + \dot{q}_m$$

Then, from the definition of  $\tilde{q}$ , (21), we obtain a relation between  $\dot{q}_s$  and  $\dot{\tilde{q}}$ , which is given by

$$\dot{q}_s = \dot{\tilde{q}} + \dot{\tilde{q}} - L_{p1} \tilde{q} + \dot{q}_m. \quad (47)$$

Finally, the definition of the inertia matrix  $M(q_s)$  implies that

$$\dot{M}(q_s) = \frac{d}{dt} M(q_s) = \frac{\partial M(q_s)}{\partial q_s} \dot{q}_s$$

hence, by property (5) and since  $q_s$  appears like argument of sinusoidal functions in  $M(q_s)$ , we can conclude that

$$M_{s,pm} \|\dot{q}_s\| \leq \|\dot{M}(q_s)\| \leq M_{s,pM} \|\dot{q}_s\| \quad (48)$$

where

$$M_{s,pm} \leq \left\| \frac{\partial M(q_s)}{\partial q_s} \right\| \leq M_{s,pM}.$$

Then, (46-48), properties (5), (6), and taking into account Assumption 3, imply that  $\beta(y, \dot{q}_s, \ddot{q}_m)$  is upperbounded by

$$\beta(y, \dot{q}_s, \ddot{q}_m) \leq \beta_o(y_N, V_M, A_M) \quad (49)$$

with

$$\begin{aligned} \beta_o(y_N, V_M, A_M) = & \varepsilon_o \|\dot{\tilde{q}}\|^2 \{ \lambda_o \|\tilde{q}\| (M_{s,pM} - C_{s,M}) + C_{s,M} L_{p1,M} \|\tilde{q}\| \} + \\ & + \|\hat{e}_\eta\|^2 \{ -2M_{s,m}^{-1} C_{s,M} (\|\dot{\tilde{q}}\| + \|\dot{\tilde{q}}\| - L_{p1,M} \|\tilde{q}\| + V_M - \frac{\|\hat{e}_\eta\|}{2}) + \\ & + \gamma_o (M_{s,m}^{-1} C_{s,M} + 1) \} + \mu_o \|\dot{\tilde{q}}\|^2 - \varepsilon_o C_{s,M} L_{p1,m}^2 \|\tilde{q}\|^2 (\lambda_o \|\tilde{q}\| + \|\dot{\tilde{q}}\|) + \\ & + \varepsilon_o \lambda_o \|\dot{\tilde{q}}\| \|\tilde{q}\| \left( \|\dot{\tilde{q}}\| (M_{s,pM} - C_{s,M}) + L_{p1,M} \|\tilde{q}\| (2C_{s,M} - M_{s,pm}) \right) + \\ & + \varepsilon_o \lambda_o \|\dot{\tilde{q}}\| \|\tilde{q}\| V_M (M_{s,pM} - C_{s,M}) + \varepsilon_o C_{s,M} L_{p1,M} \|\dot{\tilde{q}}\| \|\tilde{q}\| (\|\dot{\tilde{q}}\| + V_M) + \\ & - \varepsilon_o C_{s,M} \|\dot{\tilde{q}}\| \|\hat{e}_\eta\| (\|\dot{\tilde{q}}\| + V_M) + \varepsilon_o \lambda_o C_{s,M} L_{p1,M} \|\tilde{q}\| \|\tilde{q}\| (\|\dot{\tilde{q}}\| + V_M) + \\ & - \varepsilon_o \lambda_o C_{s,M} \|\tilde{q}\| \|\hat{e}_\eta\| (\|\dot{\tilde{q}}\| + V_M) - A_M (\|\dot{\tilde{q}}\| - \mu_o \|\tilde{q}\|) + \\ & + 2\gamma_o M_{s,m}^{-1} C_{s,M} \|\hat{e}_\eta\| (\|\dot{\tilde{q}}\| + \|\dot{\tilde{q}}\| + V_M \|\hat{e}_\eta\|). \end{aligned} \quad (50)$$

where the vector  $y_N \in \mathbb{R}^6$  is defined as

$$y_N^T := \left[ \|\dot{\tilde{q}}\| \quad \|\tilde{q}\| \quad \|\dot{\tilde{q}}\| \quad \|\tilde{q}\| \quad \|\hat{e}_\eta\| \quad \|\hat{e}_\eta\| \right]. \quad (51)$$

#### 4.2.2 Negative definiteness of $\dot{V}(y)$

From the upperbound of  $\beta(y, \dot{q}_s, \ddot{q}_m)$  (49), the upperbound of  $\mu(\tilde{q})$ ,  $\gamma(\hat{e}_\eta)$  (38), and considering the vector  $y_N$  defined by (51), it follows that  $\dot{V}(y)$  (41) can be upperbounded by

$$\dot{V}(y) \leq -y_N^T Q_N y_N + \beta_1(y_N, V_M, A_M) \quad (52)$$

where the matrix  $Q_N = Q_N^T$  is given by

$$Q_N = \begin{bmatrix} Q_{11_N} & Q_{12_N} & Q_{13_N} \\ Q_{12_N}^T & Q_{22_N} & Q_{23_N} \\ Q_{13_N}^T & Q_{23_N}^T & Q_{33_N} \end{bmatrix} \quad (53)$$

with the block matrices

$$\begin{aligned} Q_{11_N} &= \varepsilon_o \begin{bmatrix} K_{d,m} - \lambda_o M_{s,M} & -\frac{1}{2} \lambda_o V_M (M_{s,pM} - C_{s,M}) \\ -\frac{1}{2} \lambda_o V_M (M_{s,pM} - C_{s,M}) & \lambda_o K_{p,m} \end{bmatrix} \\ Q_{12_N} &= \frac{\varepsilon_o}{2} \begin{bmatrix} -M_{s,M} L_{p1,M} & K_{p,m} - K_{d,M} L_{p1,M} + M_{s,m} L_{p1,m}^2 - C_{s,M} L_{p1,M} V_M \\ -\lambda_o M_{s,M} L_{p1,M} & \lambda_o (K_{p,m} - K_{d,M} L_{p1,M} + M_{s,m} L_{p1,m}^2 - C_{s,M} L_{p1,M} V_M) \end{bmatrix} \\ Q_{13_N} &= \frac{\varepsilon_o}{2} \begin{bmatrix} -K_{d,M} + C_{s,M} V_M - 2\gamma_o M_{s,m}^{-1} C_{s,M} \varepsilon_o^{-1} & 0 \\ -\lambda_o K_{d,M} + \lambda_o C_{s,M} V_M & 0 \end{bmatrix} \\ Q_{22_N} &= \begin{bmatrix} -\mu_o & \frac{1}{2} (M_{s,m}^{-1} K_{p,m} + \mu_o L_{p1,m}) \\ \frac{1}{2} (M_{s,m}^{-1} K_{p,m} + \mu_o L_{p1,m}) & \mu_o M_{s,m}^{-1} K_{p,m} + L_{p2,m} (\mu_o + L_{p1,m}) \end{bmatrix} \\ Q_{23_N} &= \begin{bmatrix} -\gamma_o M_{s,m}^{-1} C_{s,M} & \frac{1}{2} M_{s,m}^{-1} K_{p,m} \\ \frac{1}{2} (M_{s,m}^{-1} K_{p,m} + L_{p2,m}) & \frac{1}{2} ((\mu_o + \gamma_o) M_{s,m}^{-1} K_{p,m} + \gamma_o L_{p2,m}) \end{bmatrix} \\ Q_{33_N} &= \begin{bmatrix} -\gamma_o (1 + M_{s,m}^{-1} C_{s,M}) + 2M_{s,m}^{-1} C_{s,M} V_M & \frac{1}{2} (M_{s,m}^{-1} K_{p,m} + \gamma_o L_{p1,m}) - \gamma_o M_{s,m}^{-1} C_{s,M} V_M \\ \frac{1}{2} (M_{s,m}^{-1} K_{p,m} + \gamma_o L_{p1,m}) - \gamma_o M_{s,m}^{-1} C_{s,M} V_M & \gamma_o M_{s,m}^{-1} K_{p,m} + L_{p2,m} (\gamma_o + L_{p1,m}) \end{bmatrix} \end{aligned}$$

and  $\beta_1(y_N, V_M, A_M)$  given by

$$\begin{aligned} \beta_1(y_N, V_M, A_M) &= \varepsilon_o \|\dot{\tilde{q}}\|^2 \{ \lambda_o \|\tilde{q}\| (M_{s,pM} - C_{s,M}) + C_{s,M} L_{p1,M} \|\tilde{q}\| \} + \\ &\quad - 2M_{s,m}^{-1} C_{s,M} \|\hat{e}_\eta\|^2 \left( \|\dot{\tilde{q}}\| + \|\tilde{q}\| - L_{p1,M} \|\tilde{q}\| - \frac{\|\hat{e}_\eta\|}{2} \right) + \\ &\quad - \varepsilon_o C_{s,M} L_{p1,m}^2 \|\tilde{q}\|^2 (\lambda_o \|\tilde{q}\| + \|\dot{\tilde{q}}\|) + \varepsilon_o C_{s,M} \|\dot{\tilde{q}}\| \|\tilde{q}\| (L_{p1,M} \|\tilde{q}\| - \|\hat{e}_\eta\|) + \\ &\quad + \varepsilon_o \lambda_o \|\dot{\tilde{q}}\| \|\tilde{q}\| \left( \|\dot{\tilde{q}}\| (M_{s,pM} - C_{s,M}) + L_{p1,M} \|\tilde{q}\| (2C_{s,M} - M_{s,pM}) \right) + \\ &\quad + \varepsilon_o \lambda_o C_{s,M} \|\tilde{q}\| \|\dot{\tilde{q}}\| \left( L_{p1,M} \|\tilde{q}\| - \|\hat{e}_\eta\| \right) - A_M \left( \|\dot{\tilde{q}}\| - \mu_o \|\tilde{q}\| \right). \quad (54) \end{aligned}$$

If the gains  $K_d, K_p, L_{p1}, L_{p2}$  and the constants  $\varepsilon_o, \lambda_o, \mu_o, \gamma_o$  satisfy conditions (18) and (19), then  $Q_N$  given by (53) is positive definite. Therefore (52) and (54) imply that

$$\dot{V}(y) \leq \|y_N\| \left( \alpha_0 - Q_{Nm} \|y_N\| + \alpha_2 \|y_N\|^2 \right) \quad (55)$$

where  $Q_{Nm} > 0$  is the minimum eigenvalue of  $Q_N$ , and  $\alpha_0, \alpha_2$  are given by

$$\alpha_0 = (1 + \sqrt{\mu_o}) \sqrt{A_M} \quad (56)$$

$$\begin{aligned} \alpha_2 = & (\varepsilon_o + \sqrt{\varepsilon_o}) \left( \sqrt{C_{s,M} L_{p1,M}} + \sqrt{\lambda_o (M_{s,pM} + C_{s,M})} \right) + 2M_{s,m}^{-1} C_{s,M} \left( \sqrt{L_{p1,M}} + \frac{3\sqrt{2}}{2} \right) + \\ & + \left( 1 + \sqrt{\lambda_o} \right) \left( \varepsilon_o C_{s,M} L_{p1,M}^2 + \sqrt{\varepsilon_o C_{s,M}} \right) + \\ & + \sqrt{\varepsilon_o \lambda_o L_{p1,M}} (\sqrt{C_{s,M}} + \sqrt{2C_{s,M} + M_{s,pM}}) \end{aligned} \quad (57)$$

Then the right-hand side in (55) corresponds to (34), and together with (40) and proposition 8, allow us to conclude uniformly ultimately boundedness of  $y_N$  (51) and consequently of  $y$  (23), and thus, by (24) we can conclude that the original state  $x$  given by (22) is uniformly ultimately bounded.

Moreover,  $\alpha_2$  depends explicitly on  $L_{p1,M}$ , such that  $y_2$  defined as in proposition 8, can be made small by a proper chose of  $L_{p1,M}$ , and thus the upperbound for the closed loop errors  $\dot{e}_s, e_s, \hat{e}, \hat{e}_\eta, \hat{e}_\eta$  can be made small. Notice that the minimum value for  $y_2$  is given by  $Q_{Nm}/(2\alpha_2)$ , such that this minimum value depends on the minimum eigenvalue of  $Q_N$  (53), which depends on  $K_{p,m}$ .

On the other hand, a region of attraction is given by

$$B = \left\{ x \in \mathbb{R}^{6n} \mid \|x\| < \frac{y_2}{\|T\|} \sqrt{\frac{P_m}{P_M}} \right\}. \quad (58)$$

where  $T$  is given by (25),  $P_m, P_M$  are defined by (40), and  $y_2$  as in proposition 8, with (34) given by (55). The region of attraction  $B$  (58) is proportional to  $y_2$ , such that the region  $B$  can be expanded by increasing  $y_2$ . ■

## 5 Simulations

The master (m) and slave (s) robots considered in the simulations are planar manipulators  $q_i \in \mathbb{R}^2$ ,  $i = m, s$ , with revolute joints, working in the x-z plane. The dynamic model is given in Spong and Vidyasagar (1989), and their parameters are listed in the following table

Table 1. Parameters of the master (m) and slave (s) robots.

	$m$ (mass) [Kg]	$l_c$ (mass centre) [m]	$i$ (inertia) [Kgm <sup>2</sup> ]	$l$ (length) [m]
link 1 (m)	10	0.54	0.02	1.0
link 2 (m)	7	0.42	0.01	0.8
link 1 (s)	12	0.6	0.05	1
link 2 (s)	5	0.5	0.03	0.8

The controller for the master robot  $\tau_m$  is the adaptive control law proposed by Slotine and Li (1987). The desired trajectory for the master robot is given by

$$q_d(t) = \begin{bmatrix} 1 + 0.25 \sin(\omega t) \\ 0.8 + 0.25 \cos(\omega t) \end{bmatrix} \quad [\text{rad}]$$

with  $\omega = 0.5$ .

The initial conditions for both robots and the observers (10), (12) are listed in tables 2 and 3.

Table 2. Joint initial conditions.

	joint 1 (m)	joint 2 (m)	joint1(s)	joint2 (s)
$q(0)$ [rad]	0.8	1	1.8	0.1
$\dot{q}(0)$ [rad/s]	0.4	0	-0.2	0.5

Table 3. Initial conditions for observers.

	joint 1	joint 2
$\hat{e}_s(0)$ [rad]	0.5	0.8
$\hat{e}_s(0)$ [rad/s]	0.7	0.4
$\hat{q}_s(0)$ [rad]	0.5	0.7
$\hat{q}_s(0)$ [rad/s]	0.4	0.2

The gain matrices, involved in the controller (9), and both observers (10), (12), are considered to be of the form  $kI$ , where  $k$  is a scalar and  $I \in \mathbb{R}^{2 \times 2}$ . The scalars associated with these gain matrices are chosen to be

Table 4. Controller gains.

$K_p$	$K_d$	$\Lambda_1$	$\Lambda_2$	$L_{p1}$	$L_{p2}$
100	10	50	50	50	50

As it is shown in figures 3, 4 the tracking errors are uniformly bounded, as well as the estimation errors, which are shown in figures 6 - 9. On the other hand the simulations were run for different values of the gains, it was observed that by increasing the gains  $K_p, L_{p1}$ , the bound of the closed loop system can be made arbitrarily small, at the same time by increasing  $K_d$ , the convergence time can be decreased. And thus, we can conclude that the performance showed in the simulations agrees with the stability result obtained in Section 4.

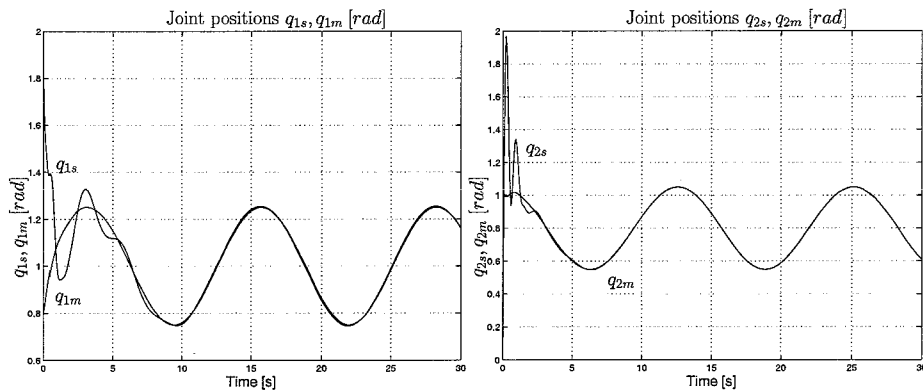


Figure 1: Joint positions  $q_{1s}, q_{1m}$  and  $q_{2s}, q_{2m}$ .

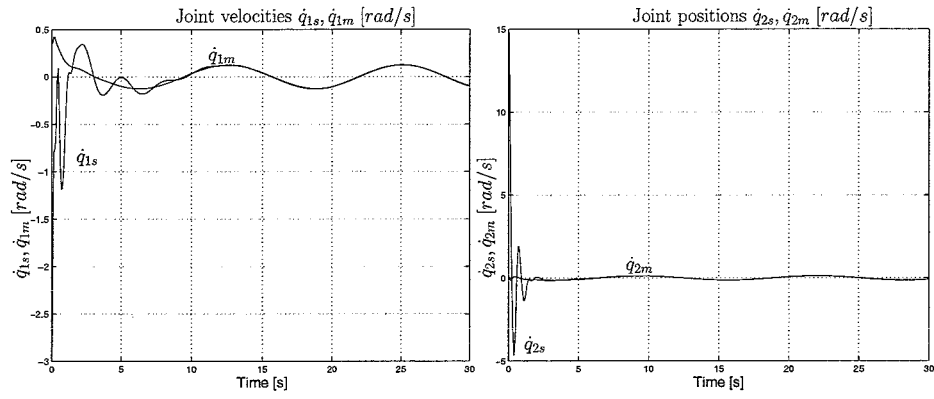


Figure 2: Joint velocities  $\dot{q}_{1s}, \dot{q}_{1m}$  and  $\dot{q}_{2s}, \dot{q}_{2m}$ .

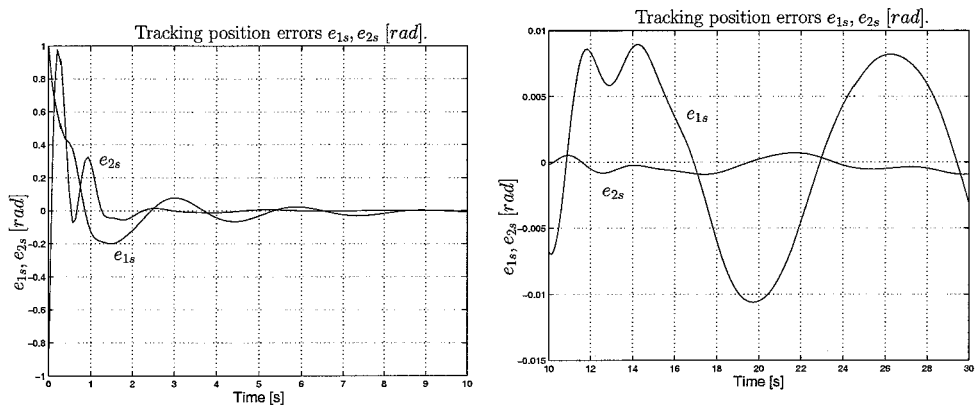


Figure 3: Tracking position errors  $e_{1s}, e_{2s}$ .

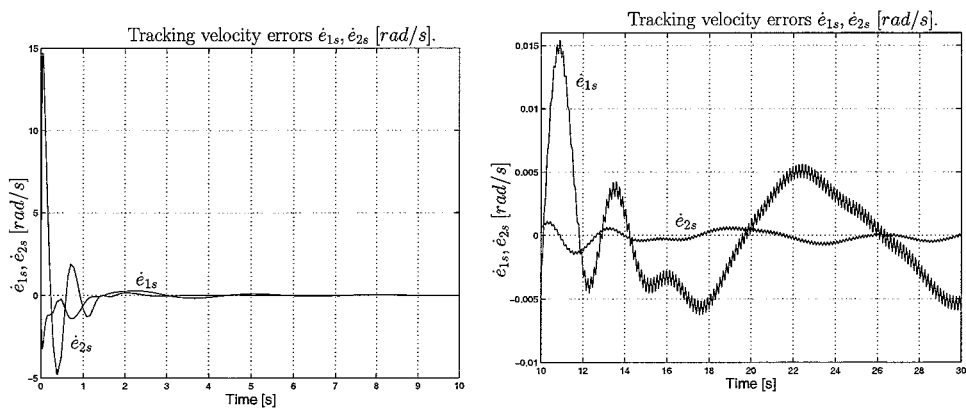


Figure 4: Tracking velocity errors  $\dot{e}_{1s}, \dot{e}_{2s}$ .

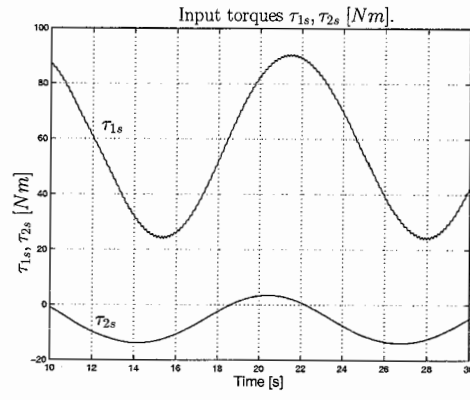
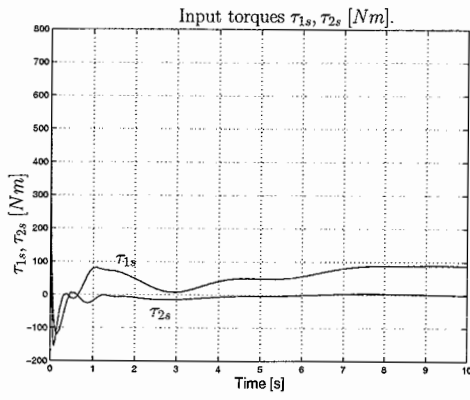


Figure 5: Input torques  $\tau_{1s}, \tau_{2s}$ .

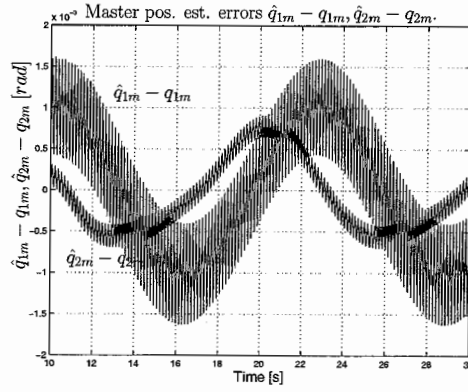
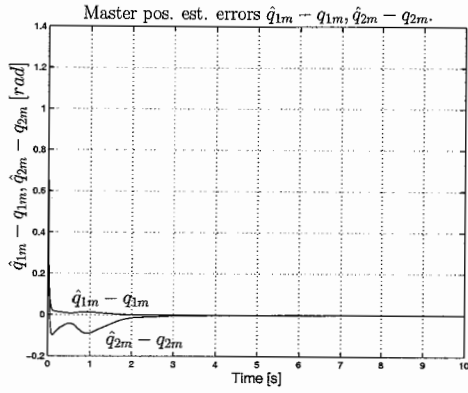


Figure 6: Master position estimation errors  $\hat{q}_{1m} - q_{1m}, \hat{q}_{2m} - q_{2m}$ .

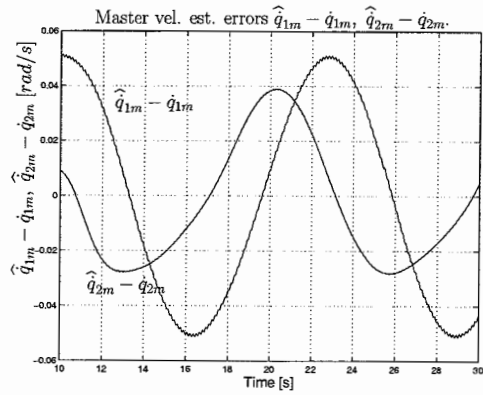
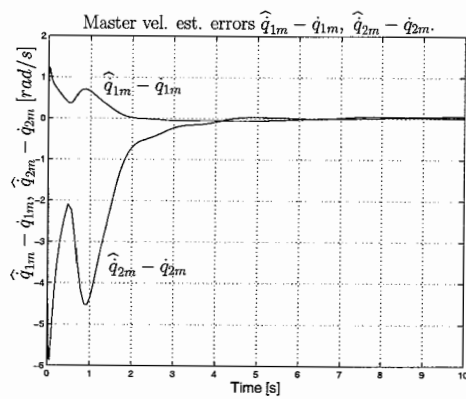


Figure 7: Master velocity estimation errors  $\hat{q}_{1m} - \dot{q}_{1m}, \hat{q}_{2m} - \dot{q}_{2m}$ .



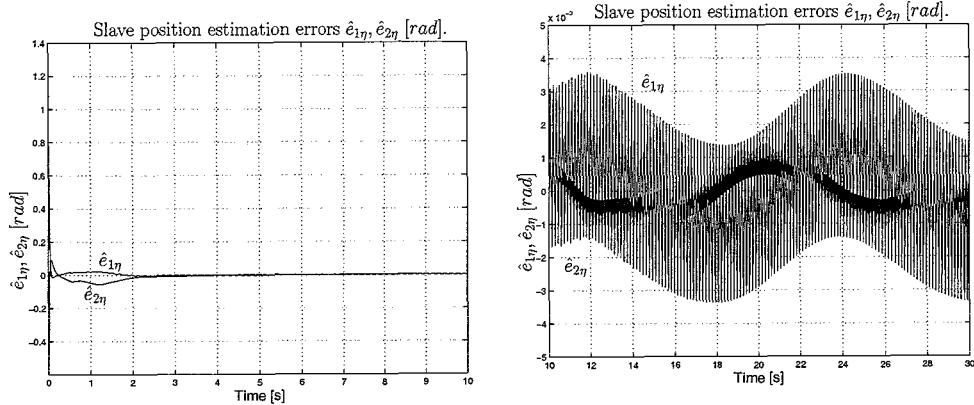


Figure 8: Slave position estimation errors  $\hat{e}_{1\eta}, \hat{e}_{2\eta}$ .

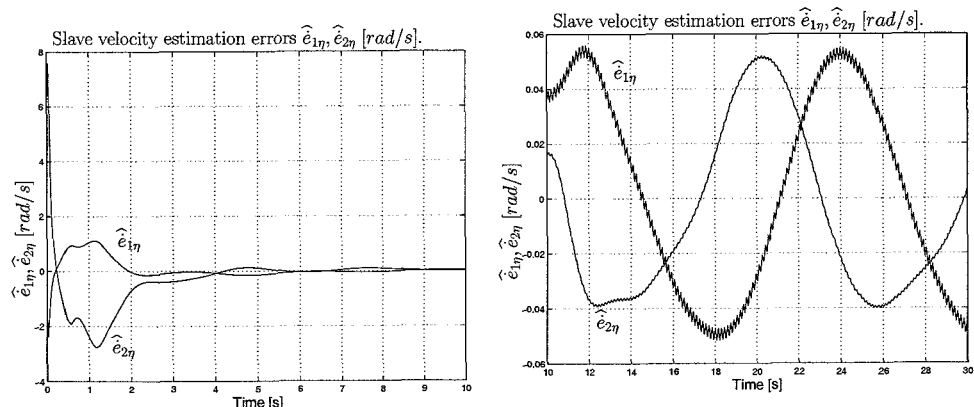


Figure 9: Slave velocity estimation errors  $\hat{e}_{1\eta}, \hat{e}_{2\eta}$ .

## 6 Remarks and discussion.

- It is important to notice that the proposed control law gives rise to coordination in the joint space. Coordination in the Cartesian space is obtained only in the case in which the length of the links of the slave robot are equal to the corresponding links in the master robot.
- In the state space representation (69), (70), the state  $\eta_1$  has been partially substituted, this is done as to take advantage of the available information in the system, i.e. the position measurement  $q_s$ .
- The variables  $\tilde{q}, \dot{\tilde{q}}$ , defined by (20), can be interpreted as the estimation error in the joint variables of the master robot  $q_m, \dot{q}_m$ , and thus,  $\tilde{q}, \dot{\tilde{q}}$  give an idea of how good the estimation of the master robot variables can be made based on measured and estimated variables of the slave robot. So, the slave robot, under the proposed controller, can be considered as a physical estimator for the master robot dynamics.
- The uniform ultimately boundedness result is of local nature, with region of attraction given by (58). This region of attraction and the bound for the closed loop errors

depend on  $y_2$  (see proposition 8 and Subsection 4.2.2) in a proportional way, such that by expanding the region of attraction, the upperbound for the closed loop errors increases, and thus a compromise has to be done.

- In order to fulfill conditions (18) and (19), some bounds of the slave robot structure and the master velocity and acceleration must be determined. Even without knowledge of these bounds, the closed loop system can be made uniformly ultimately bounded, by selecting the control gains large enough. However, such high gain implementations are not always desirable in practical circumstances.

## 7 Conclusions

In the present paper we have designed a control scheme for coordination of robot manipulators that requires only position measurements. The control scheme is formed by a feedback controller, which utilizes estimates for the tracking errors, as well as for the velocity and acceleration variables, these estimates are obtain by two nonlinear observers. The resulting closed loop system was proved to be uniformly ultimately bounded. Also a relation between the bound of the errors and the design parameters was given, which can be used to guarantee the desired tracking accuracy.

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## Appendix A

Consider the matrix  $Q_N$  given by (53);  $\Delta Q_i$  represents the determinant of the  $i - th$  leading minor of  $Q_N$ . Sufficient conditions for positive definiteness of  $Q_N$  are given by (18-19), with  $\varepsilon_{q6}$ ,  $L_{p2q4}$ ,  $L_{p2q5}$ ,  $L_{p2q6}$ ,  $L_{p1q5}$ ,  $K_{pq2}$ ,  $K_{pq3}$ ,  $K_{dq1}$ ,  $K_{dq3}$ ,  $K_{dq5}$ ,  $K_{dq6}$  given by

$$\begin{aligned} K_{dq1} &= \lambda_o M_{s,M}, \\ K_{pq2} &= \frac{\lambda_o V_M^2 (M_{s,pM} - C_{s,M})^2}{K_{d,m} - \lambda_o M_M}, \\ K_{pq3} &= \frac{-\varepsilon_o \lambda_o L_{p1,M}^2 M_{s,M}^2}{8\mu_o}, \end{aligned}$$

$K_{dq3}$ : denotes the solution of the equation  $\Delta Q_3 = b_1 K_{dq3} + b_2 = 0$ , with  $b_1, b_2$  the resultant coefficients in the factorization of  $K_{d,m}$  in  $\Delta Q_3$ , and  $K_{d,m}$  substituted by  $K_{dq3}$ .

$L_{p2q4}$ : denotes the solution of the equation  $\Delta Q_4 = a_1 L_{p2q4} + a_2 = 0$ , with  $a_1, a_2$  the resultant coefficients in the factorization of  $L_{p2,m}$  in  $\Delta Q_4$ , and  $L_{p2,m}$  substituted by  $L_{p2q4}$ .

$$L_{p1q5} = \frac{\sqrt{8\varepsilon_o \lambda_o K_{p,m} \mu_o}}{\varepsilon_o \lambda_o M_{s,m}}$$

$K_{dq5}$ : denotes the solution of the equation  $\Delta Q_5 = c_1 K_{dq5} + c_2 = 0$ , with  $c_1, c_2$  the resultant coefficients in the factorization of  $K_{d,m}$  in  $\Delta Q_5$ , and  $K_{d,m}$  substituted by  $K_{dq5}$ .

$L_{p2q5}$ : denotes the largest solution of the equation  $c_1 = d_0 + d_1 L_{p2q5} + d_2 L_{p2q5}^2 = 0$ , with  $c_1$  as in  $K_{dq5}$ ;  $d_0, d_1, d_2$  the resultant coefficients in the factorization of  $L_{p2,m}$  in  $c_1$ , and  $L_{p2,m}$  substituted by  $L_{p2q5}$ .

$K_{dq6}$ : denotes the solution of the equation  $\Delta Q_6 = r_1 K_{dq6} + r_2 = 0$ , with  $r_1, r_2$  the resultant coefficients in the factorization of  $K_{d,m}$  in  $\Delta Q_6$ , and  $K_{d,m}$  substituted by  $K_{dq6}$ .

$\varepsilon_{q6}$ : denotes the solution of the equation  $r_1 = s_1\varepsilon_{q6} + s_2 = 0$ , with  $r_1$  as in  $K_{dq6}$ ;  $s_1, s_2$  the resultant coefficients in the factorization of  $\varepsilon_o$  in  $r_1$ , and  $\varepsilon_o$  substituted by  $\varepsilon_{q6}$ .

$L_{p2q6}$ : denotes the largest solution of the equation  $s_1 = t_0 + t_1L_{p2q6} + t_2L_{p2q6}^2 + t_3L_{p2q6}^3 = 0$ , with  $s_1$  as in  $\varepsilon_{q6}$ ;  $t_0, t_1, t_2, t_3$  the resultant coefficients in the factorization of  $L_{p2,m}$  in  $s_1$ , and  $L_{p2,m}$  substituted by  $L_{p2q6}$ .

## Appendix B

First, we obtain the error dynamics in terms of the tracking errors  $(e_s, \dot{e}_s)$ , the estimation tracking errors  $(\hat{e}, \hat{\dot{e}})$ , and the estimation position and velocity errors  $(\hat{e}_\eta, \hat{\dot{e}}_\eta)$ , and second we consider the coordinate transformation defined by (20), (21).

### Tracking error dynamics

Substitution of the control law  $\tau_s$  (9) in the slave robot dynamics (1) yields the closed loop error equation

$$M_s(q_s)\ddot{q}_s + C_s(q_s, \dot{q}_s)\dot{q}_s = M_s(q_s)\hat{\ddot{q}}_m + C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{q}}_m - K_d\hat{\dot{e}}_s - K_p e_s$$

by adding and subtracting  $K_d\dot{e}_s + M_s(q_s)\ddot{q}_m + C_s(q_s, \dot{q}_s)\dot{q}_m$ , and considering the tracking errors defined by (8), this equation results in

$$\begin{aligned} M_s(q_s)\ddot{e}_s + C_s(q_s, \dot{q}_s)\dot{e}_s + K_d\dot{e}_s + K_p e_s &= M_s(q_s) \left( \hat{\ddot{q}}_m - \ddot{q}_m \right) + C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{q}}_m + \\ &- C_s(q_s, \dot{q}_s)\dot{q}_m - K_d \left( \hat{\dot{e}}_s - \dot{e}_s \right). \end{aligned} \quad (59)$$

From (8), (11), (13), and (14), the following equalities can be established

$$\begin{aligned} \hat{q}_m - q_m &= \hat{e} - \hat{e}_\eta \\ \hat{\dot{q}}_m - \dot{q}_m &= \hat{\dot{e}} - \hat{\dot{e}}_\eta \\ \hat{\ddot{q}}_m - \ddot{q}_m &= \hat{\ddot{e}} - \hat{\ddot{e}}_\eta \end{aligned} \quad (60)$$

Considering (11), (13), (60) and property (4), it follows that

$$\begin{aligned} C_s(q_s, \hat{\dot{q}}_s)\hat{\dot{q}}_m - C_s(q_s, \dot{q}_s)\dot{q}_m &= C_s(q_s, \dot{q}_s)\hat{\dot{e}} - 2C_s(q_s, \dot{q}_s)\hat{\dot{e}}_\eta + C_s(q_s, \hat{\dot{e}}_\eta)\hat{\dot{e}}_\eta + \\ &+ C_s(q_s, \hat{\dot{e}}_\eta)\dot{e}_s - C_s(q_s, \hat{\dot{e}}_\eta)\hat{\dot{e}}. \end{aligned} \quad (61)$$

Substitution of (61) in (59), and considering (11), (60), yields

$$\begin{aligned} M_s(q_s)\ddot{e}_s + C_s(q_s, \dot{q}_s)\dot{e}_s + K_d\dot{e}_s + K_p e_s &= M_s(q_s) \left( \hat{\ddot{e}} - \hat{\ddot{e}}_\eta \right) + C_s(q_s, \dot{q}_s)\hat{\dot{e}} + \\ &- 2C_s(q_s, \dot{q}_s)\hat{\dot{e}}_\eta + C_s(q_s, \hat{\dot{e}}_\eta)\hat{\dot{e}}_\eta + \\ &+ C_s(q_s, \hat{\dot{e}}_\eta)\dot{e}_s - C_s(q_s, \hat{\dot{e}}_\eta)\hat{\dot{e}} + \\ &+ K_d\hat{\dot{e}} \end{aligned} \quad (62)$$

### Estimation tracking error dynamics

If the states  $x_1, x_2 \in \mathbb{R}^n$  are defined as  $x_1 := e_s$ ,  $x_2 := \dot{e}_s$ , then (62) has the state space representation

$$\frac{d}{dt}x_1 = x_2 \quad (63)$$

$$\begin{aligned}
\frac{d}{dt}x_2 &= M_s(q_s)^{-1}\{-C_s(q_s, \dot{q}_s)x_2 - K_d x_2 - K_p x_1 + M_s(q_s) (\widehat{\ddot{e}} - \widehat{\ddot{e}}_\eta) + \\
&\quad + C_s(q_s, \dot{q}_s)\widehat{\dot{e}} - 2C_s(q_s, \dot{q}_s)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)x_2 + \\
&\quad - C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}} + K_d \widehat{\dot{e}}\}
\end{aligned} \tag{64}$$

and the estimation tracking errors (11) in  $x_1, x_2$ , are given by

$$\hat{e} = x_1 - \hat{e}_s, \quad \widehat{\dot{e}} = x_2 - \widehat{\dot{e}}_s. \tag{65}$$

Therefore, from (63), (64) and the nonlinear Luenberger observer (10), the estimation tracking error dynamics are given by

$$\begin{aligned}
\frac{d}{dt}\hat{e} &= \widehat{\dot{e}} - \Lambda_1 \hat{e} \\
\frac{d}{dt}\widehat{\dot{e}} &= M_s(q_s)^{-1}\{-C_s(q_s, \dot{q}_s)x_2 - K_d x_2 - K_p x_1 + M_s(q_s) (\widehat{\ddot{e}} - \widehat{\ddot{e}}_\eta) + \\
&\quad + C_s(q_s, \dot{q}_s)\widehat{\dot{e}} - 2C_s(q_s, \dot{q}_s)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)x_2 + \\
&\quad - C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}} + K_d \widehat{\dot{e}} + C_s(q_s, \widehat{\dot{q}}_s)\widehat{\dot{e}}_s + K_d \widehat{\dot{e}}_s + K_p \widehat{e}_s\} - \Lambda_2 \widehat{\dot{e}}.
\end{aligned}$$

Considering (13), (60), (65) and after a straightforward computation, these equations reduce to

$$\frac{d}{dt}\hat{e} = \widehat{\dot{e}} - \Lambda_1 \hat{e} \tag{66}$$

$$\frac{d}{dt}\widehat{\dot{e}} = M_s(q_s)^{-1}\{-K_p \hat{e} - 2C_s(q_s, \dot{q}_s)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}}_\eta\} - \Lambda_2 \widehat{\dot{e}} \tag{67}$$

### Estimation velocity error dynamics

From the definition of the tracking errors (8), it follows that

$$\ddot{q}_s = \ddot{e}_s + \ddot{q}_m. \tag{68}$$

Therefore, if the states  $z_1, z_2 \in \mathbb{R}^n$  are defined as  $z_1 := q_s, z_2 := \dot{q}_s$ , then from (62) we obtain the state space representation

$$\frac{d}{dt}z_1 = z_2 \tag{69}$$

$$\begin{aligned}
\frac{d}{dt}z_2 &= M_s(q_s)^{-1}\{-C_s(q_s, z_2)\dot{e}_s - K_d \dot{e}_s - K_p e_s + M_s(q_s) (\widehat{\ddot{e}} - \widehat{\ddot{e}}_\eta) + \\
&\quad + C_s(q_s, z_2)\widehat{\dot{e}} - 2C_s(q_s, z_2)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\dot{e}_s + \\
&\quad - C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}} + K_d \widehat{\dot{e}}\} + \ddot{q}_m
\end{aligned} \tag{70}$$

and the estimation velocity errors (13) in  $z_1, z_2$  are given by

$$\hat{e}_\eta = z_1 - \widehat{q}_s, \quad \widehat{\dot{e}}_\eta = z_2 - \widehat{\dot{q}}_s. \tag{71}$$

So, from (69), (70) and observer (12), the estimation position and velocity error dynamics are given by

$$\begin{aligned}
\frac{d}{dt}\hat{e}_\eta &= \widehat{\dot{e}}_\eta - L_{p1} \hat{e}_\eta \\
\frac{d}{dt}\widehat{\dot{e}}_\eta &= M_s(q_s)^{-1}\{-C_s(q_s, z_2)\dot{e}_s - K_d \dot{e}_s - K_p e_s + M_s(q_s) (\widehat{\ddot{e}} - \widehat{\ddot{e}}_\eta) + \\
&\quad + C_s(q_s, z_2)\widehat{\dot{e}} - 2C_s(q_s, z_2)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}}_\eta + C_s(q_s, \widehat{\dot{e}}_\eta)\dot{e}_s + \\
&\quad - C_s(q_s, \widehat{\dot{e}}_\eta)\widehat{\dot{e}} + K_d \widehat{\dot{e}} + C_s(q_s, \widehat{\dot{q}}_s)\widehat{\dot{e}}_s + K_d \widehat{\dot{e}}_s + K_p e_s\} - L_{p2} \widehat{\dot{e}}_\eta + \ddot{q}_m
\end{aligned}$$

considering (13), (60), and (71), these equations reduce to

$$\frac{d}{dt}\hat{e}_\eta = \hat{e}_\eta - L_{p1}\hat{e}_\eta \quad (72)$$

$$\frac{d}{dt}\hat{e} = \frac{d}{dt}(\hat{e} - \hat{e}_\eta) + M_s(q_s)^{-1}\{-2C_s(q_s, z_2)\hat{e}_\eta + C_s(q_s, \hat{e}_\eta)\hat{e}_\eta\} - L_{p2}\hat{e}_\eta + \ddot{q}_m. \quad (73)$$

Finally, from (67) and (73), it follows that

$$\frac{d}{dt}\hat{e}_\eta = \hat{e}_\eta - L_{p1}\hat{e}_\eta \quad (74)$$

$$\frac{d}{dt}\hat{e} = M_s(q_s)^{-1}\{-2K_p\hat{e} - 2C_s(q_s, \dot{q}_s)\hat{e}_\eta + C_s(q_s, \hat{e}_\eta)\hat{e}_\eta\} - 2\Lambda_2\hat{e} + L_{p2}\hat{e}_\eta - \ddot{q}_m \quad (75)$$

where the fact that  $z_2 = \dot{q}_s$  has been used.

### Coordinate transformations

Consider the coordinate transformation defined by (20), subtraction of (67, 74) from (66, 75) gives rise to the dynamics for  $\tilde{q}, \dot{\tilde{q}}$

$$\begin{aligned} \frac{d}{dt}\tilde{q} &= \dot{\tilde{q}} - L_{p1}\tilde{q} \\ \frac{d}{dt}\dot{\tilde{q}} &= -M_s(q_s)^{-1}K_p(\tilde{q} + \hat{e}_\eta) - L_{p2}\tilde{q} - \ddot{q}_m \end{aligned}$$

where Assumption 1 has been used.

From (67), (74), it follows that

$$\begin{aligned} \frac{d}{dt}\hat{e}_\eta &= \hat{e}_\eta - L_{p1}\hat{e}_\eta \\ \frac{d}{dt}\hat{e} &= M_s(q_s)^{-1}\{-K_p(\tilde{q} + \hat{e}_\eta) - 2C_s(q_s, \dot{q}_s)\hat{e}_\eta + C_s(q_s, \hat{e}_\eta)\hat{e}_\eta\} - L_{p2}(\tilde{q} + \hat{e}_\eta) \end{aligned}$$

And on the other hand, the tracking error dynamics (62) in  $\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}$  is given by

$$\begin{aligned} M_s(q_s)\ddot{\tilde{q}} + C_s(q_s, \dot{q}_s)\dot{\tilde{q}} + K_d\dot{\tilde{q}} + K_p\tilde{q} &= M_s(q_s)\ddot{\tilde{q}} + C_s(q_s, \dot{q}_s)\dot{\tilde{q}} + \\ &\quad -C_s(q_s, \dot{q}_s)\hat{e}_\eta - C_s(q_s, \hat{e}_\eta)\dot{\tilde{q}} + \\ &\quad +C_s(q_s, \hat{e}_\eta)\dot{\tilde{q}} + K_d(\dot{\tilde{q}} + \hat{e}_\eta). \end{aligned} \quad (76)$$

Then, by adding and subtracting  $K_p\tilde{q} + C_s(q_s, \dot{q}_s)L_{p1}\tilde{q} + K_dL_{p1}\dot{\tilde{q}} + M(q_s)(L_{p1}\dot{\tilde{q}} - L_{p1}L_{p1}\tilde{q})$  from (76), and considering the coordinate transformation defined by (21), it results in

$$\begin{aligned} M_s(q_s)\ddot{\tilde{q}} + C_s(q_s, \dot{q}_s)\dot{\tilde{q}} + K_d\dot{\tilde{q}} + K_p\tilde{q} &= -C_s(q_s, \dot{q}_s)\hat{e}_\eta + C_s(q_s, \hat{e}_\eta)\dot{\tilde{q}} + \\ &\quad -C_s(q_s, \hat{e}_\eta)L_{p1}\tilde{q} + C_s(q_s, \dot{q}_s)L_{p1}\tilde{q} + \\ &\quad +K_d\hat{e}_\eta + K_dL_{p1}\tilde{q} - K_p\tilde{q} + \\ &\quad +M(q_s)(L_{p1}\dot{\tilde{q}} - L_{p1}L_{p1}\tilde{q}). \end{aligned}$$

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