Further analysis of the local defect correction method

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Abstract

We analyze a special case of the Local Defect Correction (LDC) method introduced in [4]. We restrict ourselves to finite difference discretizations of elliptic boundary value problems. The LDC method uses the discretization on a uniform global coarse grid and on one or more uniform local fine grids for approximating the continuous solution. We will prove a close correspondence between this LDC method and the Fast Apative Composite grid (FAC) method from [6,7]. This result makes it possible to explain important properties of the LDC method, e.g. concerning the size of the discretization error and the convergence rate.

A.M.S. Classifications: 65N22, 65N50
Keywords : local defect correction, fast adaptive composite grid method.

1 Introduction

Many practical boundary value problems produce solutions that contain several high activity regions. In these regions the solution varies much more rapidly than in the remaining part of the domain. This behaviour of the solution may be caused by the differential operator itself, by the forcing term in the differential equation, by the boundary conditions or by an irregular boundary (e.g. a re-entrant corner).

If one wants to discretize such a boundary value problem on a uniform grid, then due to the large variations of the solution in the high activity regions, a relatively small grid size is required to obtain a sufficiently accurate approximation of the solution. However, outside the high activity regions the behaviour of the solution is much more smooth and therefore a (much) larger grid size seems to be sufficient in that part of the domain. So approximating the continuous solution on a single uniform grid is often computationally inefficient for boundary value problems that produce solutions that contain high activity regions.

Instead, the solution can be approximated using *several uniform grids with different grid sizes that cover different parts of the domain* [1, 4, 7]. At least one grid should cover the entire domain. The grid size of this *global coarse grid* is chosen in agreement with the smooth behaviour of the solution outside the high activity regions. Besides a global grid several *local grids* are used that are uniform too. Each of them covers only a (small) part of the domain and contains a high activity region. The grid size of each of these grids is chosen in agreement with the behaviour of the solution in the corresponding high activity region. In this way every part of the domain is covered by a (locally) uniform grid whose grid size is in agreement with the behaviour of the continuous solution in that part of the domain. This refinement strategy is known as *local uniform grid refinement.* The solution is approximated on the *composite grid* which is the union of the uniform subgrids.

In [4] Hackbusch introduced the *local defect correction method* (LDC) for approximating the continuous solution on a composite grid. In this iterative process a basic global discretization is improved by local discretizations defined in the subdomains. At every step this iterative process yields a discrete approximation of the continuous solution on the composite grid.

The *fast adaptive composite grid method* (FAC) by McCormick [6, 7, 8] is an iterative method for solving a given discrete problem on the composite grid. Approximations of the solution of this discrete problem are computed by solving (discrete) problems on the global and local grids. In [7] it is noted that " in essence FAC is very similar to LDC but differs in several simple but important respects".

In this paper we present a further analysis of the LDC method. In [4] an overlap parameter $d \geq 0$ is introduced and an analysis of the LDC method for the case $d > 0$ (independent of the grid size) is given. In this paper we analyze the LDC method for the case with minimal overlap, i.e. $d = 0$. The main result of this paper is that under certain (reasonable) assumptions the *LDC and FAC methods are equivalent:* the resulting iterates of both methods are the same (although the algorithms are different!). This result has some interesting consequences for the analysis of the LDC method. For example, using the underlying composite grid system (that is not used in the LDC algorithm) bounds for the discretization error, in a finite difference setting, can be derived. Also the convergence theory of the FAC method yields an indication of convergence properties of the LDC iteration.

The remainder of this paper is organized as follows. In Section 2 we describe a model situation of a problem on a composite grid. In Section 3 we present the LDC and FAC

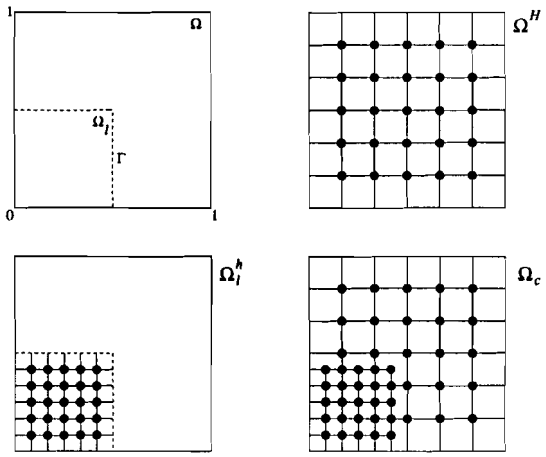

Figure 1: Examples of Ω_l, Ω^H, Ω_l^h and Ω_c.

methods. In Section 4 we prove an equivalence between these two methods. In Section 5 we discuss certain implications of this equivalence result and we present some numerical results.

2 Model Situation

In this section we introduce notation and describe a model case with a global coarse and a local fine grid. In Remark 2.1 we discuss possible generalizations.
We consider *Dirichlet boundary value problems*

$$\begin{aligned} \mathcal{L}U &= f \quad \text{in } \Omega, \\ U &= g \quad \text{at } \partial\Omega, \end{aligned} \tag{2.1}$$

with $\Omega = (0,1) \times (0,1)$, $\partial\Omega$ the boundary of Ω and $\mathcal{L}$ a scalar linear elliptic second-order differential operator. Dirichlet boundary conditions are chosen for ease of presentation. We assume that problem (2.1) is such that the continuous solution varies very rapidly in some (small) part of the domain, which is contained in the region $\Omega_l \subset \Omega$. In the remaining part of the domain the continuous solution is assumed to behave much more smoothly. The boundary $\partial\Omega_l$ of Ω_l consists of two parts. A part that coincides with $\partial\Omega$ and a remaining part. The latter part is called the *interface* $\Gamma = \partial\Omega_l \backslash \partial\Omega$ (see Figure 1). We note that we may have $\partial\Omega_l \cap \partial\Omega = \{\emptyset\}$, in which case the interface Γ coincides with $\partial\Omega_l$.

In order to compute a numerical approximation of the solution U we discretize (2.1) with respect to some discretization grid using *finite differences*. We assume that the finite difference matrices that appear in this section and in the following sections are all regular.

We use two *uniform* grids, a *global* one and a *local* one. The *global coarse grid* Ω^H is a uniform grid with grid size H that covers the domain Ω. The local fine grid Ω_l^h is a uniform grid with grid size h that covers the region Ω_l (see Figure 1). The *space of grid functions* on Ω^H (Ω_l^h) is denoted by $\mathcal{F}^H$ ($\mathcal{F}_l^h$). Since the continuous solution varies (much) more rapidly in Ω_l than in the remainder of Ω, a (much) smaller grid size is needed in Ω_l than in the remainder of Ω to provide the required level of resolution: $h \ll H$.

We assume that the interface Γ coincides with grid lines of Ω^H. Also we assume that all grid points of $\Omega^H \cap \Omega_l$ belong to Ω_l^h. We note that Ω_l^h does not contain grid points on the

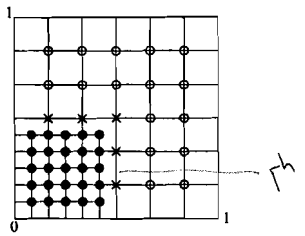

Figure 2: Partitioning of Ω_c; $\circ$: grid point of Ω_C,$\bullet$: grid point of Ω_l^h, $\times$: grid point of Γ^H.

interface Γ (see Figure 1). These fine interface grid points generate the fine interface grid Γ^h. The coarse interface grid Γ^H consists of all coarse interface grid points $\mathbf{x} \in \Omega^H \cap \Gamma$. The corresponding spaces of grid functions are denoted by $\mathcal{F}_\Gamma^h$ and $\mathcal{F}_\Gamma^H$ respectively.

Below we will also use a so called *composite grid.* The *composite grid* Ω_c is a nonuniform grid that covers the domain Ω. It is the union of the global coarse grid Ω^H and the local fine grid Ω_l^h (see Figure 1,2). The space of grid functions on the composite grid is denoted by $\mathcal{F}_c$.

Related to these grids we now introduce discrete operators and appropriate intergrid transfer operators.

First we discretize (2.1) with respect to the global coarse grid Ω^H. At each grid point $\mathbf{x} \in \Omega^H$ the differential operator in (2.1) is replaced by a *finite difference approximation.* This yields the *basic coarse grid problem*:

$$L^H u^H = f^H \text{ on } \Omega^H, \tag{2.2}$$

with $u^H, f^H \in \mathcal{F}^H$ and $L^H : \mathcal{F}^H \to \mathcal{F}^H$. The Dirichlet boundary values in (2.1) are incorporated in f^H.

For a given $v^H \in \mathcal{F}^H$ (e.g. $v^H = u^H$) we consider a corresponding fine grid problem on Ω_l^h. For this problem *artificial* Dirichlet boundary values are specified at all grid points $\mathbf{x} \in \Gamma^h$. Related to this we introduce the *trivial injection* r_Γ:

$$(r_\Gamma v^H)(\mathbf{x}) = v^H(\mathbf{x}), \qquad \mathbf{x} \in \Gamma^H, \tag{2.3}$$

with v^H a grid function that is defined on Γ^H (but not necessarily on the whole grid Ω^H), and an *interpolation operator*

$$p_\Gamma : \mathcal{F}_\Gamma^H \to \mathcal{F}_\Gamma^h. \tag{2.4}$$

In practice one will use linear or quadratic interpolation (cf. Section 5). A finite difference approximation on Ω_l^h using boundary values derived from v^H then results in a system

$$L_l^h v_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma v^H \text{ on } \Omega_l^h, \tag{2.5}$$

with $v_l^h, f_l^h \in \mathcal{F}_l^h$, $L_l^h : \mathcal{F}_l^h \to \mathcal{F}_l^h$, $L_\Gamma^h : \mathcal{F}_\Gamma^h \to \mathcal{F}_l^h$. *Natural* Dirichlet boundary conditions on $\partial\Omega_l \cap \partial\Omega$ are incorporated into f_l^h. The incorporation of the *artificial* Dirichlet boundary conditions on Γ in the system is given explicitly by the term $-L_\Gamma^h p_\Gamma r_\Gamma v^H$.

Finally we consider a discretization on the composite grid Ω_c. At all grid points $\mathbf{x} \in \Omega_c$ the differential operator in (2.1) is replaced by a certain finite difference approximation. We denote the resulting *composite grid problem* by

$$L_c u_c = f_c \text{ on } \Omega_c, \tag{2.6}$$

with $u_c, f_c \in \mathcal{F}_c$ and $L_c : \mathcal{F}_c \rightarrow \mathcal{F}_c$.
The composite grid is partitioned in the following way (see Figure 2)

$$\Omega_c = \Omega_C \cup \Gamma^H \cup \Omega_l^h. \tag{2.7}$$

We assume that at all grid points $\mathbf{x} \in \Omega_C$ the same finite difference formula is used as in the discretization process on the global coarse grid in (see (2.2)). Define the *trivial injection* $r_c : \mathcal{F}_c \rightarrow \mathcal{F}^H$ by

$$(r_c w_c)(\mathbf{x}) = w_c(\mathbf{x}), \quad w_c \in \mathcal{F}_c, \ \mathbf{x} \in \Omega^H. \tag{2.8}$$

Then we have:

$$(L_c u_c)(\mathbf{x}) = (L^H r_c u_c)(\mathbf{x}), \quad \mathbf{x} \in \Omega_C, \tag{2.9}$$

$$f_c(\mathbf{x}) = f^H(\mathbf{x}), \quad \mathbf{x} \in \Omega_C, \tag{2.10}$$

with L^H and f^H as in (2.2).
Define the *trivial injection* $r_{cl} : \mathcal{F}_c \rightarrow \mathcal{F}_l^h$ by

$$(r_{cl} w_c)(\mathbf{x}) = w_c(\mathbf{x}), \quad w_c \in \mathcal{F}_c, \ \mathbf{x} \in \Omega_l^h. \tag{2.11}$$

We assume that in $\mathbf{x} \in \Omega_l^h$ the composite grid discretization is of the following form:

$$(L_c u_c)(\mathbf{x}) = (L_l^h r_{cl} u_c)(\mathbf{x}) + (L_\Gamma^h p_\Gamma r_\Gamma r_c u_c)(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^h, \tag{2.12}$$

$$f_c(\mathbf{x}) = f_l^h(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^h, \tag{2.13}$$

with L_l^h, L_Γ^h, f_l^h and r_Γ as in (2.5) and p_Γ as in (2.4).

There are several options for choosing the composite grid discretization in the interface points $\mathbf{x} \in \Gamma^H$. However, the general form of the FAC iteration applied to the composite grid system does not depend on this choice (cf. Section 3, (3.18)). In Section 4 we will consider one particular choice which results from the analysis of the LDC method.

Remark 2.1. In [4] a much more general setting is presented. For example, for the LDC method it is not necessary that the local fine grid is a refinement of the coarse grid ($(\Omega^H \cap \Omega_l) \subset \Omega_l^h$). Also the setting in [4] allows the use of a variety of discretization methods. Due to this generality the analysis in [4] uses several technical assumptions which may be hard to verify in concrete situations. In this paper we restrict ourselves to the specific situation described above. This makes it possible to give a detailed analysis of the LDC method without technical assumptions.

3 LDC and FAC methods

In this section we describe a Local Defect Correction method (LDC) and a Fast Adaptive Composite Grid method (FAC). Both are iterative methods for computing a discrete approximation of the continuous solution on a composite grid. An important property of both methods is that essentially the iterative process only uses *uniform* grids (a global coarse one and a local fine one).

In the LDC iteration the global coarse grid Ω^H and the local fine grid Ω_l^h are used to compute a numerical approximation of the continuous solution U of (2.1). At each iteration

step a discrete problem on Ω^H and a discrete problem on Ω_l^h are solved. The local defect correction iteration was introduced by Hackbusch in [4]. There Hackbusch introduces a parameter $d \geq 0$ which is the measure for a certain overlap. In this paper we only consider the case $d = 0$ (cf. Remark 4.3).

We introduce the following notation. We use a *local coarse grid*

$$\Omega_l^H := \Omega_l \cap \Omega^H, \tag{3.1}$$

and the space of grid functions on Ω_l^H is denoted by $\mathcal{F}_l^H$.
We define the *trivial injection* $r_l : \mathcal{F}_l^h \to \mathcal{F}_l^H$ by

$$(r_l v_l^h)(\mathbf{x}) = v_l^h(\mathbf{x}), \quad v_l^h \in \mathcal{F}_l^h, \mathbf{x} \in \Omega_l^H. \tag{3.2}$$

We will use the characteristic function $\chi : \mathcal{F}_l^H \to \mathcal{F}^H$ given by

$$(\chi w)(\mathbf{x}) := \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ 0 & \mathbf{x} \in \Omega^H \backslash \Omega_l^H \end{cases} . \tag{3.3}$$

In LDC one starts with solving the basic coarse grid problem (2.2). The resulting u^H is used to define boundary values for a local fine grid problem, i.e. we solve (2.5) with $v^H = u^H$, resulting in a local fine grid approximation u_l^h. By solving the local fine grid problem we aim at improving the approximation of the continuous solution U in the region Ω_l. However, the Dirichlet boundary conditions on Γ^h result from the basic global coarse grid problem and the approximation u_l^h can be no more accurate than the approximation u^H *at the interface.* In general, local phenomena cause the approximations $u^H(\mathbf{x})$ to be relatively inaccurate at all grid points $\mathbf{x} \in \Omega^H$. Therefore the results of this simple two step process usually do not achieve an accuracy that is in agreement with the added resolution (see e.g. [2], [4]). In the local defect correction iteration coarse and fine processing steps are reused to quickly obtain such accuracy.

In the next step of the LDC iteration the approximation u_l^h is used to *update the global coarse grid problem* (2.2). The right hand side of (2.2) is updated at grid points that are part of Ω_l^H. The updated global coarse grid problem is given by

$$L^H \bar{u}^H = \bar{f}^H \tag{3.4a}$$

with

$$\bar{f}^H(\mathbf{x}) := \begin{cases} (L_l^H r_l u_l^h)(\mathbf{x}) + (L_\Gamma^H r_\Gamma u^H)(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ f^H(\mathbf{x}) & \mathbf{x} \in \Omega^H \backslash \Omega_l^H \end{cases} . \tag{3.4b}$$

The operators $L_l^H : \mathcal{F}_l^H \to \mathcal{F}_l^H$ and $L_\Gamma^H : \mathcal{F}_\Gamma^H \to \mathcal{F}_l^H$ are coarse grid analogues of L_l^h and L_Γ^h in (2.5) and they satisfy:

$$(L^H w^H)(\mathbf{x}) = (L_l^H w^H|_{\Omega_l^H})(\mathbf{x}) + (L_\Gamma^H w^H|_{\Gamma^H})(\mathbf{x}), \quad w^H \in \mathcal{F}^H, \ \mathbf{x} \in \Omega_l^H. \tag{3.5}$$

Using (3.3) we can rewrite (3.4a), (3.4b) as follows:

$$L^H \bar{u}^H = f^H + \chi(L_l^H r_l u_l^h + L_\Gamma^H r_\Gamma u^H - f^H). \tag{3.6}$$

So the right hand side of the global coarse grid problem is *corrected* by the *defect* of a *local fine grid approximation.* Once we have solved (3.6) we can update the local fine grid problem:

$$L_l^h \bar{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \bar{u}^H. \tag{3.7}$$

The approximations $\bar{u}^H$ and $\bar{u}_l^h$ of U are used to define an approximation of U on the composite grid:

$$\bar{u}_c(\mathbf{x}) := \begin{cases} \bar{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \bar{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases}.$$

In [2] an error analysis for this approximation that results after only *one* LDC step is given.

In the LDC iteration global problems like (3.6) and local problems like (3.7) are combined in the way described above.

LDC

Start: exact solution of the global problem

$$L^H u_0^H = f^H \text{ on } \Omega^H$$

exact solution of the local problem

$$L_l^h u_{l,0}^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u_0^H \text{ on } \Omega_l^h$$

computation of a composite grid approximation

$$u_{c,0}(\mathbf{x}) := \begin{cases} u_{l,0}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_0^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases}$$

$i = 1, 2, \ldots$:

a) computation of the right hand side of the global problem

$$\bar{f}^H := (1-\chi) f^H + \chi L_l^H r_l u_{l,i-1}^h + \chi L_\Gamma^H r_\Gamma u_{i-1}^H \tag{3.8a}$$

b) exact solution of the global problem

$$L^H u_i^H = \bar{f}^H \text{ on } \Omega^H \tag{3.8b}$$

c) exact solution of the local problem

$$L_l^h u_{l,i}^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u_i^H \text{ on } \Omega_l^h \tag{3.8c}$$

d) computation of a composite grid approximation

$$u_{c,i}(\mathbf{x}) := \begin{cases} u_{l,i}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_i^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases} \tag{3.8d}$$

Remark 3.1. In the LDC iteration it is not necessary to compute the composite grid approximation explicitly. (3.8d) is added for reasons that will become clear later on.
- In practice the systems in (3.8b,3.8c) will be solved approximately by a fast iterative method. Then one can take advantage of the fact that one has to solve problems on *uniform* grids.

Any fixed point $(\hat{u}^H, \hat{u}_l^h)$ of the iterative process (3.8) is characterized by the system (see [4])

$$\begin{array}{ll} L^H\hat{u}^H - \chi L_\Gamma^H r_\Gamma \hat{u}^H - \chi L_l^H r_l \hat{u}_l^h = (1-\chi) f^H & \text{on } \Omega^H, \\ L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H & \text{on } \Omega_l^h. \end{array} \tag{3.9}$$

Below we describe the FAC method. Note that the LDC method is most naturally interpreted as an *iterative discretization method.* The FAC method is an *iterative solver* for an a priori given discrete problem on the composite grid.

We consider a given composite grid problem as in (2.6) with L_c such that (2.9), (2.10), (2.12), (2.13) hold. In the FAC method approximations of u_c from (2.6) are computed in an iterative way. At each iteration step a discrete problem on the uniform global coarse grid and a discrete problem on the uniform local fine grid are solved exactly and the resulting solutions are used to improve the current iterate.

First we introduce a restriction operator $\tilde{r}_c : \mathcal{F}_c \to \mathcal{F}^H$. We assume that $\tilde{r}_c$ is the trivial injection at $\Omega_C \cup \Gamma^H$:

$$(\tilde{r}_c w_c)(\mathbf{x}) = w_c(\mathbf{x}), \qquad w_c \in \mathcal{F}_c, \ \ \mathbf{x} \in \Omega_C \cup \Gamma^H. \tag{3.10}$$

Let $\tilde{u}_c$ be an approximation of u_c. Inserting $\tilde{u}_c$ into the system $L_c u_c - f_c = 0$ yields

$$d_c := f_c - L_c \tilde{u}_c. \tag{3.11}$$

The *composite grid defect* d_c is restricted to the global coarse grid and to the local fine grid:

$$d^H := \tilde{r}_c d_c, \tag{3.12}$$

$$d_l^h := r_{cl} d_c. \tag{3.13}$$

In (3.13) r_{cl} is the trivial injection from (2.11).
An approximation $v^H \in \mathcal{F}^H$ of v_c is computed by solving the global coarse grid problem

$$L^H v^H = d^H, \tag{3.14}$$

with L^H as in (2.2) (cf. (2.9)).
Also an approximation $v_l^h \in \mathcal{F}_l^h$ of v_c is computed. The approximation v^H of v_c that results from (3.14) is used to define Dirichlet boundary conditions on the interface in the following local fine grid problem (cf. (2.5), (2.12))

$$L_l^h v_l^h = d_l^h - L_\Gamma^h p_\Gamma r_\Gamma v^H, \tag{3.15}$$

with L_l^h, L_Γ^h, r_Γ as in (2.5) and p_Γ as in (2.4).
The approximation v_l^h from (3.15) is used to correct the approximation $\tilde{u}_c$ of u_c at grid points of Ω_l^h:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v_l^h(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^h. \tag{3.16}$$

At grid points $\mathbf{x} \in \Omega_c \backslash \Omega_l^h$ the approximation v^H from (3.14) is used to correct the approximation $\tilde{u}_c$:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v^H(\mathbf{x}), \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h. \tag{3.17}$$

The FAC iteration is an iterative process that combines local and global discrete problems in the way described above.

FAC

Start: Initial composite grid approximation $u_{c,0}$ given.

$i = 1, 2, \ldots$:

a1) computation of the composite grid defect

$$d_c := f_c - L_c u_{c,i-1} \tag{3.18a}$$

a2) restriction of the composite grid defect to the global coarse grid

$$d^H := \tilde{r}_c d_c \tag{3.18b}$$

a3) restriction of the composite grid defect to the local fine grid

$$d_l^h := r_{cl} d_c \tag{3.18c}$$

b) exact solution of the global problem

$$L^H v^H = d^H \text{ on } \Omega^H \tag{3.18d}$$

c) exact solution of the local problem

$$L_l^h v_l^h = d_l^h - L_\Gamma^h p_\Gamma r_\Gamma v^H \text{ on } \Omega_l^h \tag{3.18e}$$

d) correction of the composite grid approximation

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + v_l^h(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^h \tag{3.18f}$$

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + v^H(\mathbf{x}), \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h \tag{3.18g}$$

Remark 3.2. Iteration (3.18) was introduced by McCormick in [8] as the *fast adaptive composite grid method in its delayed correction form.*
- The fixed point of this iterative process is given by the exact solution of the composite grid problem (2.6).
- For $i \geq 2$ the composite grid defect is equal to zero at all grid points that do not belong to the interface.
- The FAC method as in (3.18) is not applicable to nonlinear problems. In case L_c is nonlinear, the method should be used in FAS-form (see [8]).

We note that in the FAC method $u_{c,0}$ is not specified yet. A possible choice for this initial approximation is the approximation that results from the starting procedure in the LDC method.

4 Comparison of LDC and FAC

The local defect correction iteration and the fast adaptive composite grid iteration have been described in the previous sections as *discretization and solution methods* for a boundary value problem whose solution contains a *high activity region.*

From (3.8) and (3.18) it is clear that *computationally* the iterative processes are very similar. At each iteration step a discrete problem on the global coarse grid Ω^H is solved exactly. The resulting solution is used to define a discrete problem on the local fine grid Ω_l^h and this discrete problem is solved exactly. The solutions of these problems are used to compute a new approximation of the continuous solution U of (2.1) on the composite grid.

The starting point for FAC is a composite grid problem $L_c u_c = f_c$ that results from discretizing (2.1) on the composite grid Ω_c. This discrete problem has to be defined a priori. At each iteration step an approximation of u_c is computed by solving discrete problems on the uniform grids Ω^H and Ω_l^h (not on the composite grid!).

In LDC the discretization process and the solution process are coupled. The 'limit discrete problem'

$$L^H \hat{u}^H - \chi L_\Gamma^H r_\Gamma \hat{u}^H - \chi L_l^H r_l \hat{u}_l^h = (1-\chi) f^H \quad \text{on } \Omega^H \tag{4.1a}$$

$$L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H \quad \text{on } \Omega_l^h \tag{4.1b}$$

is an implicit result of the iterative process. At each iteration step an approximation of this limit discrete problem is computed. In LDC we only discretize with respect to *uniform* grids (in contrast with FAC). In the solution process only discrete problems on uniform grids have to be solved (as in the FAC iteration).

In the remainder of this section we assume that the discretization processes on the global coarse grid and on the local fine grid are given. Then the results of the LDC iteration only depend on the choice for the prolongation operator p_Γ. The results of the FAC iteration depend on the finite difference scheme that is used at the interface grid points and on the choice for the prolongation operator p_Γ, the restriction operator $\tilde{r}_c$ and the initial composite grid approximation $u_{c,0}$. As stated in Section 2 we consider linear problems and we assume that the finite difference operators L^H, L_l^H, L_l^h and L_c are nonsingular. In this section we also assume that L^H, L_l^H, L_l^h correspond to 9-point stencils.

In this section the main result of this paper is presented, namely that under certain reasonable assumptions the LDC iterates and the FAC iterates are the same.

First we prove a result for the limit value $(\hat{u}^H, \hat{u}_l^h)$ of the LDC iteration. At grid points $\mathbf{x} \in \Omega_l^H$ two approximations exist: $\hat{u}^H(\mathbf{x})$ and $\hat{u}_l^h(\mathbf{x})$. We show that these approximations are identical.

Lemma 4.1. The limit value $(\hat{u}^H, \hat{u}_l^h)$ of the LDC method satisfies

$$\hat{u}^H(\mathbf{x}) = \hat{u}_l^h(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^H. \tag{4.2}$$

Proof. According to (4.1a) and (3.3) we have

$$(L^H \hat{u}^H)(\mathbf{x}) - (L_\Gamma^H r_\Gamma \hat{u}^H)(\mathbf{x}) - (L_l^H r_l \hat{u}_l^h)(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_l^H.$$

From (2.3) and (3.5) it follows that

$$(L^H \hat{u}^H)(\mathbf{x}) = (L_l^H \hat{u}^H|_{\Omega_l^H})(\mathbf{x}) + (L_\Gamma^H r_\Gamma \hat{u}^H)(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^H.$$

Thus

$$L_l^H \hat{u}^H|_{\Omega_l^H} = L_l^H r_l \hat{u}_l^h.$$

Since we have assumed that L_l^H is regular we have that

$$\hat{u}^H(\mathbf{x}) = r_l \hat{u}_l^h(\mathbf{x}) \overset{(3.2)}{=} \hat{u}_l^h(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^H.$$

□

As a consequence of Lemma 4.1 the coupled system (4.1) can be represented as a certain composite grid system. To show this, we first introduce some further notation related to the composite grid.
Define $\hat{u}_c \in \mathcal{F}_c$ by

$$\hat{u}_c(\mathbf{x}) := \begin{cases} \hat{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \hat{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h = \Omega^H \backslash \Omega_l^H \end{cases}. \tag{4.3}$$

Define

$$\tilde{\Gamma}^H := \{\mathbf{x} \in \Omega_l^H \mid \text{distance}(\mathbf{x}, \Gamma) = H\}$$

The space of grid functions on $\Omega^H \backslash \Omega_l^H$, $\tilde{\Gamma}^H$ is denoted by $\mathcal{F}_o^H$ and $\mathcal{F}_{\tilde{\Gamma}}^H$ respectively. Note that $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$. We introduce $L_o^H : \mathcal{F}_o^H \to \mathcal{F}_o^H$ and $L_{\tilde{\Gamma}}^H : \mathcal{F}_{\tilde{\Gamma}}^H \to \mathcal{F}_o^H$ such that (cf. (3.5))

$$(L^H w^H)(\mathbf{x}) = (L_o^H w^H|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) + (L_{\tilde{\Gamma}}^H w^H|_{\tilde{\Gamma}^H})(\mathbf{x}), \quad w^H \in \mathcal{F}^H, \ \mathbf{x} \in \Omega^H \backslash \Omega_l^H. \tag{4.4}$$

Let the *trivial injection* $r_{\tilde{\Gamma}} : \mathcal{F}_l^h \to \mathcal{F}_{\tilde{\Gamma}}^H$ be defined by

$$(r_{\tilde{\Gamma}} w_l^h)(\mathbf{x}) := w_l^h(\mathbf{x}), \quad w_l^h \in \mathcal{F}_l^h, \ \mathbf{x} \in \tilde{\Gamma}^H. \tag{4.5}$$

Now we can represent $\hat{u}_c$ from (4.3) as the solution of a composite grid system.

Lemma 4.2. The composite grid approximation $\hat{u}_c$ from (4.3) satisfies:

$$\hat{L}_c \hat{u}_c = \hat{f}_c \tag{4.6a}$$

with

$$\hat{L}_c = \begin{pmatrix} L_l^h & L_\Gamma^h p_\Gamma r_\Gamma \\ L_{\tilde{\Gamma}}^H r_{\tilde{\Gamma}} & L_o^H \end{pmatrix}, \tag{4.6b}$$

$$\hat{u}_c = \begin{pmatrix} \hat{u}_l^h \\ \hat{u}^H|_{\Omega^H \backslash \Omega_l^H} \end{pmatrix}, \tag{4.6c}$$

$$\hat{f}_c = \begin{pmatrix} f_l^h \\ f^H|_{\Omega^H \backslash \Omega_l^H} \end{pmatrix}. \tag{4.6d}$$

Proof. First note that

$$L_l^h \hat{u}_l^h + L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H|_{\Omega^H \backslash \Omega_l^H} = f_l^h$$

holds due to (4.1b).
Also note that for $\mathbf{x} \in \Omega^H \backslash \Omega_l^H$ we have

$$\begin{aligned}
(L_{\tilde{\Gamma}}^H r_{\tilde{\Gamma}} \hat{u}_l^h + L_o^H \hat{u}^H|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) &\overset{(4.5)}{=} (L_{\tilde{\Gamma}}^H \hat{u}_l^h|_{\tilde{\Gamma}^H} + L_o^H \hat{u}^H|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) \\
&\overset{(4.2)}{=} (L_{\tilde{\Gamma}}^H \hat{u}^H|_{\tilde{\Gamma}^H} + L_o^H \hat{u}^H|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) \\
&\overset{(4.4)}{=} (L^H \hat{u}^H)(\mathbf{x}) \\
&\overset{(4.1a)}{=} f^H(\mathbf{x})
\end{aligned}$$

□.

Remark 4.3. The composite grid problem (4.6) is such that (2.9), (2.10), (2.12), (2.13) hold. Furthermore we have specified the discretization in the interface points $\mathbf{x} \in \Gamma^H$ (cf. (4.4)). From Lemma 4.2 we see that the LDC process induces a natural corresponding composite grid system. For this result to hold the assumption $d = 0$ is essential.

In the remainder of this section we assume that $\hat{L}_c$ is regular. In a finite difference setting this assumption will be satisfied in general (cf. Section 5 and [3]).

Below, in Theorem 4.7, we derive an expression for the iteration matrix of the FAC method applied to the composite grid problem in (4.6). First we introduce notation for the restrictions $\tilde{r}_c$ and r_{cl} from (3.10) and (2.11) respectively. Also we introduce corresponding prolongations. We use a block partitioning corresponding to $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$, $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$. Then the restrictions $\tilde{r}_c : \mathcal{F}_c \to \mathcal{F}^H$ and $r_{cl} : \mathcal{F}_c \to \mathcal{F}_l^h$ can be represented as

$$\tilde{r}_c = \begin{bmatrix} \tilde{r} & \emptyset \\ \emptyset & I \end{bmatrix}, \quad r_{cl} = [I \ \emptyset], \tag{4.7}$$

with $\tilde{r} : \mathcal{F}_l^h \to \mathcal{F}_l^H$ a given restriction operator.
For the prolongation $\tilde{p}_c : \mathcal{F}^H \to \mathcal{F}_c$ we assume a given prolongation operator $\tilde{p} : \mathcal{F}_l^H \to \mathcal{F}_l^h$ (e.g. $\tilde{p} = \tilde{r}^T$). We use the following prolongations:

$$\tilde{p}_c = \begin{bmatrix} \tilde{p} & \emptyset \\ \emptyset & I \end{bmatrix}, \quad {r_{cl}}^T = \begin{bmatrix} I \\ \emptyset \end{bmatrix}. \tag{4.8}$$

Below the following operators $P_i : \mathcal{F}_c \to \mathcal{F}_c$ play an important role:

$$P_1 := \tilde{p}_c (L^H)^{-1} \tilde{r}_c \hat{L}_c, \quad P_2 := {r_{cl}}^T (L_l^h)^{-1} r_{cl} \hat{L}_c. \tag{4.9}$$

In Remark 4.9 below we will see that the error propagation of the FAC method is determined by the operator $(I - P_2)(I - P_1)(I - P_2)$.
First we note that this operator does not depend on $\tilde{r}$, $\tilde{p}$ from (4.7), (4.8).

Lemma 4.4. The operator $(I - P_2)(I - P_1)(I - P_2)$ is independent of the choice of $\tilde{r}$, $\tilde{p}$ in (4.7), (4.8).

Proof. First note that $I - P_2$ does not depend on $\tilde{r}$, $\tilde{p}$. The operator $I - P_2$ is of the form $\begin{bmatrix} \emptyset & \star \\ \emptyset & \star \end{bmatrix}$ so $(I - P_2)\tilde{p}_c$ does not depend on $\tilde{p}$.

Further we note that $\hat{L}_c(I - P_2)$ is of the form $\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \star \end{bmatrix}$ so $\tilde{r}_c\hat{L}_c(I - P_2)$ does not depend on $\tilde{r}$. □

The following lemma will be used in the proofs of Theorem 4.7 and Theorem 4.8.
We use block partitioning corresponding to $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$, $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$.

Lemma 4.5. The following holds

$$(I - P_2)P_1 = \begin{bmatrix} \emptyset & -(L_l^h)^{-1}L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} (L^H)^{-1}\tilde{r}_c\hat{L}_c. \tag{4.10}$$

Proof. Note that

$$\begin{aligned} (I - P_2)P_1 &= (I - {r_{cl}}^T(L_l^h)^{-1}r_{cl}\hat{L}_c)\tilde{p}_c(L^H)^{-1}\tilde{r}_c\hat{L}_c \\ &= \begin{bmatrix} \emptyset & -(L_l^h)^{-1}L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} \begin{bmatrix} \tilde{p} & \emptyset \\ \emptyset & I \end{bmatrix} (L^H)^{-1}\tilde{r}_c\hat{L}_c \\ &= \begin{bmatrix} \emptyset & -(L_l^h)^{-1}L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} (L^H)^{-1}\tilde{r}_c\hat{L}_c \end{aligned}$$

□.

Remark 4.6. From the result in (4.10) it is clear that $(I - P_2)(I - P_1)$ does not depend on the choice of $\tilde{p}$ in (4.8).

Theorem 4.7. If the FAC iteration (3.18) is applied to the composite grid system in (4.6), then the iterates $u_{c,i}$ $(i \geq 1)$ satisfy

$$u_{c,i} = Mu_{c,i-1} + (I - M)(\hat{L}_c)^{-1}\hat{f}_c, \tag{4.11a}$$

with

$$M = (I - P_2)(I - P_1). \tag{4.11b}$$

Proof. Clearly the FAC method applied to (4.6) is a linear iteration, with fixed point $\hat{u}_c$. So it is of the form as in (4.11a).
Using (3.18) we get:

$$\begin{aligned} u_{c,i} &= u_{c,i-1} + {r_{cl}}^T(L_l^h)^{-1}(r_{cl}d_c - L_\Gamma^h p_\Gamma r_\Gamma v^H) + \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} v^H \\ &= u_{c,i-1} + P_2(\hat{u}_c - u_{c,i-1}) + \left(\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} - \begin{bmatrix} \emptyset & (L_l^h)^{-1}L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & \emptyset \end{bmatrix} \right) (L^H)^{-1}\tilde{r}_c\hat{L}_c(\hat{u}_c - u_{c,i-1}). \end{aligned}$$

Using Lemma 4.5 results in

$$\begin{aligned} u_{c,i} - \hat{u}_c &= u_{c,i-1} - \hat{u}_c + P_2(\hat{u}_c - u_{c,i-1}) + (I - P_2)P_1(\hat{u}_c - u_{c,i-1}) \\ &= (I - P_2)(I - P_1)(u_{c,i-1} - \hat{u}_c). \end{aligned}$$

This proves the expression for the iteration matrix in (4.11b). □

We note that a result as in Theorem 4.7, but then in a variational setting, is also proved in

e.g. [8].

Theorem 4.8. The iterates $u_{c,i}$ $(i \geq 1)$ from the LDC method (3.8) satisfy

$$u_{c,i} = M u_{c,i-1} + (I - M)(\hat{L}_c)^{-1} \hat{f}_c, \tag{4.12}$$

with M as in (4.11b).

Proof. Note that, due to Lemma 4.2, (4.12) is equivalent with

$$u_{c,i} - \hat{u}_c = M(u_{c,i-1} - \hat{u}_c). \tag{4.13}$$

In LDC we use a trivial injection operator r_l; so with r_c as in (2.8), i.e. $r_c = \begin{bmatrix} r_l & \emptyset \\ \emptyset & I \end{bmatrix}$, we have (cf. (3.8a))

$$\begin{aligned} \tilde{f}^H &= (1-\chi) f^H + \chi L_l^H r_l u_{l,i-1}^h + \chi L_\Gamma^H r_\Gamma u_{i-1}^H \\ &= (1-\chi) f^H + \chi L^H r_c u_{c,i-1} \\ &= L^H r_c u_{c,i-1} + (1-\chi)(f^H - L^H r_c u_{c,i-1}). \end{aligned} \tag{4.14}$$

We now consider the term $(1-\chi)(f^H - L^H r_c u_{c,i-1})$. We will show that for all $\mathbf{x} \in \Omega^H$ the following holds:

$$((1-\chi)(f^H - L^H r_c u_{c,i-1}))(\mathbf{x}) = (\tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}). \tag{4.15}$$

For $\mathbf{x} \in \Omega_l^H$ the left hand side in (4.15) equals zero due to the definition of χ. On the other hand, for $\mathbf{x} \in \Omega_l^H$, we also have

$$\begin{aligned} (\tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}) &= ([\tilde{r} \ \emptyset](\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}) \\ &= (\tilde{r}(f_l^h - L_l^h u_{l,i-1}^h - L_\Gamma^h p_\Gamma r_\Gamma u_{i-1}^H))(\mathbf{x}) \ = \ 0, \end{aligned}$$

due to (3.8c). So (4.15) holds for $\mathbf{x} \in \Omega_l^H$.
For $\mathbf{x} \in \Omega_C = \Omega^H \backslash (\Omega_l^H \cup \Gamma^H)$ we have

$$\begin{aligned} ((1-\chi)(f^H - L^H r_c u_{c,i-1}))(\mathbf{x}) &= (f^H - L^H r_c u_{c,i-1})(\mathbf{x}) \\ &= (f^H - L^H u_{i-1}^H)(\mathbf{x}) \ = \ 0 \end{aligned}$$

due to (3.8a,b).
On the other hand, for $\mathbf{x} \in \Omega_C$, we also have

$$(\tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}) = (f^H - L^H u_{i-1}^H)(\mathbf{x}) = 0.$$

So (4.15) holds for $\mathbf{x} \in \Omega_C$.
Finally, for $\mathbf{x} \in \Gamma^H$, we have

$$\begin{aligned} ((1-\chi)(f^H - L^H r_c u_{c,i-1}))(\mathbf{x}) &= (f^H - L^H r_c u_{c,i-1})(\mathbf{x}) \\ &= (\tilde{r}_c \hat{f}_c)(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x}) \\ &\overset{(4.4)}{=} (\tilde{r}_c \hat{f}_c)(\mathbf{x}) - (L_o^H u_{c,i-1}|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) - (L_{\tilde{\Gamma}}^H u_{c,i-1}|_{\tilde{\Gamma}^H})(\mathbf{x}) \\ &= (\tilde{r}_c \hat{f}_c)(\mathbf{x}) - (\hat{L}_c u_{c,i-1})(\mathbf{x}) \\ &\overset{(3.10)}{=} (\tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}). \end{aligned}$$

So (4.15) also holds for $\mathbf{x} \in \Gamma^H$.
Combination of (4.14), (4.15) yields

$$\bar{f}^H = L^H r_c u_{c,i-1} + \tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}). \tag{4.16}$$

For $u_{c,i}$ in the LDC method we have:

$$\begin{aligned}
u_{c,i} - \hat{u}_c &= \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} u_i^H + r_{cl}{}^T (L_l^h)^{-1}(f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u_i^H) - \hat{u}_c \\
&= -(I - P_2)\hat{u}_c + \left(\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} - \begin{bmatrix} \emptyset & (L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & \emptyset \end{bmatrix} \right) (L^H)^{-1} \bar{f}^H \\
&\overset{(4.16)}{=} -(I - P_2)\hat{u}_c + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} (L^H)^{-1} \tilde{r}_c \hat{L}_c (\hat{u}_c - u_{c,i-1}) \\
&\quad + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c u_{c,i-1} \\
&= -(I - P_2)\hat{u}_c + (I - P_2) P_1 (\hat{u}_c - u_{c,i-1}) \\
&\quad + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c u_{c,i-1},
\end{aligned} \tag{4.17}$$

where in the last equality we used Lemma 4.5. Now note that

$$\begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c = \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} \begin{bmatrix} r_l & \emptyset \\ \emptyset & I \end{bmatrix} = I - P_2.$$

Using this in (4.17) we get

$$\begin{aligned}
u_{c,i} - \hat{u}_c &= -(I - P_2)\hat{u}_c + (I - P_2) P_1 (\hat{u}_c - u_{c,i-1}) + (I - P_2) u_{c,i-1} \\
&= (I - P_2)(I - P_1)(u_{c,i-1} - \hat{u}_c),
\end{aligned}$$

which proves (4.12). □

From Theorem 4.7 and Theorem 4.8 it is clear that there is a close correspondence between the FAC method applied to the composite grid system in (4.6) and the LDC method. Firstly, both methods are linear iterative methods with the same unique fixed point $\hat{u}_c$. Secondly, both methods have the same iteration matrix $M = (I - P_2)(I - P_1)$. Of course, this implies that both methods have the same convergence rate. In the FAC method we are still free to choose the initial composite grid approximation $u_{c,0}$. A possible choice is the approximation that results from the starting procedure in the LDC method (3.8). With this initial approximation both methods yield identical iterates.

Remark 4.9. The iteration matrix of both the FAC method applied to (4.6) and the LDC method is given by $M = (I - P_2)(I - P_1)$.
Often it is advantageous to analyze the more symmetric operator $\tilde{M} = (I - P_2)(I - P_1)(I - P_2)$. Such an analysis immediately yields (sharp) results for M, because due to $(I - P_2)^2 = (I - P_2)$ we have $\sigma(\tilde{M}) = \sigma(M)$ and $\|M^k\| = \|\tilde{M}^{k-1}(I - P_1)\|$ for $k \geq 2$.

5 Implications of the Equivalence Between LDC and FAC

In this section we discuss some consequences of the main result in Section 4: the equivalence between LDC and FAC. Some of these consequences are illustrated by means of numerical results.

We recall that the main assumptions for this equivalence to hold are the following:

- for the overlap parameter d in the LDC method from [4] we take $d = 0$;
- for the FAC method we use a composite grid system as in (4.6). So for discretization an interface grid point is treated as a coarse grid point;
- for the FAC method we use the same starting procedure as for the LDC method.

Below we will illustrate certain phenomena using numerical results for the following model problem:

$$-\Delta U = f \quad \text{in } \Omega = (0,1) \times (0,1), \tag{5.1a}$$

$$U = g \quad \text{on } \partial U, \tag{5.1b}$$

with f and g such that the solution U is given by

$$U(x,y) = \tfrac{1}{2}\{\tanh(25(x+y-\tfrac{15}{8})) + 1\}. \tag{5.1c}$$

The solution U is shown in Figure 3. Clearly this solution varies very rapidly in a small part

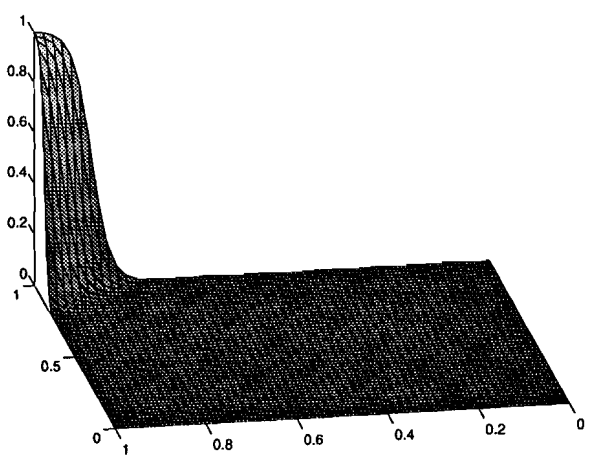

Figure 3: The solution U from (5.1c).

of the domain and is relatively smooth in the remaining part of the domain. For Ω_l we take

$$\Omega_l = \{(x,y) \mid |x-1| \leq \frac{1}{4} \wedge |y-1| \leq \frac{1}{4}\}. \tag{5.2}$$

The discretization is as follows. Using the LDC approach we only have to specify a global (coarse grid) discretization (cf. (2.2)), an interpolation operator p_Γ (cf. (2.4)) and a local

(fine grid) discretization (cf. (2.5)). Both for the global and local problem we use the standard 5-point stencil for the Laplacian. For the interpolation p_Γ we use piecewise quadratic interpolation (motivation below).

Below we discuss certain aspects concerning *implementation*, *discretization error* and *convergence rate*.

Implementation. With respect to implementation the LDC approach has an important advantage compared to the FAC method: in the LDC algorithm we do not use a composite grid operator (L_c in FAC). In LDC we only need a global coarse grid discretization, a local fine grid discretization and an operator p_Γ for interpolating artificial boundary values on Γ.

Discretization error. An important consequence of the results proved in Section 4 is that for the discretization error of the limit of the LDC iteration it is sufficient to analyze the composite grid discretization in (4.6) (which may be easier than an analysis of the coupled system in (4.1)). Note that in this discretization the treatment of the interface points is rather unusual. It turns out that, at least for the finite difference case, stability and reasonable error estimates can be proved. A detailed analysis of the finite difference discretization on composite grids for problems as in (5.1) is presented in another paper [3]. Here we just give a typical error estimate from [3]. By $d_{h,H}(\mathbf{y})$ we denote the local discretization error in the grid point $\mathbf{y}$. By $\Gamma_h^\star$ we denote the set of grid points in Ω_l^h next to the interface Γ, i.e.

$$\Gamma_h^\star = \{\mathbf{x} \in \Omega_l^h \mid dist(\mathbf{x}, \Gamma) = h\}.$$

Due to the interpolation needed on Γ the local discretization errors in points $\mathbf{y} \in \Gamma_h^\star$ depend on $\sigma := H/h$. For the composite grid discretization as in (4.6) applied to a problem as in (5.1) the following estimates are valid:

$$\max_{\mathbf{y} \in \Omega^H \setminus \Omega_l^H} |d_{h,H}(\mathbf{y})| \leq C_1 H^2, \tag{5.3a}$$

$$\max_{\mathbf{y} \in \Omega_l^h \setminus \Gamma_h^\star} |d_{h,H}(\mathbf{y})| \leq C_2 h^2, \tag{5.3b}$$

$$\max_{\mathbf{y} \in \Gamma_h^\star} |d_{h,H}(\mathbf{y})| \leq C_3 \sigma^2 H^j, \tag{5.3c}$$

with $j = 0, 1$ if p_Γ corresponds to piecewise linear or quadratic interpolation respectively. The constants C_i depend on higher derivatives of U and due to the local high activity we have $C_2 \gg C_1$, $C_2 \gg C_3$. In [3] it is proved that the following global discretization error estimate holds:

$$\|\hat{u}_c - U\|_\infty \leq C(C_1 H^2 + C_2 h^2 + C_3 H^{j+1}), \tag{5.4}$$

with C a small constant that does not depend on U, h, H.
As usual in finite difference estimates, the result in (5.4) has the disadvantage that high (fourth order) derivatives are involved. However, the estimate in (5.4) nicely separates the influence of the high activity region ($C_2 h^2$), the low activity region ($C_1 H^2$), and the interpolation on the interface ($C_3 H^{j+1}$). Note that all constants in (5.4) are independent of $\sigma = H/h$. Based on (5.4) we used piecewise quadratic interpolation in the experiments below.
We conclude that the LDC approach results in a stable composite grid discretization for which an error estimate as in (5.4) holds. Based on (5.4) we expect the following. If we take $H = H_0$ fixed then halving h should result in quadratic convergence until a certain threshold value $h = h_{min}$ is reached. Then halving H and using a constant refinement factor $\sigma = H/h$ equal

to $\sigma_0 := H_0/h_{min}$ should again result in quadratic convergence. This behaviour is confirmed by the results of Experiment 1.

Convergence rate. First we note that the asymptotic (for $H \downarrow 0$) analysis in [4] is not valid here, because in [4] the assumption $d > 0$ (independent of H) is crucial. A rigorous analysis for the case $d = 0$ is not available (yet).
However, there exists convergence theory, in a variational setting, for the FAC method (e.g. in [8]). In spite of the fact that the composite grid problem in (4.6) does not fit into a variational setting, the equivalence between LDC and FAC leads us to the claim that the LDC method will have convergence properties similar to those of the variational FAC method. Therefore (cf. [7, 8]) we expect a convergence rate that is independent of H and of $\sigma = H/h$. This claim is supported by the results of Experiment 2.

Experiment 1 We solved the problem in (5.1) with Ω_l as in (5.2). The composite grid discretization is as described above. In Table 1 we show the discretization error $\|\hat{u}_c - U\|_\infty$ for several values of H and σ. The results are in agreement with the estimate in (5.4).

Experiment 2. We solved the composite grid problem described above, using the LDC (or equivalently FAC) method. Below in Table 2 we give the average error reduction per iteration (in $\| \cdot \|_\infty$) in the first 4 iterations, for several values of H and $\sigma = H/h$. The convergence rate appears to be independent of both H and σ.

$H = 1/20$					$H = 1/40$
$\sigma = 1$	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$\sigma = 16$	$\sigma = 16$
$1.22e-01$	$1.60e-02$	$3.68e-03$	$8.59e-04$	$5.79e-04$	$1.42e-04$

Table 1: Discretization error $\|\hat{u}_c - U\|_\infty$ in Experiment 1.

$H = 1/20$			$\sigma = 2$		
$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$H = 1/20$	$H = 1/40$	$H = 1/80$
$1.0e-02$	$1.4e-02$	$1.5e-02$	$1.0e-02$	$1.0e-02$	$7.4e-03$

Table 2: Error reduction in Experiment 2.

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