Some remarks on subdesigns of symmetric designs

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1. Introduction

Let $D(v,k,\lambda)$ be a symmetric design containing a symmetric design $D_1(v_1,k_1,\lambda_1)$ $(k_1 < k)$. We call D_1 a subdesign of D. Assume

$$D = \begin{pmatrix} D_1 & R \\ S & T \end{pmatrix}$$

and let $x = \dfrac{(k-k_1)v_1}{v-v_1}$. We show that $k \geq (k_1 - x)^2 + \lambda$ (theorem 1). If equality holds, D_1 is called a tight subdesign of D. In the special case $\lambda_1 = \lambda$, our inequality reduces to that of R.C. Bose and S.S. Shrikhande [3] and tight subdesigns then correspond to their notion of Baer subdesigns. We give examples of tight subdesigns. We divide the possibilities for (v,k,λ) designs, having Baer subdesigns into three cases, and give examples for each case.

2. Main results

Theorem 1. Let $D_1(v_1,k_1,\lambda_1)$ be a subdesign of $D(v,k,\lambda)$. Let $x = \dfrac{(k-k_1)v_1}{(v-v_1)}$. Then $k \geq (k_1 - x)^2 + \lambda$.

Proof. Let $D = \begin{pmatrix} D_1 & R \\ S & T \end{pmatrix}$. Then x = average row sum of S. Form

$$A = \begin{pmatrix} 0 & D \\ D^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & D_1 & R \\ 0 & 0 & S & T \\ D_1^t & S^t & 0 & 0 \\ R^t & T^t & 0 & 0 \end{pmatrix}$$

where D^t denotes the transpose of D. Next, we construct the matrix M consisting of the average row sums of A corresponding to the given blocking. Then,

$$M = \begin{pmatrix} 0 & 0 & k_1 & k-k_1 \\ 0 & 0 & x & k-x \\ k_1 & k-k_1 & 0 & 0 \\ x & k-x & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} k_1 & k-k_1 \\ x & k-x \end{pmatrix}$ are k and $k_1 - x$. Hence the eigenvalues of M are $\pm k$ and $\pm(k_1 - x)$. The eigenvalues of A are $\pm k$ and $\pm\sqrt{k-\lambda}$. Using the result of [9] on the interlacing of the eigenvalues of M and A, then gives $\sqrt{k-\lambda} \ge (k_1 - x)$. This yields $k \ge (k_1 - x)^2 + \lambda$. □

Remark. It can be proved that if $k = (k_1 - x)^2 + \lambda$, then S has constant row sums.

Definition. $D_1(v_1,k_1,\lambda_1)$ is a tight subdesign of $D(v,k,\lambda)$ if $k = (k_1 - x)^2 + \lambda$.

Corollary ([3] or [10]). Let $D_1(v_1,k_1,\lambda_1)$ be a subdesign of $D(v,k,\lambda)$. Then $k \ge (k_1 - 1)^2 + \lambda$.

This follows immediately upon noting that in this case $x \le 1$.

If $D_1(v_1,k_1,\lambda)$ is a subdesign of $D(v,k,\lambda)$ and $k = (k_1 - 1)^2 + \lambda$, then D_1 is called a Baer subdesign of D ([3]). For $\lambda = 1$, Baer subdesigns are just Baer subplanes of projective planes. In this case many things have been investigated [6].

Example 1. Let D be the design formed by the points and hyperplanes of $PG(n,q)$, $n > 3$. Let X and Y be m and $n-m-1$ dimensional subspaces of $PG(n,q)$, respectively, which do not have a point in common. The points of X and the hyperplanes containing Y form a subdesign of D. This subdesign is not tight.

Example 2. Let H_1 be a regular Hadamard matrix of size $4n^2$. Then H_1 is equivalent to a symmetric design $D_1(4n^2, n(2n-1), n(n-1))$. Put

$$H = \begin{pmatrix} H_1 & -H_1 & -H_1 & -H_1 \\ -H_1 & H_1 & -H_1 & -H_1 \\ -H_1 & -H_1 & H_1 & -H_1 \\ -H_1 & -H_1 & -H_1 & H_1 \end{pmatrix}.$$

Then H is a regular Hadamard matrix of size $16n^2$ and is equivalent to a symmetric design $D(16n^2, 2n(4n+1), 2n(2n+1))$. It is easily checked that D_1 is a tight subdesign of D. For examples of regular Hadamard matrices cf. [8].

Remarks. Let $D_1(v_1, k_1, \lambda_1)$ be a tight subdesign of $D(v,k,\lambda)$. Then

(i) $k - \lambda$ is a square.

(ii) The complement of D_1 is a tight subdesign of the complement of D.

Using $\bar{D}$ to denote the complement of D, we then have

Example 3. Let $D_1(v_1, k_1, 1)$ be a Baer subplane of $D(v,k,1)$. Then $\bar{D}_1(v_1, v_1 - k_1, v_1 - 2k_1 + 1)$ is a tight subdesign of $\bar{D}(v, v-k, v-2k+1)$.

Theorem 2. Let $D_1(v_1, k_1, \lambda)$ be a Baer subdesign of $D(v,k,\lambda)$. Then, one of the following holds:

a) $v = \lambda(\lambda^2 - 2\lambda + 2)$, D has parameters $(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ and D_1 is the trivial design $(\lambda, \lambda, \lambda)$.

b) $v = \lambda^2(\lambda + 2)$, D has parameters $(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ and D_1 is the trivial design $(\lambda+2, \lambda+1, \lambda)$.

c) $v > \lambda^2(\lambda + 2)$.

Proof. Let D be a non-trivial design having a Baer subdesign D_1. Then $k < v - 1$ or equivalently $\lambda < k - 1$. Since D_1 is a Baer subdesign of D, we have

$$x = \frac{(k - k_1)v_1}{(v - v_1)} = 1 \ .$$

This gives

(1) $\qquad v = v_1(k - k_1 + 1)$

and

(2) $\qquad k = (k_1 - 1)^2 + \lambda \ .$

If D_1 is trivial then $v_1 = k_1 = \lambda_1$ or $v_1 = k_1 + 1 = \lambda_1 + 2$. Using (1) and (2) we see that these two trivial cases lead to (a) and (b), respectively. If $v_1 > k_1 + 1$ then (1) and (2) give

$$v > (k_1 + 1)((k_1 - 1)(k_1 - 2) + \lambda) \ .$$

Using $k_1 > \lambda + 1$ we obtain $v > \lambda^2(\lambda + 2)$. □

We now give examples to show that in each of the above cases, there exist symmetric designs with Baer subdesigns.

Example 4. A symmetric design $D(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ has the parameters of the symmetric design on the points and planes of $PG(3, \lambda - 1)$ which exist for all prime powers $\lambda - 1$. Moreover the points on a given line and all planes containing it form a Baer subdesign $D_1(\lambda, \lambda, \lambda)$.

Example 5. From Ahrens and Szekeres [1], the existence of symmetric designs D with parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ is known for all prime powers λ. From their construction it can be easily seen that D has a Baer subdesign $D_1(\lambda + 2, \lambda + 1, \lambda)$, corresponding to a clique of size $\lambda + 2$ in the corresponding graph.

Before giving an example to show the existence of a design satisfying (c) of theorem 2, we make some observations:
If we consider designs (v,k,λ) with $v > \lambda^2(\lambda + 2)$, then according to [5], p. 105 the only known examples are projective planes of prime power order and biplanes (= symmetric designs with $\lambda = 2$) on 37, 56 and 79 points; as far as we know meanwhile one other example is found, a (71,15,3) design, see [2].

Note that if $D_1(v_1, k_1, \lambda)$ is a Baer subdesign of $D(v,k,\lambda)$, then v cannot be prime. Thus if we are to find a Baer subdesign $D_1(v_1, k_1, \lambda)$ of $D(v,k,\lambda)$ which is not a Baer subplane, it is easily seen from above that D(56,11,2) is the only possible candidate. Any Baer subdesign of D has parameters (7,4,2), whose of the complement of the Fano plane (7,3,1). The next example shows that there is a (56,11,2) design with a Baer subdesign.

Example 6. We follow [7] Denniston who gives constructions of (56,11,2) designs some of which are based on Cameron's description [4] of biplanes. Namely, one block b^* is fixed and all the other blocks are in 1-1 correspondence with the unordered pairs of points of b^*. Each point not on b^* is represented by a disjoint union of polygons on the points of b^*. The block represented by {p,q} is incident with p and q and with the points represented by graphs in which p and q are joined.

Let us represent the points of b^* by $0,\dots,10$ then according to [7] in at least two of the constructed biplanes (the "nice" one due to Gewirtz, Hall, Lane and Wales, and another design due to Assmus and others), there exist three points off b^* whose polygons are

$$(9\ 8\ 10\ 9)\ (0\ 2\ 4\ 6\ 0)\ (1\ 3\ 5\ 7\ 1)$$
$$(0\ 4\ 10\ 0)\ (9\ 2\ 8\ 6\ 9)\ (1\ 3\ 7\ 5\ 1)$$
$$(2\ 6\ 10\ 2)\ (9\ 4\ 8\ 0\ 9)\ (1\ 7\ 3\ 5\ 1)\ .$$

It is easily seen that these 3 points together with the points 1,3,5 and 7 from b^* form a (7,4,2) design which is a Baer subdesign of the (56,11,2) design.

Using the above example and remark ii) we have

Example 7. There exists a D(56,45,36) which has the Fano plane (= (7,3,1) design) as a tight subdesign.

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