A mathematical theory connecting scattering and diffraction phenomena, including Bragg-type interferences

A MATHEMATICAL THEORY CONNECTING SCATTERING AND DIFFRACTION PHENOMENA, INCLUDING BRAGG-TYPE INTERFERENCES

by

Prof. dr. H. Bremmer

Department of Electrical Engineering
Eindhoven University of Technology
Eindhoven, The Netherlands

A MATHEMATICAL THEORY CONNECTING SCATTERING AND DIFFRACTION PHENOMENA, INCLUDING BRAGG-TYPE INTERFERENCES

by

Prof. Dr. H. Bremmer

TH-Report 71-E-26

december 1971

ISBN 90 6144 026 2

Summary

This paper deals with the waves which can be produced inside, as well as in front of and behind a scattering slab, with parallel boundaries, which contains a slightly inhomogeneous medium. The refractive index of the latter is given as a function of the three space variables and the time, without considering possible statistical properties. The theory treating the contributions generated by an obliquely incident wave and due to a successive number of scatterings, is developed with the aid of symbolic expressions depending on partial derivatives with respect to the space-time variables; in particular the twoo-dimensional Laplace operator of the refractive-index distributions in planes parallel to the boundaries is shown to play a dominant role. A resulting representation of the Born approximation (accounting for first-order scatterings only) clearly shows the transition from the mainly scattering behaviour of a thin slab to the predominantly diffracting behaviour of a thick slab. The main waves generated the individual travelling waves (termed acoustic waves) that compose the four-dimensional distribution of the refractive index according to a Fourier synthesis, are connected with an axtension of Bragg's relation for stationary periodic structures to corresponding moving structures. It is shown how this relation is also useful for the interpretation of the higher-order scattering contributions (plural scattering). The property that the cooperation of all contributions, associated with any number of scatterings, leads to a diffusion type of propagation here results from a simple symbolic representation of the wave equation inside the scatterer. It further appears that the contribution from a special number of scatterings can be represented by a multi-dimensional Fourier integral the integrand of which has a denominator again connected with the mentioned extension of Bragg's relation. The special cases of a stationary scatterer, and of the conventional approximations related to predominantly forward scattering, are included in the discussions.

A mathematical theory connecting scattering and diffraction phenomena, including Bragg-type interferences

A mathematical theory connecting scattering and diffraction phenomena, in particular Bragg-type interferences

1. Introduction

Both scattering and diffraction phenomena occur when any kind of waves passes through an inhomogeneous medium with a given structure. The distinction between both phenomena is rather vague, but a general characteristic of scattering consists of its connection with local volume elements whereas structures on a larger scale cause diffraction effects. This difference appears from an increase of scattering effects at high frequencies whereas diffraction phenomena, as occurring for instance in shadow regions, are best observable at low frequencies. In this paper the direct connection between scattering and diffraction will result from mathematical expressions depending on functionals of differential operators, in particular on the Laplace operator of the two-dimensional distribution of the refractive index of the inhomogeneous medium in planes parallel to the boundaries of the considered slab that contains the scattering medium. Moreover, the mathematical procedure involves simple relations for the contributions depending on a specified number of scatterings by successive volume elements (terms of the Born series). In this way the transition from plural scattering to the diffusion properties of multiple scattering becomes clear in particular. After introducing the model of our analysis, a plane parallel slab containing the inhomogeneous also time dependent medium, we shall deal with the waves generated inside it by an obliquely incident plane wave, as well as with the waves leaving the slab both at its front side and its back. Conventional saddlepoint approximations for the Born approximation of these latter waves will be deduced in a very simple way from special representations of the Born approximation.

2. Description of a model for investigating scattering properties

We assume a slab $0 < z < D$ containing a medium with a given distribution

$$n(x,y,z,t) = 1 + \delta n(x,y,z,t)$$

for the refractive index referring to the waves to be considered, the homogeneous medium outside the slab being characterized by the normalized refractive index n = 1. In practice we have in mind applications for which $|\delta n| \ll 1$. Therefore, the underlying scalar wave equation

$$\nabla^2 u - \frac{n^2(x,y,z,t)}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

will be replaced by

$$\nabla^2 u - \frac{1 + 2\,\delta n(x,y,z,t)}{c^2}\;\frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

In view of the scalar treatment, our theory will not be applicable at once to vectorial situations, such as associated with Maxwell's equations; however, we know how even then special quantities satisfy, at least in a fair approximation, a scalar equation of the above type. As distinct from many scattering theories, we include a possible time dependence of the refractive index.

We observe that the scattering theory to be developed may also refer to particles, for instance to the scattering of the electrons of an incident beam by the individual molecules and atoms of the slab medium. In such cases the behaviour of the particles is to be described in terms of waves of a Schrödinger type while the scattering by a single particle can then be ascribed to a locally concentrated medium the refractive index of which has a spatial distribution to be determined such as to yield the given scattering properties of the particle in question. The final refractive index of the medium then results from a spatial convolution product of this local distribution and that of the density of the scattering particles.

The wave incident from the half space $z < 0$ is assumed as a plane travelling one, and may as such in a normalized form be given by

$$u_{pr}(x,y,z,t) = e^{i(k_{ox}x + k_{oy}y + k_{oz}z - \omega_o t)} =$$

$$= e^{i(\vec{k}_o \vec{r} - \omega_o t)} \;. \tag{2}$$

3. The Born series for an inhomogeneous non stationary medium

The role of the scattering time-dependent medium appears most clearly from the following alternative representation of the wave equation (1):

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{2}{c^2}\,\delta n(x,y,z,t)\,\frac{\partial^2 u}{\partial t^2}. \tag{3}$$

The corresponding Born series is to be derived from the well-known solution of the complete wave equation $(\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2)\,u = -\rho(x,y,z,t)$ in terms of retarded functions, in contrast with the Born series in the conventional treatment of stationary media which is connected with the three-dimensional Helmholtz equation. In the present situation we thus arrive at the integral equation

$$u(P,t) = u_{pr}(P,t) - \frac{1}{2\pi c^2}\int d\tau_q\,\frac{\delta n(Q,t-\frac{PQ}{c})\cdot\frac{\partial^2 u}{\partial t^2}(Q,t-\frac{PQ}{c})}{PQ}, \tag{4}$$

where (P,t) is the abbreviation for the coordinates x_p, y_p, z_p and the time t of the point of observation and, similarly, (Q,t) for the points in space over which the integration has to be extended, $d\tau_q = dx_q\,dy_q\,dz_q$ being the corresponding volume element; further PQ indicates the distance from P to Q. As a matter of fact, the integration only concerns the slab $o < z_q < D$, δn being zero outside the latter.

The representation (4) leads straightforwardly to a Neumann-Liouville expansion (Born series)

$$u(P,t) = \sum_{N=0}^{\infty} u_N(P,t) \tag{5}$$

of the solution, with $u_0(P,t) = u_{pr}(P,t)$, while the fu[illegible] terms are mutually connected according to the recurrence relation:

$$u_N(P,t) = -\frac{1}{2\pi c^2}\int d\tau_q\,\frac{\delta n(Q,t-\frac{PQ}{c})\cdot\frac{\partial^2 u_{N-1}}{\partial t^2}(Q,t-\frac{PQ}{c})}{PQ}. \tag{6}$$

It is well known how U_N represents the effect of N sucressive scatterings and that in its explicit representation U_N depends on a 3N-dimensional integral the integrand of which contains N factors δn.

It is a basic feature of our theory that, for the present, we shall perform the integration in (6) only with respect to the transverse coordinates x_q and y_q, which integration then comprises the complete plane $z = z_q$, that is the domain $-\infty < x_q, y_q < \infty$. The method will be the same as applied earlier[1] to the simpler stationary medium.

For convenience we introduce the abbreviation

$$\varphi_{N-1}(P,t) \equiv \delta n(P,t) \;\; \frac{\partial^2 u_{N-1}}{\partial t^2}(P,t). \tag{7}$$

The corresponding factor in the integrand of (6) can then represented by a Taylor expansion with respect to x_q, y_q, t which reads as follows in a symbolic form:

$$\varphi_{N-1}\left(Q, t - \frac{PQ}{c}\right) = \\ = e^{(x_q - x_p)\frac{\partial}{\partial x_p} + (y_q - y_p)\frac{\partial}{\partial y_p} + (z_q - z_p)\frac{\partial}{\partial z_p}} \varphi_{N-1}(x_p, y_p, z_q, t). \tag{8}$$

In fact, the elementary expansion of the exponential here leads to the Taylor series in question. Symbolic expressions like the present one, and those to be derived later, can always be converted into ordinary non-symbolic ones by substituting a Fourier integral for the operand, in this case

$$\varphi_{N-1}(P,t) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; e^{i(k_x x_p + k_y y_p)} \phi_{N-1}(k_x, k_y; z_p, t).$$

The application of any operator $g(\partial/\partial x_p, \partial/\partial y_p)$ then amounts to a multiplication of the integrand of such a Fourier integral by the factor $g(ik_x, ik_y)$. We emphasize that, henceforth, all symbolic expressions are to be applied (if not indicated otherwise) to the complete part of the integrand following it.

The substitution of the mentioned symbolic form (8) into (6) yields, remembering (7),

$$u_N(P,t) =$$

$$-\frac{1}{2\pi c^2}\int_0^D dz_q \int_{-\infty}^{\infty} dx_q \int_{-\infty}^{\infty} dy_q \frac{1}{PQ} e^{(x_q-x_p)\frac{\partial}{\partial x_p}+(y_q-y_p)\frac{\partial}{\partial y_p}-\frac{PQ}{c}\frac{\partial}{\partial t}} \varphi_{N-1}(x_p,y_p,z_q,t).$$

We pass from x_q and y_q to polar coordinates according to

$$x_q - x_p = \lambda \cos\varphi \quad , \quad y_q - y_p = \lambda \sin\varphi ,$$

so as to obtain:

$$u_N(P,t) =$$

$$-\frac{1}{2\pi c^2}\int_0^D dz_q \int_0^{2\pi} d\varphi \int_0^{\infty} d\lambda\, \lambda \frac{e^{\lambda\left(\cos\varphi\frac{\partial}{\partial x_p}+\sin\varphi\frac{\partial}{\partial y_p}\right)-\frac{1}{c}\sqrt{\lambda^2+(z_q-z_p)^2}\frac{\partial}{\partial t}}}{\sqrt{\lambda^2+(z_q-z_p)^2}} \varphi_{N-1}(x_p,y_p,z_q,t).$$

The differential operators entering here have nothing to do with the integrations for which they can just be considered as parameters. The integration with respect to φ involves a zero-order Bessel function which results in

$$u_N(P,t) = -\frac{1}{c^2}\int_0^D dz_q \int_0^{\infty} d\lambda\, \lambda \frac{J_0\left(i\lambda\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}\right)}{\sqrt{\lambda^2+(z_q-z_p)^2}} e^{-\frac{1}{c}\sqrt{\lambda^2+(z_q-z_p)^2}\frac{\partial}{\partial t}} \varphi_{N-1}(x_p,y_p,z_q,t).$$

In order to facilitate the integration with respect λ we prefer the alternative representation

$$u_N(P,t) = -\frac{1}{c^2}\int_0^D dz_q \int_0^{\infty} d\lambda\, \lambda \frac{J_0\left(\lambda\sqrt{-\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_p^2}}\right)}{\sqrt{\lambda^2-(i|z_p-z_q|)^2}} e^{-\sqrt{\lambda^2-(i|z_p-z_q|)^2}\cdot\frac{1}{c}\frac{\partial}{\partial t}} \varphi_{N-1}(x_p,y_p,z_q,t),$$

which enables an immediate application of Sommerfeld's integral[2)]

$$\int_0^\infty d\lambda\, \lambda \frac{J_0(\lambda\rho)}{\sqrt{\lambda^2-k^2}} e^{-\sqrt{\lambda^2-k^2}\,z} = \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}. \qquad (z>0) \tag{9}$$

We thus obtain:

$$u_N(P,t) = -\frac{1}{c^2}\int_0^D dz_q \frac{e^{-|z_p-z_q|\sqrt{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_p^2}}}}{\sqrt{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_p^2}}}\, \varphi_{N-1}(x_p,y_p,z_q,t).$$

In applications of Sommerfeld's integral to complex ρ and z, as needed here, the square root $\sqrt{\rho^2+z^2}$ should have a real positive part in view of the principle of analytic continuation. The same therefore holds for

$$\sqrt{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_p^2}}$$

when this quantity is converted into an ordinary one in its application to the three-dimensional Fourier integral of $\varphi_{N-1}(x_p,y_p,z_q,t)$ with respect to x_p,y_p,t. We may also substitute

$$\sqrt{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_p^2}} = -i\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}, \tag{10}$$

provided the new square root is defined as having a positive (or zero) imaginary part. We then arrive, yet substituting the definition (7) for φ_{N-1}, at the following recurrence relation connecting the contributions associated with two successive numbers of scattering:

$$u_N(P,t) = -\frac{i}{c^2}\int_0^D dz_q \frac{e^{i|z_p-z_q|\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}}{\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}\left(\delta n\,\frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{x_p,y_p,z_q,t}.$$

$$(\mathrm{Im}\sqrt{} \geq 0) \tag{11}$$

4. The splitting into forward and backward travelling contributions

The formula (11) expresses how a N-th order wave contribution results from the preceding $(N-1)^{th}$ contribution. Let us assume $k_{oz} > 0$ for the primary wave (2) which thus may be considered (if $\omega_o > 0$) as a wave propagating towards $z \to +\infty$. Therefore, we shall call this direction "forward", in the sense of which the half space $z < 0$ constitutes the front side, and the half space $z > 0$ the back of the scattering slab. In the former half space U_N represents a contribution originating from the slab and thus porpagating always backwards, which is in agreement with the invariable sign of $z_p - z_q$ in (11), viz $|z_p - z_q| = -(z_p - z_q)$. At the back of the slab $(z > D)$, on the contrary, we have to do with the situation $|z_p - z_q| = z_p - z_q$, thus leading again to a single sign of $z_p - z_q$ for this forward travelling wave. On the other hand, inside the slab this quantity may have different signs within the interval $0 < z_q < D$. This leads to two contributions of (11) according to

$$u_N(P,t) = \overset{\uparrow}{u}_N(P,t) + \overset{\downarrow}{u}_N(P,t), \tag{12}$$

the individual terms of which are to be defined by:

$$\overset{\uparrow}{u}_N(P,t) = -\frac{i}{c^2}\int_0^{z_p} dz_q \, \frac{e^{i(z_p - z_q)\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}}{\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \left(\delta n \, \frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{x_p, y_p, z_q, t} , \tag{13a}$$

$$\overset{\downarrow}{u}_N(P,t) = -\frac{i}{c^2}\int_{z_p}^{D} dz_q \, \frac{e^{i(z_q - z_p)\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}}{\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \left(\delta n \, \frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{y_p, z_q, t} . \tag{13b}$$

The first contribution $\overset{\uparrow}{U}_N$ may be interpreted as a forward travelling wave, the sign of $z_p - z_q$ being the same as that of the purely forward wave U_N at the back of the slab. Similarly, $\overset{\downarrow}{U}_N$ represents a backward travelling wave. Obviously, we could substitute

$$\frac{\partial^2 u_{N-1}}{\partial t^2} = \frac{\partial^2 \overset{\uparrow}{u}_{N-1}}{\partial t^2} + \frac{\partial^2 \overset{\downarrow}{u}_{N-1}}{\partial t^2}$$

in the integrands of (13a) and (13b). We then infer that the Nth forward scattering contribution originates partly from the $(N-1)^{st}$ forward travelling wave, and partly from the $(N-1)^{st}$ backward travelling wave, the corresponding holding for the N-th backward scattering contribution.

It is interesting to derive new differential relations from (13). The following equations can be verified without difficulty:

$$\left\{\frac{\partial}{\partial z_p} - i\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right\} \overset{\uparrow}{u}_N(P,t) = -\frac{i}{c^2} \frac{1}{\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \left(\delta n \, \frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{P,t}, \tag{14a}$$

$$\left\{\frac{\partial}{\partial z_p} + i\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right\} \overset{\downarrow}{u}_N(P,t) = \frac{i}{c^2} \frac{1}{\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \left(\delta n \, \frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{P,t}. \tag{14b}$$

$$(\mathrm{Im}\sqrt{} \geq 0)$$

Moreover, we can introduce the total forward travelling and the total backward travelling wave defined by:

$$\overset{\uparrow}{u} = \sum_{N=0}^{\infty} \overset{\uparrow}{u}_N \qquad , \qquad \overset{\downarrow}{u} = \sum_{N=0}^{\infty} \overset{\downarrow}{u}_N . \tag{15}$$

A summation of (14a) and (14b) over N = 1,2,3,... then yields, taking into account that $\overset{\uparrow}{u}_o = u_o$ is identical with the primary wave (2), while $\overset{\downarrow}{u}_o = 0$,

$$\left\{\frac{\partial}{\partial z_p} - i\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right\}\overset{\uparrow}{u}(P,t) = -\frac{i}{c^2\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}\left(\delta n\,\frac{\partial^2 u}{\partial t^2}\right)_{P,t}, \tag{16a}$$

$$\left\{\frac{\partial}{\partial z_p} + i\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right\}\overset{\downarrow}{u}(P,t) = \frac{i}{c^2\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}\left(\delta n\,\frac{\partial^2 u}{\partial t^2}\right)_{P,t}. \tag{16b}$$

$$(\mathrm{Im}\sqrt{} \geq 0).$$

Finally, an addition of these latter relations gives:

$$\frac{\partial u(P,t)}{\partial z_p} = i\sqrt{\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\left\{\overset{\uparrow}{u}(P,t) - \overset{\downarrow}{u}(P,t)\right\}. \tag{17}$$

Obviously, the equations (16) and (17) take account of all multiple-scattering effects.

In the case of a stationary medium, that is $\partial\,\delta n/\partial t = 0$, all time dependence is contained in the factor $\exp(-i\omega_o t) = \exp(-ik_o ct)$ which enters in all quantities like u_{pr}, u, $\partial^2 u/\partial t^2$ etc. The equations (16) then become:

$$\left(\frac{\partial}{\partial z_p} - i\sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}\right)\overset{\uparrow}{u}(P,t) = \frac{ik_o^2}{\sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}}\left(\delta n\; u\right)_{P,t}, \tag{18a}$$

$$\left(\frac{\partial}{\partial z_p} + i\sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}\right)\overset{\downarrow}{u}(P,t) = -\frac{ik_o^2}{\sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}}\left(\delta n\; u\right)_{P,t}. \tag{18b}$$

5. The forward-scattering approximations

In many practical circumstances all scattering processes are predominantly in the forward direction so that $\overset{\downarrow}{U}$ becomes very small compared to $\overset{\uparrow}{U}$. This situation admits a considerable simplification of the operator

$$\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\,.$$

Let us first consider the application of the corresponding operator

$$\sqrt{k_0^2+\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}$$

for the stationary case to a special component of the Fourier synthesis of, e.g., the function

$$\delta n(P)=\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z\, e^{i(k_x x+k_y y+k_z z)}\, G_3(k_x,k_y,k_z)\,. \tag{19}$$

The application in question results in an additional factor $(k_o^2 - k_x^2 - k_y^2)^{\frac{1}{2}}$ in the integrand. It is well known that forward scattering occurs when $\lambda_o \ll \ell$ holds for all relevant scale lengths ℓ contained in the distribution of $\delta n(x,y,z)$, $\lambda_o = 2\pi/k_o$ being the wavelength of the incident wave. These scale lengths correspond to the dominant components in the spatial Fourier spectrum (19) which means that the wave numbers k_x, k_y, k_z of these components should have a magnitude of the order of ℓ^{-1}. We then have

$$\frac{k_x^2+k_y^2}{k_0^2}\approx\frac{1}{k_0^2\ell^2}\approx\frac{\lambda_0^2}{\ell^2},$$

which quantity thus proves to be very small. Therefore, the following approximation can be used under forward-scattering conditions:

$$\sqrt{k_0^2+\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}=k_0\sqrt{1+\frac{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}{k_0^2}}\sim k_0+\frac{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}{2k_0}, \tag{20}$$

while, also:

$$\frac{1}{\sqrt{k_0^2+\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}}=\frac{1}{k_0}\left(1+\frac{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}{k_0^2}\right)^{-1/2}\sim\frac{1}{k_0}-\frac{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}}{2k_0^3}. \tag{21}$$

The application of these approximations to the equations (18a) and (18b) yields the following wave equations for stationary forward-scattering conditions:

$$\left(k_0+\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{2k_0}+i\frac{\partial}{\partial z}\right)\overset{\uparrow}{u}=\left(-k_0+\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{2k_0}\right)(\delta n\ u), \tag{22a}$$

$$\left(k_0+\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{2k_0}-i\frac{\partial}{\partial z}\right)\overset{\downarrow}{u}=\left(-k_0+\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{2k_0}\right)(\delta n\ u). \tag{22b}$$

The form of these equations, which only contain the first derivative of the variable z, is typical for a propagation of diffusion type in the z direction. This might be yet clearer from the corresponding equations for the quantities:

$$\overset{\uparrow}{v}=e^{-ik_0 z}\,\overset{\uparrow}{u} \qquad \text{and} \qquad \overset{\downarrow}{v}=e^{ik_0 z}\,\overset{\downarrow}{u}.$$

In the recent literature such forward-scattering approximations have been applied to stationary scatterers, e.g., by <u>Molyneux</u>[3)] and <u>De Wolf</u>[4)].

Considering next a non stationary medium, the parameter $(1/c^2)\partial^2/\partial t^2$ will in general involve quantities of the same order as k_o^2 since, in practical situations, the main time dependence results from that of the primary wave; this presumes that the relevant travelling waves contained in the four-dimensional Fourier spectrum of δn, viz.

$$\delta n(x,y,z,t)=\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z\int_{-\infty}^{\infty}d\omega\, e^{i(k_x x+k_y y+k_z z-\omega t)}\,\mathcal{G}_4(k_x,k_y,k_z,\omega)=$$

$$=\int d\vec{k}\int d\omega\, e^{i(\vec{k}\vec{r}-\omega t)}\,\mathcal{G}_4(\vec{k},\omega), \tag{23}$$

do have frequencies ω which are much smaller than ω_o. We then can apply approximations such as (compare (10)):

$$\sqrt{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}} = i\sqrt{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}} =$$

$$= \frac{i}{c}\frac{\partial}{\partial t}\left(1 - c^2\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{\frac{\partial^2}{\partial t^2}}\right)^{1/2} \sim \frac{i}{c}\frac{\partial}{\partial t} - \frac{ic}{2}\,\frac{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}{\frac{\partial}{\partial t}}. \tag{24}$$

6. Some expressions for the Born approximation

This approximation, viz. $u = u_o + u_1$, only takes into account single-scattering effects as described by u_1. Usually, it is directly derived from some equation equivalent to our (3) in which u is then to be replaced by u_{pr} in the right-hand side. In our analysis the scattering part of the Born approximation also results from taking N = 1 in the relevant expressions, while substituting the primary field (2) for u_o. We thus obtain from (6) and (11) respectively:

$$u_1(P,t) = \frac{k_0^2}{2\pi} e^{-i\omega_0 t} \int d\tau_q \frac{\delta n\left(Q, t-\frac{PQ}{c}\right)}{PQ} e^{i(\vec{k}_0 \vec{r}_q + k_0 PQ)} , \tag{25}$$

and

$$u_1(P,t) = i k_0^2 \int_0^D dz_q \frac{e^{i|z_p - z_q|\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}}{\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \delta n(x_p, y_p, z_q, t)\, e^{i(k_{0x} x_p + k_{0y} y_p + k_{0z} z_q - \omega_0 t)} . \tag{25a}$$

Splitting off the primary wave as given by (2) the latter expression can further be reduced to:

$$u_1(P,t) = i k_0^2 e^{i(\vec{k}_0 \vec{r}_p - \omega_0 t)} \int_0^D dz_q \frac{e^{i\left\{k_{0z}(z_q - z_p) + |z_p - z_q|\sqrt{\left(\frac{\partial}{\partial x_p}+ik_{0x}\right)^2+\left(\frac{\partial}{\partial y_p}+ik_{0y}\right)^2-\frac{1}{c^2}\left(\frac{\partial}{\partial t}-i\omega_0\right)^2}\right\}}}{\sqrt{\left(\frac{\partial}{\partial x_p}+ik_{0x}\right)^2+\left(\frac{\partial}{\partial y_p}+ik_{0y}\right)^2-\frac{1}{c^2}\left(\frac{\partial}{\partial t}-i\omega_0\right)^2}} \delta n(x_p, y_p, z_q, t).$$

$$(\mathrm{Im}\sqrt{\ } \geq 0) \tag{26a}$$

Using the alternative square root (see (10)) we obtain:

$$u_1(P,t)=$$
$$k_0^2 e^{i(\vec{k}_0\vec{r}_p-\omega_0 t)}\int_0^D dz_q \frac{e^{-ik_{0z}(z_p-z_q)-|z_p-z_q|\sqrt{(k_{0x}-i\frac{\partial}{\partial x_p})^2+(k_{0y}-i\frac{\partial}{\partial y_p})^2-\frac{1}{c^2}(\omega_0+i\frac{\partial}{\partial t})^2}}}{\sqrt{(k_{0x}-i\frac{\partial}{\partial x_p})^2+(k_{0y}-i\frac{\partial}{\partial y_p})^2-\frac{1}{c^2}(\omega_0+i\frac{\partial}{\partial t})^2}}\,\delta n(x_p,y_p,z_q,t)$$

$$\left(\mathrm{Re}\sqrt{}\geq 0\right). \tag{26b}$$

These operational expressions transform into ordinary ones if we substitute for δn the Fourier synthesis (23) in terms of "travelling waves". For convenience we introduce the term "acoustic waves" for the latter, thus reminding the practical property that the phase velocity $\omega/k = \omega/|\vec{k}|$ of the individual components $\vec{k},\omega$ is in general very small compared to the propagation velocity $c = \omega_0/k_0$ of an incident electromagnetic wave (compare the end of section 5). The function G_4 giving the spectrum of the acoustic waves has to account for the fact that δn vanishes outside the slab $0 < z < D$. The substitution of (23) into (26a) first gives:

$$u_1(P,t)=ik_0^2 e^{i(\vec{k}_0\vec{r}_p-\omega_0 t)}\int_0^D dz_q \frac{e^{-ik_{0z}(z_p-z_q)+i|z_p-z_q|\sqrt{(\frac{\partial}{\partial x_p}+ik_{0x})^2+(\frac{\partial}{\partial y_p}+ik_{0y})^2-\frac{1}{c^2}(\frac{\partial}{\partial t}-i\omega_0)^2}}}{\sqrt{(\frac{\partial}{\partial x_p}+ik_{0x})^2+(\frac{\partial}{\partial y_p}+ik_{0y})^2-\frac{1}{c^2}(\frac{\partial}{\partial t}-i\omega_0)^2}}$$
$$\int d\vec{k}\int d\omega\, G_4(\vec{k},\omega)\, e^{i(\vec{k}\vec{r}_p-\omega t)-ik_z(z_p-z_q)},$$

and next, after an evaluation of the differential operators,

$$u_1(P,t)=ik_0^2 e^{i(\vec{k}_0\vec{r}_p-\omega_0 t)}\int d\vec{k}\int d\omega \frac{G_4(\vec{k},\omega)\; e^{i(\vec{k}\vec{r}_p-\omega t)}}{\sqrt{(\frac{\omega_0+\omega}{c})^2-(k_{0x}+k_x)^2-(k_{0y}+k_y)^2}}\times$$
$$\times\int_0^D dz_q\, e^{i|z_p-z_q|\sqrt{(\frac{\omega_0+\omega}{c})^2-(k_{0x}+k_x)^2-(k_{0y}+k_y)^2}-ik_{0z}(z_p-z_q)}$$

$$\left(\mathrm{Im}\sqrt{}\geq 0\right) \tag{27}$$

7. The discrimination between thin and thick scatterers

In order to illustrate the usefulness of symbolic expressions by an example, we consider the Born approximation for a stationary medium in the case of a perpendicularly incident wave

$$u_{M} = e^{i(k_o z - \omega_c t)}$$

($K_{ox} = K_{oy} = 0$). We then obtain from (26a) for the forward-scattering part for the single-scattering contribution:

$$u_1(P,t) = i k_o^2 u_M(P,t) \int_0^{z_p} dz_q \, \frac{e^{i(z_p - z_q)\left(-k_o + \sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}\right)}}{\sqrt{k_o^2 + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}}} \, \delta n(x_p, y_p, z_q).$$

We further assume the applicability of the forward-scattering approximation and thus get, using (20) in the exponent while replacing the denominator by k_o:

$$u_1(P,t) = i k_o u_M(P,t) \int_0^{z_p} dz_q \, e^{\frac{i}{2k_o}(z_p - z_q)\left(\frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial y_p^2}\right)} \, \delta n(x_p, y_p, z_q). \qquad (28)$$

For a special scale length ℓ the order of magnitude of the exponent appears to be that of $(z_p - z_q)/(k_o \ell^2)$, with the maximal value $d/(k_o \ell^2)$ if $d = z_p$ constitutes the penetration depth at which we are observing. We can therefore discriminate between small penetration depths for which $d \ll k_o \ell^2 \sim \ell^2/\lambda_o$ for all relevant scale lengths ℓ, and large penetration depths for which this is not the case. The scatterer itself may be called "thin" if even

$$D \ll \frac{\min(\ell^2)}{\lambda_o},$$

$\min \ell$ being the smallest relevant scale length. It is then allowed to neglect the effect of the exponent in (28) altogether, and we arrive at the well known approximation for the "geometric optical region" or Fresnel zone, viz.

$$\left\{u_1(P,t)\right\}_{Fresnel} = ik_o\, u_{pr}(P,t)\int_0^{z_p} dz_q\;\; \delta n(x_p, y_p, z_q). \tag{29}$$

The wave function here only depends on the refractive-index distribution along the primary ray (x_p and y_p constant) arriving at the point of observation. We here recognize a pure shadow effect while the factor i in

$$\frac{u_{pr}+u_1}{u_{pr}} \sim e^{\,ik_o\int_0^{z_p} dz_q\;\; \delta n(x_p, y_p, z_q)}$$

indicates an almost pure phase modulation for a non-absorting object (δn real).

For thicker objects we have to take into account at least a few terms of the expansion of the exponential in (28). Writing out the first terms we find:

$$u_1(P,t) = u_{pr}(P,t)\left\{ik_o\int_0^{z_p} dz_q\;\; \delta n(x_p,y_p,z_q) - \frac{1}{2}\int_0^{z_p} dz_q\,(z_p - z_q)\cdot\left(\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}\right)\delta n(x_p,y_p,z_q) + \cdots\right\}; \tag{30}$$

the second term represents focussing and defocussing effects which are independent of the frequency in a non dispersive scatterer (see also the discussion in reference 1, page 420).

For increasing penetration depths, which can only exist in "thick objects" for which D is at least of the order of ℓ^2/λ_o for the relevant scale lengths, an increasing number of terms in (30) is needed. We then enter gradually into the diffraction or Fraunhofer region. There, the typical diffraction phenomena depend first of all on the magnitude of d λ_o/ℓ^2 and these phenomena thus become less spectacular for increasing frequency (decreasing wave length). This is in contrast with the order of magnitude of $k_o\, d(\langle\delta n^2\rangle)^{\frac{1}{2}}$ which determines, in view of (29), the order of amplitude in the Fresnel region; the variance $\langle\delta n^2\rangle$, expressing the average value of δn^2, is a measure there for the possible values of δn. In this latter region the scattered wave increases with increasing frequency, and we have to do there with "pure scattering".

A special application of the geometric-optical approximation (29) may concern Röntgen diffraction by a thin object composed of a number of parallel sheats all of which contain an identical periodic structure which, however, may have a random lateral displacement in the consecutive sheats. In spite of this granular structure (with a predominating periodicity) the periodicity in the transverse coordinates x and y, existing in $\delta n(x,y,z)$ for each individual z value, is maintained in the integral

$$\int_0^D \delta n(x,y,z)\, dz$$

which fines the wave-function distribution across the outer edge $z = D$ of the object. Therefore, the diffraction observed outside is still dominated by the present periodicity though the latter may be less apparent due to the mentioned lateral shifts. An explanation for this situation has been discussed by <u>von Laue</u>[5] when considering a corresponding model.

The consequences of the transition from thin to thick objects in electron microscopy have been described elsewhere[6].

8. A further symbolic expression for the Born approximation

The expression to be derived in this section is in particular useful for computation of the scattered waves observed at great distances in front or behind the scattering slab (see the next section).

We start from formula (25) in which the time shift in the δn-distribution may be accounted for as follows by introducing a delta function:

$$u_1(P,t) = \frac{k_0^2}{2\pi} \int d\tau_Q \, \frac{e^{i\vec{k}_0 \vec{r}_Q}}{PQ} \int_{-\infty}^{\infty} dt' \, \delta n(Q,t') e^{-i\omega_0 t'} \, \delta\!\left(t - \frac{PQ}{c} - t'\right). \tag{31}$$

We next need the four-dimensional Fourier transform of the normalized Green function $\delta(t-r/c)/r$ associated with the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = -4\pi \, \delta(x)\, \delta(y)\, \delta(z)\, \delta(t).$$

This transform, the derivation of which is given in the appendix, reads:

$$\frac{\delta\left(t-\frac{\sqrt{x^2+y^2+z^2}}{c}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{1}{4\pi^3}\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \int_{-\infty}^{\infty} ds_3 \int_{-\infty-ic'}^{\infty-ic'} du \, \frac{e^{i(s_1x+s_2y+s_3z+ut)}}{s_1^2+s_2^2+s_3^2-\frac{u^2}{c^2}} \, . \qquad (c'>0) \qquad (32)$$

In order to apply this formula to (31) we substitute $x = x_q - x_p$, $y = y_q - y_p$, $z = z_q - z_p$, so that the vector with components (x,y,z) can be represented by $\overrightarrow{Pq}$; moreover, $\vec{s}$ may be short for the vector with components s_1, s_2, s_3. Replacing further t by $t - t'$, we thus get:

$$\frac{\delta\left(t-\frac{PQ}{c}-t'\right)}{PQ} = \frac{1}{4\pi^3}\int d\vec{s}\int_{-\infty-ic'}^{\infty-ic'} du \, \frac{e^{i\vec{s}\,\overrightarrow{PQ}+iu(t-t')}}{s^2-\frac{u^2}{c^2}} \, . \qquad (c'>0)$$

The formula (31) can then be transformed into:

$$u_1(P,t) = \frac{k_0^2}{8\pi^4}\int d\tau_q \int_{-\infty}^{\infty} dt' \, e^{i(\vec{k}_0\vec{r}_q-\omega_0 t')} \, \delta n(Q,t') \int d\vec{s}\int_{-\infty-ic'}^{\infty-ic'} du \, \frac{e^{i\{\vec{s}(\vec{r}_p-\vec{r}_q)+u(t-t')\}}}{s^2-\frac{u^2}{c^2}} \, ,$$

or, inverting the orders of integration, into

$$u_1(P,t) = 2k_0^2\int d\vec{s}\int_{-\infty-ic'}^{\infty-ic'} du \, \frac{e^{i(\vec{r}_p\vec{s}+tu)}}{s^2-\frac{u^2}{c^2}} \, \frac{1}{(2\pi)^4}\int d\tau_q \int dt' \, e^{-i\{(\vec{s}-\vec{k}_0)\vec{r}_q+(\omega_0+u)t'\}} \, \delta n(Q,t') .$$

However, the Fourier inversion of (23) yields:

$$g_4(\vec{k},\omega) = \frac{1}{(2\pi)^4}\int d\tau_q \int_{-\infty}^{\infty} dt' \, e^{-i(\vec{k}\vec{r}_q-\omega t')} \, \delta n(Q,t') \, ,$$

so that the single-scattering contribution u_1 can be represented as follows in terms of the spectral function G_4 of the "acoustic waves":

$$u_1(P,t) = 2k_0^2 \int d\vec{s} \int_{-\infty - ic'}^{\infty - ic'} du \, \frac{e^{i(\vec{r}_p \vec{s} + tu)}}{s^2 - \frac{u^2}{c^2}} \, G_4(\vec{s} - \vec{k}_0, -\omega_0 - u). \qquad (c' > 0) \tag{33}$$

The coordinates and time of the observation point (P,t) only occur in the exponential according to which the spatial and time derivatives are equivalent with the following operators:

$$\frac{\partial}{\partial \vec{r}_p} = \left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_p}, \frac{\partial}{\partial z_p}\right) = i(s_x, s_y, s_z) = i\vec{s},$$

$$\frac{\partial}{\partial t} = iu.$$

This involves the following symbolic expression replacing (33):

$$u_1(P,t) = 2k_0^2 \, G_4\left(-\vec{k}_0 - i\frac{\partial}{\partial \vec{r}_p}, -\omega_0 + i\frac{\partial}{\partial t}\right) \int d\vec{s} \int_{-\infty - ic'}^{\infty - ic'} du \, \frac{e^{i(\vec{r}_p \vec{s} + tu)}}{s^2 - \frac{u^2}{c^2}}.$$

In view of (32) the four-dimensional integral here equals

$$4\pi^3 \, \frac{\delta\left(t - \frac{r_p}{c}\right)}{r_p},$$

so that, also,

$$u_1(P,t) = 8\pi^3 k_0^2 \, G_4\left(-\vec{k}_0 - i\frac{\partial}{\partial \vec{r}_p}, -\omega_0 + i\frac{\partial}{\partial t}\right) \frac{\delta\left(t - \frac{r_p}{c}\right)}{r_p}.$$

In practical circumstances the frequency spectrum of u_1 will be concentrated around the incident frequency ω_o. A convenient representation of this spectrum is then obtained by substitution of the following Fourier transform with respect to the time variable:

$$\frac{\delta(t-\frac{r_p}{c})}{r_p} = \frac{1}{2\pi r_p}\int_{-\infty+ic'}^{\infty+ic'} d\omega \; e^{-i(\omega_o+\omega)(t-\frac{r_p}{c})},$$

in which c' may assume any possible value. We then arrive at the following final formula, to be applied in the next section:

$$u_1(P,t) = 4\pi^2 k_o^2 \int_{-\infty+ic'}^{\infty+ic'} d\omega \quad G_4\left(-\vec{k}_o - i\frac{\partial}{\partial \vec{r}_p}, \omega\right) \frac{e^{-i(\omega_o+\omega)(t-\frac{r_p}{c})}}{r_p}. \tag{34}$$

$$(c'>0)$$

9. The Born approximation at great distances

The operators entering in (34) in the acoustica-wave spectrum G_4 are only effective on the integrand of the ω-integral. We find, e.g.,

$$\frac{\partial}{\partial x_p}\frac{e^{-i(\omega_o+\omega)(t-r_p/c)}}{r_p} =$$

$$= \frac{i(\omega_o+\omega)}{c}\frac{x_p}{r_p}\left\{1 - \frac{1}{(i/c)(\omega_o+\omega)r_p}\right\}\frac{e^{-i(\omega_o+\omega)(t-r_p/c)}}{r_p}.$$

Provided that

$$r_p \gg \frac{c}{\min|\omega_o+\omega|}, \tag{35}$$

we thus get the approximation:

$$\frac{\partial}{\partial x_p}\frac{e^{-i(\omega_o+\omega)(t-r_p/c)}}{r_p} \sim \frac{i(\omega_o+\omega)}{c}\frac{x_p}{r_p}\frac{e^{-i(\omega_o+\omega)(t-r_p/c)}}{r_p},$$

according to which $\partial/\partial x_p$ amounts to a multiplication by the factor

$$i\frac{(\omega_0+\omega)}{c}\frac{x_p}{r_p}.$$

A combination with the corresponding expressions for $\partial/\partial y_p$ and $\partial/\partial z_p$ leads to the following vectorial representation of these approximations:

$$\frac{\partial}{\partial \vec{r}_p} \sim \frac{i(\omega_0+\omega)}{c}\vec{u}_r, \tag{35a}$$

if $\vec{u}_r$ marks the unit vector in the radial direction, that is the vector with components

$$\left(\frac{x_p}{r_p}, \frac{y_p}{r_p}, \frac{z_p}{r_p}\right).$$

Moreover, we have

$$\frac{\partial}{\partial t} = -i(\omega_0+\omega),$$

and thus obtain the following for-distance approximation of (34) if (35) is fulfilled for all relevant ω-values:

$$u_1(P,t) \sim 4\pi^2 k_0^2 \int_{-\infty+ic'}^{\infty+ic'} d\omega \int_V \left(-\vec{k}_0+\frac{\omega_0+\omega}{c}\vec{u}_r, \omega\right)\frac{e^{-i(\omega_0+\omega)(t-\frac{r_p}{c})}}{r_p}. \tag{36}$$

In view of the path of integration and the positive value of ω_o, the condition (35) amounts for $c' > 0$ to:

$$r_p \gg \frac{c}{\omega_0+c'}.$$

Therefore, the ω integration may even refer to real frequencies ($c'=0$) provided that $r_p >> c/\omega_o$.

The interpretation of (36), insofar as referring to the contribution of a particular frequency ω of the acoustic-wave spectrum G_4, involves the wellknown property that only the effect of one single component of this spectrum is observed at great distances, namely the component characterized by the wave-number vector:

$$\vec{k} = -\vec{k}_0 + \frac{\omega_0+\omega}{c}\,\vec{u}_r\;; \tag{37}$$

the significance of this relation will yet be discussed later (see section 11). We further notice that the geometrical situation $0 < z < D$ of the scattering slab, as well as its thickness D, only enter implicitly in the function G_4 which has to account for the vanishing of δn outside the slab. The different numerical behaviour of the scattered wave on the front side and at the back of the slab results from the fact that the sign of the z component of $\vec{u}_r$ is negative in the former case and positive in the latter one.

Results equivalent to (36) are usually derived from a saddlepoint approximation of integrals like (27). The present derivation has the adventage that the expression (34), to which the final approximation has to be applied, is yet rigorous. This facilitates the discussion of the reliability of the approximation, and also the derivation of corrections to it. As a matter of fact the latter depend on a Taylor expansion of G_4 starting with:

$$G_4\left(-\vec{k}_0 - i\frac{\partial}{\partial\vec{r}_p}, -\omega_0 + i\frac{\partial}{\partial t}\right) = G_4\left(-\vec{k}_0 + \frac{\omega_0+\omega}{c}\vec{u}_r, \omega\right) +$$

$$+\frac{\partial G_4}{\partial\vec{k}}\left(-\vec{k}_0+\frac{\omega_0+\omega}{c}\vec{u}_r, \omega\right).\left(-i\frac{\partial}{\partial\vec{r}_p} - \frac{\omega_0+\omega}{c}\vec{u}_r\right) + \frac{\partial G_4}{\partial\omega}\left(-\vec{k}_0+\frac{\omega_0+\omega}{c}\vec{u}_r, \omega\right).\left(-\omega_0-\omega+i\frac{\partial}{\partial t}\right)$$

$$+ \cdots$$

10. Waves inside the scattering slab according to the Born approximation

The distinction between forward and backward travelling waves (see section 4) can also be applied to the single-scattering contribution u_1 as represented by (27). Obviously, this leads to a splitting inside the travelling slab $0 < z_p < D$ according to

$$u_1 = u_1^{\uparrow} + u_1^{\downarrow},$$

with

$$u_1^{\uparrow}(P,t) = i k_0^2 e^{i(\vec{k}_0\vec{r}_p - \omega_0 t)} \int d\vec{k} \int d\omega \frac{G_4(\vec{k},\omega)\; e^{i(\vec{k}\vec{r}_p - \omega t)}}{\sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2}} \times$$

$$\times \int_0^{z_p} dz_q\, e^{i(z_p - z_q)\left\{\sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2} - k_{0x} - k_x\right\}},$$

$$u_1^{\downarrow}(P,t) = i k_0^2 e^{i(\vec{k}_0\vec{r}_p - \omega_0 t)} \int d\vec{k} \int d\omega \frac{G_4(\vec{k},\omega)\; e^{i(\vec{k}\vec{r}_p - \omega t)}}{\sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2}} \times$$

$$\times \int_{z_p}^{D} dz_q\, e^{i(z_p - z_q)\left\{\sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2} + k_{0x} + k_x\right\}}.$$

$$(\mathrm{Im}\sqrt{} \geqslant 0)$$

The integration with respect to z_q is elementary, and yields, after introducing the abbreviations:

$$\alpha_1^{\uparrow} \equiv k_{0x} + k_x - \sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2}, \qquad (\mathrm{Im}\,\alpha_1^{\uparrow} \leqslant 0)$$

$$\alpha_1^{\downarrow} \equiv k_{0x} + k_x + \sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2}, \qquad (\mathrm{Im}\,\alpha_1^{\downarrow} \geqslant 0)$$

$$\beta_1 \equiv \sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2}, \qquad (\mathrm{Im}\,\beta_1 \geqslant 0) \tag{38}$$

the following remaining four-dimensional integrals:

$$u_1^{\uparrow}(P,t) = k_0^2 e^{i(\vec{k}_0\vec{r}_p - \omega_0 t)} \int d\vec{k} \int d\omega \frac{G_4(\vec{k},\omega) e^{i(\vec{k}\vec{r}_p - \omega t)}\left(1 - e^{-i\alpha_1^{\uparrow} z_p}\right)}{\alpha_1^{\uparrow}\beta_1},$$

$$u_1^{\downarrow}(P,t) = k_0^2 e^{i(\vec{k}_0\vec{r}_p - \omega_0 t)} \int d\vec{k} \int d\omega \frac{G_4(\vec{k},\omega) e^{i(\vec{k}\vec{r}_p - \omega t)}\left(e^{i(D - z_p)\alpha_1^{\downarrow}} - 1\right)}{\alpha_1^{\downarrow}\beta_1}. \tag{39}$$

We thus infer the existence of four different waves contained in the representations:

$$u_1^{\uparrow}(P,t) = k_0^2 \int d\vec{k} \int d\omega \, \frac{G_4(\vec{k},\omega)}{\alpha_1^{\uparrow}\beta_1} e^{i\{\vec{k}_{fm}\vec{r}_P - (\omega_0+\omega)t\}} - k_0^2 \int d\vec{k} \int d\omega \, \frac{G_4(\vec{k},\omega)}{\alpha_1^{\uparrow}\beta_1} e^{i\{\vec{k}_{fd}\vec{r}_P - (\omega_0+\omega)t\}} ,$$

$$u_1^{\downarrow}(P,t) = k_0^2 \int d\vec{k} \int d\omega \, \frac{G_4(\vec{k},\omega)}{\alpha_1^{\downarrow}\beta_1} e^{i\{D\alpha_1^{\downarrow} + \vec{k}_{bd}\vec{r}_P - (\omega_0+\omega)t\}} - k_0^2 \int d\vec{k} \int d\omega \, \frac{G_4(\vec{k},\omega)}{\alpha_1^{\downarrow}\beta_1} e^{i\{\vec{k}_{bm}\vec{r}_P - (\omega_0+\omega)t\}} ,$$

(40)

where we have introduced the following individual wave-number vectors:

$$\vec{k}_{fm} = \vec{k}_{bm} = \vec{k}_0 + \vec{k} ,$$

$$\vec{k}_{fd} = \vec{k}_0 + \vec{k} - \alpha_1^{\uparrow}\, \vec{u}_z ,$$

$$\vec{k}_{bd} = \vec{k}_0 + \vec{k} - \alpha_1^{\downarrow}\, \vec{u}_z$$

(41)

($\vec{u}_z$ = unit vector in the z direction).

As a matter of fact, each individual acoustic wave, fixed by the set $(\vec{k},\omega)$, thus generates four different waves characterized by their exponential functions. However, the first and the fourth one, both being proportional to

$$e^{i\{(\vec{k}_0+\vec{k})\vec{r}_P - (\omega_0+\omega)t\}} ,$$

have the same wave-number vector $\vec{k}_{fm} = \vec{k}_{bm}$. These waves can be termed "modulated waves" since they originate from a direct modulation of a pa[illegible] the incident primary wave by the acoustic wave in question. For the rest, these two waves result from a forward and a backward scattering contribution respectively, which is expressed by the first subscript of the corresponding labels fm and dm, while the common second label "m" refers to the concept of modulation.

The two remaining waves in (40), proportional to

$$e^{i\{\vec{k}_{fd}\,\vec{r}_p - (\omega_0+\omega)t\}} \quad \text{and} \quad e^{i\{\vec{k}_{bd}\,\vec{r}_p - (\omega_0+\omega)t\}},$$

will be called "diffracted waves", according to which the second subscript "d" has been introduced. The other subscript again refers to the fact that the first of these waves results from forward scattering, but the second one from backscattering. The length k of the wave-number vector of these diffracted waves becomes very simple; for the "f d" wave it can be derived as follows:

$$k_{fd}^2 = \vec{k}_{fd}\,\vec{k}_{fd} = (\vec{k}_0+\vec{k})(\vec{k}_0+\vec{k}) - 2\overset{\uparrow}{\alpha}_1(\vec{k}_0+\vec{k})\,\vec{u}_z + \overset{\uparrow}{\alpha}_1{}^2 =$$

$$= 2(k_{0z}+k_z)^2 - 2\overset{\uparrow}{\alpha}_1(k_{0z}+k_z) + \left(\frac{\omega_0+\omega}{c}\right)^2 - 2(k_{0z}+k_z)\beta_1 = \left(\frac{\omega_0+\omega}{c}\right)^2.$$

The corresponding computation for the "b d" wave shows that

$$k_{fd} = k_{bd} = \frac{|\omega_0+\omega|}{c}$$

holds for both types of diffracted waves. We infer that these waves are similar to a wave propagating through vacuum with a frequency $\omega_0 + \omega$. Moreover, the relation

$$\vec{k}_{fd} - \vec{k}_{bd} = (\overset{\uparrow}{\alpha}_1 - \overset{\downarrow}{\alpha}_1)\,\vec{u}_z = 2\beta_1\,\vec{u}_z$$

indicates that the endpoints of the wave-number vectors for the two diffracted waves, when plotted in a $\vec{k}$-space, are connected by a line parallel to the z-axis. The further relation

$$(\vec{k}_{fd})_x = (\vec{k}_{bd})_x = (\vec{k}_{fm})_x = k_{0x} + k_x,$$

and the corresponding one for the y components shows that the latter line will also pass through the endpoint of the vector $\vec{k}_{fm} = \vec{k}_{bm}$. We thus have explained the $\vec{k}$-space diagram of fig. 1, in which $\overset{\uparrow}{\alpha}_1$, $\overset{\downarrow}{\alpha}_1$ and β_1 are illustrated by the lengths indicated in the right-hand part of the figure.

11. Resonances inside the scattering slab according to the Born approximation

The expressions (39) and (40) show that a resonance, that is a very high excitation of one of the single-scattered waves by a special acoustic wave, might occur when $\overset{\uparrow}{\alpha}_1 = 0$, $\overset{\downarrow}{\alpha}_1 = 0$ or $\beta_1 = 0$. The latter case has very little practical importance since, usually, the length k of the wave-number for the relevant acoustic waves is much smaller than the corresponding value ω_o/c for the incident electromagnetic wave. The condition $\beta_1 = 0$ then involves approximately

$$\left(\frac{\omega_o+\omega}{c}\right)^2 \sim k_{ox}^2 + k_{oy}^2,$$

or in turn

$$\left(\frac{\omega_o}{c}\right)^2 \sim k_{ox}^2 + k_{oy}^2,$$

that is $k_{oz} \sim 0$. This situation can only occur for a nearly grazing incident wave.

Let us now consider the case $\overset{\uparrow}{\alpha}_1 = 0$. The corresponding factor in the integrand of the first expression (36), viz.

$$\frac{1-e^{-i\overset{\uparrow}{\alpha}_1 z_p}}{\overset{\uparrow}{\alpha}_1},$$

tends to i z_p for $\overset{\uparrow}{\alpha}_1 \to 0$. The associated contribution to the scattered wave would thus become infinite for $z_p \to \infty$. However, (39) has been derived for points inside the slab ($z_p < D$) while, even in the case of large values of z_p inside (occuring for very large D), we find the following limit for the factor in question:

$$\lim_{z_p\to\infty} \frac{1-e^{-i\overset{\uparrow}{\alpha}_1 z_p}}{\overset{\uparrow}{\alpha}_1} = i\int_0^\infty e^{-i\overset{\uparrow}{\alpha}_1 z}\,dz =$$

$$= \begin{cases} \dfrac{1}{\overset{\uparrow}{\alpha}_1} & \text{for } \mathrm{Im}\,\overset{\uparrow}{\alpha}_1 < 0 \\ i\pi\delta(\overset{\uparrow}{\alpha}_1) + P\dfrac{1}{\overset{\uparrow}{\alpha}_1} & \text{for } \mathrm{Im}\,\overset{\uparrow}{\alpha}_1 = 0 \end{cases}; \qquad (42)$$

the symbol P indicates that, if necessary, a principal value should be taken when the integration is performed with this factor in the integrand. This evaluation, however, will never lead to a diverging integral in (39). The corresponding considerations hold with respect to $\overset{\downarrow}{\alpha}_1$.

The role of resonances $\overset{\uparrow}{\alpha}_1 = 0$ and $\overset{\downarrow}{\alpha}_1 = 0$ thus does not look very striking. Nevertheless its importance can be recognized from the far-distance approximation (36) which showed that, of all acoustic waves with a special frequency ω, the only one observable at great distances is characterized by the wave-number vector of (37). This vector satisfies the relation

$$\left| \vec{k}_0 + \vec{k} \right| = \frac{|\omega_0 + \omega|}{c}, \tag{43}$$

in which the modulus sign on the left-hand side refers to the length of $\vec{k}_o + \vec{k}$. On the other hand, the relations $\overset{\uparrow}{\alpha}_1 = 0$ and $\overset{\downarrow}{\alpha}_1 = 0$ prove to lead to the same condition when working out the equation:

$$\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{0x}+k_x)^2 - (k_{0y}+k_y)^2 = \left(k_{0z}+k_z-\overset{\uparrow}{\alpha}_1\right)^2 = \left(k_{0z}+k_z-\overset{\downarrow}{\alpha}_1\right)^2.$$

The far field, both at the front side and at the back of the scattering slab, thus depends on the resonance condition $\overset{\uparrow}{\alpha}_1 = 0$ or $\overset{\downarrow}{\alpha}_1 = 0$.

The condition (43) turns out to be nothing else than an extension of the well known Bragg relation for reflection or scattering against a crystal lattice if the latter, instead of being stationary, is replaced here by the set of parallel wave fronts W on which the amplitude of a single travelling acoustic wave assumes, e.g., its maximal values. These wavefronts, perpendicular to the wave-number vector $\vec{k}$ (of length k), have a separation d given by $k = 2\pi/d$. In fact, remembering the relation $k_o^2 = \omega_o^2/c^2$, the formula (43) proves to be equivalent with:

$$k^2 + 2\,\vec{k}_0\,\vec{k} = \frac{\omega^2}{c^2} + \frac{2\,\omega_0\,\omega}{c^2}. \tag{44}$$

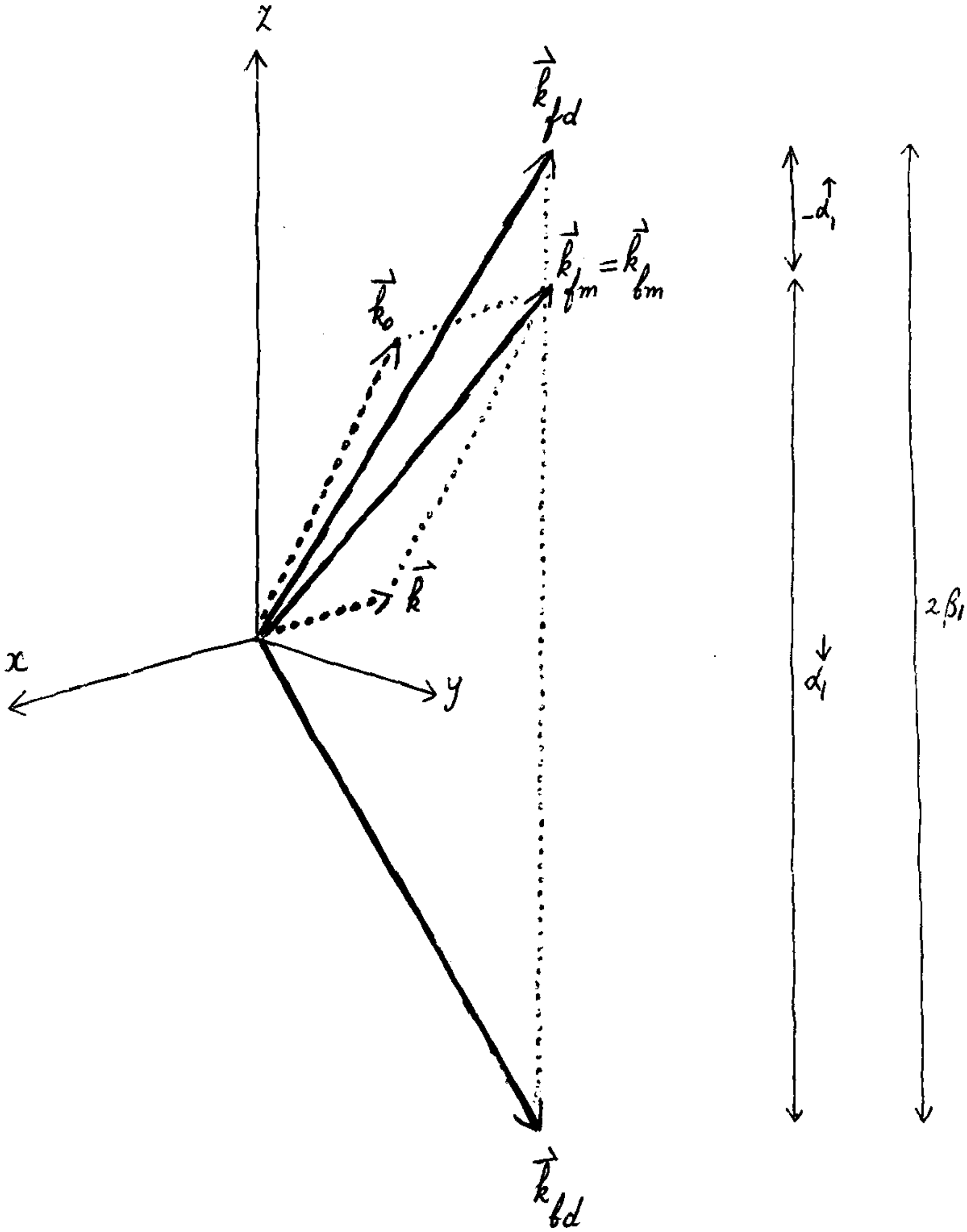
z
$\vec{k}_{fd}$
$\vec{k}_{fm}=\vec{k}_{bm}$
$\vec{k}_0$
$\vec{k}$
x
y
$\vec{k}_{bd}$
$-\vec{\alpha}_1$
$\vec{\alpha}_1$
$2\beta_1$

FIGURE 1.

After introducing the phase velocity $v = \omega/k$ of the acoustic wave, the wave length $\lambda_o = 2\pi/k_o$ of the incident wave, and the angle of incidence i_o (as represented in fig. 2) given by $\vec{k}_o \vec{k} = - k_o k \sin i_o$, we find that (44) can be transformed into:

$$\lambda_0 \left(1 - \frac{v^2}{c^2}\right) = 2d\left(\sin i_0 + \frac{v}{c}\right), \tag{45}$$

which reduces in the stationary case (v=0) to the ordinary Bragg relation.

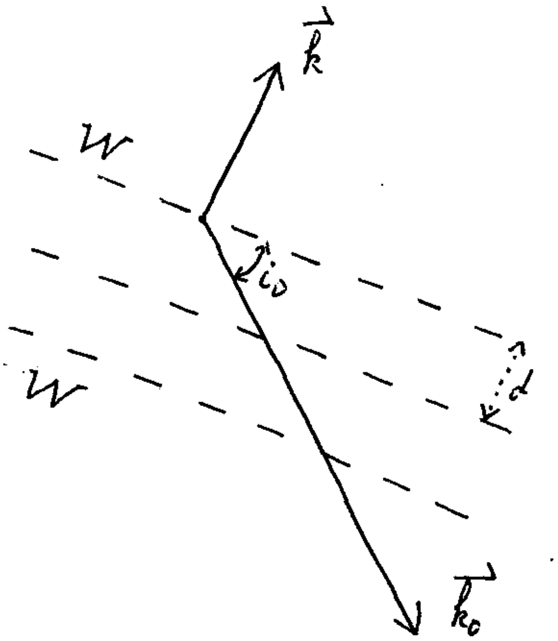

Fig. 2

We yet emphasize that the simultaneous occurence of modulated and diffracted waves, which, in view of (41), leads to the discussed resonance condition if they should be identical, completely depends on the assumption that the scattering body has a finite extent in one special direction, viz. the z direction. As a matter of fact the diffracted waves would disappear altogether if the scattering medium were infinite in all directions. This is at once clear from fig. 1 according to which, the modulated waves being given, the orientation of the diffracted waves depends on a line parallel to the z-axis in the diagram for the wave-number vectors; therefore, this orientation becomes indefinite when the z direction loses its special significance for a body occupying the entire space. The same conclusion is arrived at if, in (27), we replace the finite integration interval $0 < z_q < D$ by the infinite interval $-\infty < z_q < \infty$ in order to take account of a scatterer of unrestricted size in all directions. The evaluation of the corresponding z_q integral in (27), with the aid of the formulas (compare (42)):

$$\int_0^\infty d\zeta\, e^{\pm i\alpha\zeta} = \begin{cases} \pm \dfrac{i}{\alpha} & \text{for } \mathrm{Im}\,\alpha \gtrless 0 \\ \pi\,\delta(\alpha) \pm P\dfrac{1}{\alpha} & \text{for } \mathrm{Im}\,\alpha = 0 \end{cases}, \tag{46}$$

then leads to the following expression for the single-order scattering due to a primary wave $u_{pr} = \exp(i\,\vec{k}_o\,\vec{r}_p - \omega_o t)$ that travels through an infinite scattering medium

$$u_1(\vec{r}_p,t) = ik_o^2\int d\vec{k}\int d\omega\, G_4(\vec{k},\omega)\,\frac{e^{i\{(\vec{k}_o+\vec{k})\vec{r}_p-(\omega_o+\omega)t\}}}{\beta_1}\left[\pi\{\delta(\overset{\uparrow}{\alpha}_1)+\delta(\overset{\downarrow}{\alpha}_1)\}-2iP\frac{\beta_1}{\overset{\uparrow}{\alpha}_1\overset{\downarrow}{\alpha}_1}\right]; \qquad (47)$$

the δ-function contributions only occur for $\vec{k}$ values leading to a real β_1 and therefore are to be omitted when β_1 is complex throughout (e.g. for ω_o with a small positive imaginary part).

In fact, in (47) we only recognize the occurence of modulated waves.

12. The higher-order terms of the Born series

According to the analysis of section 9 the primary wave generates, by a single scattering act, four new waves propagating through the inhomogeneous slab (see equ. (40)). It is then to be expected that, in turn, each of these waves will again generate four further waves, this being the result of a second scattering. Obviously, this process will be continued ad infinitum, that is, each of the waves $U_{N,j}$ produced after N successive scatterings will generate four new waves all of which belong to the class of $U_{N+1,j}$ waves that are generated after N+1 scatterings. Starting from the four waves $U_{1,j}$ (j=1,2,3,4) given in (40) we thus would obtain sixteen waves $U_{2,j}$ (j=1,2,..,16), sixty-four waves $U_{3,j}$ (j=1,2,...,64), and so on. However, we infer from the results of section 9 that the actual number of different $U_{1,j}$ waves reduced from 4 to 3 since the wave-number vector is the same for the contributions due to forward modulation and backward modulation respectively. Such reductions also occur for the higher-order scattering contributions, the number of independent $U_{2,j}$ waves thus becoming only 5 instead of 16 (see the next section).

A general theory for the generation of all these new waves after each individual scattering is to be obtained from the recurrence relation (11). For its application we first need a representation of U_N in terms of its composing waves $U_{N,j}$. Since each of these waves originates after a number of successive scatterings from waves of a lower scattering order, the wave-number vector $\vec{k}_{N,j}$ corresponding to $U_{N,j}$ will depend on the wave-number vectors associated with the chain of preceding waves from which $U_{N,j}$ results ultimately. These preceding waves $U_{m,j}$ (m=1,2,..,N-1) are fixed by the wave-number components which constitute the integration variables of the 4m-fold Fourier integral that results from N successive applications of the recurrence relation (11), while introducing the Fourier integrals (23) for each factor δn in the relevant integrand. We thus arrive at the following representaion for U_N:

$$u_N(P,t) = \int d\vec{k}_1 \int d\vec{k}_2 \cdots \int d\vec{k}_N \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \cdots \int_{-\infty}^{\infty} d\omega_N \, G(\vec{k}_1,\omega_1)\, G(\vec{k}_2,\omega_2)\cdots \cdot G(\vec{k}_N,\omega_N) \sum_j c_{N,j}\, e^{i\{\vec{k}_{N,j}\cdot\vec{r}_P-(\omega_0+\omega_1+\cdots+\omega_N)t\}} \tag{48}$$

This formula expresses how the generation of a special N,j wave depends on N integrations over the complete scattering medium though the 3N integrations over the spatial coordinates are replaced here, in the corresponding Fourier space, by those over the components of wave-number vectors $\vec{k}_1,\ldots,\vec{k}_N$. Each vector $\vec{k}_{N,j}$ depends in general on the complete set of variables contained in the combination of the components $\vec{k}_1, \vec{k}_2,\ldots,\vec{k}_N$ and of the frequencies $\omega_1, \omega_2,\ldots,\omega_N$.

Returning to the recurrence relation (11) we observe that the operator in its integrand acts on the quantity $\delta n.\partial^2 U_{N-1}/\partial t^2$ as observed at the point (x_p, y_p, z_q) at the time t. The two factors of this quantity can be represented as follows, introducing (23) for δn, and the second-order time derivative of (48) in order to get $\partial^2 U_{N-1}/\partial t^2$. We thus obtain:

$$\delta n(x_p,y_p,z_q,t) = \int d\vec{k}_N \int_{-\infty}^{\infty} d\omega_N \; G(\vec{k}_N,\omega_N)\, e^{i(k_{Nx}x_p+k_{Ny}y_p+k_{Nz}z_q-\omega_N t)},$$

$$\left(\frac{\partial^2 u_{N-1}}{\partial t^2}\right)_{x_p,y_p,z_q,t} = -\int d\vec{k}_1\cdots\int d\vec{k}_{N-1}\int_{-\infty}^{\infty} d\omega_1\cdots\int_{-\infty}^{\infty} d\omega_{N-1}\, G(\vec{k}_1,\omega_1)\cdots G(\vec{k}_{N-1},\omega_{N-1})\times$$
$$\times(\omega_0+\omega_1+\cdots+\omega_{N-1})^2 e^{-i(\omega_0+\cdots+\omega_{N-1})t}\sum_j c_{N-1,j}\, e^{i\{k_{N-1,j,x}x_p+k_{N-1,j,y}y_p+k_{N-1,j,z}z_q\}}.$$

The exponential entering in the product of these two expressions constitutes the only factor in the integrand of (11) which depends on the variables x_p, y_p, z_q, t. Therefore, the operator $\partial^2/\partial x_p^2 + \partial^2/\partial y_p^2 - (1/c^2)\partial^2/\partial t^2$ in (11) only affects this very exponential so that this operator may be replaced by the following non symbolic quantity:

$$\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\left(\frac{\omega_0+\omega_1+\cdots+\omega_N}{c}\right)^2-\left(k_{Nx}+k_{N-1,j,x}\right)^2-\left(k_{Ny}+k_{N-1,j,y}\right)^2=$$

$$=\left(\frac{\omega_0+\omega_1+\cdots+\omega_N}{c}\right)^2-\left|\left(\vec{k}_N+\vec{k}_{N-1,j}\right)_\perp\right|^2=\beta_{N,j}^2 \text{ say,} \tag{49}$$

where $\perp$ refers to the transverse component (perpendicular to the z-axis) of the vector $\vec{k}_N + \vec{k}_{N-1,j}$ the modulus of which is only relevant.

Working out the substitutions, mentioned here, in the expression (11) we arrive at:

$$u_N(P,t)=\frac{i}{c^2}\int d\vec{k}_1\cdots\int d\vec{k}_N\int d\omega_1\cdots\int d\omega_N\, G(\vec{k}_1,\omega_1)\cdots G(\vec{k}_N,\omega_N)\,(\omega_0+\omega_1+\cdots+\omega_{N-1})^2 \times$$

$$\times e^{-i(\omega_0+\omega_1+\cdots+\omega_N)t}\sum_j C_{N-1,j}\, e^{i\{(k_{Nx}+k_{N-1,j,x})x_p+(k_{Ny}+k_{N-1,j,y})y_p\}}\times$$

$$\times\int_0^D dz_q\,\frac{e^{i\{|z_p-z_q|\beta_{N,j}+(k_{Nz}+k_{N-1,j,z})z_q\}}}{\beta_{N,j}}.$$

The elementary evaluation of the integration over z_q leads to the following relation expressing the sum U_N of all $k_{N,j}$ waves in terms of the preceding $k_{N-1,j}$ waves:

$$u_N(\vec{r},t)=\frac{1}{c^2}\int d\vec{k}_1\cdots\int d\vec{k}_N\int d\omega_1\cdots\int d\omega_N\; G(\vec{k}_1,\omega_1)\cdots G(\vec{k}_N,\omega_N)\;(\omega_0+\omega_1+\cdots+\omega_{N-1})^2\times$$

$$\times e^{-i(\omega_0+\cdots+\omega_N)t}\sum_j \frac{c_{N-1,j}}{\beta_{N,j}}\Bigg[\left\{\frac{1}{k_{Nz}+k_{N-1,j,z}-\beta_{N,j}}-\frac{1}{k_{Nz}+k_{N-1,j,z}+\beta_{N,j}}\right\}e^{i(\vec{k}_N+\vec{k}_{N-1,j})\vec{r}_P}-$$

$$-\frac{e^{i\{(k_{Nx}+k_{N-1,j,x})x_P+(k_{Ny}+k_{N-1,j,y})y_P+\beta_{N,j}z_P\}}}{k_{Nz}+k_{N-1,j,z}-\beta_{N,j}}+\frac{e^{i\{(k_{Nz}+k_{N-1,j,z}+\beta_{N,j})D+(k_{Nx}+k_{N-1,j,x})x_P+(k_{Ny}+k_{N-1,j,y})y_P-\beta_{N,j}z_P\}}}{k_{Nz}+k_{N-1,j,z}+\beta_{N,j}}\Bigg] \tag{50}$$

According to the nomenclature introduced in section 9 the four contributions corresponding to the four terms occurring in the squared brackets of (50) can be interpreted as resulting from N successive scatterings the last of which is due to a "forward modulation" (fm), a "backward modulation" (bm), a "forward diffraction" (fd), and a "backward diffraction" (bd) respectively. In view of the definition of the coefficients c_{Nj} and of the wave-number vectors $\vec{k}_{Nj}$ of the individual waves, as given by (48), the information contained in (50) can be represented as follows:

$$\begin{aligned}
c_{N,\ldots,fm} &= \left(\frac{\omega_0+\omega_1+\cdots+\omega_{N-1}}{c}\right)^2\frac{1}{(k_{Nz}+k_{N-1,j,z}-\beta_{N,j})}\;\frac{c_{N-1,j}}{\beta_{N,j}},\\[2ex]
c_{N,\ldots,bm} &= -\left(\frac{\omega_0+\omega_1+\cdots+\omega_{N-1}}{c}\right)^2\frac{1}{(k_{Nz}+k_{N-1,j,z}+\beta_{N,j})}\;\frac{c_{N-1,j}}{\beta_{N,j}},\\[2ex]
c_{N,\ldots,fd} &= -\left(\frac{\omega_0+\omega_1+\cdots+\omega_{N-1}}{c}\right)^2\frac{1}{(k_{Nz}+k_{N-1,j,z}-\beta_{N,j})}\;\frac{c_{N-1,j}}{\beta_{N,j}},\\[2ex]
c_{N,\ldots,bd} &= \left(\frac{\omega_0+\omega_1+\cdots+\omega_{N-1}}{c}\right)^2\frac{e^{i(k_{Nz}+k_{N-1,j,z}+\beta_{N,j})D}}{(k_{Nz}+k_{N-1,j,z}+\beta_{N,j})}\;\frac{c_{N-1,j}}{\beta_{N,j}},
\end{aligned} \tag{51}$$

$$\vec{k}_{Nj,\ldots,fm} = \vec{k}_{Nj,\ldots,bm} = \vec{k}_N + \vec{k}_{N-1,j} \quad ,$$
$$\vec{k}_{Nj,\ldots,fd} = \left(\vec{k}_N + \vec{k}_{N-1,j}\right)_\perp + \beta_N \, \vec{u}_z ,$$
$$\vec{k}_{Nj,\ldots,bd} = \left(\vec{k}_N + \vec{k}_{N-1,j}\right)_\perp - \beta_N \, \vec{u}_z . \tag{51a}$$

The subscript j is an abbreviation here for the labels left open on the left-hand sides and which have to indicate the types of scattering (fm,bm,fd or bd) associated with the first N-1 scatterings.

The higher-order terms of the Born series could be computed in sucession with the aid of the recurrence relations represented by (51) and (51a), when startinf from u_1 (see the next section). We yet observe how the diffraction determining the final N-th scattering in the case of the last two waves of (51a) involves the relation:

$$\left| \vec{k}_{Nj,\ldots,d} \right|^2 = \left(\frac{\omega_0 + \omega_1 + \cdots + \omega_N}{c}\right)^2 .$$

13. <u>Waves contained in the second-order term of the Born series</u>

We first observe that an application of the general recurrence relations (51) and (51a) for N = 1 yield the scattering contribution u_1, as it should, if we identify u_0 with the normalized primary wave (for which $c_0 = 1$); for this verification we have to remember how the coefficients c_{Nj} and the wave-number vectors $\vec{k}_{Nj}$ are fixed by (48), while the definitions (38) and the relation $\omega_0 = k_0 c$ are also to be taken into account. The first-order scattering contributions are then fully represented by:

$$c_{1;fm} = \frac{k_0^2}{\alpha_1^{\uparrow}\beta_1}, \qquad c_{1;bm} = -\frac{k_0^2}{\alpha_1^{\downarrow}\beta_1},$$

$$c_{1;fd} = -\frac{k_0^2}{\alpha_1^{\uparrow}\beta_1}, \qquad c_{1;bd} = \frac{k_0^2 e^{iD\alpha_1^{\downarrow}}}{\alpha_1^{\downarrow}\beta_1},$$

$$\vec{k}_{1;fm} = \vec{k}_{1;bm} = \vec{k}_0 + \vec{k}_1, \tag{52}$$

$$\vec{k}_{1;fd} = (\vec{k}_0 + \vec{k}_1)_{\perp} + \beta_1 \vec{u}_x,$$

$$\vec{k}_{1;bd} = (\vec{k}_0 + \vec{k}_1)_{\perp} - \beta_1 \vec{u}_x.$$

With the aid of these expressions we can evaluate the recurrence formulas (51) and (51a) for N = 2. As an example we consider the determination of $c_{2;\,bd,\,fm}$ and $\vec{k}_{2;\,bd,\,fm}$ which quantities fix the amplitude and the wave-number vector of the contribution originating from a first scattering of the "back diffraction" type, followed by a second scattering of the "forward modulation" type. A straightforward application of (51) and (51a) first yields:

$$c_{2;\,bd,\,fm} = \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{1}{k_{2x} + k_{1,\,bd,\,x} - \beta_{2,\,bd}} \, \frac{c_{1;bd}}{\beta_{2;bd}},$$

$$\vec{k}_{2;\,bd,\,fm} = \vec{k}_2 + \vec{k}_{1,\,bd} \quad ,$$

in which, in view of (49),

$$\beta_{2;bd} = \sqrt{\left(\frac{\omega_0+\omega_1+\omega_2}{c}\right)^2 - \left(\vec{k}_2 + \vec{k}_{1;bd}\right)_{\perp}^2}\,.$$

With the aid of (52) these expressions can next be transformed into:

$$c_{2;\,bd,\,fm} = \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{e^{iD\alpha_1^{\downarrow}}}{(k_{2x} - \beta_1 - \beta_2)} \, \frac{k_0^2}{\alpha_1^{\downarrow}\beta_1\beta_2},$$

$$\vec{k}_{2;\,bd,\,fm} = \left(\vec{k}_0+\vec{k}_1+\vec{k}_2\right)_\perp + \left(k_{2x}-\beta_1\right)\vec{u}_x = \vec{K}_0+\vec{K}_1+\vec{K}_2-\alpha_1^{\downarrow}\vec{u}_x,$$

if we yet introduce:

$$\beta_2 = \sqrt{\left(\frac{\omega_0+\omega_1+\omega_2}{c}\right)^2 - \left(\vec{k}_0+\vec{k}_1+\vec{k}_2\right)_\perp^2} .$$

All other contributions to the second-order Born term u_2 can be evaluated in the same way. It is then convenient to use the further abbreviations:

$$\overset{\downarrow}{\alpha}_2 = k_{0x}+k_{1x}+k_{2x}-\beta_2 ,$$

$$\alpha_2^{\downarrow} = k_{0x}+k_{1x}+k_{2x}+\beta_2 .$$

We thus get the following representations in which all contributions associated with an identical wave-number vector have been put together with curly braces:

$$\left.\begin{aligned}
C_{2;\,fm,\,fm} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}\alpha_2^{\uparrow}\beta_1\beta_2},\\
C_{2;\,fm,\,bm} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}\alpha_2^{\downarrow}\beta_1\beta_2},\\
C_{2;\,bm,\,fm} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\downarrow}\alpha_2^{\uparrow}\beta_1\beta_2}\\
C_{2;\,bm,\,bm} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\downarrow}\alpha_2^{\downarrow}\beta_1\beta_2}
\end{aligned}\right\}\quad \vec{k} = \vec{k}_0+\vec{k}_1+\vec{k}_2 ,$$

$$\left.\begin{aligned}
C_{2;\,fm,\,fd} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}\alpha_2^{\uparrow}\beta_1\beta_2}\\
C_{2;\,bm,\,fd} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\downarrow}\alpha_2^{\uparrow}\beta_1\beta_2}\\
C_{2;\,fd,\,fd} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}\left(\alpha_2^{\uparrow}-\alpha_1^{\uparrow}\right)\beta_1\beta_2}\\
C_{2;\,bd,\,fd} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\downarrow}\left(\alpha_2^{\uparrow}-\alpha_1^{\downarrow}\right)\beta_1\beta_2}
\end{aligned}\right\}\quad \vec{k} = \left(\vec{k}_0+\vec{k}_1+\vec{k}_2\right)_\perp + \beta_2\vec{u}_x ,$$

$$\left.\begin{aligned}
\zeta_{2;\,fm,\,bd} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD\alpha_2^{\downarrow}}}{\alpha_1^{\uparrow}\alpha_2^{\downarrow}\beta_1\beta_2} \\
\zeta_{2;\,bm,\,bd} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD\alpha_2^{\downarrow}}}{\alpha_1^{\downarrow}\alpha_2^{\downarrow}\beta_1\beta_2} \\
\zeta_{2;\,fd,\,bd} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD(\alpha_2^{\downarrow}-\alpha_1^{\uparrow})}}{\alpha_1^{\uparrow}(\alpha_2^{\downarrow}-\alpha_1^{\uparrow})\beta_1\beta_2} \\
\zeta_{2;\,bd,\,bd} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD\alpha_2^{\downarrow}}}{\alpha_1^{\downarrow}(\alpha_2^{\downarrow}-\alpha_1^{\downarrow})\beta_1\beta_2}
\end{aligned}\right\}\quad \vec{k} = \left(\vec{k}_0+\vec{k}_1+\vec{k}_2\right)_{\perp} - \beta_2\,\vec{u}_z\,,$$

$$\left.\begin{aligned}
\zeta_{2;\,fd,\,fm} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}(\alpha_2^{\uparrow}-\alpha_1^{\uparrow})\beta_1\beta_2} \\
\zeta_{2;\,fd,\,bm} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2}{\alpha_1^{\uparrow}(\alpha_2^{\downarrow}-\alpha_1^{\uparrow})\beta_1\beta_2}
\end{aligned}\right\}\quad \vec{k} = \vec{k}_0+\vec{k}_1+\vec{k}_2-\alpha_1^{\uparrow}\,\vec{u}_z\,,$$

$$\left.\begin{aligned}
\zeta_{2;\,bd,\,fm} &= \left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD\alpha_1^{\downarrow}}}{\alpha_1^{\downarrow}(\alpha_2^{\uparrow}-\alpha_1^{\downarrow})\beta_1\beta_2} \\
\zeta_{2;\,bd,\,bm} &= -\left(\frac{\omega_0+\omega_1}{c}\right)^2 \frac{k_0^2\, e^{iD\alpha_1^{\downarrow}}}{\alpha_1^{\downarrow}(\alpha_2^{\downarrow}-\alpha_1^{\downarrow})\beta_1\beta_2}
\end{aligned}\right\}\quad \vec{k} = \vec{k}_0+\vec{k}_1+\vec{k}_2-\alpha_1^{\downarrow}\,\vec{u}_z\,.$$

We thus infer in particular that, instead of sixteen second-order waves, only five types with different k vectors remain.

14. The Born approximation for a general infinite scatterer

The occurence of many types of waves, as described in the last three sections, is first of all a consequence of the finite size of the scattering body in the z-direction. As remarked at the end of section 11, the "diffracted waves" disappear in a scatterer extending up to infinity in all directions. This suggests to consider those contributions of the first few Born terms that remain when leaving out all diffracted waves. Assuming β_1 complex throughout, we find from (49) applying the identity:

$$\overset{\uparrow}{\alpha}_1 \overset{\downarrow}{\alpha}_1 = \left| \vec{k}_0 + \vec{k} \right|^2 - \left(\frac{\omega_0 + \omega}{c} \right)^2,$$

the following representation for the first Born term:

$$u_1(P,t) = 2k_0^2 \int d\vec{k} \int d\omega \, \frac{g_4(\vec{k},\omega)}{\left|\vec{k}_0 + \vec{k}\right|^2 - \left(\frac{\omega_0+\omega}{c}\right)^2} \, e^{i\left\{(\vec{k}_0+\vec{k})\vec{r}_p - (\omega_0+\omega)t\right\}}$$

The question arises whether such an expression might also hold for finite scatterers, thus as those considered sofar which only occupy the space $0 < z < D$. In fact, the latter can be considered as a special case of an infinite one for which δn happens to vanish outside the mentioned space. From what follows the above formula proves to be applicable quite generally indeed, provided that the integration path for ω is chosen properly; we remind that the integration over $\vec{k}$ concerns that of each of its components k_x, k_y, k_z along the corresponding real axis.

For the sake of generality we shall investigate simultaneously the two integrals

$$J_{\pm} = 2k_0^2 \int d\vec{k} \int_{-\infty \pm i0}^{\infty \pm i0} d\omega \, \frac{g_4(\vec{k},\omega)}{\left|\vec{k}_0+\vec{k}\right|^2 - \left(\frac{\omega_0+\omega}{c}\right)^2} \, e^{i\left\{(\vec{k}_0+\vec{k})\vec{r}_p - (\omega_0+\omega)t\right\}}, \qquad (53)$$

with an ω path just above or just below the real axis respectively.

After applying a splitting into partial fractions, while substituting the inversion of the four-dimensional Fourier integral (23) according to:

$$\mathcal{G}_4(\vec{k},\omega) - \frac{1}{(2\pi)^4}\int d\vec{\xi}\int_{-\infty}^{\infty} d\tau\ \delta n(\vec{\xi},\tau)\, e^{-i(\vec{k}\vec{\xi}-\omega\tau)}, \tag{54}$$

we can reduce (53) to the following formula, yet inverting the orders of integration,

$$\mathcal{J}_{\pm} = \frac{k_0^2 c\, e^{-i\omega_0 t}}{(2\pi)^4}\int d\vec{k}\,\frac{e^{i(\vec{k}_0+\vec{k})\vec{r}_P}}{|\vec{k}_0+\vec{k}|}\int d\vec{\xi}\int_{-\infty}^{\infty} d\tau\ \delta n(\vec{\xi},\tau)\, e^{-i\vec{k}\vec{\xi}}\ \times$$
$$\times\int_{-\infty\pm i0}^{\infty\pm i0} d\omega\, e^{-i\omega(t-\tau)}\left\{\frac{1}{\omega+\omega_0+c|\vec{k}_0+\vec{k}|}-\frac{1}{\omega+\omega_0-c|\vec{k}_0+\vec{k}|}\right\}. \tag{55}$$

The integration path of the two ω contributions can be closed by adding either an integration along the upper half of the infinite circle in the ω-plane when $t-\tau<0$, or along the corresponding lower half of this circle when $t-\tau>0$. In the former case the contour for J+ does not enclose any of the two poles at $\omega = -\omega_0 \mp c\,|K_0+K|$, and the integral vanishes, the poles in question being situated on the real axis (we assume ω_0 as real here); on the other hand, the contour for J- then encloses both poles which leads to an addition of the associated residues. In the case of $t-\tau>0$, on the contrary, the contour for J+ encloses both poles, but that of J- involves a vanishing integral. The corresponding elementary evaluations of the complete ω-integral in (59) in the four cases of J+ and J-, and of $t-\tau<0$ and $t-\tau>0$, can be summarized by the single expression:

$$\int_{-\infty\pm i0}^{\infty\pm i0} d\omega\, e^{-i\omega(t-\tau)}\left\{\frac{1}{\omega+\omega_0+c|\vec{k}_0+\vec{k}|}-\frac{1}{\omega+\omega_0-c|\vec{k}_0+\vec{k}|}\right\}=$$
$$= \pm 4\pi\, e^{i\omega_0(t-\tau)}\sin\{c|\vec{k}_0+\vec{k}|(t-\tau)\}\ U\{\pm(t-\tau)\},$$

U(x) being Heaviside's unit function.

A substitution of this result into (55) yields, after a further inversion of the order of integration,

$$\mathcal{J}_{\pm} = \pm \frac{k_o^2 c}{4\pi^3} \int d\vec{\xi} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega_o\tau}\, \delta n(\vec{\xi},\tau)\; U\{\pm(t-\tau)\} \times$$

$$\times \int d\vec{k}\, \frac{e^{i(\vec{k}_o+\vec{k})\vec{r}_p}}{|\vec{k}_o+\vec{k}|}\, e^{-i\vec{k}\vec{\xi}} \sin\{c\,|\vec{k}_o+\vec{k}|(t-\tau)\}. \tag{56}$$

The $\vec{k}$ integral, K say, can be represented as follows by introducing the new integration vector $\vec{s} = \vec{K}_o+\vec{K}$, with length $S = |\vec{K}_o+\vec{K}|$,

$$K = e^{i\vec{k}_o\vec{\xi}} \int d\vec{s}\, e^{i(\vec{r}_p-\vec{\xi})\vec{s}}\, \frac{\sin\{c(t-\tau)s\}}{s}.$$

This latter integral is of the type I as investigated in the appendix, and thus proves to result in:

$$K = 4\pi\, \frac{e^{i\vec{k}_o\vec{\xi}}}{|\vec{r}_p-\vec{\xi}|} \int_0^{\infty} ds\, \sin\left(|\vec{r}_p-\vec{\xi}|s\right)\; \sin\{c(t-\tau)s\}.$$

In turn, this new integral can be transformed into two integrals of the form

$$\int_0^{\infty} d\lambda\, \cos(cA\lambda) = \pi\, \delta(cA) = \frac{\pi}{c}\, \delta(A),$$

thus yielding:

$$K = \frac{2\pi}{c^2}\, \frac{e^{i\vec{k}_o\vec{\xi}}}{|\vec{r}_p-\vec{\xi}|} \left\{ \delta\left(t-\tau-\frac{|\vec{r}_p-\vec{\xi}|}{c}\right) - \delta\left(t-\tau+\frac{|\vec{r}_p-\vec{\xi}|}{c}\right) \right\}.$$

The substitution of this expression into (56) first gives:

$$\mathcal{J}_{\pm} = \pm \frac{k_o^2}{2\pi} \int d\vec{\xi}\, \frac{e^{i\vec{k}_o\vec{\xi}}}{|\vec{r}_p-\vec{\xi}|} \int d\tau\, \delta n(\vec{\xi},\tau)\, e^{-i\omega_o\tau}\, U\{\pm(t-\tau)\} \left\{ \delta\left(\tau-t+\frac{|\vec{r}_p-\vec{\xi}|}{c}\right) - \delta\left(\tau-t-\frac{|\vec{r}_p-\vec{\xi}|}{c}\right) \right\},$$

but, applying the properties

$$h(x)\ \delta(x-x_0) = h(x_0)\ \delta(x-x_0),$$

$$U\left(\frac{|\vec{r}_P-\vec{\xi}|}{c}\right)=1 \quad , \quad U\left(-\frac{|\vec{r}_P-\vec{\xi}|}{c}\right)=-1,$$

this can further be reduced to:

$$J_\pm = \frac{k_0^2}{2\pi}\int d\vec{\xi}\,\frac{e^{i\vec{k}_0\vec{\xi}}}{|\vec{r}_P-\vec{\xi}|}\int_{-\infty}^{\infty} d\tau\ \delta n(\vec{\xi},\tau)\,e^{-i\omega_0\tau}\,\delta\left(\tau-t\pm\frac{|\vec{r}_P-\vec{\xi}|}{c}\right) =$$

$$= \frac{k_0^2}{2\pi}e^{-i\omega_0 t}\int d\vec{\xi}\,\frac{e^{i(\vec{k}_0\vec{\xi}\pm k_0|\vec{r}_P-\vec{\xi}|)}}{|\vec{r}_P-\vec{\xi}|}\ \delta n\left(\vec{\xi},t\mp\frac{|\vec{r}_P-\vec{\xi}|}{c}\right).$$

Next introducing, as in section 3, the integration point Q and the observation point P with the components of $\vec{\xi}$ and $\vec{r}$, respectively, as coordinates, we arrive at:

$$J_\pm = \frac{k_0^2}{2\pi}e^{-i\omega_0 t}\int \frac{d\tau_Q}{PQ}\,e^{i(\vec{k}_0\vec{r}_Q\pm k_0\,PQ)}\ \delta n\left(Q,t\mp\frac{PQ}{c}\right). \tag{57}$$

Hence, J+ proves to be identical with the expression (25) for the Born approximation u_1. In other words, according to (53), this approximation can now be represented by

$$u_1(P,t) = 2k_0^2\int d\vec{k}\int_{-\infty+i0}^{\infty+i0} d\omega\,\frac{G_4(\vec{k},\omega)}{|\vec{k}_0+\vec{k}|^2-\left(\frac{\omega_0+\omega}{c}\right)^2}\,e^{i\{(\vec{k}_0+\vec{k})\vec{r}_P-(\omega_0+\omega)t\}} . \tag{58}$$

We here infer that the corresponding integral for J- would represent u_1 if Maxwell's equations should be solved with the aid of advanced potentials instead of retarded potentials, because the former are in accordance with the occurrence of the argument t + PQ/c instead of t - PQ/c in the δ function of (57). Finally we observe that the difference J+ - J-, that is the difference of the values of u_1 when associated with retarded and advanced potentials respectively, amounts to the sum of the residues belonging to the poles $\omega = -\omega_o \pm c|\vec{K}_o+\vec{K}|$ in the above integrals. Of course, it is also possible to evaluate the residues at each of these poles individually. This leads to the following result:

Residue at $\omega = -\omega_o \mp c\,|\vec{k}_o+\vec{k}|$ of

$$2k_o^2\int d\vec{k}\int d\omega\,\frac{g_4(\vec{k},\omega)}{|\vec{k}_o+\vec{k}|^2-\left(\frac{\omega_o+\omega}{c}\right)^2}\,e^{i\{(\vec{k}_o+\vec{k})\vec{r}_P-(\omega_o+\omega)t\}}:$$

$$\frac{1}{2}\left(J_- - J_+\right)\mp\frac{ik_o^2}{2\pi^2c}\,P\int d\vec{\xi}\int d\tau\,\frac{\delta n(\vec{\xi},\tau)\,e^{i(\vec{k}_o\vec{\xi}-\omega_o\tau)}}{(t-\tau)^2-\frac{|\vec{r}_P-\vec{\xi}|^2}{c^2}},$$

P referring to a principal value. From the difference of these two expressions we would obtain that the integration along a lemmiscate-shaped curve in the ω plane which encircles both poles in opposite direction will lead to the latter double integral. This integral constitutes a superposition of the elementary solutions

$$\frac{1}{t^2-\frac{r^2}{c^2}}$$

of the wave equation for vacuum the individual contributions of which result after a shift τ in time and a displacement $\vec{\xi}$ of the origin at $r = 0$. Apparently such solutions, though connected with the existence of the characteristics of the hyperbolic differential equation, never play a dominant role in propagation theories.

15. The higher-order Born terms for a general infinite scatterer

The integral (58) for the first-order Born term was originally obtained by leaving out the "diffracted" waves in (40). Similarly, we may look for the second-order term u_2 as investigated in section 13, omitting then all terms which are either completely or partly of the "diffracted" type. We thus have to add all "full modulated" contributions, that is those with the wave-number vector $\vec{k}_o + \vec{k}_1 + \vec{k}_2$. The necessary summation over the coefficients of the four contributions of this type is facilitated by an application of the following elementary identity:

$$\frac{1}{\alpha_1^{\uparrow}\alpha_2^{\uparrow}} - \frac{1}{\alpha_1^{\uparrow}\alpha_2^{\downarrow}} - \frac{1}{\alpha_1^{\downarrow}\alpha_2^{\uparrow}} + \frac{1}{\alpha_1^{\downarrow}\alpha_2^{\downarrow}} =$$

$$= \frac{4\beta_1\beta_2}{\left\{|\vec{k}_0+\vec{k}_1|^2-\left(\frac{\omega_0+\omega_1}{c}\right)^2\right\}\left\{|\vec{k}_0+\vec{k}_1+\vec{k}_2|^2-\left(\frac{\omega_0+\omega_1+\omega_2}{c}\right)^2\right\}} .$$

The summation over the corresponding waves leads to the formula:

$$u_2(P,t) = 4k_0^2\int d\vec{k}_1\int d\vec{k}_2\int d\omega_1\int d\omega_2\,\frac{G_4(\vec{k}_1,\omega_1)\;G_4(\vec{k}_2,\omega_2)\;\left(\frac{\omega_0+\omega_1}{c}\right)^2}{\left\{|\vec{k}_0+\vec{k}_1|^2-\left(\frac{\omega_0+\omega_1}{c}\right)^2\right\}\left\{|\vec{k}_0+\vec{k}_1+\vec{k}_2|^2-\left(\frac{\omega_0+\omega_1+\omega_2}{c}\right)^2\right\}}\times$$

$$\times\, e^{i\{(\vec{k}_0+\vec{k}_1+\vec{k}_2)\vec{r}_P-(\omega_0+\omega_1+\omega_2)t\}} .$$

A comparison of this expression with (58) suggests that the following formula might represent all terms of the Born series:

$$u_N(P,t) = 2^N k_0^2\int d\vec{k}_1\ldots d\vec{k}_N\int_{-\infty+i0}^{\infty+i0} d\omega_1\ldots d\omega_N\;G_4(\vec{k}_1,\omega_1)\ldots G_4(\vec{k}_N,\omega_N)\times$$

$$\times\frac{\left\{\left(\frac{\omega_0+\omega_1}{c}\right)\left(\frac{\omega_0+\omega_1+\omega_2}{c}\right)\ldots\left(\frac{\omega_0+\omega_1+\ldots+\omega_{N-1}}{c}\right)\right\}^2}{\left\{|\vec{k}_0+\vec{k}_1|^2-\left(\frac{\omega_0+\omega_1}{c}\right)^2\right\}\ldots\left\{|\vec{k}_0+\vec{k}_1+\ldots+\vec{k}_N|^2-\left(\frac{\omega_0+\omega_1+\ldots+\omega_N}{c}\right)^2\right\}}\times$$

$$\times\, e^{i\{(\vec{k}_0+\vec{k}_1+\ldots+\vec{k}_N)\vec{r}_P-(\omega_0+\omega_1+\ldots+\omega_N)t\}} , \qquad (59)$$

where the integration path for all frequencies $\omega_1,\omega_2,\ldots,\omega_N$ has been chosen in accordance with the single one occurring in (58).

In order to verify the general validity of (59) it suffices to give a proof by induction, showing how its validity for a special u_N involves that for u_{N+1}. The proof is then complete since we know already its correctness for $N = 1$. Therefore, we now assume the validity of (59) for a given N value and then apply the recurrence relation (11) in a version holding for a general infinite scatterer not restricted to the domain $0 < z < D$.

Obviously this version reads as follows for N replaced by N+1:

$$u_{N+1}(P,t) = -\frac{i}{c^2}\int_{-\infty}^{\infty} dz_q \, \frac{e^{i|z_p - z_q|\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}}{\sqrt{\frac{\partial^2}{\partial x_p^2}+\frac{\partial^2}{\partial y_p^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}} \left(\delta n \frac{\partial^2 u_N}{\partial t^2}\right)_{x_p, y_p, z_q, t}$$

$$(\mathrm{Im}\sqrt{} \geqslant 0). \tag{60}$$

We may now substitute the expression for $\partial^2 u_N/\partial t^2$ that follows from (59); it has to be multiplied by δn (x_p, y_p, z_q, t) for which we take the following representation for its four-dimensional Fourier integral:

$$\delta n(x_p, y_p, z_q, t) = \int d\vec{k}_{N+1} \int_{-\infty+i0}^{\infty+i0} d\omega_{N+1} \, G_4(\vec{k}_{N+1}, \omega_{N+1}) \, e^{i(k_{N+1,x} x_p + k_{N+1,y} y_p + k_{N+1,z} z_q - \omega_{N+1} t)}.$$

We thus obtain:

$$\left(\delta n \frac{\partial^2 u_N}{\partial t^2}\right)_{x_p, y_p, z_q, t} = -2^N k_0^2 c^2 \int d\vec{k}_1 \ldots d\vec{k}_{N+1} \int_{-\infty+i0}^{\infty+i0} d\omega_1 \ldots d\omega_{N+1} \times$$

$$\times \frac{G_4(\vec{k}_1, \omega_1) \ldots G_4(\vec{k}_{N+1}, \omega_{N+1}) \left\{\left(\frac{\omega_0+\omega_1}{c}\right) \ldots \left(\frac{\omega_0+\ldots+\omega_N}{c}\right)\right\}^2}{\left\{|\vec{k}_0+\vec{k}_1|^2 - \left(\frac{\omega_0+\omega_1}{c}\right)^2\right\} \ldots \left\{|\vec{k}_0+\ldots+\vec{k}_N|^2 - \left(\frac{\omega_0+\ldots+\omega_N}{c}\right)^2\right\}} \times$$

$$\times e^{i\{(k_{0x}+\ldots+k_{N+1,x})x_p + (k_{0y}+\ldots+k_{N+1,y})y_p + (k_{0z}+\ldots+k_{N+1,z})z_q - (\omega_0+\ldots+\omega_{N+1})t\}}. \tag{61}$$

The effect of the operator given by the square root in (60) amounts to the equivalent quantity

$$\sqrt{} = \sqrt{-(k_{0x}+\cdots+k_{N+1,x})^2-(k_{0y}+\cdots+k_{N+1,y})^2+\left(\frac{\omega_0+\cdots+\omega_{N+1}}{c}\right)^2},$$

$$(\mathrm{Im}\sqrt{}>0)$$

where Im $\sqrt{}$ will always differ from zero since all frequency variables $\omega_1,\omega_2,\ldots,\omega_{N+1}$ were assumed with a positive imaginary part ε before passing to the limit $\varepsilon \to 0$ in the integrals

$$\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\omega \ldots$$

In fact, whereas the radicand is always real for $\varepsilon = 0$, it now becomes complex throughout which enables the possibility of a square root with a positive imaginary part. The substitution of (61) into (60) thereupon yields:

$$u_{N+1}(P,t) = i\,2^N k_0^2 \int d\vec{k}_1 \ldots d\vec{k}_{N+1} \int_{-\infty+i0}^{\infty+i0} d\omega_1 \ldots d\omega_{N+1}\, \mathcal{G}_y(\vec{k}_1,\omega_1)\ldots \mathcal{G}_y(\vec{k}_{N+1},\omega_{N+1}) \times$$

$$\times \frac{\left\{\left(\frac{\omega_0+\omega_1}{c}\right)\ldots\left(\frac{\omega_0+\cdots+\omega_N}{c}\right)\right\}^2 e^{i\{(k_{0x}+\cdots+k_{N+1,x})x_p+(k_{0y}+\cdots+k_{N+1,y})y_p-(\omega_0+\cdots+\omega_{N+1})t\}}}{\left\{|\vec{k}_0+\vec{k}_1|^2-\left(\frac{\omega_0+\omega_1}{c}\right)^2\right\}\cdots\cdot\left\{|\vec{k}_0+\cdots+\vec{k}_N|^2-\left(\frac{\omega_0+\cdots+\omega_N}{c}\right)^2\right\}} \times$$

$$\times \int_{-\infty}^{\infty} dx_q \frac{e^{i\left\{|x_p-x_q|\sqrt{\left(\frac{\omega_0+\cdots+\omega_{N+1}}{c}\right)^2-(k_{0x}+\cdots+k_{N+1,x})^2-(k_{0y}+\cdots+k_{N+1,y})^2}+(k_{0x}+\cdots+k_{N+1,x})x_q\right\}}}{\sqrt{\left(\frac{\omega_0+\cdots+\omega_{N+1}}{c}\right)^2-(k_{0x}+\cdots+k_{N+1,x})^2-(k_{0y}+\cdots+k_{N+1,y})^2}}.$$

$$(\mathrm{Im}\sqrt{}>0)$$

(62)

The integral over z_q (without the constant denominator) can be split into its forward and backward scattering parts, viz. $z_q < z_p$ and $z_q > z_p$. With the aid of the substitutions $\zeta = z_p - z_q$ and $\zeta = z_q - z_p$ respectively, the z_q integral, without the denominator, proves to reduce to the following sum:

$$e^{i(k_{0z}+\cdots+k_{N+1,z})z_p}\left\{\int_0^\infty d\zeta\, e^{i(\sqrt{}-k_{0z}-\cdots-k_{N+1,z})\zeta}+\int_0^\infty d\zeta\, e^{i(\sqrt{}-k_{0z}-\cdots-k_{N+1,z})\zeta}\right\}.$$

The new integrals are extremely simple since they converge at the upper limit in view of the negative sign of the real part of $i\sqrt{}$, and therefore also of the ζ coefficient in both exponents. The z_q integral in (62), including its constant denominator, can thus be reduced to:

$$\frac{e^{i(k_{0z}+\cdots+k_{N+1,z})z_p}}{\sqrt{}}\left\{\frac{-1}{i(\sqrt{}-k_{0z}-\cdots-k_{N+1,z})}+\frac{-1}{i(\sqrt{}+k_{0z}+\cdots+k_{N+1,z})}\right\}=$$

$$=\frac{2i\,e^{i(k_{0z}+\cdots+k_{N+1,z})z_p}}{(\sqrt{})^2-(k_{0z}+\cdots+k_{N+1,z})^2}=$$

$$=\frac{2i\,e^{i(k_{0z}+\cdots+k_{N+1,z})z_p}}{\left(\frac{\omega_0+\cdots+\omega_{N+1}}{c}\right)^2-\left|\vec{k}_0+\cdots+\vec{k}_{N+1}\right|^2}.$$

The resulting expression for (62) itself then proves to be equal to (59) for N replaced by N+1. This completes the proof of the very general representation (59).

16. Derivation of the various waves inside the scatterer from the expression for an infinite one

Apparently the formula (59) for this expression reveals only a single type of waves, namely with the wave-number vector $\vec{k}_o+\ldots+\vec{k}_N$ and the frequency $\omega_o+\ldots+\omega_N$, such in contrast with the variety of waves discussed in the sections 10 and 13 for u_1 and u_2. It is true that these latter refer to a finite scatterer, whereas (59) has been derived for an infinite scatterer. On the other hand, any finite scatterer can also be considered as an infinite one since the finite size can be accounted for in the Fourier expression (23) for δn in which G_4 then has to be such to yield vanishing δn values for $z < 0$ and $z > D$. The question arises whether this property can be made clear by a convenient representation of G_4. This proves to be possible indeed by Fourier transforming the relevant relation

$$\delta n(x,y,z,t) = \delta n(x,y,z,t)\left\{U(z) - U(z-D)\right\},$$

$U(x)$ (unity for $x > 0$, vanishing for $x < 0$) being again Heaviside's unit function. Here we can apply, for the variable z, the theorem that the Fourier transform of a product of two functions equals the convolution of the Fourier transforms of each of its factors. The evaluation of this convolution in the present case leads to the formula:

$$G_4(k_x,k_y,k_z,\omega) = \int_{-\infty}^{\infty} dk_z'\; G_4(k_x,k_y,k_z',\omega)\,\frac{1-e^{-iD(k_z-k_z')}}{2\pi i(k_z-k_z')}. \tag{63}$$

This expression could be substituted in (59) for each of its factors $G_4(\vec{K}_j,\omega_j)$; the integration, thereafter, with respect to the wave-number components k_{zj} would be elementary, though tedious. We thus could recover the many types of waves such as derived before for u_1 and u_2. This will be verified below for the case of u_1.

The expression (58) proves to be equivalent with the following one when using the parameters $\overset{\uparrow}{\alpha}_1(\vec{k},\omega)$, $\overset{\downarrow}{\alpha}_1(\vec{k},\omega)$ and $\beta_1(k_x,k_y,\omega)$ introduced in section 10:

$$u_1(P,t) = k_0^2 \int d\vec{k} \int_{-\infty+i0}^{\infty+i0} d\omega \, \frac{G_y(\vec{k},\omega)}{\beta_1} \left(\frac{1}{\overset{\uparrow}{\alpha}_1} - \frac{1}{\overset{\downarrow}{\alpha}_1} \right) e^{i\{(\vec{k}_0+\vec{k})\vec{r}_P - (\omega_0+\omega)t\}} .$$

After substitution of (63), while inverting the order of integration with respect to k_z', we arrive at:

$$u_1(P,t) =$$

$$\frac{i k_0^2}{2\pi} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z' \int_{-\infty+i0}^{\infty+i0} d\omega \, G_y(k_x,k_y,k_z',\omega) \frac{e^{i\{(k_{0x}+k_x)x_P + (k_{0y}+k_y)y_P - (\omega_0+\omega)t\}}}{\beta_1} J, \quad (64)$$

in which J is short for the integral over k_z, viz.

$$J \equiv \int_{-\infty}^{\infty} dk_z \, e^{i(k_{0z}+k_z)z_P} \left\{ \frac{1}{k_{0z}+k_z-\beta_1} - \frac{1}{k_{0z}+k_z+\beta_1} \right\} \frac{\{e^{-iD(k_z-k_z')} - 1\}}{(k_z-k_z')} .$$

We pass to the new integration variable $s = k_{oz} + k_z$ and determine the partial fractions for each of the four factors obtained when working out the integrand. This leads to the alternative form:

$$J = \frac{1}{\overset{\uparrow}{\alpha}{}_1'} \left[e^{iD(k_{0z}+k_z')} \left\{ -\int_{-\infty}^{\infty} ds \, \frac{e^{-i(D-z_P)s}}{s-\beta_1} + \int_{-\infty}^{\infty} ds \, \frac{e^{-i(D-z_P)s}}{s-k_{0z}-k_z'} \right\} + \int_{-\infty}^{\infty} ds \, \frac{e^{iz_P s}}{s-\beta_1} - \int_{-\infty}^{\infty} ds \, \frac{e^{iz_P s}}{s-k_{0z}-k_z'} \right] +$$

$$+ \frac{1}{\overset{\downarrow}{\alpha}{}_1'} \left[e^{iD(k_{0z}+k_z')} \left\{ \int_{-\infty}^{\infty} ds \, \frac{e^{-i(D-z_P)s}}{s+\beta_1} - \int_{-\infty}^{\infty} ds \, \frac{e^{-i(D-z_P)s}}{s-k_{0z}-k_z'} \right\} - \int_{-\infty}^{\infty} ds \, \frac{e^{iz_P s}}{s+\beta_1} + \int_{-\infty}^{\infty} ds \, \frac{e^{iz_P s}}{s-k_{0z}-k_z'} \right] , \quad (65)$$

in which $\overset{\uparrow}{\alpha}{}_1'$ and $\overset{\downarrow}{\alpha}{}_1'$ refer to k_z' instead of k_z so that, e.g.,

$$\overset{\uparrow}{\alpha}{}_1' = k_{0z} + k_z' - \beta_1(k_x,k_y).$$

All the integrals entering here can be evaluated by closing the integration path either by the upper or by the lower half of the infinite circle of the complex S-plane. Moreover, we again use the fact that $\omega_o + \omega$ is complex with infinitesimal positive imaginary part (ω_o also being provided with such an imaginary part) so that the radicant of

$$\beta_1 = \sqrt{\left(\frac{\omega_0+\omega}{c}\right)^2 - (k_{oz}+k_z)^2 - (k_{oy}+k_y)^2}$$

will have a positive imaginary part as well, like β_1 itself. The poles at $S = K_{oz} + K'_z$ are only apparent since they merely result from the mentioned splitting into partial fractions; therefore, it is allowed to take the principal values at these poles. The six different integrals then reduce to complete or half residues at the poles $S = \beta_1$, $-\beta_1$, $k_{oz} + k'_z$, or just vanish, depending on whether or not the poles β_1 and $-\beta_1$ are enclosed by the contour resulting from the mentioned closing of the integration path. We thus find, taking into account that $D - Z_p$ and z_p are positive inside the scatterer,

$$\int_{-\infty}^{\infty} ds \frac{e^{-i(D-z_p)s}}{s-\beta_1} = 0 \quad ; \quad \int_{-\infty}^{\infty} ds \frac{e^{-i(D-z_p)s}}{s-k_{oz}-k'_z} = -\pi i e^{-i(D-z_p)(k_{oz}+k'_z)}$$

$$\int_{-\infty}^{\infty} ds \frac{e^{iz_p s}}{s-\beta_1} = 2\pi i e^{i\beta_1 z_p} \quad ; \quad \int_{-\infty}^{\infty} ds \frac{e^{iz_p s}}{s-k_{oz}-k'_z} = \pi i e^{iz_p(k_{oz}+k'_z)}$$

$$\int_{-\infty}^{\infty} ds \frac{e^{-i(D-z_p)s}}{s+\beta_1} = -2\pi i e^{i\beta_1(D-z_p)} \quad ; \quad \int_{-\infty}^{\infty} ds \frac{e^{iz_p s}}{s+\beta_1} = 0 .$$

These results are to be substituted into (65). The final expression for (64) then constitutes an integral which, after replacing the integration variable k'_z by k_z, proves to be identical with the sum of $\overset{\uparrow}{u}_1$ and $\overset{\downarrow}{u}_1$ in (40).

Hence the four types of waves discussed in section 10 are recovered by the expression for u_1 according to the general formula (59). In the same way it would be possible to obtain again the five wave types of u_2 by substituting (63) for both $G(\vec{k}_1,\omega_1)$ and $G(\vec{k}_2,\omega_2)$ in the expression (59) for N = 2.

17. Comparison of the various representations for the waves inside the scatterer. One-dimensional illustration

In the preceding section it appeared how the first-order scattering contribution u_1 can just as well be considered as the superposition, according to (40), of four different wave types (waves either forward or backward, and either modulated or diffracted), as well as the superposition of only one type of waves represented by the special case N = 1 of (59), viz. the formula (58). The first representation has the adventage to show an explicit connection with the finite size ($0 < z < D$) of the scatterer; the much simpler second representation could be obtained at the cost of an ω-integration in the complex plane instead of along the real axis, though the integration path should tend to the real axis in view of the limits $\pm \infty + io$. In other words, each individual acoustic travelling wave, fixed by special values of $\vec{k}$ and ω, may be considered as causing either four different types of undamped waves (ω real) or as causing only one type of travelling waves with negative damping, in the limit of vanishing damping (ω complex, $\mathrm{Im}\,\omega \to +0$). Such a situation might yet be illustrated for a much simpler case, the one-dimensional wave propagation through an inhomogeneous stationary stratified medium in a direction perpendicular to the stratification.

Let δn be a function of z only, with $\delta n \neq 0$ for $0 < z < D$, while the incident wave $e^{-iko(z-ct)}$ travels in the z direction. The wave equation (1) then reduces to

$$\frac{d^2u}{dz^2} + k_0^2\left\{1 + 2\,\delta n(z)\right\}u = 0,$$

which may be represented, in analogy to (3), by

$$\frac{d^2u}{dz^2} + k_0^2 u = -\varphi(z), \tag{66}$$

where

$$\varphi(z) \equiv 2k_0^2\, \delta n(z)\; u. \tag{67}$$

The corresponding Green function, satisfying the equation

$$\frac{d^2\mathcal{G}}{dz^2} + k_0^2\mathcal{G} = -\delta(z),$$

as well as the radiation condition at infinity, reads

$$\mathcal{G}(z) = \frac{i}{2k_0}\, e^{ik_0|z|},$$

if we assume k_o with a (possibly infinitesimal) positive imaginary part. We thus find the following particular solution of (66):

$$u(z) = \int_{-\infty}^{\infty} d\zeta\; \varphi(\zeta)\; \mathcal{G}(z-\zeta) = \frac{i}{2k_0}\int_{-\infty}^{\infty} d\zeta\; \varphi(\zeta)\, e^{ik_0|z-\zeta|}.$$

In view of (67), and the primary field with the normalized amplitude e^{ik_oz}, we next arrive at the following integral equation for u:

$$u(z) = e^{ik_0z} + ik_0\int_{-\infty}^{\infty} d\zeta\; \delta n(\zeta)\; u(\zeta)\, e^{ik_0|z-\zeta|}.$$

The contribution u_1 contained in the corresponding Born approximation $e^{ik_oz} + u_1$ reads:

$$u_1(z) = ik_0\int_{-\infty}^{\infty} d\zeta\; \delta n(\zeta)\; e^{ik_0(\zeta+|z-\zeta|)}. \tag{68}$$

From this latter expression we can derive a representation in terms of a single type of travelling waves, corresponding to the previous "modulated" waves, and another one containing both "modulated" and "diffracted" waves.

We shall start with the derivation of the former and therefore introduce the following Fourier integral

$$\delta n(z) = \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} dk\; G(k)\, e^{ikz}, \tag{69}$$

which is just as well an inversion of

$$G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\;\; \delta n(z)\; e^{-ik_0 z},$$

as well as of the simpler expression with $\varepsilon = 0$; the only assumption to be made here is that G(k) given for Im k = 0 admits an analytic continuation up to Im k = ε. We next split (68) into its "forward" and "backward" contributions (with $\zeta < z$ and $\zeta > z$ respectively):

$$u_1(z) = ik_0\left\{e^{ik_0 z}\int_{-\infty}^{z} d\zeta\; \delta n(\zeta) + e^{-ik_0 z}\int_{z}^{\infty} d\zeta\; \delta n(\zeta)\, e^{2ik_0\zeta}\right\}, \tag{70}$$

and then substitute (69) for $\delta n(\zeta)$. A further inversion of the orders of integration gives:

$$u_1(z) = ik_0\int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} dk\; G(k)\left\{e^{ik_0 z}\int_{-\infty}^{z} d\zeta\, e^{ik\zeta} + e^{-ik_0 z}\int_{z}^{\infty} d\zeta\, e^{i(2k_0+k)\zeta}\right\}.$$

The first ζ integral only converges, with the value e^{ikz}/ik, provided that Im k = $-\varepsilon < 0$. The corresponding condition for the convergence of the second integral to

$$i\,\frac{e^{i(2k_0+k)z}}{2k_0+k}$$

requires Im k > -2 Im k_0. The combined condition

$$-2\,\mathrm{Im}\,k_0 < \mathrm{Im}\,k = -\varepsilon < 0$$

can always be satisfied if Im $k_o > 0$ where Im K_o may become infinitesimal (Im K_o = +0). The substitution of the above values of the two ζ integrals leads to the final expression:

$$u_1(z) = 2k_0^2 \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} dk \; \mathcal{G}(k) \frac{e^{i(k_0+k)z}}{k(2k_0+k)} \cdot \qquad (0 < \varepsilon < 2 \,\mathrm{Im}\, k_0) \tag{71}$$

Obviously, the integration path in this representation in terms of "modulated" waves only has to be complex in order to avoid the poles at k=0 and $k=-2k_o$. It could just as well be replaced by an integration along the real axis, apart from an identation below k=0 and above $k=-2k_o$. This property can be used to arrive at the second representation for u_1, along lines corresponding to those of the derivation in the preceding section of the formula (40), containing three wave types, from the formula (58) only depending on "modulated" waves. We just have to substitute in (71) the one-dimensional analogon of (63), that is the formula

$$\mathcal{G}(k) = \int_{-\infty}^{\infty} dk' \; \mathcal{G}(k') \; \frac{1-e^{-iD(k-k')}}{2\pi i\,(k-k')},$$

which accounts for the finite size $0 < z < D$ of the scattering medium. The substitution in question yields, after another inversion of the orders of integration,

$$u_1(z) = \frac{k_0^2}{\pi i} \int_{-\infty}^{\infty} dk' \; \mathcal{G}(k') \int dk \frac{e^{i(k_0+k)z}}{k(2k_0+k)} \; \frac{1-e^{-iD(k-k')}}{(k-k')}, \tag{72}$$

where the path of integration for k has to pass, as mentioned, below the pole $k = 0$ and above $k = -2k_o$.

With the aid of the identity

$$\frac{1}{k(2k_o+k)(k-k')} = -\frac{1}{2k_o k'}\frac{1}{k} + \frac{1}{2k_o(2k_o+k')}\frac{1}{(2k_o+k)} + \frac{1}{k'(2k_o+k')}\frac{1}{(k-k')},$$

it is possible to split the k integral in (72) into six contributions each of which can be evaluated as a residue after closing the path of integration, either along the upper or along the lower half of the infinite circle in the k-plane, remembering the above mentioned indentations; in doing so it is irrelevant whether the integration paths pass above or below k = k', or, whether the principal value is taken there because k = k' constitutes no singularity of the k integrand. The final evaluation leads to the following remaining integration over k' (which may then be replaced by k):

$$u_1(z) = \int_{-\infty}^{\infty} dk\; G(k) \left\{ \frac{2k_o^2}{k(2k_o+k)} e^{i(k_o+k)z} - \frac{k_o}{k} e^{ik_o z} + \frac{k_o e^{iD(2k_o+k)}}{(2k_o+k)} e^{-ik_o z} \right\}. \qquad (73)$$

In contrast to (72) the integration can now be performed along the real axis itself, the complete integrand having no singularities at k = 0 or $k = -2k_o$. In terms of the terminology of section 10 the first contribution of (73) can be termed "modulated", the second and third one "forward diffracted" and "backward diffracted" respectively. We yet remark the the same expression (73) also results from (70) by substituting there the Fourier integral (69) for $\varepsilon = 0$; the integrations to be performed with respect to ζ are then most elementary. Summarizing we have shown in this section, by a very simple example, how the possibility of various representations by special combinations of waves may be connected with proper choices of the integration path for a relevant variable.

18. The total scattered wave

From the various representations discussed in the preceding sections it will be clear that the total wave, resulting from the summation over all scattering contributions of any order, can also be expressed in different ways. In the sections 10 and 13 we have considered the superposition of partly "modulated" and partly "diffracted" contributions. On the other hand, the analysis of the sections 14, 15 and 16 has shown that all waves can be considered as "modulated" provided that the integration path for the frequencies is not taken directly along the real axis. A further possibility for the classification of all contributions is obtained by adding together all waves travelling in a special direction independent of the associated number of scatterings. The corresponding representation is arrived at by passing in the integral (59) for the N-th order term u_N to the new integrations variables:

$$\vec{\lambda}_j = \vec{k}_0 + \vec{k}_1 + \cdots + \vec{k}_j \quad , \qquad \Omega_j = \omega_0 + \omega_1 + \cdots + \omega_j \,.$$
$$(j = 1, 2, \ldots, N)$$

This transformation has a Jacobi determinant with the value 1, while the following relation can be applied when working out the transformation:

$$\mathcal{G}(\vec{k}_1, \omega_1)\ \mathcal{G}(\vec{k}_2, \omega_2) \cdots \mathcal{G}(\vec{k}_N, \omega_N) =$$
$$= \mathcal{G}(\vec{\lambda}_1 - \vec{k}_0, \Omega_1 - \omega_0)\ \mathcal{G}(\vec{\lambda}_2 - \vec{\lambda}_1, \Omega_2 - \Omega_1)\ \mathcal{G}(\vec{\lambda}_3 - \vec{\lambda}_2, \Omega_3 - \Omega_2) \cdots \times$$
$$\times \cdots \mathcal{G}(\vec{\lambda}_{N-1} - \vec{\lambda}_{N-2}, \Omega_{N-1} - \Omega_{N-2})\ \mathcal{G}(\vec{k} - \vec{\lambda}_{N-1}, \omega - \Omega_{N-1}).$$

The summation over N = 1,2,3,... after the evaluation of u_N in terms of the new variables then results in the following total wave when, moreover, $\vec{\lambda}_N$ and Ω_N are replaced in all terms by the common variables $\vec{k}$ and ω, respectively:

$$u(P,t) = e^{i(\vec{k}_0 \vec{r}_p - \omega_0 t)} + k_0^2 \int_{-\infty}^{\infty} d\vec{k} \int_{-\infty+i0}^{\infty+i0} d\omega \, \frac{e^{i(\vec{k}\vec{r}_p - \omega t)}}{|\vec{k}|^2 - \frac{\omega^2}{c^2}} \times$$

$$\times \sum_{N=1}^{\infty} 2^N \int_{-\infty}^{\infty} d\vec{\lambda}_1 \cdots d\vec{\lambda}_{N-1} \int_{-\infty+i0}^{\infty+i0} d\Omega_1 \cdots d\Omega_{N-1} \frac{\left(\frac{\Omega_1}{c} \frac{\Omega_2}{c} \cdots \frac{\Omega_{N-1}}{c}\right)^2}{\left\{|\vec{\lambda}_1|^2 - \left(\frac{\Omega_1}{c}\right)^2\right\}\left\{|\vec{\lambda}_2|^2 - \left(\frac{\Omega_2}{c}\right)^2\right\} \cdots \left\{|\vec{\lambda}_{N-1}|^2 - \left(\frac{\Omega_{N-1}}{c}\right)^2\right\}} \times$$

$$\times \mathcal{G}(\vec{\lambda}_1 - \vec{k}_0, \Omega_1 - \omega_0)\ \mathcal{G}(\vec{\lambda}_2 - \vec{\lambda}_1, \Omega_2 - \Omega_1) \cdots \mathcal{G}(\vec{\lambda}_{N-1} - \vec{\lambda}_{N-2}, \Omega_{N-1} - \Omega_{N-2})\ \mathcal{G}(\vec{k} - \vec{\lambda}_{N-1}, \omega - \Omega_{N-1}).$$

(74)

In view of its importance we yet give an alternative derivation of (74) which starts with the recurrence relation (6), to be represented here as the following four-dimensional convolution:

$$u_N(\vec{r},t) = -\frac{1}{2\pi c^2}\left\{\frac{\delta(t-\frac{r}{c})}{r}\right\} * \left\{\delta n(\vec{r},t)\ \frac{\partial^2}{\partial t^2} u_{N-1}(\vec{r},t)\right\}. \tag{75}$$

The convolution of two space-time functions $h_1(r,t)$ and $h_2(r,t)$ is defined here as:

$$h_1(\vec{r},t) * h_2(\vec{r},t) = \iiiint_{-\infty}^{\infty} d\xi_x\, d\xi_y\, d\xi_z\, d\tau\ h_1(\xi_x,\xi_y,\xi_z,\tau)\ h_2(x-\xi_x, y-\xi_y, z-\xi_z, t-\tau) =$$
$$= \int d\vec{\xi} \int d\tau\ h_1(\vec{\xi},\tau)\ h_2(\vec{r}-\vec{\xi}, t-\tau)$$

Obviously, N successive applications of (75) lead to the following explicit expression for the N-th term of the Born series:

$$u_N(\vec{r},t) = -\frac{1}{(2\pi c^2)^N}\ \frac{\delta(t-\frac{r}{c})}{r} * \left[\delta n(\vec{r},t)\ \frac{\partial^2}{\partial t^2}\left\{\frac{\delta(t-\frac{r}{c})}{r} * \ldots\right]\right]^{N-1}\left\{\delta n(\vec{r},t)\ \frac{\partial^2}{\partial t^2} u_0(\vec{r},t)\right\}, \tag{76}$$

where the (N-1)th power refers to N-1 steps of the type

$$\delta n(\vec{r},t)\ \frac{\partial^2}{\partial t^2}\left\{\frac{\delta(t-\frac{r}{c})}{r} * \varphi(\vec{r},t)\right\}.$$

The further reduction of (76) will be based on the well-known convolution properties:

$$\left\{h_1(\vec{r},t) * h_2(\vec{r},t)\right\}_F(\vec{k},\omega) = (2\pi)^4\, h_{1,F}(\vec{k},\omega)\ h_{2,F}(\vec{k},\omega),$$

$$\left\{h_1(\vec{r},t)\cdot h_2(\vec{r},t)\right\}_F(\vec{k},\omega) = \int d\vec{k'} \int d\omega'\ h_{1,F}(\vec{k'},\omega')\ h_{2,F}(\vec{k}-\vec{k'}, \omega-\omega'), \tag{77}$$

in which the symbol F defines Fourier transforms according to:

$$h(\vec{r},t) = \int d\vec{k} \int d\omega\, e^{i(\vec{k}\vec{r}-\omega t)}\ h_F(\vec{k},\omega). \tag{78}$$

Apart from these results, and the elementary way in which the second-order time derivatives can be accounted for, we also need the formula (32) for which we use the following representation (after the substitutions $\vec{S} \to k$, $u \to -\omega$, while taking $c' = +0$):

$$\frac{\delta(t-\frac{r}{c})}{r} = \frac{1}{4\pi^3}\int d\vec{k} \int_{-\infty+i0}^{\infty+i0} d\omega \; \frac{e^{i(\vec{k}\vec{r}-\omega t)}}{|\vec{k}|^2 - \frac{\omega^2}{c^2}} \tag{79}$$

It thus proves possible to arrive at a general formula related to the operator entering in (76), viz.

$$\left[\frac{\partial^2}{\partial t^2}\left\{\frac{\delta(t-\frac{r}{c})}{r} * \varphi(\vec{r},t)\right\}\right]_F (\vec{k},\omega) = -4\pi \frac{\omega^2}{|\vec{k}|^2 - \frac{\omega^2}{c^2}} \varphi_F(\vec{k},\omega).$$

Next, with the aid of the transform (23) for δn, we also find:

$$\left[\delta n(\vec{r},t)\frac{\partial^2}{\partial t^2}\left\{\frac{\delta(t-\frac{r}{c})}{r} * \varphi(\vec{r},t)\right\}\right]_F (\vec{k},\omega) =$$

$$= -4\pi \int d\vec{k}_1 \int_{-\infty+i0}^{\infty+i0} d\omega_1 \frac{\omega_1^2}{|\vec{k}_1|^2 - \frac{\omega_1^2}{c^2}} \; \mathcal{G}_4(\vec{k}+\vec{k}_1, \omega+\omega_1)\; \varphi_F(-\vec{k}_1, -\omega_1), \tag{80}$$

where all integrations over ω, here and henceforth, are to be performed along a line just above the real axis in the complex ω-plane, in view of the singularity of the integrand on this axis itself.

Starting from the elementary relation:

$$\left\{\delta n(\vec{r},t)\;\frac{\partial^2}{\partial t^2} u_0(\vec{r},t)\right\}_F (\vec{k},\omega) = -\omega_0^2\left\{\delta n(\vec{r},t)\, e^{i(\vec{k}_0\vec{r}-\omega_0 t)}\right\}_F(\vec{k},\omega) =$$

$$= -\omega_0^2 \mathcal{G}_4(\vec{k}-\vec{k}_0, \omega-\omega_0),$$

we find after a first application of (80):

$$\left(\delta n(\vec{r},t)\;\frac{\partial^2}{\partial t^2}\left[\frac{\delta(t-\frac{r}{c})}{r} * \left\{\delta n(\vec{r},t)\;\frac{\partial^2}{\partial t^2} u_0(\vec{r},t)\right\}\right]\right)_F (\vec{k},\omega) =$$

$$= 4\pi\omega_0^2 \int d\vec{k}_1 \int_{-\infty+i0}^{\infty+i0} d\omega_1 \frac{\omega_1^2}{|\vec{k}_1|^2 - \frac{\omega_1^2}{c^2}} \mathcal{G}_4(\vec{k}+\vec{k}_1, \omega+\omega_1)\; \mathcal{G}_4(-\vec{k}_0-\vec{k}_1, -\omega_0-\omega_1).$$

Applying (80) also to the further steps in the evaluation of (76) we finally obtain the general expression:

$$\{u_N(\vec{r},t)\}_F(\vec{k},\omega) = \frac{\omega_0^2\left(-\frac{2}{c^2}\right)^N}{|\vec{k}|^2-\frac{\omega^2}{c^2}} \int d\vec{k}_1 \dots d\vec{k}_{N-1} \int_{-\infty+i0}^{\infty+i0} d\omega_1 \dots d\omega_{N-1} \times$$

$$\times \frac{\omega_1^2\omega_2^2\dots\omega_{N-1}^2}{\left(|\vec{k}_1|^2-\frac{\omega_1^2}{c^2}\right)\dots\left(|\vec{k}_{N-1}|^2-\frac{\omega_{N-1}^2}{c^2}\right)} \mathcal{G}_4(\vec{k}+\vec{k}_1,\omega+\omega_1)\, \mathcal{G}_4(\vec{k}_2-\vec{k}_1,\omega_2-\omega_1)\dots\times$$

$$\times \dots \mathcal{G}_4(\vec{k}_{N-1}-\vec{k}_{N-2},\omega_{N-1}-\omega_{N-2})\, \mathcal{G}_4(-\vec{k}_0-\vec{k}_{N-1},-\omega_0-\omega_{N-1}).$$

A reversal of both the signs of the variables $\vec{k}_1,\dots,\vec{k}_{N-1}$ and $\omega_1,\dots,\omega_{N-1}$, and of their numbering from 1 to N-1, leads to a Fourier transform which proves to be identical with that of the N-th term in (74). This general formula has thus been verified again.

The total wave expressed by (74) constitutes the rigorous solution of an integral equation such as considered in a vectorial version in Born and Wolf [7] (see chapter XII) when dealing with scattering due to a single real acoustic or ultrasonic wave. A direct application to such an isolated wave requires prudence in view of the ω-integrations along a line just above the real axis. For a single plane travelling acoustic wave for which $\delta n \neq 0$ starts at $t = 0$, the following representation could be used in order to find the associated G_4 transform:

$$e^{i(\vec{k}_0\vec{r}-\omega t)}\, \mathcal{U}(t) = \int d\vec{k} \int_{-\infty+i0}^{\infty+i0} d\omega\, e^{i(\vec{k}\vec{r}-\omega t)} \frac{i}{2\pi} \frac{\delta(\vec{k}-\vec{k}_0)}{\omega-\omega_0}$$

19. The higher-order Born terms at great distances (stationary scatterer)

So far, an approximation applicable at great distances beyond or in front of the scattering slab $0 < z < D$ has only been considered in section 9 for the Born approximation. A corresponding much less explicit approximation can be derived for the N-th order contribution u_N, either from (74) or (59).

In this section we shall derive such an approximation for the simplified case of a stationary medium.

For such a medium δn can be fixed by the three-domensional Fourier integral

$$\delta n(\vec{r}) = \int d\vec{k}\; G_3(\vec{k})\, e^{i\vec{k}\vec{r}},$$

the transform G_3 of which enters in the special representation:

$$G_4(\vec{k},\omega) = G_3(\vec{k})\;\delta(\omega). \tag{81}$$

The expression (59) for u_N now reduces to:

$$u_N(P,t) = 2^N k_0^{2N} e^{-i\omega_0 t}\int d\vec{k}_1\ldots d\vec{k}_N \frac{G_3(\vec{k}_1)\,G_3(\vec{k}_2)\ldots G_3(\vec{k}_N)}{\left(|\vec{k}_0+\vec{k}_1|^2-k_0^2\right)\ldots\left(|\vec{k}_0+\ldots+\vec{k}_N|^2-k_0^2\right)}\times$$

$$\times e^{i(\vec{k}_0+\vec{k}_1+\ldots+\vec{k}_N)\vec{r}_P}. \tag{82}$$

The approximation yielding the simplification at great distances, viz. (35a), reduces for the stationary medium to:

$$\frac{\partial}{\partial \vec{r}_P} = \nabla_P \sim i k_0\, \vec{u}_r,$$

$\vec{u}_r$ again fixing the direction of observation. Splitting off the last integration in (82), we find:

$$\int d\vec{k}_N \frac{G_3(\vec{k}_N)}{|\vec{k}_0+\ldots+\vec{k}_N|^2-k_0^2}\, e^{i(\vec{k}_0+\ldots+\vec{k}_N)\vec{r}_P} =$$

$$= G_3(-\vec{k}_0-\ldots-\vec{k}_{N-1}-i\nabla_P)\int d\vec{k}_N \frac{e^{i(\vec{k}_0+\ldots+\vec{k}_N)\vec{r}_P}}{|\vec{k}_0+\ldots+\vec{k}_N|^2-k_0^2},$$

where the last integral equals, in view of formula (A.1) of the appendix, the quantity $(2\pi^2/r_p)e^{ik_o r_p}$, at least if we assume k_o with an infinitesimal positive imaginary part (radiation condition). Applying these remarks to (82) we obtain:

$$u_N(P,t) \sim 2^{N+1}\pi^2 k_o^{2N} e^{-i\omega_o t}\frac{e^{ik_o r_p}}{r_p}\int d\vec{k}_1 \cdots d\vec{k}_{N-1}\,\frac{G_3(\vec{k}_1)\cdots G_3(\vec{k}_{N-1})\,G_3(k_o\vec{u}_r-\vec{k}_o-\cdots-\vec{k}_{N-1})}{\{|\vec{k}_o+\vec{k}_1|^2-k_o^2\}\cdots\{|\vec{k}_o+\cdots+\vec{k}_{N-1}|^2-k_o^2\}}$$

$$(N \geq 2). \tag{83}$$

The corresponding expression for the Born approximation in a stationary medium proves to read, applying (81) to (36),

$$u_1(P,t) \sim 4\pi^2 k_o^2 e^{-i\omega_o t}\frac{e^{ik_o r_p}}{r_p}\,G_3(k_o\vec{u}_r-\vec{k}_o); \tag{84}$$

on the other hand, the next term of the Born series, that is (83) for N=2, already leads to an integral, viz.

$$u_2(P,t) \sim 8\pi^2 k_o^4 e^{-i\omega_o t}\frac{e^{ik_o r_p}}{r_p}\int d\vec{k}\,\frac{G_3(\vec{k})\,G_3(k_o\vec{u}_r-\vec{k}_o-\vec{k})}{|\vec{k}_o+\vec{k}|^2-k_o^2}. \tag{85}$$

In such expressions the infinitesimal positive imaginary part of k_o can also be accounted for by taking the integration path for the $\vec{k}$ variables just underneath the real axis.

A comparison of all these formulas shows that the approximation outside the scatterer can be interpreted, for each term u_j of the Born series, as representing a radiation pattern in which, in the factors occuring next to the spherical wave $e^{ik_o r_p}/r_p$, the dependence on the direction of observation enters through the occurrence of the unit vector $\vec{u}_r$ that fixes this direction. According to (84) only a single acoustic wave contributes to the Born approximation at great distances, whereas for all higher-order terms a continous wave-number spectrum of such waves contributes to the intensity observed in the direction of observation. As a matter of fact, the wave-number spectrum for u_N results from a vectorial

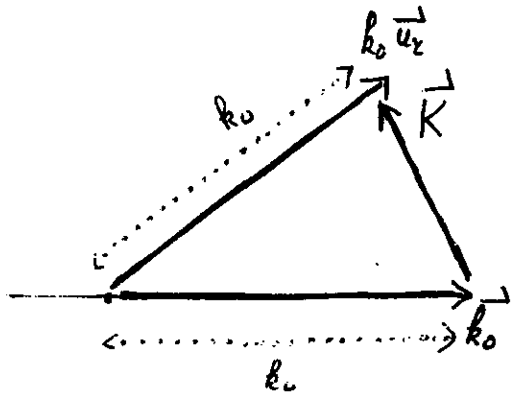

Fig. 3

addition of the wave-number vectors of those individual acoustic waves, for which the sum of these vectors amounts to a single vector

$$\vec{K} = \vec{k_1} + \cdots + \vec{k_{N-1}} + (k_0 \vec{u_r} - \vec{k_0} - \cdots - \vec{k_{N-1}}) =$$

$$= k_0 \vec{u_r} - \vec{k_0}. \tag{86}$$

This latter vector fits to the vectorial diagram of fig. 3, the geometrical configuration of which is well known from scattering theories based only on the Born approximation.

20. Final remarks

The aim of this article has been to show the usefulness of expressions containing differential operators, with their special applications to scattering and diffraction phenomena. The latter can be interpreted as interactions between an incident wave and travelling waves of acoustic type. This interaction has been first investigated by Brillouin[8)], by working out the Born approximation at great distances in the case of a scatterer of limited size, in a vectorial version based on the Maxwell equations. Though represented in a quite different form, the resonance condition derived by this author proves to be equivalent to our equation (45)[*)].

*) this can be verified by evaluating the relation $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$ in formula (28) of Brillouin's paper.

Our multiple-scattering analysis is in particular characterized by the formula (59) or the equivalent (74). The denominator of (59) shows how all possible resonance effects are contained in the relation

$$\left|\vec{k_0} + \vec{K}\right|^2 = \left(\frac{\omega_0 + \Omega}{c}\right)^2, \tag{87}$$

where

$$\vec{K} = \vec{k_1} + \vec{k_2} + \cdots + \vec{k_j} \quad , \quad \Omega = \omega_1 + \omega_2 + \cdots + \omega_j , \tag{88}$$

j being any positive integer fixing the number of scatterings (see also the remarks at the end of section 19 for a stationary medium). The j-th order scatterings thus result from the interaction of the incident wave with fictitious travelling waves the wave-number vector $\vec{K}$ for each of which is the vectorial sum of those of j different acoustic waves. In other words, multiple-scattering effects can be ascribed to the cooperation of different acoustic waves, a cooperation which might be compared with the one occurring in nonlinear phenomena depending on such waves. In this connection it is striking to notice the analogy of the equations (88) to those describing, e.g., resonance conditions for the so-called wave-wave interaction in nonlinear plasma theory; main attention is given there to the case j = 2 which has to do with the interaction of three waves. An essential difference between our situation and that of non-linear theories concerns the absence in the former of a dispersion relation. Such a relation restricts the possibilities for resonance as discussed in textbooks, for instance in chapter I of the book of Sagdeev and Galeev[9].

The existence of resonance conditions is also apparent in the observation of the individual effect of a special single acoustic wave at great distances, either at the front side or at the back of the scattering slab. In fact, our equation (37) shows how single scattering due to an acoustic wave with a special wave-number vector $\vec{k}$ is only observable if the corresponding vector $\vec{k}_o + \vec{k}$ for the scattered wave is parallel to the unit vector $\vec{u}_r$ fixing the direction of observation. The relation (37) implies the single-scattering resonance relation (43). The extension of this property to multiple scattering is obvious in view of the equation (87) then replascing (43), and from the fact that exp $\{i(\vec{K}\,\vec{r} - \Omega t)\}$ constitutes the exponential for a multiple-scattered wave as occurring in the integrand of (59). Multiple scattering associated with the relations (88) is observable at great distances provided that the direction of observation fixed by $\vec{u}_r$ satisfies the following extension of (37):

$$\vec{k}_o + \vec{K} = \frac{\omega_o + \Omega}{c};$$

the special case of this relation for a stationary scattering is given by (86). The extension of (45) to multiple scattering is obtained by replacing d by $2\pi/|\vec{K}|$, that is by the separation of consecutive wavefronts perpendicular to the effective wave-number vector $\vec{K}$; moreover, i_o then has to refer to the angle of incidence with respect to these wave fronts.

We finally emphasize that no statistical considerations whatever have been included in this paper, though many of its results are adapted to statistical applications. As an example we mention how a statistical average of the expression (85) for the observation of second-order scattering at great distances will depend on the average of the product of the two G_3 functions entering there. The property that the statistical average of $\langle G_3(\vec{k}_1)G_3(\vec{k}_2)\rangle$ is known to be proportional to $\delta(\vec{k}_1+\vec{k}_2)$, at least in the case of homogeneous turbulence, here leads to the result that the average $\langle u_2\rangle$ becomes proportional to $\delta(k_o\vec{u}_r - \vec{k}_o)$. In other words, at great distances second-order scattering only becomes observable when due to those acoustic waves which are also the only ones that contribute, according to (84), to the first-order scattering.

Appendix. Derivation of the equation (32)

We first consider the general integral

$$J = \int d\vec{s}\; e^{i(\vec{a}\,\vec{s})}\; f(s),$$

in which the integration should extend over the entire space for which the components S_1, S_2, S_3 of the vector $\vec{S}$ constitute rectangular coordinates; the function f(s) is assumed to depend only on the length S of this vector.

We take the S_3-axis in the direction of $\vec{a}$ so that $(\vec{a}\,\vec{s})$ reduces to $S_3\,a$, a being the length of $\vec{a}$. Introducing next polar coordinates according to

$$S_1 = S \sin\vartheta \cos\varphi \quad , \quad S_2 = S \sin\vartheta \sin\varphi \quad , \quad S_3 = S\cos\vartheta,$$

we get:

$$J = \int_0^\infty ds\, s^2 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin\vartheta\, e^{ias\cos\vartheta} f(s) = \frac{4\pi}{a}\int_0^\infty ds\; s\,\sin(as)\; f(s).$$

Taking $f(s) = (s^2-u^2/c^2)^{-1}$, we find:

$$\int d\vec{s}\,\frac{e^{i\vec{a}\vec{s}}}{s^2-\frac{u^2}{c^2}} = \frac{4\pi}{a}\int_0^\infty ds\; s\;\frac{\sin(as)}{s^2-\frac{u^2}{c^2}} =$$

$$= -\frac{i\pi}{a}\left\{\int_{-\infty}^{\infty} ds\,\frac{e^{ias}}{s-\frac{u}{c}} - \int_{-\infty}^{\infty} ds\,\frac{e^{-ias}}{s-\frac{u}{c}}\right\}.$$

Since a constitutes a positive quantity the integration path of the first integral can be closed at infinity along the upper half of the infinite circle in the complex S-plane, whereas the same is possible for the second integral with respect to the lower half of this circle. Assuming

Im u < 0 the only singularity of the integrands, the pole at s = u/c, is enclosed by the contour integration of the second integral, but not by that of the first one. The residue at this pole then yields:

$$\int d\vec{s}\,\frac{e^{i\vec{a}\vec{s}}}{s^2-\frac{u^2}{c^2}} = \frac{2\pi^2}{a}\,e^{-ia\frac{u}{c}}\quad,\qquad (\mathrm{Im}\,u<0)$$

We also give for reference the corresponding expression for u replaced by $-k_o c$, viz.

$$\int d\vec{s}\,\frac{e^{i\vec{a}\vec{s}}}{s^2-k_o^2} = \frac{2\pi^2}{a}\,e^{ik_o a}\;.\qquad (\mathrm{Im}\,k_o>0) \tag{A1}$$

The integral to be evaluated in (32), that is

$$K=\frac{1}{4\pi^3}\int d\vec{s}\int_{-\infty-ic'}^{\infty-ic'} du\,\frac{e^{i(\vec{r}\vec{s}+tu)}}{s^2-\frac{u^2}{c^2}},\qquad (c'>0;\; r=\sqrt{x^2+y^2+z^2})$$

now reduces in view of the above result, yet inverting the order of integration (while taking $\vec{a}=\vec{r}$), to

$$K=\frac{1}{4\pi^3}\int_{-\infty-ic'}^{\infty-ic'} du\,e^{itu}\,\frac{2\pi^2}{r}\,e^{-ir\frac{u}{c}}=$$

$$=\frac{1}{2\pi r}\int_{-\infty-ic'}^{\infty-ic'} du\,e^{iu(t-\frac{r}{c})}=\frac{\delta(t-\frac{r}{c})}{r},$$

that is the relation which had to be proved.

References

1) Bremmer, H.: "Symposium on Quasi-Optics", p. 415. Polytechnic Institute of Brooklyn, 1964.

2) Sommerfeld, A.: Ann. der Physik und Chemie (4) 28, 682 (1909); see also Watson, G.N.: "Theory of Bessel Functions", Cambridge 1944, p. 416.

3) Molyneux, J. Opt. Soc. Am. 61, 248 and 369 (1971).

4) de Wolf, D.A.: "Wave Propagation and multiple Scattering in a random continuum", thesis, Eindhoven (Netherlands) 1968.

5) Laue, M. von: Naturwissenschaften 30, 205 (1942).

6) Bremmer, H. and Van Dorsten, A.C., Z. angewandte Phys. 27, 219 (1969).

7) Born, M and Wolf, E.: "Principles of Optics", Oxford, 1965.

8) Brillouin, L.: Ann. Phys. (París) 9, 88 (1921).

9) Sagdeev, R.Z. and Galeev, A.A.: "Non linear plasma theory", New York and Amsterdam, 1969.

EINDHOVEN UNIVERSITY OF TECHNOLOGY
THE NETHERLANDS
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

1) Dijk, J., Jeuken, M., Maanders, E.J.
AN ANTENNA FOR A SATELLITE COMMUNICATION GROUNDSTATION (Provisional Electrical Design)
TH-report 68-E-01, March 1968. ISBN 90 6144 001 7.

2) Veefkind A., Blom J.H. and Rietjens L.H.Th.:
THEORETICAL AND EXPERIMENTAL INVESTIGATION OF A NON-EQUILIBRIUM PLASMA IN A MHD CHANNEL
TH-report 68-E-02, March 1968. Submitted to the Symposium on a Magnetohydrodynamic Electrical Power Generation, Warsaw, Poland, 24-30 July, 1968. ISBN 90 6144 002 5

3) Boom, A.J.W. van den and Melis J.H.A.M.:
A COMPARISON OF SOME PROCESS PARAMETER ESTIMATION SCHEMES
TH-report 68-E-03, September 1968 ISBN 90 6144 003 3

4) Eykhoff, P., Ophey P.J.M. Severs J. and Oome J.O.M.:
AN ELECTROLYTIC TANK FOR INSTRUCTIONAL PURPOSES REPRESENTING THE COMPLEX-FREQUENCY PLANE
TH-report 68-E-04, September 1968 ISBN 90 6144 004 1

5) Vermij L. and Daalder J.E.:
ENERGY BALANCE OF FUSING SILVER WIRES SURROUNDED BY AIR
TH-report 68-E-05, November 1968. ISBN 90 6144 005 X

6) Houben J.W.M.A. and Massee P.:
MHD POWER CONVERSION EMPLOYING LIQUID METALS
TH-report 69-E-06, February 1969. ISBN 90 6144 006 8

7) Heuvel, W.M.C. van den and Kersten W.F.J.:
VOLTAGE MEASUREMENT IN CURRENT ZERO INVESTIGATIONS
TH-report 69-E-07, September 1969. ISBN 90 6144 007 6

8) Vermij, L.:
SELECTED BIBLIOGRAPHY OF FUSES
TH-report 69-E-08, September 1969. ISBN 90 6144 008 4

9) Westenberg, J.Z.:
SOME IDENTIFICATION SCHEMES FOR NON-LINEAR NOISY PROCESSES
TH-report 69-E-09, December 1969. ISBN 90 6144 009 2

10) Koop, H.E.M., Dijk, J. and Maanders E.J.:
ON CONICAL HORN ANTENNAS.
TH-report 70-E-10, February 1970. ISBN 90 6144 010 6

11) Veefkind, A.:
NON-EQUILIBRIUM PHENOMENA IN A DISC-SHAPED MAGNETOHYDRODYNAMIC GENERATOR
TH-report 70-E-11, March 1970. ISBN 90 6144 011 4

12) Jansen, J.K.M. Jeuken, M.E.J. and Lambrechtse, C.W.:
THE SCALAR FEED TH-report 70-E-12, December 1969. ISBN 90 6144 012 2

13) Teuling, D.J.A.:
ELECTRONIC IMAGE MOTION COMPENSATION IN A PORTABLE TELEVISION CAMERA
TH-report 70-E-13, 1970. ISBN 90 6144 013 0

14) Lorencin, M.:
AUTOMATIC METEOR REFLECTIONS RECORDING EQUIPMENT.
TH-report 70-E-14, November 1970. ISBN 90 6144 014 9

15) Smets, A.J.:
THE INSTRUMENTAL VARIABLE METHOD AND RELATED IDENTIFICATION SCHEMES.
TH-report 70-E-15. November 1970. ISBN 90 6144 015 7

EINDHOVEN UNIVERSITY OF TECHNOLOGY
THE NETHERLANDS
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

16) White, Jr. R.C.:
A SURVEY OF RANDOM METHODS FOR PARAMETER OPTIMATIZATION
TH-report 70-E-16, February 1971. ISBN 90 6144 016 5

17) Talmon. J.L.:
APPROXIMATED GAUSS-MARKOV ESTIMATORS AND RELATED SCHEMES
TH-report 71-E-17, February 1971. ISBN 90 6144 017 3

18) Kalásek, V.:
MEASUREMENT OF TIME CONSTANTS ON CASCADE D.C. ARC IN NITROGEN
TH-report 71-E-18, February 1971. ISBN 90 6144 018 1

19) Hosselet, L.M.L.F.:
OZONBILDUNG MITTELS ELEKTRISCHER ENTLADUNGEN
TH-report 71-E-19, March 1971. ISBN 90 6144 019 X

20) Arts, M.G.J.:
ON THE INSTANTANEOUS MEASUREMENT OF BLOODFLOW BY ULTRASONIC MEANS
TH-report 71-E-20, May 1971. ISBN 90 6144 020 3

21) Roer, Th.G. van de:
NON-ISO THERMAL ANALYSIS OF CARRIER WAVES IN A SEMICONDUCTOR
TH-report 71-E-21, August 1971. ISBN 90 6144 021 1

22) Jeuken, P.J. Huber, C. and Mulders, C.E.:
SENSING INERTIAL ROTATION WITH TUNING FORKS
TH-report 71-E-22, September 1971. ISBN 90 6144 022 X

23) Dijk, J. and Maanders E.J.:
APERTURE BLOCKING IN CASSEGRAIN ANTENNA SYSTEMS. A REVIEW
TH-report 71-E-23, September 1971. ISBN 90 6144 023 8 (In preparation)

24) Kregting, J. and White, Jr. R.C.:
ADAPTIVE RANDOM SEARCH.
TH-report 71-E-24, October 1971. ISBN 90 6144 024 6 (In preparation)