

Literature study on the output regulation problem

Citation for published version (APA): Pavlov, A. (2001). *Literature study on the output regulation problem*. (DCT rapporten; Vol. 2001.045). Technische Universiteit Eindhoven.

Document status and date: Published: 01/01/2001

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

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Literature Study on the Output Regulation Problem

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Eindhoven, October 2001

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THE OUTPUT REGULATION PROBLEM Report on literature study.

Alexei Pavlov

October, 2001

Abstract

This report is devoted to the problem of asymptotic regulation of the output of a dynamic system, which is subject to disturbances generated by an exosystem. As an introduction, some results on the output regulation problem for linear systems are reviewed. For nonlinear systems necessary and sufficient conditions for the solvability of the regulator problem are considered under the assumption of neutral stability of the exosystem. The basic concepts involved in these conditions such as the regulator equations and the internal model are described. The extension of these results for nonlinear systems with uncertainties are discussed further. Several approaches towards approximate solutions of the output regulation problem are reviewed. Finally, two examples for which the neutral stability assumption does not hold, but the output regulation problem is still solvable are considered.

1 Introduction

In the context of the output regulation problem we consider systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ y &= h_m(x, w) \\ e &= h_r(x, w) \end{aligned}$$

with state $x \in X \subset \mathbb{R}^n$, input $u \in U \subset \mathbb{R}^m$, measured output $y \in \mathbb{R}^l$, regulated output $e \in \mathbb{R}^p$ and exogenous disturbance input $w \in W \subset \mathbb{R}^r$ generated by the exosystem

$$\dot{w} = s(w). \tag{2}$$

We assume that f(x, u, w), $h_m(x, w)$, $h_r(x, w)$ and s(w) are C^k functions (for some large k) of their arguments and also that f(0, 0, 0) = 0, $h_m(0, 0) = 0$, $h_r(0, 0) = 0$ and s(0) = 0.

In general the regulator problem can be formulated as follows: Find a feedback controller of the form

$$\dot{\xi} = \eta(\xi, y)$$
 (3)
 $u = \theta(\xi)$

such that for the closed loop system (1)-(3) the regulated output converges to zero: $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

In many cases the additional requirement of asymptotic stability of the closed loop system (1), (3) with $w \equiv 0$ is set. In this case the problem is referred to as the *output regulation problem with internal stability*. This problem will be considered throughout the rest of the report. In the presence of parametric uncertainties either in equations (1) or (2) the regulation problem is called *structurally stable* or *robust output regulation problem* as will be specified in the sequel.

A lot of control problems can be put in the framework of the regulator problem. A simple example is asymptotic tracking of certain classes of reference outputs and asymptotic rejection of undesired disturbances. In this case system (2) generates both the reference signals and disturbances, and e is the tracking error (for example e might be equal to x - w, if both x and w belong to the same space).

A physical example of the output regulation problem is the automatic smooth vertical landing of an aircraft (or a helicopter) on the deck of a ship, which, due to the waves, is subject to a motion. The problem is to synchronize the vertical position of the aircraft with the unknown periodic motion of the vessel. The aircraft dynamics can be described by nonlinear equations of the form $\dot{x} = f(x, u, \mu)$, $y_a = h_a(x)$, where u is the input (for example the thrust), y_a is the vertical position of the aircraft and μ is a vector of uncertain parameters (mass, inertia, etc.). The motion of the ship can be approximated by a sum of a fixed number of sinusoids with unknown frequencies, phases and amplitudes. Thus its dynamics can be described by linear equations of the form $\dot{w} = S_{\omega}w$, $y_s = h_s w$, where y_s is the vertical position of the deck of the vessel. All the eigenvalues of S_{ω} are simple and lie on the imaginary axis (because the system generates only sinusoids and constants). Such exosystem satisfies the neutral stability assumption, which plays an important role in the theory presented in the sequel and will be explained later. In terms of the output regulation problem this equation is an exosystem. The matrix S_{ω} depends on the uncertainty vector ω , which corresponds to the unknown frequencies of the sinusoids. The initial conditions of the exosystem correspond to unknown phases and amplitudes of the sinusoids. The measurement available for feedback is the distance between the aircraft and the deck of the ship $(y = y_a - y_s)$. The problem is to find a controller of the form (3) rendering the distance between the aircraft and the deck tend to zero, i.e. $e(t) = y_a(t) - y_s(t) \to 0$ as $t \to \infty$, for all uncertainties μ and ω from given sets \mathcal{M} , Ω and for all initial conditions of the aircraft and ship dynamics from a given set I. This problem has been solved for the case of a vertical takeoff and landing aircraft [6] and it has been approached for the case of a helicopter in [8]. The problem presented above is still abstract, i.e. some conditions are not taken into account. In particular in practice an additional requirement that e(t) must remain nonnegative must be set (otherwise we allow the plane to crash onto the deck).

The rest of the report is organized as follows. The linear regulator problem is considered in Section 2. Section 3 is devoted to the problem of the local output regulation of nonlinear systems. The problem of approximate output regulation is considered in Section 4. The output regulation problem for nonlinear systems with uncertainties is discussed in Section 5. Section 6 covers examples of solutions to the regulator problem for complex dynamical systems. Conclusions in Section 7 finish the report.

The sources of the report are the following. Section 2 is based mostly on [1] and [2]. Sections 3, 5 and Appendix are based on [1]. Section 4 is based on [1] and [10]. The results of Section 6 are taken from [5].

2 Output regulation for linear systems

In the case of linear systems equations (1), (2) take the form

$$\dot{x} = Ax + Bu + Pw$$

$$y = C_m x + Q_m w$$

$$e = C_r x + Q_r w,$$

$$(4)$$

$$\dot{w} = Sw.$$
 (5)

The problem of output regulation is to find, if possible, a feedback law

$$\dot{\xi} = F\xi + Gy$$
 (6)
 $u = H\xi$

such that

(a) the equilibrium $(x,\xi) = (0,0)$ of the unforced closed loop system

$$\dot{x} = Ax + BH\xi$$

$$\dot{\xi} = F\xi + GC_m x$$

$$(7)$$

is asymptotically stable (internal stability),

(b) the forced closed loop system

$$\dot{x} = Ax + BH\xi + Pw$$

$$\dot{\xi} = F\xi + GC_m x + GQ_m w$$

$$\dot{w} = Sw$$

$$e = C_r x + Q_r w$$
(8)

is such that $e(t) \to 0$ as $t \to \infty$ for every initial condition $(x(0), \xi(0), w(0))$ (output regulation).

If (7) is required to be asymptotically stable, then the matrix

$$J = \begin{pmatrix} A & BH \\ GC_m & F \end{pmatrix}$$
(9)

must have all eigenvalues with negative real part. It can be shown that necessarily (A, B) must be stabilizable and (C_m, A) must be detectable. On the other hand, asymptotic decay of the regulated output requires a more subtle condition, which is based on the following result.

Lemma 1 Consider the closed loop system (8) and suppose that matrix (9) has all eigenvalues with negative real part. Suppose the eigenvalues of the matrix S have nonnegative real part. Then,

$$\lim_{t \to \infty} e(t) = 0 \tag{10}$$

for each initial condition $(x(0), \xi(0), w(0))$ if and only if there exist matrices Π and Σ satisfying

$$\Pi S = A\Pi + BH\Sigma + P$$

$$\Sigma S = F\Sigma + GC_m\Pi + GQ_m$$

$$0 = C_r\Pi + Q_r.$$
(11)

Remark 1. The condition on the spectrum of S is not restrictive, since the stable modes of S corresponding to its eigenvalues with negative real part decay exponentially and thus they do not affect the asymptotic behaviour of the error e. Hence we can neglect their effect.

Remark 2. In the case that y = e (i.e. $C_m = C_r =: C$ and $Q_m = Q_r =: Q$) equations (11) take the form

$$\Pi S = A\Pi + BH\Sigma + P$$

$$\Sigma S = F\Sigma$$

$$0 = C\Pi + Q.$$
(12)

The idea lying behind Lemma 1 is simple. The closed loop system (8) has two complimentary invariant subspaces R_{st} and R_{ust} . The first one is the stable subspace $R_{st} = \{(x, \xi, w) | w = 0\}$ corresponding to the eigenvalues of the stable matrix J. The second subspace corresponds to the eigenvalues of S and it is equal to

$$R_{ust} = \{(x,\xi,w) | x = \Pi w, \xi = \Sigma w, w \in \mathbb{R}^r \}.$$

The first two equations of (11) mean that R_{ust} is invariant for the system (8), while the last equation says that the error e is zero on R_{ust} . R_{ust} is globally attractive. Thus any solution of (8) converges to R_{ust} on which the error e is zero.

Lemma 1 illustrates the basic idea of solving the regulator problem: the closed loop system (8) ((1)-(3) in the nonlinear case) must have an invariant attractive set such that the regulation error on this set is zero. In the case of linear systems this set is the invariant subspace R_{ust} , which is globally defined and globally attractive. In the nonlinear setting of the problem such set can be a manifold and it can be only locally defined and locally attractive.

Now consider the problem of *robust* output regulation. Within this part we assume that y = e (i.e. $C_m = C_r =: C$ and $Q_m = Q_r =: Q$) and that the number of inputs and outputs is the same $(u \in \mathbb{R}^m, e \in \mathbb{R}^m)$.

Robust linear regulator. A fixed controller of the form (6) is a robust regulator at $\{A_0, B_0, C_0, P_0, Q_0\}$ if: (i) it solves the problem of output regulation for the plant characterized by the nominal set of parameters $\{A_0, B_0, C_0, P_0, Q_0\}$,

(ii) it solves the problem of output regulation for each perturbed set of parameters $\{A, B, C, P, Q\}$, as long as the latter is such that the corresponding unforced closed loop system remains stable, i.e. is such that the matrix

$$\left(\begin{array}{cc}
A & BH\\
GC & F
\end{array}\right)$$
(13)

has all eigenvalues with negative real part.

Notice, that if (F, G, H) are the matrices of a robust regulator, then by Lemma 1 equations (12) must be solvable for any perturbed $\{A, B, C, P, Q\}$ as long as (13) remains stable. If we denote $\Gamma = H\Sigma$ and consider only the first and the last equation of (12), we conclude that the equations

$$\Pi S = A\Pi + B\Gamma + P$$

$$0 = C\Pi + Q$$
(14)

must be solvable for any $\{A, B, C, P, Q\}$ specified in (ii). In particular they must be solvable for the fixed matrices ($\{A_0, B_0, C_0\}$ and any P, Q. Equations (14) are called the *regulator equations*.

The main result concerning the existence of a robust regulator is contained in the following theorem.

Theorem 1 Consider the plant

$$\dot{x} = Ax + Bu + Pw \tag{15}$$
$$e = Cx + Qw$$

with exosystem (5). Suppose the eigenvalues of S have nonnegative real parts. There exists a robust regulator at $\{A_0, B_0, C_0, P_0, Q_0\}$ if and only if the following conditions hold

A1 the pair (A_0, B_0) is stabilizable, the pair (C_0, A_0) is detectable

A2 the matrix

$$\begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix}$$
(16)

has independent rows for each λ , which is an eigenvalue of S.

Condition A1 is a natural necessary condition discussed above. It is known from linear control theory that A2 is equivalent to the solvability of equations (14) for the fixed matrices $\{A_0, B_0, C_0\}$ and any P, Q. Thus it readily follows that A2 is a necessary condition of a robust regulator existence. The proof of sufficiency of conditions A1, A2 is omitted here, see [1].

Remark. Conditions A1, A2 concern the existence of solutions of the regulator equations (14) for fixed $\{A_0, B_0, C_0\}$ and any P, Q. The solvability of the regulator equations for some pair P, Q can be expressed in geometric terms as a certain property of the largest controlled invariant subspace V^* of the extended system

$$egin{array}{rcl} \dot{x}&=&Ax&+&Pw+Bu\ \dot{w}&=&&Sw\ e&=&Cx&+&Qw=C_e\left(egin{array}{c}x\w\end{array}
ight) \end{array}$$

contained in $ker(C_e)$. Details can be found in [1]. See also [11] for extensive treatment of the geometric approach to linear multivariable control.

We proceed with the construction of a robust regulator. Without loss of generality, suppose that the matrix S has been transformed into a block-diagonal matrix of the form

$$S = \left(egin{array}{cc} * & 0 \\ 0 & S_{min} \end{array}
ight)$$

in which S_{min} is a matrix whose characteristic polynomial coincides with the minimal polynomial¹ of S. Let q denote the dimension of S_{min} and let Φ be a $qm \times qm$ matrix defined as

$$\Phi = \left(egin{array}{ccccc} S_{min} & 0 & \cdots & 0 \ 0 & S_{min} & \cdots & 0 \ dots & dots & \ddots & dots \ 0 & 0 & \cdots & S_{min} \end{array}
ight).$$

(the block S_{min} is repeated m times in Φ , where m is the number of input and output components of (4). Let N and Γ be matrices, of dimensions $qm \times m$ and $m \times qm$ respectively, such that the pair (Φ, N) is controllable and the pair (Γ, Φ) is observable. Such matrices exist due to the special structure of Φ . Finally, choose matrices K, L and M such that

$$\left(\begin{array}{cc} A_0 & B_0(\Gamma \ M) \\ \left(\begin{array}{c} N \\ L \end{array}\right) C_0 & \left(\begin{array}{c} \Phi & 0 \\ 0 & K \end{array}\right) \end{array}\right)$$

has all eigenvalues with negative real part. Such matrices exist due to the assumptions A1, A2 and the construction of N, Γ (this is not trivial, but the explanation is omitted here). Using matrices Φ , N, Γ and K, L, M thus defined, we construct a robust regulator of the form (6) with

$$F = \left(egin{array}{cc} \Phi & 0 \\ 0 & K \end{array}
ight), \quad G = \left(egin{array}{c} N \\ L \end{array}
ight), \ H = (\Gamma \quad M).$$

One may notice that the regulator consists of two components:

$$\dot{\xi}_1 = \Phi \xi_1 + Ne \tag{17}$$

$$= K\xi_2 + Le \tag{18}$$

$$u = 1\xi_1 + M\xi_2.$$

The component (17) provides a control signal capable of keeping x in the set $\{x|x = \Pi w\}$ (if the system has been initiated on this set), on which the error is zero. The second component (18) makes the trajectories on this set globally asymptotically attractive. The matrix Φ contains m "copies" of the exosystem, in other words it incorporates an "internal model" of the exosystem.

 ξ_2

Theorem 1 can be extended to the case of different number of input and output components. In this case a robust regulator may exist only if the number of output components does not exceed the number of input components. In general the so-called *Internal Model Principle* holds [11]:

A regulator synthesis is structurally stable only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably duplicated model of the dynamic structure of the exogeneous signals which the regulator is required to process. Thus a robust regulator exists only if the measured variable includes the regulated variable. The treatment of the linear regulator problem for the case when the regulated and the measured outputs do not coincide can be found in [11].

The internal model principle shows the limitations of linear controllers. Namely, if the exosystem contains uncertainties (which is the same to say that the exosystem is an unknown representative of a known

¹A polynomial $d(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0$ is called minimal for a square matrix S if $d(S) = S^k + a_{k-1}S^{k-1} + \cdots + a_0I = 0$ and $d(\lambda)$ has the lowest degree among the polynomials having this property. A minimal polynomial always exists and is unique. Its degree is less than or equal to the dimension of S. See [3] or any other good book on linear algebra for details.

family of exosystems), there exist no linear controller solving the problem of robust output regulation. Otherwise such controller must have contained internal models of all the exosystems from the family. This, if the family has an infinite number of entries, is not possible for a finite dimensional linear controller. Exosystems generating harmonic signals of unknown frequencies is a simple example of such situation. Nonlinear controllers may allow to overcome this problem. For example in [9] an adaptive controller for tracking harmonic signals of unknown frequencies is presented.

3 Output regulation for nonlinear systems

In this section we consider the problem of local output regulation of nonlinear systems given by the equations

$$\dot{x} = f(x, u, w)$$

$$e = h(x, w).$$
(19)

It is assumed that e, being the regulated and measured output, has the same number of components as the control input $(u \in \mathbb{R}^m, e \in \mathbb{R}^m)$. The disturbance w is generated by the exosystem

$$\dot{w} = s(w). \tag{20}$$

The major assumption on the exosystem used in the sequel is that (20) is *neutrally stable*. This means that the equilibrium w = 0 is a stable equilibrium (in the sense of Lyapunov) of (20) and the system is Poisson stable. Recall that a system is called Poisson stable if for any w_0 the solution $\phi_t(w_0)$ starting at w_0 is defined for all $t \in \mathbb{R}$ and for every neighborhood V of w_0 and every T > 0 there exist $t_1 > T$, $t_2 < -T$ such that $\phi_{t_1}(w_0), \phi_{t_2}(w_0) \in V$. In particular neutral stability implies that the exosystem has a critically stable linearization. An important representative of neutrally stable exosystems is a linear system generating harmonic signals w(t).

The problem of Local Output Regulation can be formally posed in the following terms. Given a nonlinear system of the form (19) with exosystem (20) find, if possible, a controller of the form

$$\xi = \eta(\xi, e)$$
(21)
$$u = \theta(\xi)$$

such that:

(a) the equilibrium $(x,\xi) = (0,0)$ of the unforced closed loop system

$$\dot{x} = f(x, \theta(\xi), 0)$$

$$\dot{\xi} = \eta(\xi, h(\xi, 0))$$
(22)

is locally asymptotically stable in the first approximation (internal stability), (b) the forced closed loop system

$$\dot{x} = f(x, \theta(\xi), w)$$

$$\dot{\xi} = \eta(\xi, h(\xi, w))$$

$$\dot{w} = s(w)$$

$$(23)$$

is such that

 $\lim_{t\to\infty} e(t) = 0$

for initial conditions $(x(0), \xi(0), w(0))$ in a neighborhood of the equilibrium (0, 0, 0) (local output regulation).

To proceed further define the following matrices

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} \end{bmatrix}_{(0,0,0)} \quad B = \begin{bmatrix} \frac{\partial f}{\partial u} \end{bmatrix}_{(0,0,0)} \quad C = \begin{bmatrix} \frac{\partial h}{\partial x} \end{bmatrix}_{(0,0)}$$

$$P = \begin{bmatrix} \frac{\partial f}{\partial w} \end{bmatrix}_{(0,0,0)} \quad Q = \begin{bmatrix} \frac{\partial h}{\partial w} \end{bmatrix}_{(0,0)} \quad S = \begin{bmatrix} \frac{\partial s}{\partial w} \end{bmatrix}_{(0)}$$

$$F = \begin{bmatrix} \frac{\partial \eta}{\partial \xi} \end{bmatrix}_{(0,0)} \quad G = \begin{bmatrix} \frac{\partial \eta}{\partial \varepsilon} \end{bmatrix}_{(0,0,0)} \quad H = \begin{bmatrix} \frac{\partial \theta}{\partial \xi} \end{bmatrix}_{(0)},$$
(24)

and denote the dimension of the state space of controller (21) as ν , i.e. $\xi \in \mathbb{R}^{\nu}$.

It is readily seen that the first approximation of (22) is given by

$$\dot{x} = Ax + BH\xi$$

$$\dot{\xi} = F\xi + GCx.$$
(25)

Since by condition (a) system (25) must be stable, i.e. the matrix

$$J = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$$
(26)

is required to have all eigenvalues with negative real part, then (A, B) must be stabilizable and (C, A) must be detectable. The output regulation property of the forced closed loop system (23) relies on the following Lemma, which is a nonlinear analog of Lemma 1.

Lemma 2 Consider the closed loop system (23). Suppose the exosystem is neutrally stable. Suppose the Jacobian matrix (26) has all the eigenvalues with negative real part. Then

$$\lim_{t\to\infty}e(t)=0$$

for each initial condition $(x(0), \xi(0), w(0))$ in a neighborhood of the equilibrium (0, 0, 0) if and only if there exist mappings $\pi : W_0 \to \mathbb{R}^n$ and $\sigma : W_0 \to \mathbb{R}^{\nu}$ (where $W_0 \subset W$ is a neighborhood of w = 0), with $\pi(0) = 0$ and $\sigma(0) = 0$ such that

$$\frac{\partial \pi}{\partial w}s(w) = f(\pi(w), \theta(\sigma(w)), w)$$

$$\frac{\partial \sigma}{\partial w}s(w) = \eta(\sigma(w), 0)$$

$$0 = h(\pi(w), w)$$
(27)

for all $w \in W_0$.

The idea of this lemma is based on Center Manifold Theory. Indeed, consider the forced closed loop system (23) and note that the Jacobian of the right-hand side, at the equilibrium $(x, \xi, w) = (0, 0, 0)$, has the following form

$$\left(egin{array}{ccc} A & BH & * \ GC & F & * \ 0 & 0 & S \end{array}
ight) = \left(egin{array}{ccc} J & * \ 0 & S \end{array}
ight)$$

where J is a matrix with all eigenvalues with negative real part, and S is a matrix with all eigenvalues with zero real part. Thus the system in question has a center manifold \mathcal{M}_c (defined in some neighborhood of the origin), which is the graph of the mapping $(x,\xi) = (\pi(w), \sigma(w))$. The first two equations of (27) are equivalent to invariance of the center manifold, while the third one claims that the regulation error e is zero on \mathcal{M}_c . Due to the assumptions of the lemma the center manifold is locally attractive. Thus every solution of (23) starting close enough to the origin converges to the set \mathcal{M}_c , on which the error eis zero.

Remark 1. Although Lemma 2 is formulated for the case when the measured and regulated outputs coincide $(y \equiv e)$, it is possible to alter the conditions of the lemma to include the case $y \neq e$.

Remark 2. There is still one open question for me in this lemma (as well as in subsequent results based on Lemma 2). The proof of sufficiency in Lemma 2 is done by presenting a center manifold being the graph of the mapping $(x, \xi) = (\pi(w), \sigma(w))$, on which the error e is zero. Since the center manifold is attractive, the claim is that all solutions will converge to this manifold. But in general a center manifold is not unique. What happens if there is another center manifold on which the error is not zero? On which of these manifolds will a solution converge (since all of them are attractive)?

One possible way to avoid this unpleasant situation in the proof of the sufficiency is to demand that all

solutions of the first two equations in (27), satisfying $\pi(0) = 0$, $\sigma(0) = 0$, must also satisfy the third equation. Or instead, as a simple corollary of the previous statement, one can demand uniqueness of solutions of the first two equations in (27), satisfying $\pi(0) = 0$, $\sigma(0) = 0$. But such corrections make the sufficient conditions much harder to check.

Notice, that if there exists a controller solving the local output regulation problem then, by Lemma 2, there necessarily exist mappings $\pi : W_0 \to \mathbb{R}^n$ and $c : W_0 \to \mathbb{R}^m$ (where $W_0 \subset W$ is a neighborhood of w = 0), with $\pi(0) = 0$ and c(0) = 0 such that

$$\frac{\partial \pi}{\partial w}s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$
(28)

for all $w \in W_0$. To conclude that, it suffices to set $c(w) = \theta(\sigma(w))$ in the first equation of (27).

Equations (28) are called the *regulator equations*. The first of these equations expresses the property that the graph of the mapping $x = \pi(w)$ is an invariant manifold of the composite system

$$\dot{x} = f(x, c(w), w)$$

$$\dot{w} = s(w),$$
(29)

while the second expresses the property that the error map e = h(x, w) is zero at each point of this invariant manifold. Let $w^*(t)$ denote the exogenous output corresponding to the initial condition w^* . If the initial state of the plant is precisely

$$x^* = \pi(w^*)$$

and the input to the plant is precisely equal to

$$u^*(t) = c(w^*(t))$$

then $x(t) = \pi(w^*(t))$ and due to the second equation in (28) we have e(t) = 0 for all $t \ge 0$. This argument shows that the control input generated by the autonomous system

$$\dot{w} = s(w)$$
 (30)
 $u = c(w)$

from the initial state $w(0) = w^*$ is precisely the input needed to obtain, for the corresponding exogenous input $w^*(t)$, a response producing an identically zero error (provided that the initial condition of the plant is appropriately set, i.e. $x^* = \pi(w^*)$).

This interpretation leads to the intuition that a controller solving the problem of local output regulation must generate an input consisting of two components: the first component $u^*(t) = c(w^*(t))$ capable of yielding e(t) = 0 for all t whenever the initial state of the system is appropriately set (namely $x^* = \pi(w^*)$), and the second component capable of rendering the particular trajectory $x^*(t) = \pi(w^*(t))$ locally exponentially stable.

The property of a controller to be able to generate its output u equal to the one generated by the system (30) can be formalized in the notion of *immersion*. Let $\{X, f, h\}$ denote the autonomous system

$$\dot{x} = f(x)$$
 (31)
 $y = h(x),$

with state $x \in X$ and output $y \in \mathbb{R}^m$, in which we suppose f to be a smooth vector field and h a smooth mapping, with f(0) = 0 and h(0) = 0.

System immersion. System $\{X, f, h\}$ is immersed into the system $\{X', f', h'\}$ if there exist a smooth mapping $\tau : X \to X'$, satisfying $\tau(0) = 0$ and

$$h(x) \neq h(z) \rightarrow h'(\tau(x)) \neq h'(\tau(z)),$$

such that

$$egin{array}{rcl} rac{\partial au}{\partial x}f(x)&=&f'(au(x))\ h(x)&=&h'(au(x)) \end{array}$$

for all $x \in X$.

The two conditions in this definition express the property that any output response generated by $\{X, f, h\}$ is also an output response of $\{X', f', h'\}$. Recalling the discussion above we can conclude, that system (30) must be immersed into any controller solving the local output regulation problem.

Now we are ready to formulate necessary and sufficient conditions for the solvability of the local output regulation problem.

Theorem 2 Consider the plant (19) with exosystem (20). Suppose the exosystem is neutrally stable. The problem of local output regulation is solvable if and only if there exist mappings $x = \pi(w)$ and u = c(w), with $\pi(0) = 0$ and c(0) = 0, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$
(32)

for all $w \in W_0$ and such that the autonomous system with output

$$\dot{w} = s(w)$$
 (33)
 $u = c(w)$

is immersed into a system

$$\dot{\xi} = \phi(\xi)$$

$$u = \gamma(\xi),$$
(34)

defined on a neighborhood Ξ_0 of the origin in \mathbb{R}^{ν} , in which $\phi(0) = 0$ and $\gamma(0) = 0$ and the two matrices

$$\Phi = \left[\frac{\partial\phi}{\partial\xi}\right]_{\xi=0}, \quad \Gamma = \left[\frac{\partial\gamma}{\partial\xi}\right]_{\xi=0} \tag{35}$$

are such that the pair

$$\begin{pmatrix} A & 0 \\ NC & \Phi \end{pmatrix}, \quad \begin{pmatrix} B \\ 0 \end{pmatrix}$$
(36)

is stabilizable for some choice of the matrix N, and the pair

$$(C \quad 0), \quad \left(\begin{array}{cc} A & B\Gamma \\ 0 & \Phi \end{array}\right) \tag{37}$$

is detectable.

Remark 1. Stabilizability/detectability conditions on the pairs (36) and (37) implicitly include conditions of stabilizability of the pairs (A, B), (Φ, N) and detectability of the pairs (C, A) and (Γ, Φ) .

Remark 2. Solvability of the regulator equations (32) can be expressed as a certain property of the zero dynamics of the extended system

$$\dot{x} = f(x, u, w)$$

 $\dot{w} = s(w)$
 $e = h(x, w).$

See Appendix or [1] for details.

The controller solving the problem is constructed in the following way. Let N be defined as in the theorem and the matrices K, L, M are chosen such that

$$\left(\begin{array}{cc} \begin{pmatrix} A & B\Gamma \\ NC & \Phi \end{pmatrix} & \begin{pmatrix} B \\ 0 \end{pmatrix} M \\ L(C & 0) & K \end{array}\right)$$

has all eigenvalues with negative real part. Such matrices exist due to conditions (36), (37). Then the controller is constructed as

$$\dot{\xi}_{0} = K\xi_{0} + Le
\dot{\xi}_{1} = \phi(\xi_{1}) + Ne
u = M\xi_{0} + \gamma(\xi_{1}).$$
(38)

As it has already been mentioned, the controller consists of two parts. The first one, corresponding to ξ_1 , provides the control signal capable of keeping the state of the system on the manifold with zero error (if the system is initiated on this manifold). The second component, corresponding to ξ_0 , makes the trajectories on this manifold locally exponentially attractive.

The local nature of the obtained results is explained by the fact that in general a solution of the regulator equations (28) and the correspondent center manifold are defined only locally. If a global solution of the regulator equations and a global *internal model* (the system $\{\Xi_0, \phi, \gamma\}$ into which $\{W, s, c\}$ is immersed) are found, then the closed-loop system

$$egin{aligned} \dot{x} &= f(x,\gamma(\xi),w) \ \dot{\xi} &= \phi(\xi) \end{aligned}$$

possesses a globally defined invariant manifold on which the regulated output e is zero. Thus the problem in question reduces to the problem of rendering this invariant manifold attractive. If the set of attraction of this manifold is required to be the whole space, then the problem is referred to as the global output regulation problem. If the set of attraction must contain any *a priori fixed* set, then the problem is called the semiglobal output regulation problem. See [1] for results concerning these two problems.

4 Approximate output regulation for nonlinear systems

Theorem 2 gives necessary and sufficient conditions for the solvability of the output regulation problem in the form of 'existence'-like conditions. As it has been shown in the previous section, to construct a controller solving the problem, one has to solve the mixed partial differential/algebraic regulator equations (32) and find a correspondent internal model (34) satisfying stabilizability/detectability conditions (36), (37). In general, such construction of an internal model is not an easy task, since it requires (at least) solving the regulator equations (to solve a partial differential equation (PDE) is not an easy task by itself). A usual approach in dealing with PDEs is to find approximate instead of exact solutions. This suggests the idea of using an approximate internal model, to the purpose of obtaining *approximate* output regulation. Several results were obtained in this direction.

In [1] the approximate output regulation problem is formulated and solved in the following way. It is shown, that if the exosystem is linear ($\dot{w} = Sw$) and neutrally stable it can be shown that for any integer p > 0 it is always possible to find a *linear* internal model (of suitable dimension depending on p) with the property that the corresponding controller yields an error, which asymptotically converges to a function $\tilde{e}(t)$ satisfying an estimate of the form

$$\|\tilde{e}(t)\| \leq E(\|w(t)\|),$$

where $E: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function such that

$$\lim_{r \to 0^+} \frac{E(r)}{r^p} = 0 \tag{39}$$

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i.e. is infinitesimal of order higher than p as $r \to 0$.

The controller giving such approximate output regulation is constructed in the following way. First, let \mathcal{P} be the set of all polynomials of degree less than or equal to p in the variables w_1, \ldots, w_r with coefficients in \mathbb{R}^m and vanishing at w = 0. \mathcal{P} is indeed a finite dimensional vector space over \mathbb{R} . If s(w) is linear in w and $k(w) \in \mathcal{P}$, then

$$L_sk(w)=rac{\partial k}{\partial w}s(w)$$

is still a polynomial in \mathcal{P} . Observe that the mapping

$$D: \mathcal{P} \rightarrow \mathcal{P} \ k(w) \rightarrow rac{\partial k}{\partial w} s(w)$$

is a linear mapping from a finite dimensional vector space to itself and let

$$d(\lambda) = \lambda^q - a_{q-1}\lambda^{q-1} - \dots - a_1\lambda - a_0$$

denote its minimal polynomial. Let $\Phi_p \in \mathbb{R}^{q \times q}$ be a matrix having minimal polynomial $d(\lambda)$. Due to the structure of Φ_p it is always possible to find a vector $N_p \in \mathbb{R}^{q \times 1}$ and a vector $\Gamma_p \in \mathbb{R}^{1 \times q}$ such that the pair (Φ_p, N_p) is controllable and the pair (Γ_p, Φ_p) is observable. Using the triplet Φ_p, N_p, Γ_p thus determined, set

$$\begin{split} \Phi &= \left(\begin{array}{cccc} \Phi_p & 0 & \cdots & 0 \\ 0 & \Phi_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p \end{array} \right), \\ N &= \left(\begin{array}{cccc} N_p & 0 & \cdots & 0 \\ 0 & N_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & N_p \end{array} \right), \\ \Gamma &= \left(\begin{array}{cccc} \Gamma_p & 0 & \cdots & 0 \\ 0 & \Gamma_p & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Gamma_p \end{array} \right), \end{split}$$

where $\Phi \in \mathbb{R}^{mq \times mq}$, $N \in \mathbb{R}^{mq \times m}$, $\Gamma \in \mathbb{R}^{m \times mq}$. The triplet (Φ, N, Γ) defines a system consisting of the aggregate of *m* identical copies of the linear system characterized by the triple (Φ_p, N_p, Γ_p) .

Suppose now that the pair (A, B) is stabilizable and the pair (C, A) is detectable, and the matrix

$$\left(\begin{array}{cc}
A - \lambda I & B \\
C & 0
\end{array}\right)$$
(40)

is nonsingular for every λ which is a root of $d(\lambda)$. Then it is possible to choose K, L, M such that the matrix

$$\left(\begin{array}{cc} \left(\begin{array}{cc} A & B\Gamma \\ NC & \Phi \end{array}\right) & \left(\begin{array}{cc} B \\ 0 \end{array}\right)M \\ L(C & 0) & K \end{array}\right)$$

has all eigenvalues with negative real part.

Then the controller solving the problem of approximate local output regulation (as specified above) is given by

$$\begin{array}{rcl} \xi_1 &=& \Phi\xi_1 + Ne\\ \dot{\xi}_2 &=& K\xi_2 + Le\\ u &=& \Gamma\xi_1 + M\xi_2. \end{array}$$

The conditions, under which we can construct this controller solving the problem, are: linearity and neutral stability of the exosystem, stabilizability/detectability of the pairs (A, B) and (C, A), and non-singularity of the matrix (40) for every λ , which is a root of $d(\lambda)$.

Another approach to approximate output regulation is based on approximation of the regulator equation solutions by the Taylor series expansion [4]. The corresponding approximate solution is then used in construction of a feedback law yielding the approximate output regulation. The *approximate output regulation problem* is defined in [10] in the following way: Given $\epsilon > 0$, design a state feedback control law of the form $u = \psi(x, w)$ such that the closed-loop system

$$\dot{x}=f(x,\psi(x,w),w)$$
 $\dot{w}=s(w)$

has the property that, for all sufficiently small initial conditions x_0 and w_0 , the closed-loop system has a bounded solution for all $t \ge 0$ and

$$\limsup_{t\to\infty} \|h(x(t),w(t))\| \le \epsilon.$$

Both approaches presented above suffer from the following drawback. The controllers designed in those ways can achieve satisfactory output regulation only if w(t) is small. This is, for example in the second case, because the Taylor-theorem based approximation is valid only in a sufficiently small neighborhood of the origin. If there exist an exact solution of the regulator equation and a correspondent internal model, which are defined globally (and thus the output regulation problem can be solved globally), then a regulator, achieving approximate output regulation only locally, is not satisfactory.

A method to solve the problem of approximate output regulation avoiding (partially) the above mentioned drawback is given in [10]. It is based on a neural network approach. This approach allows to approximate a given function $f \in C^k$ over any given compact set Γ by a function \hat{f} of a special form. Finite order derivatives of \hat{f} up to order k also approximate the correspondent derivatives of f over Γ . Application of this methodology to the regulator equations gives the results presented in [10].

Prior to formulating these results consider the special form, in which approximations of the regulator equation solutions will be found. Let $\varphi \in C^k(\mathbb{R})$, $k \geq 2$, be non-constant and bounded real-valued function. Consider functions $\hat{\pi}(w)$ and $\hat{c}(w)$ in the following form:

$$\hat{\pi}(w) = \begin{bmatrix} \sum_{j=1}^{N^{\pi}} \alpha_{1j}^{\pi} \varphi(\beta_{1j}^{\pi} w + b_{1j}^{\pi}) \\ \vdots \\ \sum_{j=1}^{N^{\pi}} \alpha_{nj}^{\pi} \varphi(\beta_{nj}^{\pi} w + b_{nj}^{\pi}) \end{bmatrix}$$

$$\hat{c}(w) = \begin{bmatrix} \sum_{j=1}^{N^{c}} \alpha_{1j}^{c} \varphi(\beta_{1j}^{c} w + b_{1j}^{c}) \\ \vdots \\ \sum_{j=1}^{N^{c}} \alpha_{mj}^{c} \varphi(\beta_{mj}^{c} w + b_{mj}^{c}) \end{bmatrix},$$
(41)

where N^{π} , N^{c} are integers, α_{1j}^{π} , ..., α_{nj}^{π} , α_{1j}^{c} , ..., α_{mj}^{c} , b_{1j}^{π} , ..., b_{nj}^{c} , b_{1j}^{c} , ..., b_{mj}^{c} are scalars and β_{1j}^{π} , ..., β_{nj}^{π} , β_{1j}^{c} , ..., β_{mj}^{c} are q-dimensional row vectors (q is the dimension of the exosystem). Such mappings (41) are called three-layer feedforward neural networks, where w is the input, $\hat{\pi}$ and \hat{c} – the outputs, N^{π} and N^{c} – the number of hidden neurons, and all the rest parameters are called the weights of the neural network.

The following lemma proves the possibility of approximating the solutions of the regulator equations by functions of the form (41) with arbitrarily small inaccuracy.

Lemma 3 Let $\pi(w)$, c(w) be a solution of the regulator equations (28), defined in a neighborhood W of the origin, and $\pi(w) \in C^k(W)$, $c(w) \in C^k(W)$. Let $Q = \{\pi(w), c(w), w) | w \in W\}$, G an open, connected subset of Q relatively compact in Q, and W_G the projection of G onto W. Then, given any $\epsilon_r > 0$, there exist two functions $\hat{\pi}(w) \in C^k(W_G)$ and $\hat{c}(w) \in C^k(W_G)$ of the form (41) such that $\hat{\pi}(0) = 0$, $\hat{c}(0) = 0$ and satisfying, for all $w \in W_G$,

$$\left\|\frac{\partial \hat{\pi}(w)}{\partial w}s(w) - f(\hat{\pi}(w), \hat{c}(w), w)\right\| < \epsilon_r$$
(42)

$$\|h(\hat{\pi}(w), w)\| < \epsilon_r \tag{43}$$

By this lemma we see that $\pi(w)$ and c(w) can be approximated by the functions of the form (41) not only in some neighborhood of the origin, but in any compact set inside the domain of $\pi(w)$ and c(w). This is a more 'global' result compared to approximations based on the Taylor theorem. The inaccuracy ϵ_r can be translated into the inaccuracy of the approximate output regulation, as stated in the following theorem.

Theorem 3 Suppose the exosystem is neutrally stable, the matrix K is chosen such that A + BK is stable, where $A = \frac{\partial f}{\partial x}(0,0,0)$, $B = \frac{\partial f}{\partial u}(0,0,0)$. Such K exists if (A,B) is stabilizable. Let $\hat{\pi}(w)$, $\hat{c}(w)$ satisfy (42), (43) for some sufficiently small ϵ_r . Then,

(i) For all sufficiently small x_0 , w_0 , the closed loop system

$$\dot{x} = f(x, u, w) \tag{44}$$

 $u = \hat{c}(w) + K(x - \hat{\pi}(w))$ (45)

$$\dot{w} = s(w)$$
 (46)

has a unique bounded solution x(t) defined for all $t \ge 0$, and

(ii) There exist M > 0 such that

$$\limsup_{t \to \infty} \|h(x(t), w(t)\| < M\epsilon_r.$$
(47)

Although approximations $\hat{\pi}(w)$ and $\hat{c}(w)$ can be defined in a 'rather big' compact set W_G the result of the theorem is still local. This is because of the linear stabilizing feedback term $K(x - \hat{\pi}(w))$, which in general can stabilize a nonlinear system only locally. Probably, other stabilizing feedbacks, which act more 'globally', may bring approximate output regulation for initial conditions from an a priori given fixed set.

Lemma 3 states only the possibility to approximate $\pi(w)$ and c(w) by functions of the form (41), but does not provide a procedure how to find the approximations. In order to describe this procedure let us first introduce some notations. Denote the weights of the neural networks (41) for both $\hat{\pi}(w)$ and $\hat{c}(w)$ as γ . The dimension S_N of the vector γ is determined by the numbers N^{π} and N^c respectively. To explicitly indicate the reliance of the neural network approximations on the weights, we will adopt the notation $\hat{\pi}(w, \gamma)$ and $\hat{c}(w, \gamma)$ in the sequel. By Lemma 3, for some given $\epsilon_r > 0$, there exist numbers N^{π} , N^c and a parameter vector $\hat{\gamma} \in \mathbb{R}^{S_N}$ such that

$$\left\|\frac{\partial \hat{\pi}(w,\gamma)}{\partial w}s(w) - f(\hat{\pi}(w,\gamma),\hat{c}(w,\gamma),w)\right\| \leq \epsilon_{r} \\ \|h(\hat{\pi}(w,\gamma),w)\| \leq \epsilon_{r},$$
(48)

for all $w \in \Gamma$, where $\Gamma \subset W_G$ is some compact subset of \mathbb{R}^q .

Next let $J(\gamma, w) =$

$$\left\|\frac{\partial\hat{\pi}(w,\gamma)}{\partial w}s(w) - f(\hat{\pi}(w,\gamma),\hat{c}(w,\gamma),w)\right\|^2 + \|h(\hat{\pi}(w,\gamma),w)\|^2.$$
(49)

Clearly, if for some $\hat{\gamma}$

$$\sup_{w \in \Gamma} J(\hat{\gamma}, w) \le \epsilon_r^2 \tag{50}$$

both inequalities (48) will be satisfied.

Since $J(\gamma, w)$ depends on both γ and w, there is no effective numerical method to solve (50). To circumvent this difficulty, we discretize (50) by letting

$$Q(\gamma) = \sum_{w \in \Gamma_d} J(\gamma,w),$$

where Γ_d is a subset of Γ consisting of finitely many elements of Γ . If for some N^{π} , N^c and $\hat{\gamma} \in \mathbb{R}^{S_N}$

$$Q(\hat{\gamma}) < \epsilon_r^2$$

then we have inequalities (48) satisfied for all $w \in \Gamma_d$. When Γ_d is sufficiently dense in Γ , we have a reason to believe that thus defined $\hat{\pi}(w, \hat{\gamma})$ and $\hat{c}(w, \hat{\gamma})$ are good approximations of solutions of (48) for all $w \in \Gamma$.

Since, for each fixed N^{π} and N^{c} , $Q(\gamma)$ relies only on the parameter γ , the optimal weights that minimize $Q(\gamma)$ can be searched by any minization technique. For example, applying the steepest descent method leads to the following update law of the weight vector:

$$\gamma_{j+1} = \gamma_j - \lambda_j rac{\partial Q(\gamma_j)}{\partial \gamma},$$

j = 0, 1, ..., with λ_j being the step size. Thus, the problem of looking for the approximation solution of the regulator equations is converted into a parameter optimization problem.

Though gradient-based methods do not necessarily lead to a weight that minimize $Q(\gamma)$, there is no need, in practice, to really search for the optimal weight. What are needed are some values of N^{π} , N^{c} and weight γ that make $Q(\gamma)$ sufficiently small. Of course, the particular values of N^{π} , N^{c} are not known *a priori*. Therefore, iteration on N^{π} , N^{c} is often inevitable.

5 Output regulation for nonlinear systems with uncertainties

In many situations regulated nonlinear systems contain uncertainties. In this section we consider a nonlinear plant modeled by equations of the form (19) in which we explicitly introduce a vector $\mu \in \mathbb{R}^p$ of unknown parameters, which are constant in time, that is

$$\dot{x} = f(x, u, w, \mu)$$

$$e = h(x, w, \mu).$$
(51)

Without loss of generality, we suppose $\mu = 0$ is the nominal value of the uncertain parameters and we also assume $f(x, u, w, \mu)$ and $h(x, w, \mu)$ to be C^k functions of their arguments. Moreover we assume $f(0, 0, 0, \mu) = 0$ and $h(0, 0, \mu) = 0$ for each value of μ .

The problem of output regulation for nonlinear systems with uncertainties can be described as follows.

Structurally stable output regulation. Given a nonlinear system of the form (51) with exosystem

$$\dot{w} = s(w) \tag{52}$$

find a controller of the form (21) such that for some neighborhood \mathcal{M} of $\mu = 0$ in \mathbb{R}^p and for each $\mu \in \mathcal{M}$: (a) the equilibrium $(x,\xi) = (0,0)$ of the unforced closed loop system

$$\dot{x} = f(x, \theta(\xi), 0, \mu)$$

$$\dot{\xi} = \eta(\xi, h(\xi, 0))$$
(53)

is locally asymptotically stable in the first approximation (internal stability),

(b) the forced closed loop system

$$\dot{x} = f(x, \theta(\xi), w, \mu)$$

$$\dot{\xi} = \eta(\xi, h(\xi, w))$$

$$\dot{w} = s(w)$$
(54)

is such that

$$\lim_{t\to\infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w(0))$ in a neighborhood of the equilibrium (0, 0, 0) (local output regulation).

To formulate the main result on the structurally stable output regulation problem let us first introduce the following notation:

$$A(\mu) = \left[rac{\partial f}{\partial x}
ight]_{(0,0,0,\mu)}, \quad B(\mu) = \left[rac{\partial f}{\partial u}
ight]_{(0,0,0,\mu)}, \quad C(\mu) = \left[rac{\partial h}{\partial x}
ight]_{(0,0,\mu)}.$$

Theorem 4 Consider the plant (51) with exosystem (52). Suppose the exosystem is neutrally stable. The problem of structurally stable local output regulation is solvable if and only if there exist mappings $x = \pi^a(w,\mu)$ and $u = c^a(w,\mu)$, with $\pi^a(0,\mu) = 0$ and $c^a(0,\mu) = 0$, both defined in a neighborhood $W_0 \times \mathcal{M} \subset W \times \mathbb{R}^p$ of the origin, satisfying the conditions

$$\frac{\partial \pi^a(w,\mu)}{\partial w}s(w) = f(\pi^a(w,\mu), c^a(w,\mu), w, \mu)$$

$$0 = h(\pi^a(w,\mu), w, \mu)$$
(55)

for all $(w, \mu) \in W_0 \times \mathcal{M}$ and such that the autonomous system with output

$$\dot{w} = s(w)$$

 $\dot{\mu} = 0$ (56)
 $u = c^{a}(w,\mu)$

is immersed into a system

$$egin{array}{rcl} \dot{\xi} &=& \phi(\xi) \ u &=& \gamma(\xi), \end{array}$$

defined on a neighborhood Ξ_0 of the origin in \mathbb{R}^{ν} , in which $\phi(0) = 0$ and $\gamma(0) = 0$ and the two matrices

$$\Phi = \left[\frac{\partial\phi}{\partial\xi}\right]_{\xi=0}, \quad \Gamma = \left[\frac{\partial\gamma}{\partial\xi}\right]_{\xi=0}$$
(57)

are such that the pair

$$\begin{pmatrix} A(0) & 0 \\ NC(0) & \Phi \end{pmatrix}, \begin{pmatrix} B(0) \\ 0 \end{pmatrix}$$
(58)

is stabilizable for some choice of the matrix N, and the pair

$$(C(0) \quad 0), \quad \left(\begin{array}{cc} A(0) & B(0)\Gamma \\ 0 & \Phi \end{array}\right)$$
(59)

is detectable.

Theorem 4 is a straightforward consequence of Theorem 2. In fact one can include the vector of parameters μ into the exogeneous signal, which results in the augmented exogeneous signal $w^a = (w, \mu)^{\top}$. With this notation, the "family" of plants (51) can be viewed as a single plant modeled by equations of the form (19), namely

$$\dot{x} = f^{a}(x, u, w^{a})$$
 (60)
 $e = h^{a}(x, w^{a}).$

The augmented exosystem will have the form

$$\dot{w}^a = s^a(w^a) = \begin{pmatrix} s(w) \\ 0 \end{pmatrix}.$$
(61)

It is neutrally stable (if the initial one is neutrally stable). Thus, applying Theorem 2 to systems (60), (61) we obtain Theorem 4.

Theorem 4 establishes necessary and sufficient conditions for the existence of a controller of the form (21) which solves the problem of local output regulation for any nonlinear system in the parameterized family (51), when the parameter μ ranges over *some* neighborhood \mathcal{M} of $\mu = 0$ in the parameter space \mathbb{R}^p . The problem of *robust* local output regulation corresponds to the case of μ ranging over an a priori fixed compact set \mathcal{M}^* in the parameter space. To this end observe that if some fixed controller solves, for any μ in \mathcal{M}^* , the problem of local output regulation, then, necessary conditions of Theorem 2 must hold for every $\mu \in \mathcal{M}^*$. In particular, then, for every $\mu \in \mathcal{M}^*$, equations (55) must have a solution $x = \pi^a(w, \mu)$, $u = c^a(w, \mu)$ defined in a neighborhood W^0_{μ} of the origin in W, and the autonomous system with output

$$egin{array}{rcl} \dot{w}&=&s(w)\ \dot{\mu}&=&0\ u&=&c^a(w,\mu)\ \dot{arepsilon}&=&\phi(arepsilon) \end{array}$$

is immersed into a system

$$\dot{\xi} = \phi(\xi) \ u = \gamma(\xi),$$

(which is the same for all $\mu \in \mathcal{M}^*$) defined on a neighborhood Ξ^0 of the origin in \mathbb{R}^{ν} , with $\phi(0) = 0$ and $\gamma(0) = 0$ and the pair of matrices

$$\Gamma = \left[rac{\partial\gamma}{\partial\xi}
ight]_{\xi=0}, \quad \Phi = \left[rac{\partial\phi}{\partial\xi}
ight]_{\xi=0}$$

is detectable.

Moreover since the controller, which solves the problem, is required to stabilize for every value of μ the linear approximation of the plant at the equilibrium (x, w) = (0, 0)

$$egin{array}{rcl} \dot{x}&=&A(\mu)x+B(\mu)u\ y&=&C(\mu)x, \end{array}$$

the latter must be robustly stabilizable on \mathcal{M}^* . This means that there must exist matrices F, G, H such that

$$\left(egin{array}{ccc} A(\mu) & B(\mu)H \ GC(\mu) & F \end{array}
ight)$$

has all eigenvalues with negative real part for all $\mu \in \mathcal{M}^*$.

Sufficient conditions for the solvability of the robust local output regulation problem are based on some technical results from linear robust control theory (see [1] Lemma 4.3 for details).

A critical aspect of the design of the internal model for the purpose of achieving asymptotic regulation is the necessity of knowing exactly the parameters of the exosystem. Actually, this is the only parameter in the entire problem, to which the described method is sensitive. It is well known that, even in linear systems, if the parameters of the exosystem (for example frequencies of a harmonic oscillator) and those of the internal model do not match exactly, a sizable steady-state error may occur. New recent approaches to the design of internal models have shown that also this kind of 'sensitivity' can be eliminated and that the accurate knowledge of the parameters of exosystem is no longer a requirement (see [7], [9] for details).

6 Regulation for complex dynamical systems

The essential assumption in the local output regulation theory discussed in Sections 3-5 is the requirement of neutral stability of the exosystem. In many cases this condition is not satisfied. This happens, for instance, if the exosystem possesses a chaotic attractor in which several equilibrium points with unstable linearization are embedded. As follows from the examples below, in some cases, although the exosystem is not neutrally stable, the problem of output regulation is still solvable. Both of the following examples concern the problem of controlled synchronization, so first we shall formulate this problem.

Consider two systems

$$\dot{w} = s(w)$$
 (62)
 $y_t = h(w),$

and

$$\dot{x} = f(x, y_t, u)$$

$$y_r = h(x)$$
(63)

where both x and w are in \mathbb{R}^n . System (62) is the so-called *transmitter* and system (63) is the *receiver*. We assume that the f, s, h are sufficiently smooth and that f(0,0,0) = 0, h(0) = 0. The question is to seek a dynamic feedback of the form

$$\dot{\xi} = \eta(\xi, y_r, y_t)$$

$$u = \theta(\xi, y_r, y_t)$$

$$(64)$$

such that for the resulting closed loop dynamics (62)-(64), no matter how they are initialized, we know that asymptotically x and w will match, i.e.

$$\lim_{t \to +\infty} \|x(t) - w(t)\| = 0.$$
(65)

This problem is referred to as the *controlled synchronization problem*. If the additional requirement of closed loop stability of the unforced closed loop system

$$\dot{x} = f(x, 0, \theta(\xi, y_r, 0))$$

$$\dot{\xi} = \eta(\xi, y_r, 0)$$
(66)

is set, the problem is referred to as *controlled synchronization problem with internal stability*. Obviously, in this way controlled synchronization problems can be considered as the output regulation problems discussed in Sec. 1.

The first example concerns solving the controlled synchronization problem for Lur'e-like systems, i.e. linear systems with a nonlinear output-dependent feedback loop.

Theorem 5 Consider the transmitter

$$\dot{w} = Aw + \Psi(y_t)$$
 (67)
 $y_t = Cw$

and receiver

$$\dot{x} = Ax + \Psi(y_t) + Bu$$

$$y_r = Cx$$

$$(68)$$

with $w, x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and Ψ a mapping of appropriate dimensions, and A, B, C matrices of appropriate dimensions. Under the assumption that (C, A) is detectable and (A, B) is stabilizable, the controlled synchronization problem with internal stability is solvable.

In general, the transmitter (67) can be unstable at the origin. Thus the assumption of neutral stability of the exosystem (utilized in Sec.3 and 4) is violated, although the regulator problem is still solvable.

Another example is controlled synchronization of coupled Van der Pol systems. As transmitter dynamics we take a Van der Pol system of the form

As receiver dynamics, we take the following controlled 'copy' of (69):

$$\begin{aligned} \dot{x}_1 &= x_2 + \alpha u \\ \dot{x}_2 &= -y_t - (y_t^2 - 1)x_2 + \beta u \\ y_r &= x_1. \end{aligned}$$
 (70)

It is known that the origin is the only equilibrium point of (69), and it is an unstable focus. Moreover (69) has a unique limit cycle C that is exponentially attracting for all initial conditions $w(0) \in \mathbb{R}^2 - \{0\}$. Thus exosystem (69) does not satisfy the neutral stability assumption.

Let $\tilde{w}(t)$ be a periodic solution that starts on \mathcal{C} , and let T denote its period. Define

$$ar{p}:=rac{1}{T}\int_0^T(ilde{w}_1(t)^2-1)dt.$$

It is shown in [5] that if $\alpha = 1$, $\beta > \max\{-\bar{p}, 1\}$, then there exist such c^* that the controlled synchronization problem with internal stability is solvable by the static (high-gain) error feedback

$$u = -c(y_r - y_t)$$

for all $c > c^*$.

These two examples show that although the assumption of neutral stability of exosystem (which is an essential assumption in the theory developed in [1]) does not hold, the output regulation problem can still be solvable. This motivates further research aiming at obtaining general conditions of solvability of the output regulation problem for nonlinear systems with exosystems exhibiting complex dynamical behaviour.

7 Conclusions

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of undesired disturbances is a central problem in control theory. In this report we have reviewed some of the results regarding this problem. First, we considered the linear output regulation problem. In this case, its solvability is equivalent to the solvability of some linear matrix equations, called the regulator equations. For nonlinear systems similar results are obtained based on Center Manifold Theory. In particular, necessary and sufficient conditions for the solvability of the output regulation problem for nonlinear systems are presented. The assumption of neutral stability of the exosystem plays a crucial role in this analysis. One of these conditions is the solvability of the nonlinear regulator equations. In this case, they are mixed nonlinear algebraic/partial differential equations. Their solvability can be expressed in geometric terms, as stated in Appendix. Another concept involved in the output regulation problem for nonlinear systems is the internal model. As in the linear case, the controller solving the problem must incorporate in the feedback loop a kind of a "duplicate" of the exosystem. The same type of reasoning is applied to nonlinear systems with parametric uncertainties, resulting in necessary and sufficient conditions for the solvability of the structurally stable output regulation problem. In general, it is difficult to find a solution to the regulator equations, which is necessary for constructing the controller solving the problem. We can avoid this difficulty by solving the problem of approximate output regulation instead of exact. Several approaches for solving the approximate problem were reviewed. The assumption of neutral stability of the exosystem is crucial for the results presented. At the same time we have considered examples of nonlinear systems for which this assumption is not satisfied, but for which it is still possible to solve the output regulation problem. This fact sets further directions for research in the field of output regulation.

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Appendix

In this appendix we will consider geometric results from [1] concerning solvability of the regulator equations. For simplicity we restrict ourselves to the particular case of systems which are affine in the input:

$$\dot{x} = f(x) + g(x)u + p(x)w$$

$$e = h(x, w).$$

$$(71)$$

In this case the regulator equations have the form:

$$\frac{\partial \pi}{\partial w}s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w$$

$$0 = h(\pi(w), w).$$
(72)

It can be shown that solvability of equations (72) is closely related to the properties of the zero dynamics of the extended system Σ_e :

$$\dot{x}_e = f_e(x_e) + g_e(x_e)u$$

$$e = h_e(x_e),$$

$$(73)$$

where

$$x_e = \begin{pmatrix} x \\ w \end{pmatrix}, \quad f_e(x_e) = \begin{pmatrix} f(x) + p(x)w \\ s(w) \end{pmatrix},$$
$$g_e(x_e) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix}, \quad h_e(x_e) = h(x, w).$$
(74)

Let us first recall the notion of zero dynamics. For this purpose, consider a system affine in input:

$$\dot{x} = f(x) + g(x)u$$

$$e = h(x),$$

$$(75)$$

in which $x \in X \subset \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, f(0) = 0, h(0) = 0.

Let M be a smooth connected submanifold of X which contains the point x = 0. The submanifold M is said to be locally controlled invariant if there exist a smooth mapping $u: M \to \mathbb{R}^m$, and a neighborhood U of the origin in \mathbb{R}^n , such that the vector field $\tilde{f}(x) = f(x) + g(x)u(x)$ is tangent to M for all $x \in M \cap U$.

An output zeroing submanifold is a connected submanifold M of X which contains the origin and satisfies the following:

(i) M is locally controlled invariant

(ii) h(x) = 0 for all $x \in M$.

In other words, an output zeroing submanifold is a submanifold M of the state space with the property that for some choice of feedback control u(x) the trajectories of the closed-loop system

$$egin{array}{rcl} \dot{x}&=&f(x)+g(x)u(x)\ y&=&h(x) \end{array}$$

which start in M stay in M for some time and the corresponding output is identically zero in the meanwhile.

If M and M' are connected smooth submanifolds of X which both contain the point x = 0, we say that M locally contains M' if for some neighborhood U of the origin $M \cap U \supset M' \cap U$. An output zeroing manifold is locally maximal if, for some neighborhood U of the origin, any other output zeroing submanifold M' satisfies $M \cap U \supset M' \cap U$. The construction of a locally maximal output zeroing submanifold is illustrated in the following statement.

Proposition 1 Part 1: Define a nested sequence of subsets $M_0 \supset M_1 \supset \ldots$ of X in the following way. Set $M_0 = \{x \in X : h(x) = 0\}$. At each k > 0, suppose that, for some neighborhood U_{k-1} of 0, $M_{k-1} \cap U_{k-1}$ is a smooth manifold, let \tilde{M}_{k-1} denote the connected component of $M_{k-1} \cap U_{k-1}$ which contains the origin $(\tilde{M}_{k-1} \text{ is nonempty because } f(0) = 0)$ and define M_k as

$$M_{k} = \{x \in M_{k-1} : f(x) \in \operatorname{span}\{g_{1}(x), \dots, g_{m}(x)\} + T_{x}M_{k-1}\}.$$
(76)

Then, for some $k^* \geq 0$ and some neighborhood U_{k^*} of 0, $M_{k^*+1} = M_{k^*}$. Suppose also that

$$\dim(ext{span}\{g_1(x),\ldots,g_m(x)\})=const$$

$$\dim(\operatorname{span}\{g_1(x),\ldots,g_m(x)\}\cap T_xM_{k^*})=const$$

for all $x \in \tilde{M}_{k^*}$. Then, the manifold $Z^* = \tilde{M}_{k^*}$ is a locally maximal output zeroing submanifold.

Part 2: If, in addition,

 $\dim(ext{span}\{g_1(x),\ldots,g_m(x)\})=m$ $ext{span}\{g_1(x),\ldots,g_m(x)\}\cap T_xZ^*=0$

at x = 0, then there exists a unique smooth mapping $u^* : Z^* \to \mathbb{R}^m$ such that the vector field

 $f^*(x) = f(x) + g(x)u^*(x)$

is tangent to Z^* .

Suppose the hypothesis listed in this Proposition are satisfied. Since the vector field $f^*(x)$ is tangent to Z^* , the restriction of $f^*(x)$ to Z^* is a well-defined vector field on Z^* . In what follows by $f^*(x)$ we will always indicate the restriction of $f^*(x)$ to Z^* . The submanifold Z^* is called the (local) zero dynamics submanifold and the vector field $f^*(x)$ is called the zero dynamics vectorfield. The pair (Z^*, f^*) is called the zero dynamics of the system (75).

Let Σ denote the system

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x,0),$$

$$(77)$$

with f(x), g(x) and h(x,0) the same as in (74), that is in Σ_e . Now we can formulate the main result concerning solvability of the regulator equations (72).

Theorem 6 Suppose systems Σ and Σ_e satisfy the conditions of Proposition 1. Let (Z_e^*, f_e^*) denote the zero dynamics of Σ_e . Then there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and u = c(w), with c(0) = 0, both defined in a neighborhood $W^0 \subset W$ of 0, satisfying equations (72), if and only if the zero dynamics of Σ_e have the following properties:

i) in a neighborhood of $x_e = 0$ the set

$$M = Z_e^* \cap (X \times \{0\})$$

is a smooth submanifold.

ii) there exists a submanifold Z_s of Z_e^* , of dimension r, which contains the origin, such that

$$T_0 Z_e^* = T_0 Z_s \oplus T_0 M,$$

iii) Z_s is locally invariant under f_e^* , and the restriction of f_e^* to Z_s is locally diffeomorfic to the vector field s(w), which characterizes the exosystem.