

## Remark on a paper by J.W.P. Hirschfeld

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Remark on a paper by J.W.P. Hirschfeld

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## 1. Introduction

In the paper: Ovals in Desarguesian Planes of Even Order [1] Hirschfeld shows (theorem 4, corollary 1) that there exist no more projectively distinct ovals with representation  $D(k)$  than  $D(2)$ ,  $D(4)$  and  $D(6)$ . Below we shall show that the ovals, thus represented, are indeed mutually projectively distinct.

## 2. Preliminary considerations

In this paper we shall restrict ourselves to ovals  $O$  in  $PG(2,32)$  which have a representation of the form  $D(k)$  (for definitions and notation see [1], p. 79).

Representations  $D(k)$  of an oval  $O$  in  $PG(2,32)$  depend completely on the frame used to define a coordinate system in  $PG(2,32)$ .\*) We shall only use frames  $(P,Q,R,S)$  with  $P = (0,0,1)$ ,  $Q = (0,1,0)$ ,  $R = (1,0,0)$ ,  $S = (1,1,1)$  where  $P$ ,  $Q$ ,  $R$  and  $S$  all are points on the oval  $O$  under consideration.

Let  $O$  be an oval with representation  $D(k)$  for a suitable frame  $(P,Q,R,S)$ . Let  $a \in \gamma_0$  ( $\gamma = GF(32)$ ). Since

$$\{(1,t,t^k) \mid t \in \gamma\} = \{(1,a^{-1}t,(a^{-1}t)^k \mid t \in \gamma\} = \{(1,a^{-1}t,a^{-k}t^{-k} \mid t \in \gamma\}.$$

$D(k)$  is also the representation of  $O$  for any frame  $(P,Q,R,S')$  with  $S' \notin \{P,Q,R\}$ ,  $S'$  on  $O$  (for the frame  $(P,Q,R,S)$ :  $S' = (1,a,a^k)$ ). (1)

If  $O$  has a representation as a translation oval and this representation is  $D(k)$  then by definition ([1], p. 79) we have for  $a \in \gamma$ :

$$\{(1,t,t^k) \mid t \in \gamma\} = \{1,(t+a),(t+a)^k \mid t \in \gamma\} = \{(1,t+a,t^{k+a^k}) \mid t \in \gamma\}$$

so if  $D(k)$  is the representation of  $O$  for some frame  $(P,Q,R,S)$  it is also the representation of  $O$  for any frame  $(P,Q,R',S)$  with  $R' \in O \setminus \{P,Q,S\}$  and with (1) for any frame  $(P,Q,R',S')$  with  $\{R',S'\} \subset O \setminus \{P,Q\}$  ( $R' = (1,a,a^k)$ ). (2)

If  $O$  has representation  $D(2)$  for some frame  $(P,Q,R,S)$  then  $O$  has also representation  $D(2)$  for the frame  $(R,Q,P,S)$  since

$$\{(1,t,t^2) \mid t \in \gamma_0\} = \{(1,t^{-1},t^{-2}) \mid t \in \gamma_0\} = \{(t^2,t,1) \mid t \in \gamma_0\}$$

and the transformation  $(x_0,x_1,x_2) \rightarrow (x_2,x_1,x_0)$  belongs to the transition

\*) field automorphisms  $\varphi$  of  $\gamma$  have the form  $\varphi: t \rightarrow t^j$ ,  $1 \leq j \leq 30$ , and  $D(k) = \{(1,t,t^k) \mid t \in \gamma\} \cup \{(010), (001)\} = \{(1,t^j,(t^j)^k) \mid t \in \gamma\} \cup \{(010), (001)\} = \{(1,t^j,(t^k)^j) \mid t \in \gamma\} \cup \{(010), (001)\}.$

from frame  $(P, Q, R, S)$  to frame  $(R, Q, P, S)$ . From (1) and (2) and the fact that  $(x + y)^2 = x^2 + y^2$  for all  $x$  and  $y$  in  $\gamma$  we can conclude that  $O$  has representation  $D(2)$  for any frame  $(P', Q, R', S')$  on  $O$ . (3)

A transformation will always work on coordinates, not on the points themselves.

### 3. Representations equivalent to $D(2)$

Let  $O$  be an oval with representation  $D(k)$  for the frame  $(P_k, Q_k, R_k, S_k)$  on  $O$  and representation  $D(2)$  for the frame  $(P_2, Q_2, R_2, S_2)$  on  $O$ . We shall show that exactly one of the following three cases holds (4)

- i)  $k = 2$ ,  $Q_k = Q_2$  and  $D(k)$  is the representation of  $O$  for any frame  $(P'_k, Q_k, R'_k, S'_k)$  on  $O$ ,
- ii)  $k = 16$ ,  $R_k = Q_2$  and  $D(k)$  is the representation of  $O$  for any frame  $(P'_k, Q'_k, R_k, S'_k)$  on  $O$ ,
- iii)  $k = 30$ ,  $P_k = Q_2$  and  $D(k)$  is the representation of  $O$  for any frame  $(P_k, Q'_k, R'_k, S'_k)$  on  $O$ .

Now assume  $Q_k \neq Q_2$ . Let  $Q'_k \in O \setminus \{P_k, R_k, Q_2\}$ . We can choose  $R$  and  $S$  on  $O$  and, if necessary, change  $S_k$  (with (1)) in such a way that  $\{P_k, R_k, S_k\} = \{Q_2, R, S\}$ . Now of course  $\{Q_k, Q'_k, Q_2\} \cap \{R, S\} = \emptyset$ . We have: For the frames  $(Q_k, Q_2, R, S)$  and  $(P_k, Q_k, R_k, S_k)$  the oval  $O$  has representations  $D(2)$  and  $D(k)$ , respectively.

The transformation  $T$  which satisfies for the frame  $(Q_k, Q_2, R, S)$   $TP_k = (001)$ ,  $TQ_k = (010)$ ,  $TR_k = (100)$ ,  $TS_k = (111)$  has also  $TD(2) = D(k)$  since  $O$  has representation  $D(k)$  for the frame  $(P_k, Q_k, R_k, S_k)$ . For the frame  $(Q'_k, Q_2, R, S)$  the oval  $O$  has representation  $D(2)$  (by (3)) and  $P_k, R_k$  and  $S_k$  have the same coordinates as for the frame  $(Q_k, Q_2, R, S)$ . So, for  $(Q'_k, Q_2, R, S)$ ,  $T(P_k, Q'_k, R_k, S_k) = ((001), (010), (100), (111))$ . For this frame  $O$  has representation  $TD(2) = D(k)$ . So we can conclude:

If  $O$  has representation  $D(k)$  for some frame  $(P_k, Q_k, R_k, S_k)$  and  $Q_k \neq Q_2$  then  $O$  has this representation for any frame  $(P_k, Q'_k, R_k, S_k)$  on  $O$  with  $Q'_k \neq Q_2$ .

The same argument for the cases  $P_k \neq Q_2$  and  $R_k \neq Q_2$  leads to a similar conclusion for  $P_k$  and  $R_k$ .

Now consider the following cases:

- i)  $P_k \neq Q_2$  and  $R_k \neq Q_2$ . Now we are free to choose  $P'_k = R_k$  and  $R'_k = P_k$  and know that  $\emptyset$  has representation  $D(k)$  for the frame  $(P'_k, Q_k, R'_k, S_k)$ . This yields:

$$\forall_{t \in \gamma_0} \exists_{s \in \gamma_0} [(1, t, t^k) = (s^k, s, 1)], \text{ so } \forall_{s \in \gamma_0} [(s^{1-k})^k = s^{-k}],$$

so  $k(1-k) \equiv -k \pmod{31}$ , so  $k = 0$  or  $k = 2$ .

Since  $D(0)$  represents no oval we find  $k = 2$ .

- ii)  $P_k \neq Q_2$  and  $Q_k \neq Q_2$ . We choose  $P'_k = Q_k$  and  $Q'_k = P_k$  and get:

$$\forall_{t \in \gamma} \exists_{s \in \gamma} [(1, t, t^k) = (1, s^k, s)] \text{ so } \forall_{s \in \gamma} [s^{k^2} = s]$$

so  $k^2 \equiv 1 \pmod{31}$  so  $k = 1$  or  $k = 30$ .

Since  $D(1)$  represents no oval we find  $k = 30$  (with [1], theorem 1, p. 81).

But this result excludes that of i) so we may conclude:  $R_k = Q_2$  and in case i):  $Q_k = Q_2$ .

Now the only other possibility is:

- iii)  $P_k = Q_2$ . We choose  $Q'_k = R_k$  on  $R'_k = Q_k$  we find:

$$\forall_{t \in \gamma_0} \exists_{s \in \gamma_0} [(1, t, t^k) = (s, 1, s^k)] \text{ so } \forall_{s \in \gamma_0} [s^{-k} = s^{k-1}]$$

so  $-k \equiv k-1 \pmod{31}$  so  $k = 16$ .

To prove (4) we still have to show that  $D(16)$  and  $D(30)$  are indeed representations of  $\emptyset$ . This follows in exactly the same manner:

$$\{(1, t, t^2) \mid t \in \gamma\} = \{(1, t^{16}, t) \mid t \in \gamma\}$$

so if  $D(2)$  is the representation of  $\emptyset$  for  $(P, Q, R, S)$  then  $D(16)$  is the representation for  $(Q, P, R, S)$  and:

$$\{(1, t, t^2) \mid t \in \gamma_0\} = \{(t^{-1}, 1, t) \mid t \in \gamma_0\} = \{(t^{30}, 1, t) \mid t \in \gamma_0\}$$

so if  $D(2)$  is the representation of  $\emptyset$  for the frame  $(P, Q, R, S)$  then  $D(30)$  is the representation of  $\emptyset$  for the frame  $(R, P, Q, S)$ . q.e.d.

With (4) we have directly: The only representations  $D(k)$  equivalent with  $D(2)$  are:  $D(2)$ ,  $D(16)$  and  $D(30)$ .

4. The relation between the representations D(4) and D(6)

In this section we shall show that the assumption  $D(4) \sim D(6)$  leads to a contradiction.

Assume that there exists an oval  $O$  with representations  $D(4)$  and  $D(6)$  for the frames  $(P_4, Q_4, R_4, S_4)$  and  $(P_6, Q_6, R_6, S_6)$  respectively.

We distinguish two cases:

i)  $\{P_4, Q_4\} \cap \{P_6, Q_6, R_6\} \neq \emptyset$ .

With (1) and (4) we can choose  $R_4, S_4$  and  $S_6$  such that  $\{P_4, Q_4, R_4, S_4\} = \{P_6, Q_6, R_6, S_6\}$ .

Let  $T$  be the transformation with respect to  $(P_4, Q_4, R_4, S_4)$  with  $TP_6 = P_4, TQ_6 = Q_4, TR_6 = R_4, TS_6 = S_4$  and  $TD(4) = D(6)$ . Now  $R_4 \neq S_6$  or  $S_4 \neq S_6$ , say  $R_4 \neq S_6$ .

Then  $P_6 = R_4$  or  $Q_6 = R_4$  or  $R_6 = R_4$ , say  $P_6 = P_4$ .

Let  $P'_6 \in O \setminus \{P_6, Q_6, R_6, S_6\}$ . Now  $(P_4, Q_4, P'_6, S_4)$  is a frame for which  $O$  has representation  $D(4)$ . Also for this frame  $T(P'_6, Q_6, R_6, S_6) = (001), (101), (100), (111)$  and  $O$  has representation  $TD(4) = D(6)$  for this frame. This implies:

If  $P_6 = R_4$  then for any frame  $(P'_6, Q_6, R_6, S_6)$   $O$  has representation  $D(6)$ .

By applying the same argument we get:

If  $P_6$  ( $Q_6$  or  $R_6$ )  $\in \{R_4, S_4\}$  then  $O$  has representation  $D(6)$  for any frame  $(P'_6, Q_6, R_6, S_6)$  on  $O$  ( $(P_6, Q'_6, R_6, S_6)$  or  $(P_6, Q_6, R'_6, S_6)$  resp.). The conclusion is that in the frame that yields representation  $D(6)$  for  $O$  we can choose besides  $S_6$  (with (1)) at least one other point arbitrarily.

i1) If this point is  $R_6$  then we can transform via  $(P_6, Q_6, R_6, S_6) \rightarrow (P_6, Q_6, S_6, R_6)$  and we get:

$$\forall_{t \in Y_{01}} \exists_{s \in Y_{01}} [(1, t, t^6) = (1, 1+s, 1+s^6)], \text{ so } \forall_{s \in Y_{01}} [(1+s)^6 = 1+s^6],$$

so  $\forall_{s \in Y_{01}} [s^2 + s^4 = 0]$  and this is not true.

i2) If this point is  $P_6$ , we transform  $(P_6, Q_6, R_6, S_6) \rightarrow (S_6, Q_6, R_6, P_6)$  and find:

$$\forall_{t \in Y_1} \exists_{s \in Y_1} [(1, t, t^6) = (1+s^6, s+s^6, s^6)] \text{ so}$$

$$\forall_{s \in Y_1} \left[ \left( \frac{s+s^6}{1+s^6} \right)^6 = \frac{s^6}{1+s^6} \right], \text{ so } \forall_{s \in Y_1} [(1+s^5)^6 = (1+s^6)^5]$$

but since  $(1+s^5)^6 + (1+s^6)^5$  is a polynomial of degree less than 31 and  $(1+s^5)^6 + (1+s^6)^5 = s^6 + s^{10} + s^{20} + s^{24} \neq 0$  this cannot be true.

i3) If this point is  $Q_6$  then we transform via  $(P_6, Q_6, R_6, S_6) \rightarrow (P_6, S_6, R_6, Q_6)$  and find:

$$\forall_{t \in \gamma_1} \exists_{s \in \gamma_1} [(1, t, t^6) = (1+s, s, s+s^6)], \text{ so } \forall_{s \in \gamma_1} [(\frac{s}{1+s})^6 = \frac{s+s^6}{1+s}],$$

$$\text{so } \forall_{s \in \gamma_1} [s^5 = (1+s^5)(1+s)^5 = 1+s+s^4+s^6+s^9+s^{10}], \text{ so}$$

$\forall_{s \in \gamma_1} [1+s+s^4+s^5+s^6+s^9+s^{10} = 0]$  and also this cannot be true on the same grounds.

So we find: the assumption  $\{P_4, Q_4\} \cap \{P_6, Q_6, R_6\} \neq \emptyset$  leads to a contradiction.

ii)  $\{P_4, Q_4\} \cap \{P_6, Q_6, R_6\} = \emptyset$ .

We choose  $S_6 = P_4$ ,  $S_4 = P_6$ ,  $R_4 = R_6$  and have, for the frame  $(P_4, Q_4, R_4, S_4)$ :  $Q_6 = (1, a, a^4)$  for some  $a \in \gamma_{01}$ . The transformation T:

$$(x_0, x_1, x_2) \rightarrow ((a+a^2+a^3)x_0 + (1+a+a^2+a^3)x_1 + x_2, x_1 + x_2, x_1 a^3 + x_2)$$

transforms  $P_6$  in  $(0,0,1)$ ,  $R_6$  in  $(1,0,0)$ ,  $Q_6$  in  $(0,1,0)$  and  $S_6$  in  $(1,1,1)$ .

So it must transform  $D(4)$  in  $D(6)$ . But then we must have  $TQ_4 \in D(6)$ , so:

$$TQ_4 = (1+a+a^2+a^3, 1, a^3) = (1, s, s^6), \text{ for some } a \in \gamma, \text{ so}$$

$$\left(\frac{1}{1+a+a^2+a^3}\right)^6 = \frac{a^3}{1+a+a^2+a^3}, \text{ so } \left(\frac{1+a}{1+a}\right)^6 = \frac{a^3(1+a)}{1+a}$$

$$\text{so } (1+a^4)^5 a^3 = (1+a)^5, \text{ so } (1+a^4)^4 a^3 = 1+a, \text{ so } (1+a^{32})a^6 = (1+a)^2, \text{ so } a^6 = 1+a.$$

But  $a^6 + a + 1$  is an irreducible polynomial over  $GF(2)$  so  $a \in \gamma \cap GF(2^6) = GF(2)$  and this leads to the desired contradiction since  $a^6 + a = 0$  for  $a \in GF(2)$ .

We have now that  $D(4) \sim D(6)$  leads to a contradiction and conclude, with [2], theorem 12, corollary 1, p. 790 that there are exactly three projectively distinct  $D(k)$  over  $GF(32)$  i.e.  $D(2)$ ,  $D(4)$  and  $D(6)$ .

### References

- [1] J.W.P. Hirschfeld, Ovals in Desarguesian Planes of Even Order. Ann. di Mat. 102 (1975), pp. 79-89.
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