Note on a paper by M. Laczkovich on functions with measurable differences (Erdös' conjecture)

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Department of Mathematics

Memorandum 1978-14

Issued November 1978

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by

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Abstract.

This note is meant to simplify certain parts of M.Laczkovich'proof of Erdös'conjecture about functions with measurable differences. The (pseudo)norm occurring in Laczkovich'proof is replaced by a norm (with essentially the same properties as Laczkovich'norm) that admits easy manipulation. The other parts of Laczkovich'proof of Erdös'conjecture need hardly any alteration when Laczkovich'norm is replaced by the one introduced in this note. It is further shown that the crucial property of Laczkovich'norm (as given in [L], Theorem 2) can be derived from the corresponding property of our norm.

1. Introduction.

M.Laczkovich recently proved the following theorem ([L], Theorem 3). If f is a real-valued function defined on $\mathbb{R}$ for which $f(x+h)-f(x)$ is measurable as a function of x for every $h \in \mathbb{R}$, then f can be written as $f=g+H+S$, where g is measurable over $\mathbb{R}$, H is additive and, for every $h \in \mathbb{R}$, $S(x+h)=S(x)$ for almost every $x \in \mathbb{R}$. This theorem was conjectured in 1951 by P.Erdös in connection with work of N.G. de Bruijn concerning difference properties of certain classes of real-valued functions defined on $\mathbb{R}$ (cf. [B1] and [B2]). The proof of this theorem as presented by Laczkovich in his paper is pretty complicated, and the purpose of this note is to simplify some of the arguments used.

Laczkovich introduces for his proof a (pseudo)norm in the space S of all real-valued measurable functions defined on $\mathbb{R}$ and periodic with period 1. The (pseudo)norm is defined by

$$\|f\| := \inf\{a+\lambda(\{x \in [0,1] \mid |f(x)| \geq a\}) \mid a > 0\}$$

for $f \in S$ (cf. [L], section 2). Here λ denotes ordinary Lebesgue measure. A number of properties of $\| \ \|$ are listed in [L](section 2, (3)-(9)), and a kind of "spread" is introduced by putting

$$s(f) := \inf\{\|f-c\cdot e\| \mid c \in \mathbb{R}\}$$

for $f \in S$ (cf.[L], section 2, Lemma 1). Here e is the constant function with $e(x)=1$ for all x.

Denote for $f \in S$, $h \in \mathbb{R}$ by $T_h f$ the element of S given by $(T_h f)(x) = f(x+h)$ $(x \in \mathbb{R})$. The following fact (cf. [L], Theorem 2) turns out to be crucial in the proof of [L], Theorem 3. If $(f_n)_n$ is a sequence in S, then $s(f_n) \to 0$ $(n \to \infty)$ if and only if $\|T_h f_n - f_n\| \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$. The proof given in [L] of this property is completely elementary but complicated, and we believe that this is caused by the fact that Laczkovich'norm is somewhat uneasy to handle.

2. A suitable norm for the space S.

We introduce a norm for the space S (with essentially the same properties as Laczkovich'one) that admits easy manipulation.

Definition 1. For $f \in S, g \in S$ we put

$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\, dx, \quad \|f\|_d := d(f,0).$$

It is easy to prove that d is a (semi)metric in S (cf. [H], Ch.VIII, section 42, exercise (4)).

We list some further properties of d and $\| \, \|_d$ in the following lemma (cf.[L], section 2, (3)-(9)).

Lemma 1. Let $f \in S, g \in S$. Then we have

(i) $0 \le d(f,g) < 1$,

(ii) $d(f,g)=0$ if and only if $f=g$ (a.e.),

(iii) $\|f+g\|_d \le \|f\|_d + \|g\|_d$,

(iv) $\|T_h f\|_d = \|f\|_d$ $(h \in \mathbb{R})$,

(v) $\lambda(\{x \in [0,1] \mid |f(x)| \ge a\}) \le \|f\| \frac{1+a}{a}$ $(a > 0)$,

(vi) if $(f_n)_n$ is a sequence in S, then $\|f_n\|_d \to 0$ $(n \to \infty)$ if and only if $f_n \to 0$ $(n \to \infty)$ in measure,

(vii) $\lim_{h \to 0} \|T_h f - f\|_d = 0$.

Proof. The proofs of the properties (i) and (ii) are trivial, and as to property (iii) we note that

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{a+b}{1+a+b} \qquad (a \geq 0, b \geq 0).$$

Property (iv) follows from periodicity of f.

Property (v) is proved as follows. Let $a > 0$, and let $E := \{x \in [0,1] \mid |f(x)| \geq a\}$. Then we have

$$\|f\|_d \geq \int_E \frac{|f(x)|}{1+|f(x)|}\,dx \geq \int_E \frac{a}{1+a}\,dx = \frac{a}{1+a}\lambda(E).$$

Hence $\lambda(E) \leq \frac{1+a}{a}\|f\|_d$.

To prove property (vi), let $(f_n)_n$ be a sequence in S. If $\|f_n\|_d \to 0$ $(n\to\infty)$, then $\lambda(\{x \in [0,1] \mid |f_n(x)| \geq a\}) \to 0$ $(n\to\infty)$ for every $a > 0$ by property (v), whence $f_n \to 0$ $(n\to\infty)$ in measure. If $f_n \to 0$ $(n\to\infty)$ in measure and $a > 0$, then

$$\|f_n\|_d \leq \frac{a}{1+a} + \lambda(\{x \in [0,1] \mid |f_n(x)| \geq a\}) < a$$

if n is sufficiently large . Hence $\|f_n\|_d \to 0$ $(n\to\infty)$.

Finally property (vii). Let $(t_n)_n$ be a sequence of step functions in S with $t_n \to f$ (a.e.). Then $\|f-t_n\|_d \to 0$ $(n\to\infty)$ by property (vi), and it follows from property (iii) and (iv) that for every $n \in \mathbb{N}$

$$\|T_hf-f\|_d \leq \|T_hf-T_ht_n\|_d + \|T_ht_n-t_n\|_d + \|t_n-f\|_d =$$

$$= 2\|t_n-f\|_d + \|T_ht_n-t_n\|_d.$$

Now if $\varepsilon > 0$ is given, take $n \in \mathbb{N}$ such that $\|f-t_n\|_d < \frac{\varepsilon}{2}$. Since $\lim_{h\to 0}\|T_ht_n-t_n\|_d = 0$ we find $\|T_hf-f\|_d < \varepsilon$ if h is sufficiently small. □

The above lemma shows that our norm $\|\ \|_d$ has essentially the same properties as the norm $\|\ \|$ of Laczkovich (we do not have property (7) of [L], but in the proof of Theorem 3 of [L] property (v) of the above lemma is equally useful).

3. Main property of $\|\ \|_d$.

We derive now the main property of $\|\ \|_d$ (i.e. [L], Theorem 2 with $\|\ \|_d$ instead of $\|\ \|$). We first introduce a notion of spread for elements of S (cf. [L], section 2).

Definition 2. Let $f \in S$. We define (recall that $e(x)=1$ $(x \in \mathbb{R})$)

$$s_d(f) := \inf\{\|f-c\cdot e\|_d \mid c \in \mathbb{R}\}.$$

We have the following lemma.

Lemma 2. Let $f \in S$. Then

(i) there exists a $c_o \in \mathbb{R}$ such that $\|f-c_o\cdot e\|_d = s_d(f)$,

(ii) $s_d(T_x f)=s_d(f)$ for every $x \in \mathbb{R}$,

(iii) $s_d(f) \le \int_o^1 \|T_h f-f\|_d \, dh$,

(iv) $\|T_h f-f\|_d \le 2s_d(f)$ for every $h \in \mathbb{R}$.

Proof. As to (i) we observe that $\|f-c\cdot e\|_d$ depends continuously on $c \in \mathbb{R}$, and that $\|f-c\cdot e\|_d \to 1$ if $|c| \to \infty$ by Lebesgue's theorem on dominated convergence. Since $\|f\|_d < 1$ we can find an $A > 0$ such that $\|f-c\cdot e\|_d \ge \|f\|_d$ for $|c| \ge A$. Now $s_d(f)=\inf\{\|f-c\cdot e\|_d | c \in \mathbb{R}\}$ is attained at some point $c_o \in [-A,A]$.

Property (ii) follows at once from Lemma 1, (iv).

Now we prove property (iii). We obtain by Fubini's theorem

$$\int_o^1 \|T_h f-f\|_d \, dh = \int_o^1 \left\{ \int_o^1 \frac{|f(x+h)-f(x)|}{1+|f(x+h)-f(x)|} dx \right\} dh =$$

$$= \int_o^1 \left\{ \int_o^1 \frac{|f(x+h-f(x)|}{1+|f(x+h)-f(x)|} dh \right\} dx = \int_o^1 \|T_x f-f(x)\cdot e\|_d \, dx.$$

By definition of $s_d(f)$ and (ii) we get

$$\int_o^1 \|T_x f-f(x)\cdot e\|_d \, dx \ge \int_o^1 s_d(T_x f) dx = \int_o^1 s_d(f) dx = s_d(f).$$

Hence $\int_o^1 \|T_h f-f\|_d \, dh \ge s_d(f)$.

To prove property (iv), we note that for every $c \in \mathbb{R}$, $h \in \mathbb{R}$

$$\|T_h f-f\|_d \le \|T_h f-c\cdot e\|_d + \|f-c\cdot e\|_d = 2\|f-c\cdot e\|_d$$

by Lemma 1, (iii) and (iv). By taking the infimum over all $c \in \mathbb{R}$ at the right hand side, we get $\|T_h f-f\|_d \le 2s_d(f)$ for every $h \in \mathbb{R}$. □

We arrive now at the counterpart of [L], Theorem 2 for $\| \|_d$.

<u>Theorem 1</u>. Let $(f_n)_n$ be a sequence in S. Then we have $s_d(f_n) \to 0$ $(n \to \infty)$ if and only if $\|T_h f_n - f_n\|_d \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$.

<u>Proof</u>. First assume that $s_d(f_n) \to 0$ $(n \to \infty)$. It follows at once from Lemma 2,(iv) that $\|T_h f_n - f_n\|_d \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$.

Next assume that $\|T_h f_n - f_n\|_d \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$. Since $\|T_h f_n - f_n\|_d < 1$ $(h \in \mathbb{R},\ n \in \mathbb{N})$ we have by lemma 2,(iii) and Lebesgue's theorem on dominated convergence

$$s_d(f_n) \leq \int_0^1 \|T_h f_n - f_n\|_d \, dh \to 0 \quad (n \to \infty). \qquad \square$$

Analyzing the proof of [L], Theorem 3 we see that this proof needs no alterations if $\| \ \|_d$ instead of $\| \ \|$ is used, perhaps except in the proof of the assertion about G in (22). There [L], section 2, property (7) is employed , but we may use Lemma 1, (v) instead (the assertion about G can also be proved by noting that for every sequence $(f_n)_n$ in S with $\sum_{n=1}^{\infty} \|f_n\|_d < \infty$ we have $f_n \to 0$ (a.e.)).

<u>Remark</u>. It is equally possible to get to Laczkovich' main result (i.e. his Theorem 3) by first deriving [L], Theorem 2 (which is our Theorem 1 with $\| \ \|$ instead of $\| \ \|_d$) from our Theorem 1, and then leaving the proof of [L], Theorem 3 as it is.

Indeed, let $(f_n)_n$ be a sequence in S with $\|T_h f_n - f_n\| \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$. We shall show that $s(f_n) \to 0$ $(n \to \infty)$.

We first note that, for every sequence $(g_n)_n$ in S, $\|g_n\|_d \to 0$ $(n \to \infty)$ if and only if $\|g_n\| \to 0$ $(n \to \infty)$ by Lemma 1, (vi) and [L], section 2, (9). Hence $\|T_h f_n - f_n\|_d \to 0$ $(n \to \infty)$ for every $h \in \mathbb{R}$.

It follows from Theorem 1 that $s_d(f_n) \to 0$ $(n \to \infty)$. We can find by Lemma 2, (i) a sequence $(c_n)_n$ in $\mathbb{R}$ with $s_d(f_n) = \|f_n - c_n \cdot e\|_d$ $(n \in \mathbb{N})$. Now $\|f_n - c_n \cdot e\|_d \to 0$ $(n \to \infty)$, whence $\|f_n - c_n \cdot e\| \to 0$ $(n \to \infty)$. Since for every $n \in \mathbb{N}$

$$s(f_n) \leq \|f_n - c_n \cdot e\|$$

we conclude that $s(f_n) \to 0$ $(n \to \infty)$.

The proof of the converse statement can be given along the same lines as the proof of the first part of our Theorem 1.

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