A note on slowly oscillating functions

A NOTE ON SLOWLY OSCILLATING FUNCTIONS

BY

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1. *Introduction and results.* Let the function $L(x)$ be defined for $x \geqslant 1$, and let it be positive, continuous and such that

$$\frac{L(\mu x)}{L(x)} \to 1, \quad (x \to \infty), \tag{1.1}$$

for every $\mu > 0$. Then $L(x)$ is termed a *slowly oscillating function.* Examples of slowly oscillating functions are

$$\log x, \log \log x, \ldots,$$

and combinations like

$$(\log x)^{\alpha} \; (\log \log x)^{\beta}.$$

J. KARAMATA (See [2], and compare J. KOREVAAR and F. VAN DER BLIJ [3]) has deduced the following representation for slowly oscillating functions:

$$L(x) = L_0(x) \exp\left(\int_1^x \frac{\delta(t)}{t}\, dt\right), \tag{1.2}$$

where $L_0(x)$ and $\delta(x)$ are continuous functions, and such that $L_0(x)$ tends to a positive limit L_0 as $x \to \infty$, while $\delta(x) \to 0$ $(x \to \infty)$. From (1.2) KARAMATA finally deduced the properties

$$x^{\epsilon} L(x) \to \infty, \quad x^{-\epsilon} L(x) \to 0 \tag{1.3}$$

as $x \to \infty$ for every $\varepsilon > 0$, and

$$\left.\begin{array}{l}\text{the relation (1.1) holds uniformly on}\\ \text{every closed interval } a \leqslant \mu \leqslant b \ (a > 0)\end{array}\right\} \qquad (1.4)$$

Slowly oscillating functions play a role as comparison functions in asymptotic relations, such as occur for example in TAUBERian theorems (see G. DOETSCH [1] chapter 10). Here the property (1.4) makes it possible to pass to the limit under the integral sign in expressions like

$$\int_a^b g(\mu) \frac{L(\mu x)}{L(x)} d\mu.$$

Therefore a direct proof of (1.4) will have some value.

In section 2 we shall give a simple direct proof of (1.4). From (1.4) we then deduce (1.2), thus obtaining a much simpler proof of this representation than KARAMATA (see section 5). In section 3 we prove that (1.4) remains valid if we replace the requirement that $L(x)$ should be continuous for $x \geqslant 1$ by the condition that $L(x)$ be measurable on every interval $1 \leqslant x \leqslant A$. If moreover $\log L(x)$ is integrable over every interval $1 \leqslant x \leqslant A$ then $L(x)$ has a representation of the form (1.2), with integrable $\log L_0(x)$ and $\delta(x)$, $L_0(x)$ tending to a limit $L_0 > 0$ as $x \to \infty$ and $\delta(x)$ tending to 0 $(x \to \infty)$. (See section 5). In section 4 we give an example to show that the requirement that $L(x)$ be continuous or measurable can not be entirely dropped if we want to ensure the validity of (1.4). This example depends on the following interesting property of the additive group G of all real numbers: there exists a sequence of subgroups G_0, G_1, G_2, ... of G such that

$$G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$$
$$G = G_0 + G_1 + G_2 + \ldots + G_n + \ldots,$$

G_{n+1} being obtained from G_n by the adjunction of exactly one new element a_n $(n = 0, 1, 2, \ldots)$. It is not difficult to see that each of the sets of real numbers

$$G_0,\ G_1 - G_0,\ G_2 - G_1,\ \ldots,\ G_{n+1} - G_n,\ \ldots$$

formed with the above subgroups must be dense everywhere on $(-\infty, \infty)$.

Before concluding the introduction we shall state our problem in a slightly more convenient form. We put

$$\log L(e^x) = f(x), \ \log \mu = \lambda. \tag{1.5}$$

Then our problem is as follows. $f(x)$ is defined for $x \geqslant 0$, real, continuous, measurable or integrable, and such that

$$f(x + \lambda) - f(x) \to 0, \ (x \to \infty), \tag{1.6}$$

for every real λ. We have to prove that (1.6) holds uniformly on every finite interval of real λ. (1.3) becomes

$$f(x)/x \to 0, \ (x \to \infty); \tag{1.7}$$

and the representation (1.2) takes the form

$$f(x) = c(x) + \int_0^x \varepsilon(t)\, dt, \tag{1.8}$$

where $c(x)$ tends to a limit c as $x \to \infty$, $\varepsilon(x) \to 0$ $(x \to \infty)$.

2. *Proof of* (1.4) *for continuous* $L(x)$. We make the substitution (1.5). Then we must prove

Theorem 1. *Let* $f(x)$ *be defined for* $x \geqslant 0$. *Let* $f(x)$ *be real, continuous and such that*

$$f(x + \lambda) - f(x) \to 0, \ (x \to \infty), \tag{2.1}$$

for every real λ. *Then* (2.1) *holds uniformly on every finite interval* $a \leqslant \lambda \leqslant b$.

Proof. Let us suppose that (2.1) does not hold uniformly on $a \leqslant \lambda \leqslant b$. Then there exist a positive number ε, a sequence of points $x_n \to \infty$ and a sequence of points $\lambda_n \,\varepsilon\, [a, b]$ such that

$$| f(x_n + \lambda_n) - f(x_n) | \geqslant \varepsilon, \ (n = 1, 2, 3, \ldots.) \tag{2.2}$$

Hence, as $f(x)$ is continuous at $x_n + \lambda_n$ $(n = 1, 2, 3, \ldots.)$, there exists, to every n, a closed interval $I_n \subset [a, b]$ containing λ_n and such that

$$| f(x_n + \lambda) - f(x_n) | > \tfrac{2}{3}\varepsilon \text{ for } \lambda \,\varepsilon\, I_n. \tag{2.3}$$

Starting from the sequences $\{x_n\}$, $x_n \to \infty$, and $\{I_n\}$, $I_n \subset [a, b]$, we shall construct sequences $\{\xi_n\}$, $\xi_n \to \infty$, and $\{K_n\}$, K_n a closed sub-interval of $[a, b]$,

$$K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots, \tag{2.4}$$

such that

$$| f(\xi_n + \lambda) - f(\xi_n) | > \tfrac{1}{3}\varepsilon \text{ for } \lambda \,\varepsilon\, K_n, \tag{2.5}$$

$n = 1, 2, 3, \ldots$.

By (2.3) we may take $\xi_1 = x_1$, $K_1 = I_1$. Let us now suppose that $\xi_1, \xi_2, \ldots, \xi_p$ and $K_1 \supset K_2 \supset \ldots \supset K_p$ have already been constructed such that (2.5) holds for $n = 1, 2, \ldots, p$. We shall then construct ξ_{p+1} and K_{p+1}. Let K_p have length δ, and let the integer q be defined such that $q\delta > b - a$. Further, let A be so large that

$$| f(x + \delta) - f(x) | < \varepsilon/(3q) \text{ for } x > A. \tag{2.6}$$

We now choose a ξ_{p+1} satisfying the conditions (*i*) ξ_{p+1} belongs to the sequence $\{x_n\}$, (*ii*) $\xi_{p+1} > \xi_p$, (*iii*) $\xi_{p+1} > A - a + q\delta$. Let $\xi_{p+1} = x_k$. For $\mu \,\varepsilon\, I_k$, by (2.3),

$$| f(\xi_{p+1} + \mu) - f(\xi_{p+1}) | > \tfrac{2}{3}\varepsilon. \tag{2.7}$$

Let the integer r, $-q \leqslant r \leqslant q$, be defined such that the translation I'_k of I_k, consisting of all points $\lambda = \mu + r\delta$, $\mu \,\varepsilon\, I_k$, has a sub-interval in common with K_p. For $\lambda \,\varepsilon\, I'_k$, by (2.7) and (2.6) ($\xi_{p+1} > A - a + q\delta$),

$$\begin{aligned} | f(\xi_{p+1} + \lambda) - f(\xi_{p+1}) | &= | f(\xi_{p+1} + \mu + r\delta) - f(\xi_{p+1}) | \\ &\geqslant | f(\xi_{p+1} + \mu) - f(\xi_{p+1}) | - \\ - | f(\xi_{p+1} + \mu) - f(\xi_{p+1} + \mu \pm \delta) | &- | f(\xi_{p+1} + \mu \pm \delta) - \\ - f(\xi_{p+1} + \mu \pm 2\delta) | - \ldots - & \\ - | f(\xi_{p+1} + \mu + (r \mp 1)\delta) &- f(\xi_{p+1} + \mu + r\delta) | \\ > \tfrac{2}{3}\varepsilon - | r | \varepsilon/(3q) \geqslant \tfrac{1}{3}\varepsilon. & \end{aligned} \tag{2.8}$$

Thus we may take $K_{p+1} = I'_k \times K_p$.

Finally, let λ_0 be a point common to all intervals K_n (see (2.4)). Then by (2.5)

$$| f(\xi_n + \lambda_0) - f(\xi_n) | > \tfrac{1}{3}\varepsilon, \; (n = 1, 2, 3, \ldots),$$

where $\xi_n \to \infty$. This is impossible, however, because of (2.1).

R e m a r k. We made use of the continuity of $f(x)$ only to deduce (2.3) from (2.2). Now (2.3) can be deduced from (2.2) (for all sufficiently large n) on the weaker hypothesis that the oscillation $\sigma(f, u)$ of $f(x)$ at $x = u$ tends to zero as $u \to \infty$. For if $\sigma(f, u) \to 0$ we can define N so that $\sigma(f, x_n + \lambda_n) < \frac{1}{3}\varepsilon$ whenever $n > \mathrm{N}$. But if the oscillation of $f(x_n + \lambda)$ at $\lambda = \lambda_n$ is less than $\frac{1}{3}\varepsilon$ there exists an interval I_n containing λ_n, where the oscillation of $f(x_n + \lambda)$ is less than $\frac{1}{3}\varepsilon$. Hence (2.3) will be satisfied whenever $n > \mathrm{N}$. We have thus proved

T h e o r e m 2. *The conclusion of theorem* 1 *remains valid if the requirement that $f(x)$ be continuous is replaced by the condition* $\sigma(f, u) \to 0$ $(u \to \infty)$.

3. *Proof of* (1.4) *for measurable $L(x)$*. We make the substitution (1.5). Then we must prove

T h e o r e m 3. *Let $f(x)$ be defined for $x \geqslant 0$. Let $f(x)$ be real, measurable on every interval $0 \leqslant x \leqslant A$ and such that*

$$f(x + \lambda) - f(x) \to 0, \ (x \to \infty), \tag{3.1}$$

for every real λ. Then (3.1) *holds uniformly on every finite interval $a \leqslant \lambda \leqslant b$.*

P r o o f. We may restrict ourselves to a proof for the interval $0 \leqslant \lambda \leqslant 1$. Let us suppose that (3.1) does not hold uniformly on $0 \leqslant \lambda \leqslant 1$. Then there exist an $\varepsilon > 0$, a sequence $\{x_n\}$, $x_n \to \infty$, and a sequence $\{\lambda_n\}$, $\lambda_n \in [0, 1]$ such that

$$| f(x_n + \lambda_n) - f(x_n) | \geqslant \varepsilon, \ (n = 1, 2, 3, \ldots.) \tag{3.2}$$

Let V_n denote the set of all λ, $-1 \leqslant \lambda \leqslant 1$, satisfying

$$| f(x_n + \lambda) - f(x_n) | < \tfrac{1}{2}\varepsilon. \tag{3.3}$$

As $f(x)$ is measurable on $[x_n - 1, x_n + 1]$, V_n is measurable. Let its measure be a_n. As $n \to \infty$, V_n will tend to V, the interval $-1 \leqslant \lambda \leqslant 1$. Hence $a_n \to 2$ as $n \to \infty$ (see for example C. DE LA VALLÉE POUSSIN [4] p. 29). On the other hand, let W_n denote the set of all μ, $-1 \leqslant \mu \leqslant 1$, satisfying

$$| f(x_n + \lambda_n + \mu) - f(x_n + \lambda_n) | < \tfrac{1}{2}\varepsilon. \tag{3.4}$$

W_n is also measurable, and its measure $\beta_n \to 2$ $(n \to \infty)$. Let W'_n be the set of all points $\lambda = \lambda_n + \mu$, $\mu \varepsilon W_n$. Clearly the measure β'_n of W'_n must be equal to β_n. If $\lambda \varepsilon W'_n$, we have by (3.4)

$$| f(x_n + \lambda) - f(x_n + \lambda_n) | < \tfrac{1}{2}\varepsilon \tag{3.5}$$

and hence by (3.2)

$$| f(x_n + \lambda) - f(x_n) | > \tfrac{1}{2}\varepsilon. \tag{3.6}$$

By (3.3) and (3.6) $V_n \times W'_n$ must be empty. But $V_n \subset [-1, 1] \subset [-1, 2]$, $W'_n \subset [\lambda_n - 1, \lambda_n + 1] \subset [-1, 2]$, and both α_n and β'_n tend to 2 as $n \to \infty$!

4. *A positive function satisfying* (1.1), *for which* (1.4) *does not hold.* We make the substitution (1.5). We shall now prove

Theorem 4. *There exists a real function $f(x)$, defined for $x \geqslant 0$, which satisfies the relation*

$$f(x + \lambda) - f(x) \to 0, \ (x \to \infty), \tag{4.1}$$

for every real λ, without satisfying it uniformly on any interval $a \leqslant \lambda \leqslant b$.

For the proof we need the following lemmas.

Lemma 1. *Let G denote the additive group of all real numbers. There exists a sequence of subgroups of G,*

$$G_0 \subset G_1 \subset G_2 \subset \ldots . \subset G_n \subset \ldots .$$

such that

$$G = G_0 + G_1 + G_2 + \ldots . + G_n + \ldots .,$$

G_{n+1} being obtained from G_n by the adjunction of one new element a_n $(n = 0, 1, 2, \ldots .)$.

Proof.[1]) We define a sequence $\{a_n\}$ $(n = 0, 1, 2, \ldots)$ as follows. Let $\{\alpha_n\}$ $(n = 0, 1, 2, \ldots .)$ be a sequence of real numbers $\neq 0$ with the property that with each α_n it also contains $(1/m)\alpha_n$ $(m = 2, 3, \ldots .)$. ($\{\alpha_n\}$ may for example consist of all rational numbers $\neq 0$). Now let $\{a_n\}$ be the (necessarily infinite) subsequence derived from $\{\alpha_n\}$ by suppressing all elements α_n which are linear combinations $m_0 a_0 + m_1 a_1 + \ldots . + m_{n-1} a_{n-1}$ (m_i integers).

[1]) We express our thanks to J. de Groot, who has suggested the proof and the remark given here.

Next we define G_0. Let A denote the additive group generated by $a_0, a_1, a_2, \ldots$, and let B denote the set $A - 0$. We choose for G_0 a maximal subgroup of G which does not contain any element of the set B. (For the existence of such a maximal subgroup, see H. ZASSENHAUS [5] p. 21).

(*i*) *If* $g_0 \varepsilon G_0$, *then* $r g_0 \varepsilon G_0$ *whenever* r *is a rational number.* For if $r g_0$ did not belong to G_0, there would exist a $b \varepsilon B$, a $g_0' \varepsilon G_0$ and an integer $m \neq 0$ such that $g_0' + m.r g_0 = b$. This is an easy consequence of the maximality of G_0 with respect to B. Hence $g_0' + (mp/q) g_0 = b$ (p, q integers $\neq 0$) or $q g_0' + mp g_0 = qb$, which is impossible because $q b \varepsilon B$.

(*ii*) *Adjunction to* G_0 *of the elements of B gives G.* For let g belong to the set $G - G_0$. By the maximality of G_0 with respect to B, there exist a $g_0 \varepsilon G_0$ and an integer $m \neq 0$ such that $g_0 + mg = b$, $b \varepsilon B$. Hence $g = (-1/m) g_0 + (1/m) b$, which is an element of $G_0(B)$ because $(-1/m) g_0 \varepsilon G_0$ (see (*i*)) and $(1/m) b \varepsilon B$ (see the definition of the sequence $\{a_n\}$).

(*iii*) $G_0(B) = G_0(a_0, a_1, a_2, \ldots)$. Hence if we define $G_{n+1} = G_n(a_n)$ for each $n \geqslant 0$, $G = G_0 + G_1 + G_2 + \ldots$. We have $G_n \subset G_{n+1}$ and not $G_n = G_{n+1}$ because a_n does not belong to G_n (see the definition of the sequence $\{a_n\}$).

Remark. G_0 *is isomorphic with* G/A. For by (*ii*) every $g \varepsilon G$ can be written in the form $g = g_0 + a$ ($g_0 \varepsilon G_0$, $a \varepsilon A$), and it is clear that this representation is unique.

Lemma 2. *Let* $\{G_n\}$ ($n = 0, 1, 2, \ldots$) *be a sequence of subgroups of G satisfying the conditions of lemma* 1. *Then each of the sets of real numbers*

$$G_0,\ G_1 - G_0,\ G_2 - G_1,\ \ldots,\ G_{n+1} - G_n,\ \ldots$$

must be dense everywhere on $(-\infty, \infty)$.

Proof. Let us first consider G_0. G_0 can not be denumerable, for then G also would be denumerable. Hence G_0 must contain at least two linearly independent elements g_0 and g_0'. For if for each pair of elements $g_0 \neq 0$ and $g_0' \neq 0$ of G_0 there would exist a relation $m g_0 + m' g_0' = 0$ (m, m' integers $\neq 0$) all elements of G_0 would be rational multiples of one element g_0.

Now if g_0 and g_0' are linearly independent, the elements $m\,g_0 + m'\,g_0'$ (m, m' integers) $\varepsilon\, G_0$ lie dense everywhere on $(-\infty, \infty)$.

Further, if $g_0 \varepsilon\, G_0$ then $g_0 + a_n \varepsilon\, G_{n+1} - G_n$. Hence the set $G_{n+1} - G_n$ is also dense everywhere on $(-\infty, \infty)$.

P r o o f o f t h e o r e m 4. Let $H_0 = G_0$, $H_n = G_n - G_{n-1}$ ($n \geqslant 1$), where the sets G_n satisfy the conditions of lemma 1. Let $x \geqslant 0$. We define

$$f(x) = \exp\left(-\frac{x}{n+1}\right) \text{ if } x\, \varepsilon\, H_n \tag{4.2}$$

($n = 0, 1, 2, \ldots$). We shall prove that $f(x)$ satisfies the conditions mentioned in theorem 4.

(*i*) Let λ be a fixed real number. We may assume that $\lambda > 0$. Let $\lambda\, \varepsilon\, H_k$. If $x\, \varepsilon\, H_n$, $n > k$, then $x + \lambda\, \varepsilon\, H_n$ and hence

$$| f(x+\lambda) - f(x) | = \exp\left(-\frac{x}{n+1}\right)\left\{1 - \exp\left(-\frac{\lambda}{n+1}\right)\right\}$$

$$\leqslant \frac{\lambda}{n+1} \exp\left(-\frac{x}{n+1}\right) \leqslant \frac{\lambda}{ex}. \tag{4.3}$$

If $x\, \varepsilon\, H_n$, $n < k$, then $x + \lambda\, \varepsilon\, H_k$; if $x\, \varepsilon\, H_k$, $x + \lambda\, \varepsilon\, H_i$, where $i \leqslant k$. Hence in both cases $f(x + \lambda) \leqslant \exp\{-(x + \lambda)/(k + 1)\}$, $f(x) \leqslant \exp\{-x/(k + 1)\}$,

$$| f(x + \lambda) - f(x) | \leqslant 2 \exp\left(-\frac{x}{k+1}\right). \tag{4.4}$$

It follows from (4.3) and (4.4) that $f(x + \lambda) - f(x) \to 0$ $(x \to \infty)$.

(*ii*) (4.1) does not hold uniformly on any interval $a \leqslant \lambda \leqslant b$. For by lemma 2 we can determine an $x_n\, \varepsilon\, [n, n + 1] \times H_0$ and a $\lambda_n\, \varepsilon\, [a, b] \times H_n$ for every $n \geqslant 0$. Hence if n is sufficiently large,

$$f(x_n + \lambda_n) - f(x_n) = \exp\{-(x_n + \lambda_n)/(n+1)\} - \exp(-x_n) \to e^{-1} \tag{4.5}$$

as $n \to \infty$.

5. *The representation* (1.2). We make the substitution (1.5), and then we prove

Theorem 5. *Let $f(x)$ be defined for $x \geqslant 0$. Let $f(x)$ be real, continuous and such that*

$$f(x+\lambda) - f(x) \to 0, \ (x \to \infty), \tag{5.1}$$

for every real λ. Then $f(x)$ can be written in the form

$$f(x) = c(x) + \int_0^x \varepsilon(t)\, dt, \tag{5.2}$$

where $c(x)$ and $\varepsilon(x)$ are continuous functions of x, satisfying $c(x) \to c$, $\varepsilon(x) \to 0$ $(x \to \infty)$.

Proof. By theorem 1, (5.1) holds uniformly on $0 \leqslant \lambda \leqslant 1$. Integrating over $0 \leqslant \lambda \leqslant 1$ we obtain

$$\int_0^1 \{f(x+\lambda) - f(x)\}\, d\lambda \to 0, \ (x \to \infty),$$

or

$$\delta(x) \equiv \int_x^{x+1} f(t)\, dt - f(x) \to 0, \ (x \to \infty). \tag{5.3}$$

If we write

$$F(x) = \int_x^{x+1} f(t)\, dt, \tag{5.4}$$

$$F(x) = \int_0^1 f(t)\, dt + \int_0^x \{f(t+1) - f(t)\}\, dt. \tag{5.5}$$

Hence, putting

$$f(x+1) - f(x) \equiv \varepsilon(x), \tag{5.6}$$

we have by (5.1) $\varepsilon(x) \to 0$, $(x \to \infty)$, and by (5.5),

$$F(x) = c + \int_0^x \varepsilon(t)\, dt. \tag{5.7}$$

Finally, from (5.7), (5.4) and (5.3),

$$f(x) = F(x) - \delta(x) = \{c - \delta(x)\} + \int_0^x \varepsilon(t)\, dt, \tag{5.8}$$

which is the representation (5.2).

Remark. It is easily seen that every function $f(x)$ of the form (5.2) satisfies (5.1) and (1.7). (1.7) can also be easily derived from (5.1), however.

The argument by which we proved theorem 5 can also be used to prove

Theorem 6. *Let $f(x)$ satisfy the conditions of theorem 5, with "continuous" replaced by "integrable over every interval* $(0, A)$*". Then $f(x)$ can be written in the form* (5.2), *where $c(x)$ and $\varepsilon(x)$ are integrable, and such that $c(x) \to c$, $\varepsilon(x) \to 0$ $(x \to \infty)$.*

Proof. Use theorem 3.

Remark. Every function of the form (5.2) (with integrable $c(x)$ and $\varepsilon(x)$) is integrable and satisfies (5.1). It also satisfies (1.7).

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