On the continuity of a reduction-amplification operator in quantum mechanics

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ON THE CONTINUITY OF A REDUCTION-AMPLIFICATION OPERATOR IN QUANTUM MECHANICS

by

S.J.L. van Eijndhoven and J. de Graaf

Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands

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Abstract

In the Banach algebra $B(H)$ of bounded linear operators on a separable Hilbert space H we investigate the continuity of the linear mapping

$$A \mapsto P_c^* A = \sum_{nm\bar{n}} \langle no|A|mo\rangle \; |n\bar{n}\rangle \langle m\bar{n}|$$

with respect to the uniform, the ultra-strong, the strong and the weak topologies on $B(H)$.

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STATEMENT OF THE PROBLEM

Let H be a Hilbert space. Let $B(H)$ denote the Banach algebra of bounded linear operators on H. Informally the operator $P_c^* : B(H) \to B(H)$ is defined by means of Dirac brackets in the following way

$$P_c^* A = \sum_{nm\bar{n}} \langle no|A|mo\rangle |n\bar{n}\rangle \langle m\bar{n}| \ .$$

Here the operator A is supposed to have the matrix representation

$$A = \sum_{n\bar{n}m\bar{m}} \langle n\bar{n}|A|m\bar{m}\rangle |n\bar{n}\rangle \langle m\bar{m}| \ .$$

The problems posed by W.M. de Muynck, cf. [M], are the following:

(i) Show that for each $A \in B(H)$ the operator $P_c^* A$ is well defined and belongs to $B(H)$.

(ii) Investigate the continuity of the mapping $P_c^* : B(H) \to B(H)$ in the uniform, the strong and the weak topologies of $B(H)$.

MATHEMATICAL FORMULATION AND RESULTS

Let X be a separable Hilbert space with a fixed orthonormal basis $(v_n)_{n=0}^{\infty}$. Let $H = X \otimes X$ denote the two-fold tensor product of X. Then $(v_n \otimes v_{\bar{n}})_{n,\bar{n}=0}^{\infty}$ is an orthonormal basis in H.

In X define the operator P_{nm} by $P_{nm} f = (f,v_n)_X v_m$.

In H the operator $P_{nm} \otimes I$ can be written

$$(P_{nm} \otimes I)F = \sum_{\bar{n}=0}^{\infty} (F, v_n \otimes v_{\bar{n}})_H (v_m \otimes v_{\bar{n}}) \ .$$

Next, let A be a bounded linear operator on H, i.e., $A \in B(H)$, and define the matrix $(a_{nm})_{n,m=0}^{\infty}$ by

$$a_{nm} = (A(v_n \otimes v_0), (v_m \otimes v_0))_H \quad .$$

Our first problem is to give a mathematical meaning to the sequence

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} (P_{nm} \otimes I) \quad .$$

Let P_0 denote the projection in X defined by $P_0 f = (f,v_0)_X v_0$. The projection operator $I \otimes P_0$ on H can be written

$$(I \otimes P_0)F = \sum_{n=0}^{\infty} (F,v_n \otimes v_0)_H (v_n \otimes v_0) \quad .$$

For any operator $A \in B(H)$ the operator $(I \otimes P_0)A(I \otimes P_0) \in B(H)$. We have $\|(I \otimes P_0)A(I \otimes P_0)\|_{B(H)} \leq \|A\|_{B(H)}$.

Further $(I \otimes P_0)A(I \otimes P_0)$ maps $H = X \otimes X$ into $X \otimes \langle v_0 \rangle$ and also $X \otimes \langle v_0 \rangle$ into $X \otimes \langle v_0 \rangle$. It is clear now that $(I \otimes P_0)A(I \otimes P_0)$ can be regarded as a mapping from X into X. This so-called 'reduction' will be denoted by A_0. Finally, we are in a position to define the operator P_c^* :

$$P_c^* : B(H) \to B(H) \quad , \; P_c^* A = A_0 \otimes I.$$

Calculation of $P_c^* A(v_p \otimes v_q)$ shows that P_c^* is indeed the desired operator as mentioned at the very beginning of this notice.

<u>Remark</u>. $P_c^*(Q \otimes I) = Q \otimes I$ for all $Q \in B(X)$.

THEOREM 1

P_c^* is a bounded linear operator on $B(H)$ and

$$\|P_c^* A\|_{B(H)} \leq \|A\|_{B(H)} .$$

Hence P_c^* is continuous in the uniform topology of $B(H)$. In other words $P_c^* \in B(B(H))$. Finally $\|P_c^*\|_{B(B(H))} = 1$.

PROOF:

$\|P_c^* A\|_{B(H)} = \|A_0 \otimes I\|_{B(H)} = \|A_0\|_{B(X)} = \|(I \otimes P_0)A(I \otimes P_0)\|_{B(H)} \leq \|A\|_{B(H)}$.

For the special choice $A = Q \otimes P_0$, $Q \in B(X)$, we have

$$\|P_c^*(Q \otimes P_0)\|_{B(H)} = \|Q\|_{B(X)} \|P_0\|_{B(X)} = \|Q\|_{B(X)} = \|Q \otimes P_0\|_{B(H)} .$$

For the used properties of Hilbert norms of tensor products of Hibert spaces see Weidmann [W]. □

LEMMA 2

Consider the linear mapping $\Gamma : B(X) \to B(H) : Q \mapsto Q \otimes I$.

Γ is the so-called 'amplification map'. See Dixmier [D].

(i) Γ is continuous with respect to the uniform topologies of $B(X)$ and $B(H)$.

(ii) Γ is continuous with respect to the ultra-strong (= strongest, [N]) topologies of $B(X)$ and $B(H)$.

(iii) Γ is sequentially continuous with respect to the strong topologies of $B(X)$ and $B(H)$.

(iv) Γ is sequentially continuous with respect to the weak topologies of $B(X)$ and $B(H)$.

PROOF:

(i) Trivial because $\|Q\|_{B(X)} = \|Q \otimes I\|_{B(H)}$.

(ii) Take a sequence $(F_k)_{k=0}^{\infty} \subset H$ such that $\sum_{k=0}^{\infty} \|(Q \otimes I)F_k\|_H^2 < \infty$.

Write $F_k = \sum_{i=0}^{\infty} f_{k_i} \otimes v_i$ with $f_{k_i} \in X$. Then

$$\sum_{k=0}^{\infty} \|(Q \otimes I)F_k\|_H^2 = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \|Qf_{k_i} \otimes v_i\|_H^2 = \sum_{k=0}^{\infty} \sum_{k=i}^{\infty} \|Qf_{k_i}\|_X^2 .$$

Hence Γ maps the set $\left\{Q \,\middle|\, \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \|Qf_{k_i}\|_X^2 < \varepsilon^2\right\} \subset B(X)$ into the set

$$\left\{W \,\middle|\, \sum_{k=0}^{\infty} \|Wf_k\|_H^2 < \varepsilon^2\right\} \subset B(H) .$$

(iii) Let $(Q_n)_{n=1}^{\infty} \subset B(X)$. Suppose $Q_n \to 0$ strongly, i.e. for all $f \in X$, $\|Q_n f\|_X \to 0$. Further let $F = \sum_{k=0}^{\infty} f_k \otimes v_k \in H$. We must show that $\|(Q_n \otimes I)F\|_H \to 0$.

$$\|(Q_n \otimes I)F\|_H^2 = \sum_{k=0}^{\infty} \|Q_n f_k \otimes v_k\|_H^2 = \sum_{k=0}^{\infty} \|Q_n f_k\|_X^2 .$$

From the Banch-Steinhaus theorem it follows that

$$\left(\|Q_n\|_{B(X)} = \|Q_n \otimes I\|_{B(H)}\right)_{n=1}^{\infty}$$

is a bounded sequence. From this and the strong convergence of Q_n it follows that $\|(Q_n \otimes I)F\|_H \to 0$.

(iv) Let $(Q_n)_{n=1}^{\infty} \subset B(X)$. Suppose $Q_n \to 0$ weakly, i.e. for all $f,\phi \in X$ $|(Q_n f,\phi)_X| \to 0$. Let $F = \sum_{k=0}^{\infty} f_k \otimes v_k \in H$ and $\Phi = \sum_{\ell=0}^{\infty} \phi_\ell \otimes v_\ell \in H$. We must show that $|((Q_n \otimes I)F,\Phi)_H) \to 0$. This follows from

$$((Q_n \otimes I)F,\Phi)_H = \sum_{k=0}^{\infty} (Q_n f_k,\phi_k)_X$$

and the uniform boundedness of the sequence $(Q_n)_{n=1}^{\infty}$, in a way similar to (iii). □

THEOREM 3

The mapping $P_c^* : B(H) \to B(H) : A \mapsto A_0 \otimes I$ is

(i) continuous with respect to the uniform topology on $B(H)$,

(ii) continuous with respect to the ultra-strong topology on $B(H)$,

(iii) sequentially continuous with respect to the strong topology on $B(H)$, and

(iv) sequentially continuous with respect to the weak topology on $B(H)$.

PROOF:

We write P_c^* as a composition of linear mappings

$$P_c^* : B(H) \to B(H) \to B(X) \to B(H)$$

$$A \mapsto (I \otimes P_0)A(I \otimes P_0) \mapsto A_0 \mapsto A_0 \otimes I \ .$$

The desired continuity of the first arrow follows from Naimark [N], Ch. vii. See also [D]. The desired continuity of the second arrow follows because $A_0 = U^{-1}((I \otimes P_0)A(I \otimes P_0))U$ where $U : X \to X \otimes \langle v_0 \rangle$ is a unitary bijection. Finally, the desired continuity of the third arrow follows from Lemma 2. □

Next we want to show that the property of sequential continuity in Theorem 3, (iii) and (iv) cannot be replaced by continuity.

LEMMA 4

The amplification map $\Gamma : B(X) \to B(H) : E \mapsto E \otimes I$ is neither strongly nor weakly continuous.

PROOF:

(i) Consider the set of operators

$$S_F = \left\{Q \mid \|(Q \otimes I)F\|_H < \varepsilon \subset B(X)\right\}.$$

Here $F = \sum_{k=0}^{\infty} f_k \otimes v_k$ is taken fixed. We have $\sum_{k=0}^{\infty} \|f_k\|_X^2 < \infty$ and we take $(f_k)_{k=0}^{\infty} \subset X$ such that the span of the f_k is dense in X. Now suppose that there exists a finite sequence $(\phi_1,\ldots,\phi_p) \subset X$ and $\delta > 0$ such that the set

$$S_{\phi_1,\ldots,\phi_p} = \left\{Q \mid \|Q\phi_i\|_X < \delta\ ,\ 1 \leq i \leq p\right\}$$

is contained in S_F. By taking $Q = \alpha\Pi \in S_{\phi_1,\ldots,\phi_p}$ where Π is the projection operator onto the orthocomplement of span $\langle\phi_1,\ldots,\phi_p\rangle$ and $\alpha > 0$ sufficiently large, we get a contradiction.

(ii) Consider the set of operators

$$R_{FG} = \left\{Q \mid |((Q \otimes I)F,G)| < \varepsilon\right\} \subset B(X)$$

where $F = \sum_{k=0}^{\infty} f_k \otimes v_k$ and $G = \sum_{\ell=0}^{\infty} g_k \otimes v_k$ are taken fixed. We choose $f_k = g_k = (k+1)^{-1} v_k$. Now suppose that there exist two finite sequences $(\phi_1,\ldots,\phi_p)$, $(\psi_1,\ldots,\psi_p) \subset X$ and $\delta > 0$ such that the set

$$R_{\phi_1,\ldots,\phi_p,\psi_1,\ldots,\psi_p} = \left\{Q \mid |(Q\phi_i,\psi_i)| < \delta,\ 1 \leq i \leq p\right\}$$

is contained in R_{FG}.

By taking $Q = \beta\Pi_1 \in R_{\phi_1,\ldots,\phi_p,\psi_1,\ldots,\psi_p}$ where Π_1 is the projection operator onto the orthocomplement of span $\langle\phi_1,\ldots,\phi_p,\psi_1,\ldots,\psi_p\rangle$ and $\beta > 0$ sufficiently large, we reach a contradiction. □

Remarks. The strong topology on $B(H)$ restricted to the amplification of $B(X)$ is equal to the ultra-strong topology on $B(X)$. By Dixmier [D], the ultra-strong topology on $B(X)$ is strictly finer than the strong topology on $B(X)$ iff X is infinite dimensional. Notice also that the ultra-strong and the strong topology coincide on bounded sets.

THEOREM 5

The reduction-amplification mapping

$$P_c^* : B(H) \to B(H) \ : A \to A_0 \otimes I$$

(i) is not continuous with respect to the strong topology on $B(H)$.

(ii) is not continuous with respect to the weak topology on $B(H)$.

PROOF:

(i) Consider the set $\Sigma_F = \{B | B = Q \otimes I,\ Q \in S_F\} \subset B(H)$ with S_F as in Lemma 4. Suppose there exists a finite sequence $(\Phi_1,\dots,\Phi_p) \subset H$ and $\delta > 0$ such that P_c^* maps the set

$$U_{\Phi_1,\dots\Phi_p} = \{A | \|A\Phi_i\|_H < \delta\} \subset B(H)$$

into Σ_F. We look at operators of the form $A = Q \otimes P_0$, $Q \in B(X)$. Write $\Phi_i = \sum_{k=0}^{\infty} \phi_{ik} \otimes v_k$. Then $\|(Q \otimes P_0)\Phi_i\|_H = \|Q\phi_{i0} \otimes v_0\|_H = \|Q\phi_{i0}\|_X$. So if the operators Q are in the set $S_{\phi_{10},\dots,\phi_{p0}}$, the operators $A = Q \otimes P_0$ are in $U_{\Phi_1,\dots,\Phi_p}$.
For these special operators we have $A_0 = Q$. So it follows from Lemma 4 that P_c^* does not map $U_{\Phi_1,\dots,\Phi_p}$, as a whole, into Σ_F.

(ii) Consider the set $\Xi_{FG} = \{B | B = Q \otimes I,\ Q \in R_{FG}\} \subset B(H)$ with R_{FG} as in Lemma 4. Suppose there exist finite sequences $(\Phi_1,\dots,\Phi_p)$, $(\Psi_1,\dots,\Psi_p) \subset$ $\subset H$ and $\delta > 0$ such that P_c^* maps the set

$$V_{\Phi_1,\ldots,\Phi_p,\Psi_1,\ldots,\Psi_p} = \{A \mid |(A\Phi_i,\Phi_i)| < \delta\}$$

into Ξ_{FG}. Again we look at operators A of the form $A = Q \otimes P_0$, $Q \in B(X)$.
Write $\Psi_i = \sum_{\ell=0}^{\infty} \Psi_{i\ell} \otimes v_\ell$. Then

$$((Q \otimes P_0)\Phi_i,\Psi_i)_H = (Q\phi_{i0} \otimes v_0, \psi_{i0} \otimes v_0)_H = (Q\phi_{i0},\psi_{i0})_X \,.$$

So if the operators Q are in the set $R_{\phi_{10},\ldots,\phi_{p0},\psi_{10},\ldots,\psi_{10}}$,
then the operator $A = Q \otimes P_0$ belongs to $V_{\Phi_1,\ldots,\Phi_p,\Psi_1,\ldots,\Psi_p}$.
For these special operators we have $A = Q \otimes P_0$.
So it follows from Lemma 4 that P_c^* does not map the set
$V_{\Phi_1,\ldots,\Phi_p,\Psi_1,\ldots,\Psi_p}$, as a whole, into Ξ_{FG}. □

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