

# A time-indexed formulation for single-machine scheduling problems : characterization of facets

**Citation for published version (APA):**

Akker, van den, J. M., van Hoesel, C. P. M., & Savelsbergh, M. W. P. (1997). *A time-indexed formulation for single-machine scheduling problems : characterization of facets*. (Onderzoekmemorandum; Vol. 97002). Universiteit Maastricht.

**Document status and date:**

Published: 01/01/1997

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

A Time-Indexed Formulation for Single-  
Machine Scheduling Problems:  
Characterization of Facets

J. van den Akker, C. van Hoesel, M. Savelsbergh

RM/97/002

<http://www.rulimburg.nl/~document/FdEWB.htm>

JEL code:



**Maastricht research school of Economics  
of TEchnology and ORganizations**

Universiteit Maastricht  
Faculty of Economics and Business Administration  
P.O. Box 616  
NL - 6200 MD Maastricht

phone : ++31 43 388 3830  
fax : ++31 43 325 8544

# A Time-Indexed Formulation for Single-Machine Scheduling Problems: Characterization of Facets

J.M. van den Akker

*Department of Mathematics and Computing Science,  
Eindhoven University of Technology,  
P.O.Box 513, 5600 MB Eindhoven, The Netherlands*

C.P.M. van Hoesel

*Department of Mathematics,  
University of Limburg,  
P.O.Box 616, 6200 MD Maastricht, The Netherlands*

M.W.P. Savelsbergh

*School of Industrial and Systems Engineering,  
Georgia Institute of Technology,  
Atlanta, GA 30332-0205, USA*

## Abstract

We report new results for a time-indexed formulation of nonpreemptive single-machine scheduling problems. We give complete characterizations of all facet inducing inequalities with integral coefficients and right-hand side 1 or 2 for the convex hull of the monotone extension of the set of feasible schedules. Furthermore, we identify conditions under which these facet inducing inequalities with right-hand side 1 or 2 are also facet defining for the convex hull of the set of feasible schedules. Our results may lead to improved cutting plane algorithms for single-machine scheduling problems.

**Key words:** scheduling, polyhedral methods, facet inducing inequalities.

## 1 Introduction

Recently developed polyhedral methods have yielded substantial progress in solving many important  $\mathcal{NP}$ -hard combinatorial optimization problems. Some well-known examples are the traveling salesman problem [Padberg and Rinaldi 1991], and large-scale 0-1 integer programming problems [Crowder, Johnson and Padberg 1983]. We refer to Hoffman and Padberg [1985] and Nemhauser and Wolsey [1988] for general descriptions of the approach.

For machine scheduling problems, however, polyhedral methods have not been nearly so successful and relatively few papers have been written in this area. The investigation and development of polyhedral methods for machine scheduling problems is important because traditional combinatorial algorithms do not perform well on difficult problem types in this class.

Balas [1985] pioneered the study of scheduling polyhedra with his work on the facial structure of the job shop scheduling problem. Queyranne [1993] completely characterized the polyhedron associated with the simple nonpreemptive single-machine scheduling problem. Queyranne and Wang [1991] generalized Queyranne's results to include precedence

constraints. Wolsey [1989] compared different formulations for the problem with precedence constraints. Dyer and Wolsey [1990] examined several formulations for the single-machine scheduling problem with release dates, and Nemhauser and Savelsbergh [1992] developed a cutting plane algorithm for this problem. Sousa and Wolsey [1992] investigated a time-indexed formulation for several variants of the nonpreemptive single-machine scheduling problem. Crama and Spieksma [1993] studied the same formulation for problems in which the jobs have equal processing times. Lasserre and Queyranne [1992] presented a mixed integer programming formulation motivated by a control theoretic view of scheduling decisions.

In this paper, we report new results for the time-indexed formulation of nonpreemptive single-machine scheduling problems studied by Sousa and Wolsey [1992]. They introduced three classes of inequalities. The first class consists of inequalities with right-hand side 1, and the second and third class of inequalities with right-hand side  $k \in \{2, \dots, n\}$ . In their cutting plane algorithm, they used an exact separation method only for inequalities with right-hand side 1 and for inequalities with right-hand side 2 in the second class. They used a simple heuristic to identify violated inequalities in the third class.

Their computational experiments revealed that the bounds obtained are strong compared to bounds obtained from other mixed integer programming formulations.

These promising computational results stimulated us to study the inequalities with right-hand side 1 or 2 more thoroughly. We extended the convex hull of the set of feasible solutions by applying monotonization. We derived complete characterizations of all facet inducing inequalities with integral coefficients and right-hand side 1 or 2 for the extended polytope. We also established conditions under which the identified inequalities are also facet inducing for the original polytope. It appears that only some of the classes of inequalities used in the computational experiments by Sousa and Wolsey were facet inducing. Our results may hence lead to improved cutting plane algorithms for single-machine scheduling problems.

For reasons of brevity, in the description of the characterizations some conditions and proofs are omitted; for a complete description see Van den Akker, Van Hoesel and Savelsbergh [1993]. The development and implementation of a branch-and-cut algorithm based on the identified classes of facet inducing inequalities will be discussed in a sequel paper. Some preliminary computational results are given in this paper.

## 2 Problem formulation

The usual setting for nonpreemptive single-machine scheduling problems is as follows. A set  $J$  of  $n$  jobs has to be scheduled on a single machine. Each job  $j \in J$  requires uninterrupted processing for a period of length  $p_j$ , where  $p_j$  is some positive integer. The machine can handle no more than one job at a time.

The time-indexed formulation studied by Sousa and Wolsey [1992] is based on time-discretization, i.e., time is divided into periods, where period  $t$  starts at time  $t - 1$  and ends at time  $t$ . The planning horizon is denoted by  $T$ , which means that all jobs have to be completed by time  $T$ . The formulation is as follows:

$$\text{minimize } \sum_{j=1}^n \sum_{t=1}^{T-p_j+1} c_{jt} x_{jt}$$

subject to

$$\sum_{t=1}^{T-p_j+1} x_{jt} = 1 \quad (j = 1, \dots, n),$$

$$\sum_{j=1}^n \sum_{s=t-p_j+1}^t x_{js} \leq 1 \quad (t = 1, \dots, T),$$

$$x_{jt} \in \{0, 1\} \quad (j = 1, \dots, n; t = 1, \dots, T - p_j + 1),$$

where  $x_{jt} = 1$  if job  $j$  is started in period  $t$  and 0 otherwise. This formulation can be used to model several single-machine scheduling problems by an appropriate choice of the objective coefficients and possibly a restriction of the set of variables. For instance, if the objective is to minimize the weighted sum of the start times, we take coefficients  $c_{jt} = w_j(t - 1)$ , where  $w_j$  denotes the weight of job  $j$ ; if there are release dates  $r_j$ , i.e., job  $j$  becomes available at time  $r_j$ , then we discard the variables  $x_{jt}$  for  $t = 1, \dots, r_j$ . In the sequel, we denote the set of feasible schedules by  $S$ .

Many of the single-machine scheduling problems that can be modeled by the time-indexed formulation given above are  $\mathcal{NP}$ -hard, and therefore the integer program given by this formulation is  $\mathcal{NP}$ -hard. Crama and Spieksma [1993] prove that even when we take  $p_j = 2$  for all  $j$  and  $c_{jt} \in \{0, 1\}$  the problem is  $\mathcal{NP}$ -hard. It is well-known that the problem of minimizing the weighted sum of the start times subject to release dates on the jobs, for which we will develop a cutting plane algorithm, is also  $\mathcal{NP}$ -hard.

In the above formulation, the convex hull  $P_S$  of  $S$ , the set of feasible schedules, is not full-dimensional. As it is often easier to study full-dimensional polyhedra, we study the convex hull  $P_{S^*}$  of  $S^*$ , where  $S^*$  is the monotone extension of  $S$ . A set  $V \subseteq \{0, 1\}^n$  is called *monotone* if for all  $x, y$  we have that  $x \leq y$  and  $y \in V$  implies that  $x \in V$ . The monotone extension  $W^*$  of a set  $W \subseteq \{0, 1\}^n$  is defined as  $W^* = \{x \in \{0, 1\}^n | x \leq y \text{ for some } y \in W\}$ . A description of  $S^*$ , the monotone extension of the set of feasible schedules  $S$ , can be obtained by relaxing the equations (2) into inequalities with sense less-than-or-equal, i.e., the set  $S^*$  is described by:

$$\sum_{t=1}^{T-p_j+1} x_{jt} \leq 1 \quad (j = 1, \dots, n), \tag{1}$$

$$\sum_{j=1}^n \sum_{s=t-p_j+1}^t x_{js} \leq 1 \quad (t = 1, \dots, T), \tag{2}$$

$$x_{jt} \in \{0, 1\} \quad (j = 1, \dots, n; t = 1, \dots, T - p_j + 1)$$

Observe that the set  $S^*$  is the set of all feasible partial schedules, i.e., the set of feasible schedules in which is not all jobs have to be started. In the sequel, when we speak about a schedule, we mean a schedule that can be partial, i.e., it does not have to contain all jobs. When the schedule has to contain all jobs we call it a complete schedule. It is not hard to show that  $P_{S^*}$  is full-dimensional. In the sequel, we consider the polytope  $P_{S^*}$  unless we state otherwise.

Note that the collection of facet inducing inequalities for the polytope  $P_{S^*}$  associated with the set of partial schedules includes the collection of facet inducing inequalities for the polytope  $P_S$  associated with the set of complete schedules.

Montone 0-1 polytopes, i.e., polytopes that are the convex hull of a monotone subset of  $\{0, 1\}^n$ , have been studied by Hammer, Johnson, and Peled [1985]. They proved the following lemma.

**Lemma 1** *Let  $P$  be a monotone polytope. A facet inducing inequality  $ax \leq b$  for  $P$  with integral coefficients  $a_j$  and integral right-hand side  $b$  has either  $b > 0$  and coefficients  $a_j$  in  $\{0, 1, \dots, b\}$  or is a positive scalar multiple of  $-x_j \leq 0$  for some  $j$ .*

**Corollary 1** *A facet inducing inequality  $ax \leq b$  for  $P_{S^*}$  with integral coefficients  $a_{js}$  and integral right-hand side  $b$  has either  $b > 0$  and coefficients  $a_{js}$  in  $\{0, 1, \dots, b\}$  or is a positive scalar multiple of  $-x_{js} \leq 0$  for some  $(j, s)$  with  $1 \leq s \leq T - p_j + 1$ .*

Observe that the above corollary implies that the inequalities  $x_{js} \geq 0$  are the only facet inducing inequalities with right-hand side 0.

Before we present our analysis of the structure of facet inducing inequalities with right-hand side 1 or 2, we introduce some notation and definitions.

The index-set of variables with nonzero coefficients in an inequality is denoted by  $V$ . The set of variables with nonzero coefficients in an inequality associated with job  $j$  defines a set of time periods  $V_j = \{s | (j, s) \in V\}$ . If job  $j$  is started in period  $s \in V_j$ , then we say that job  $j$  is started in  $V$ . With each set  $V_j$  we associate two values

$$l_j = \min\{s | s - p_j + 1 \in V_j\}$$

and

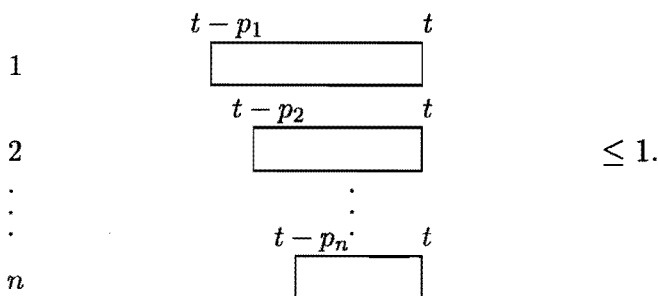
$$u_j = \max\{s | s \in V_j\}.$$

For convenience, let  $l_j = \infty$  and  $u_j = -\infty$  if  $V_j = \emptyset$ . Note that if  $V_j \neq \emptyset$ , then  $l_j$  is the first period in which job  $j$  can be finished if it is started in  $V$ , and that  $u_j$  is the last period in which job  $j$  can be started in  $V$ . Furthermore, let  $l = \min\{l_j | j \in \{1, \dots, n\}\}$  and  $u = \max\{u_j | j \in \{1, \dots, n\}\}$ .

We define an interval  $[t_1, t_2]$  as the set of periods  $\{t_1 + 1, t_1 + 2, \dots, t_2\}$ , i.e., the set of periods between time  $t_1$  and time  $t_2$ . If  $t_1 \geq t_2$ , then  $[t_1, t_2] = \emptyset$ .

For presentational convenience, we use  $x(S)$  to denote  $\sum_{(j,s) \in S} x_{js}$ . As a consequence of the Lemma 1, valid inequalities with right-hand side 1 will be denoted by  $x(V) \leq 1$  and valid inequalities with right-hand side 2 will be denoted by  $x(V^1) + 2x(V^2) \leq 2$ , where  $V = V^1 \cup V^2$  and  $V^1 \cap V^2 = \emptyset$ . Furthermore, we define  $V_j^2 = \{s | (j, s) \in V^2\}$ .

In the sequel, we shall often represent inequalities by diagrams. A diagram contains a line for each job. The blocks on the line associated with job  $j$  indicate the time periods  $s$  for which  $x_{js}$  occurs in the inequality. For example, an inequality of the form (2) can be represented by the following diagram:



### 3 Facet inducing inequalities with right-hand side 1

The purpose of this section is twofold. First, we present new results that extend and complement the work of Sousa and Wolsey [1992]. Second, we familiarize the reader with our approach in deriving complete characterizations of classes of facet inducing inequalities.

Establishing complete characterizations of facet inducing inequalities proceeds in two phases. First, we derive necessary conditions in the form of various structural properties. Second, we show that these necessary conditions on the structure of facet inducing inequalities are also sufficient. Finally, we give conditions on the horizon  $T$  under which the presented sufficient conditions are also sufficient for an inequality to be facet inducing for the original polytope  $P_S$ .

A valid inequality  $x(V) \leq 1$  is called *maximal* if there does not exist a valid inequality  $x(W) \leq 1$  with  $V \subsetneq W$ . The following lemma gives a general necessary condition and is frequently used in the proofs of structural properties.

**Lemma 2** *A facet inducing inequality  $x(V) \leq 1$  is maximal.  $\square$*

**Property 1** *If  $x(V) \leq 1$  is facet inducing, then the sets  $V_j$  are intervals, i.e.,  $V_j = [l_j - p_j, u_j]$ .*

**Proof.** Let  $j \in \{1, \dots, n\}$  and assume  $V_j \neq \emptyset$ . By definition  $l_j - p_j + 1$  is the smallest  $s$  such that  $s \in V_j$  and  $u_j$  is the largest such value. Consider any  $s$  with  $l_j - p_j + 1 < s < u_j$  and let job  $j$  be started in period  $s$ , i.e.,  $x_{js} = 1$ .

Suppose  $(i, t) \in V$  is such that  $x_{it} = x_{js} = 1$  defines a feasible schedule. If  $t < s$ , i.e., job  $i$  is started before job  $j$ , then the schedule that we obtain by postponing the start of job  $j$  until period  $u_j$  is also feasible. This schedule does not satisfy  $x(V) \leq 1$ , which contradicts the validity of the inequality. Hence no job can be started in  $V$  before job  $j$ . Similarly, we obtain a contradiction if  $t > s$ , which implies that no job can be started in  $V$  after job  $j$ .

We conclude that choosing  $x_{js} = 1$  prohibits any job from starting in  $V$ . Because of the maximality of  $x(V) \leq 1$ , we must have  $(j, s) \in V$ .  $\square$

**Property 2** *Let  $x(V) \leq 1$  be facet inducing.*

(a) *Assume  $l = l_1 \leq l_2 = \min\{l_j | j \in \{2, \dots, n\}\}$ . Then  $V_1 = [l - p_1, l_2]$  and  $V_j = [l_j - p_j, l]$  for all  $j \in \{2, \dots, n\}$ .*

(b) *Assume  $u = u_1 \geq u_2 = \max\{u_j | j \in \{2, \dots, n\}\}$ . Then  $V_1 = [u_2 - p_1, u]$  and  $V_j = [u - p_j, u_j]$  for all  $j \in \{2, \dots, n\}$ .*

**Proof.** (a) Let  $x(V) \leq 1$  be facet inducing with  $l = l_1 \leq l_2 = \min\{l_j \mid j \in \{2, \dots, n\}\}$ . Observe that Property 1 implies that  $V_1$  is an interval and that by definition its lower bound equals  $l - p_1$ . We now show that the upper bound is equal to  $l_2$ . Since  $x_{2, l_2 - p_2 + 1} = 1$  and  $x_{1s} = 1$  defines a feasible schedule for any  $s > l_2$ , we have that only one of these variables can occur in  $x(V) \leq 1$ ; as by definition  $(2, l_2 - p_2 + 1) \in V$ , it follows that the upper bound of  $V_1$  is at most  $l_2$ . Now, let  $x_{1s} = 1$  for some  $s \in [l - p_1, l_2]$ . Reasoning as in the proof of Property 1 we can show that since  $l - p_1 + 1 \in V_1$  it follows that no job can be started in  $V$  after job 1. As  $s \leq l_2 = \min\{l_j \mid j \in \{2, \dots, n\}\}$ , it is impossible to start any job in  $V$  before job 1. From the maximality of  $x(V) \leq 1$  we conclude that  $V_1 = [l - p_1, l_2]$ . Similar arguments can be applied to show that  $V_j = [l_j - p_j, l]$  for all  $j \in \{2, \dots, n\}$ .

The proof of (b) is similar to that of (a).  $\square$

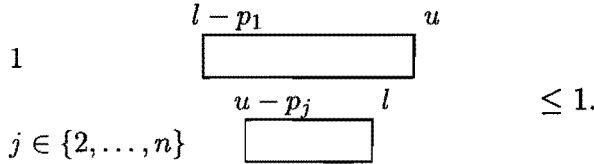
Observe that by Property 2(a) a facet inducing inequality  $x(V) \leq 1$  with  $l = l_1$  necessarily has  $u_1 = u$ . Consequently, Property 2(a) and 2(b) can be combined to give the following theorem.

**Theorem 1** *A facet inducing inequality  $x(V) \leq 1$  has the following structure:*

$$\begin{aligned} V_1 &= [l - p_1, u], \\ V_j &= [u - p_j, l] \quad (j \in \{2, \dots, n\}), \end{aligned} \tag{3}$$

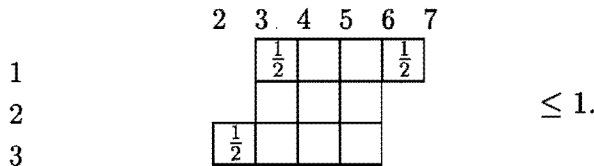
where  $l = l_1 \leq u_1 = u$ .  $\square$

This theorem says that a facet inducing inequality with right-hand side 1 can be represented by the following diagram:



Note that if  $l = u$ , the inequalities with structure (3) coincide with the inequalities (2); if  $V_j = \emptyset$  for all  $j \in \{2, \dots, n\}$ ,  $l = p_1$ , and  $u = T - p_1 + 1$ , then the inequalities with structure (3) coincide with the inequalities (1).

**Example 1** Let  $n = 3$ ,  $p_1 = 3$ ,  $p_2 = 4$  and  $p_3 = 5$ . The inequality with structure (3),  $l = l_1 = 6$  and  $u = u_1 = 7$  is given by the following diagram:



Note that the fractional solution  $x_{14} = x_{17} = x_{33} = \frac{1}{2}$  violates this inequality.



The following theorem shows that the given necessary conditions are also sufficient. The proof of this theorem uses the concept of a *counterexample*. If  $x(V) \leq 1$  is facet inducing, then since  $x(V) \leq 1$  is maximal, for any  $(j, s) \notin V$  there must be a feasible schedule such that  $x_{js} = 1$  and  $x(V) = 1$ . Such a schedule is called a *counterexample* for  $(j, s)$ .

**Theorem 2** *A valid inequality  $x(V) \leq 1$  with structure (3) that is maximal is facet inducing for  $P_{S^*}$ .  $\square$*

**Proof.** Let  $x(V) \leq 1$  be a valid inequality with structure (3) that is maximal, and let  $F = \{x \in P_{S^*} | x(V) = 1\}$ . We show that  $\dim(F) = \dim(P_{S^*}) - 1$  by exhibiting  $\dim(P_{S^*}) - 1$  linearly independent *directions* in  $F$ , where a direction is a vector  $d = x - y$  with  $x, y \in F$ . For notational convenience, a direction will be specified by its nonzero components. We give two sets of directions: unit vectors  $d_{js} = 1$  for all  $(j, s) \notin V$ , and a set of  $|V| - 1$  linearly independent directions  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V$ . Together these give  $\dim(P_{S^*}) - 1$  linearly independent directions.

If  $(j, s) \notin V$ , then since  $x(V) \leq 1$  is maximal there exists a counterexample for  $(j, s)$ , say, defined by  $x_{js} = x_{j's'} = 1$ . Clearly, this schedule is an element of  $F$  and, furthermore, the schedule  $y_{j's'} = 1$  also is an element of  $F$ . We find that  $d = x - y$  yields the direction  $d_{js} = 1$ .

We determine the set of  $|V| - 1$  directions  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V$  in such a way that the undirected graph  $G$  whose vertices are the elements of  $V$  whose edges are given by the pairs  $\{(j_1, s_1), (j_2, s_2)\}$  corresponding to the determined directions form a spanning tree. It is easy to see that this implies that the determined directions are linearly independent. Note that any unit vector  $x_{js} = 1$  with  $(j, s) \in V$  is an element of  $F$ . Now we easily obtain the following directions. We have directions  $d_{1s} = 1, d_{1, s+1} = -1$  for  $s, s+1 \in V_1$ , i.e.,  $s = l - p_1 + 1, \dots, u - 1$ . In the same way, we have directions  $d_{js} = 1, d_{j, s+1} = -1$  for  $j \in \{2, \dots, n\}$ ,  $s, s+1 \in V_j$ , i.e.,  $s = u - p_j + 1, \dots, l - 1$ . Finally, we have directions  $d_{1l-p_1+1} = 1, d_{j, u-p_j+1} = -1$  for  $j \in \{2, \dots, n\}$  with  $V_j \neq \emptyset$ . It is easy to see that these directions determine a spanning tree, and we hence have determined  $|V| - 1$  linearly independent directions.  $\square$

The class of inequalities with structure (3) was already identified by Sousa and Wolsey [1992]. They have shown that if the planning horizon  $T$  is large enough these inequalities are also facet inducing for  $P_{S^*}$ . Let  $p_{\max} = \max\{p_j | j \in \{1, \dots, n\}\}$ .

**Theorem 3** *If  $T \geq \sum_{j=1}^n p_j + 3p_{\max}$ , then a valid inequality  $x(V) \leq 1$  with structure (3) that is maximal is facet inducing for  $P_S$ .  $\square$*

It is not hard to show that a valid inequality  $x(V) \leq 1$  with structure (3) is maximal if and only if either  $V_j \neq \emptyset$  for some  $j \in \{2, \dots, n\}$ , or  $x(V) \leq 1$  coincides with one of the inequalities (1), i.e.,  $l = p_1$  and  $u = T - p_1 + 1$ . Hence inequalities (1) and (2) are facet inducing for  $P_{S^*}$ .

Note that an inequality with structure (3) is determined by one job, which w.l.o.g. is called job 1 and two time periods  $l$  and  $u$ . Since the maximality condition stating that  $V_j \neq \emptyset$  for some  $j \in \{2, \dots, n\}$  implies that  $u - p_{\max} \leq l$ , it follows that the number of facet inducing inequalities with structure (3) that does not coincide with an inequality (1) is bounded by  $nTp_{\max}$ , and hence the total number of facet inducing inequalities with structure (3) is bounded by  $nTp_{\max} + n$ , and hence is polynomial in the size of the formulation.

## 4 Facet inducing inequalities with right-hand side 2

In the previous section, we have derived a complete characterization of all facet inducing inequalities with right-hand side 1. We now derive a similar characterization of all facet inducing inequalities with right-hand side 2.

First, we study the structure of valid inequalities with right-hand side 2 and coefficients 0, 1, and 2. Consider a valid inequality  $x(V^1) + 2x(V^2) \leq 2$ . Clearly, at most two jobs can be started in  $V$ . Let  $j \in \{1, \dots, n\}$  and  $s \in V_j$ . It is easy to see that, if job  $j$  is started in period  $s$ , at least one of the following three statements is true.

- (i) It is impossible to start any job in  $V$  before job  $j$ , and at most one job can be started in  $V$  after job  $j$ .
- (ii) There exists a job  $i$  with  $i \neq j$  such that job  $i$  can be started in  $V$  before as well as after job  $j$  and any job  $j'$  with  $j' \neq j, i$  cannot be started in  $V$ .
- (iii) At most one job can be started in  $V$  before job  $j$ , and it is impossible to start any job in  $V$  after job  $j$ .

Therefore, we can write  $V = LUMUU$ , where  $L \subseteq V$  is the set of variables for which statement (i) holds,  $M \subseteq V$  is the set of variables for which statement (ii) holds, and  $U \subseteq V$  is the set of variables for which statement (iii) holds. Analogously, we can write  $V_j = L_j \cup M_j \cup U_j$ . Note that each of the sets  $L_j, M_j$ , and  $U_j$  may be empty.

If job  $j$  is started in a period in  $V_j^2$ , then it is impossible to start any job in  $V$  before or after job  $j$ . It follows that  $V_j^2 \subseteq L_j \cap U_j$  for all  $j$  and hence  $V^2 \subseteq L \cap U$ . It is not hard to see that if  $L_j \neq \emptyset$  and  $U_j \neq \emptyset$ , then the minimum of  $L_j$  is less than or equal to the minimum of  $U_j$ , and the maximum of  $L_j$  is less than or equal to the maximum of  $U_j$ . By definition  $L_j \cap M_j = \emptyset$  and  $M_j \cap U_j = \emptyset$ . The set  $M_j$  consists of periods between the maximum of  $L_j$  and the minimum of  $U_j$  and hence  $M_j$  must be empty if  $L_j \cap U_j \neq \emptyset$ . By definition of the sets  $L$  and  $U$ ,  $x(L) \leq 1$  and  $x(U) \leq 1$  are valid inequalities.

We conclude that a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  can be represented by a collection of sets  $L_j, M_j$ , and  $U_j$ . To derive necessary conditions on the structure of facet inducing inequalities with right-hand side 2, we study this LMU-structure more closely.

A valid inequality  $x(V^1) + 2x(V^2) \leq 2$  is called *nondecomposable* if it cannot be written as the sum of two valid inequalities  $x(W) \leq 1$  and  $x(W') \leq 1$ . A valid inequality  $x(V^1) + 2x(V^2) \leq 2$  is called *maximal* if there does not exist a valid inequality  $x(W^1) + 2x(W^2) \leq 2$  with  $V \subseteq W$ ,  $V^2 \subseteq W^2$ , where at least one of the subsets is a proper subset. The following lemma yields a general necessary condition and will be frequently used to prove structural properties.

**Lemma 3** *A facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable and maximal.*  
□

The remaining part of the analysis of the LMU-structure proceeds in two phases. In the first phase, we derive conditions on the structure of the sets  $L$  and  $U$  by considering them separately from the other sets. The structural properties thus derived reveal that we have to

distinguish three situations when we consider the overall LMU-structure, based on how the sets  $L$  and  $U$  can be joined. In the second phase, we investigate each of these three situations and derive conditions on the structure of the set  $M$ .

**Property 3** *If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing, then the sets  $L_j$ ,  $M_j$ , and  $U_j$  are intervals.*

**Proof.** Let  $j \in \{1, \dots, n\}$  and assume  $L_j \neq \emptyset$ . By definition  $l_j - p_j + 1$  is the smallest  $s$  such that  $s \in L_j$ . Let  $s_1$  denote the largest such value. Consider any  $s$  with  $l_j - p_j + 1 < s < s_1$  and let job  $j$  be started in period  $s$ , i.e.,  $x_{js} = 1$ .

Reasoning as in the proof of Property 1, we can show that from  $s_1 \in L_j$  it follows that no job can be started in  $V$  before job  $j$ . Suppose  $(i_1, t_1), (i_2, t_2) \in V$  with  $s < t_1$  and  $s < t_2$  are such that  $x_{js} = x_{i_1 t_1} = x_{i_2 t_2} = 1$  defines a feasible schedule. Then the schedule obtained by starting job  $j$  in period  $l_j - p_j + 1$  instead of in period  $s$  is also feasible, which contradicts  $l_j - p_j + 1 \in L_j$ . Hence at most one job can be started in  $V$  after job  $j$ .

We conclude that if  $x_{js} = 1$ , then no job can be started in  $V$  before job  $j$  and at most one job can be started in  $V$  after job  $j$ . This means that if we choose  $x_{js} = 1$ , then  $x(V^1) + 2x(V^2) \leq 1$ . Because of the maximality of  $x(V^1) + 2x(V^2) \leq 2$ , we must have  $(j, s) \in V$ . Consequently,  $s \in L_j$  and we have that  $L_j$  is an interval. Analogously, the sets  $M_j$  and  $U_j$  are intervals.  $\square$

Consider a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$ . We have seen that  $V^2 \subseteq L \cap U$ . Observe that if job  $j$  is started in  $L_j \cap U_j$ , then it is impossible to start any job in  $V$  before or after job  $j$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, this implies  $V_j^2 = L_j \cap U_j$  for all  $j$ , i.e.,  $V^2 = L \cap U$ .

**Property 4** *Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing.*

(a) *Assume  $l = l_1 \leq l_2 \leq \min\{l_j \mid j \in \{3, \dots, n\}\}$ . Then  $L_1 = [l - p_1, l_2]$  and  $L_j = [l_j - p_j, l]$  for all  $j \in \{2, \dots, n\}$ . Furthermore, there exists a  $j \in \{2, \dots, n\}$  such that  $L_j \neq \emptyset$ .*

(b) *Assume  $u = u_1 \geq u_2 \geq \max\{u_j \mid j \in \{3, \dots, n\}\}$ . Then  $U_1 = [u_2 - p_1, u]$  and  $U_j = [u - p_j, u_j]$  for all  $j \in \{2, \dots, n\}$ . Furthermore, there exists a  $j \in \{2, \dots, n\}$  such that  $U_j \neq \emptyset$ .*

**Proof.** (a) Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing with  $l = l_1 \leq l_2 \leq \min\{l_j \mid j \in \{3, \dots, n\}\}$ . Note that by definition  $l - p_1 + 1 \in V_1$ . Since the earliest possible completion time of a job started in  $V$  is  $l$ , we must have  $l - p_1 + 1 \in L_1$ . Reasoning as in the proof of Property 2, we find that  $L_1$  is an interval with lower bound equal to  $l - p_1$  and with upper bound at most equal to  $l_2$ . Now, let  $x_{1s} = 1$  for some  $s \in [l - p_1, l_2]$ . As in the proof of Property 3, we can show that from  $l - p_1 + 1 \in L_1$ , it follows that at most one job can be started in  $V$  after job 1. Since  $s \leq l_2$ , it is impossible to start any job in  $V$  before job 1. Because of the maximality of  $x(V^1) + 2x(V^2) \leq 2$ , we conclude that  $s \in L_1$  and hence  $L_1 = [l - p_1, l_2]$ . Similar arguments can be applied to show that  $L_j = [l_j - p_j, l]$  for all  $j \in \{2, \dots, n\}$ .

Now suppose  $L_j = \emptyset$  for all  $j \in \{2, \dots, n\}$ . We show that in this case  $x(V^1) + 2x(V^2) \leq 2$  can be written as the sum of two valid inequalities with right-hand side 1, which contradicts the fact that  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing. Define  $W = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(j, s) \mid j \in \{2, \dots, n\}, s \in V_j\}$  and  $W' = \{(1, s) \mid s \in V_1\}$ . We first show that  $\sum_{j=2}^n \sum_{s \in V_j} x_{js} \leq 1$  is a valid inequality. For all  $j \in \{2, \dots, n\}$  we have, since by assumption  $L_j = \emptyset$ ,  $l_j - p_j \geq l$ ,

i.e.,  $s > l$  for all  $s \in V_j$ . Consequently, if  $x_{j_1 s_1} = x_{j_2 s_2} = 1$  defines a feasible schedule such that  $\sum_{j=2}^n \sum_{s \in V_j} x_{js} = 2$ , then  $x_{1, l-p_1+1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$  also defines a feasible schedule, which contradicts the validity of  $x(V^1) + 2x(V^2) \leq 2$ . Hence  $\sum_{j=2}^n \sum_{s \in V_j} x_{js} \leq 1$  is a valid inequality and it easily follows that  $x(W) \leq 1$  is also valid. Clearly,  $x(W') \leq 1$  is a valid inequality and  $x(W) + x(W') = x(V^1) + 2x(V^2)$ . We conclude that  $x(V^1) + 2x(V^2) \leq 2$  is not facet inducing. Hence  $L_j \neq \emptyset$  for some  $j \in \{2, \dots, n\}$ .

The proof of (b) is similar to that of (a).  $\square$

As the proof of theorem (2), many of the proofs of the properties and theorems presented in this section use the concept of a counterexample. If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing, then, since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, for any  $(j, s) \notin V$  there must exist a feasible schedule such that  $x_{js} = 1$  and  $x(V^1) + 2x(V^2) = 2$ . Such a schedule is called a *counterexample* for  $(j, s)$ .

**Property 5** Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing.

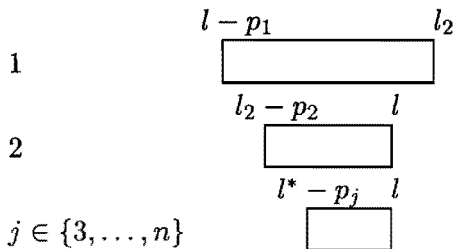
(a) Assume  $l = l_1 \leq l_2 \leq l^*$ , where  $l^* = \min\{l_j \mid j \in \{3, \dots, n\}\}$ . Then for all  $j \in \{3, \dots, n\}$  such that  $L_j \neq \emptyset$  we have  $l_j = l^*$  and for all  $j \in \{3, \dots, n\}$  such that  $L_j = \emptyset$  we have  $l^* - p_j \geq l$ , i.e.,  $L_j = [l^* - p_j, l]$  for all  $j \in \{3, \dots, n\}$ .

(b) Assume  $u = u_1 \geq u_2 \geq u^*$ , where  $u^* = \max\{u_j \mid j \in \{3, \dots, n\}\}$ . Then for all  $j \in \{3, \dots, n\}$  such that  $U_j \neq \emptyset$  we have  $u_j = u^*$  and for all  $j \in \{3, \dots, n\}$  such that  $U_j = \emptyset$  we have  $u^* \leq u - p_j$ , i.e.,  $U_j = [u - p_j, u^*]$  for all  $j \in \{3, \dots, n\}$ .

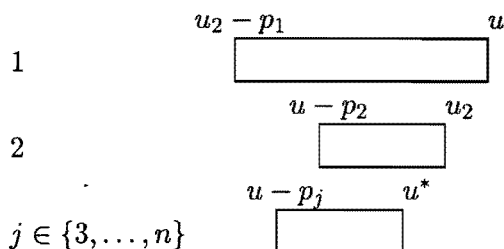
**Proof.** (a) Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing with  $l = l_1 \leq l_2 \leq l^*$ . By definition of  $l^*$  and Property 4,  $L_j \subseteq [l^* - p_j, l]$  for all  $j \in \{3, \dots, n\}$ . We assume w.l.o.g.  $l^* = l_3$ . Suppose that  $L_j \neq [l^* - p_j, l]$  for some  $j \in \{4, \dots, n\}$ , say  $L_4 \neq [l^* - p_4, l]$ . Clearly, if  $l^* - p_4 \geq l$ , then  $L_4 = \emptyset$  and hence  $L_4 = [l^* - p_4, l]$ . Consequently  $l^* - p_4 < l$  and  $l_4 > l^*$ , i.e.,  $l^* - p_4 + 1 \notin V_4$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, there is a counterexample for  $(4, l^* - p_4 + 1)$ . Let  $x_{4, l^* - p_4 + 1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$  define such a counterexample. Since  $l^* - p_4 + 1 \leq l$ , the jobs  $j_1$  and  $j_2$  are started after job 4. Clearly one of the jobs 1, 2 and 3 does not occur in  $\{j_1, j_2\}$ . Suppose job 3 does not occur. It is now easy to see that  $x_{3, l^* - p_3 + 1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$  is a feasible schedule, which contradicts the validity of  $x(V^1) + 2x(V^2) \leq 2$ . If job 1 or job 2 does not occur in  $\{j_1, j_2\}$  we obtain a contradiction in the same way.

The proof of (b) is similar to that of (a).  $\square$

Properties 4 and 5 say that if  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing and we assume  $l = l_1 \leq l_2 \leq l^*$ , then the set  $L$  can be represented by the following diagram:



Similarly, if we assume  $u = u_1 \geq u_2 \geq u^*$ , then the set  $U$  can be represented by the following diagram:



Observe that a facet inducing inequality with right-hand side 2 has at most three types of intervals  $L_j$ , each characterized by the definition of the first period of the interval, and at most three types of intervals  $U_j$ , each characterized by the definition of the last period of the interval. Stated slightly differently, with the exception of two jobs the intervals  $L_j$  have the same structure for all jobs. Similarly, the intervals  $U_j$  have the same structure for all but two jobs. It turns out that, when we study the overall LMU-structure, it suffices to consider three situations, based on the jobs with the deviant intervals  $L_j$  and  $U_j$ :

- (1a)  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$ , where  $l^* = \min\{l_j \mid j \in \{3, \dots, n\}\}$  and  $u^* = \max\{u_j \mid j \in \{3, \dots, n\}\}$ ;
- (1b)  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ , where  $l^* = \min\{l_j \mid j \in \{3, \dots, n\}\}$  and  $u^* = \max\{u_j \mid j \in \{2, 4, \dots, n\}\}$ ;
- (2)  $l = l_1$  and  $u = u_2$ .

Before we investigate each of the three situations, we prove a property that applies to case 1.

**Property 6** *If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1 < l_2 = \min\{l_j \mid j \in \{2, \dots, n\}\}$  and  $u = u_1 > u_i = \max\{u_j \mid j \in \{2, \dots, n\}\}$ , then  $l_2 < u_i$ .*

**Proof.** Suppose that  $l_2 \geq u_i$ . We show that  $x(V^1) + 2x(V^2) \leq 2$  can be written as the sum of two valid inequalities with right-hand side 1, which contradicts the fact that  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing. Let  $W = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(j, s) \mid j \in \{2, \dots, n\}, s \in V_j\}$  and  $W' = \{(1, s) \mid s \in V_1\} \cup \{(j, s) \mid j \in \{2, \dots, n\}, s \in L_j \cap U_j\}$ . Clearly  $x(W) + x(W') = x(V^1) + 2x(V^2)$  and  $x(W') \leq 1$  is a valid inequality. From  $V_j \subseteq [l_j - p_j, u_j] \subseteq [l_2 - p_j, u_i]$  for all  $j \in \{2, \dots, n\}$  and  $l_2 \geq u_i$ , it easily follows that  $\sum_{j=2}^n \sum_{s \in V_j} x_{js} \leq 1$  is a valid inequality and hence  $x(W) \leq 1$  is also valid.  $\square$

#### 4.1 Case (1a)

Observe that the conditions on  $l_j$  and  $u_j$  and Properties 4 and 5 completely determine the sets  $L$  and  $U$ . Therefore, all that remains to be investigated is the structure of the set  $M$ .

**Property 7** *If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$ , then  $M_1 = [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$ ,  $M_2 = [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$  and  $M_j = [u_2 - p_j, l_2] \cap [l, u - p_j]$  for  $j \in \{3, \dots, n\}$ .*

**Proof.** Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$ . We derive the structure of the set  $M$  from that of  $L$  and  $U$ .

If job 1 is started in  $M_1$ , then, since  $l_2 \leq l^*$  and  $u_2 \geq u^*$ , it is possible to start job 2 in  $V$  before as well as after job 1, which implies that  $M_1 \subseteq [l_2, u_2 - p_1]$ . Furthermore, it is impossible to start any job  $j \in \{3, \dots, n\}$  in  $V$  and hence  $M_1 \subseteq [u^* - p_1, l^*]$ . We conclude that  $M_1 \subseteq [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$ . If job 1 is started in period  $s \in [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$ , then, since  $s \in [l_2, u_2 - p_1]$ ,  $[l_2, u_2 - p_1] \subset [l - p_1, u]$ , and  $L_2 \cap U_2 = [u - p_2, l]$ , job 2 cannot be started in  $L_2 \cap U_2$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, it follows that  $M_1 = [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$ .

By definition  $M_2 \subseteq [l_2 - p_2, u_2]$ . If job 2 is started in  $M_2$ , then, since  $l = l_1$  and  $u = u_1$ , it should be possible to start job 1 in  $V$  before as well as after job 2, which implies that  $M_2 \subseteq [l, u - p_2]$ . Furthermore, it is impossible to start any job  $j \in \{3, \dots, n\}$  in  $V$  and hence  $M_2 \subseteq [u^* - p_2, l^*]$ . We conclude that  $M_2 \subseteq [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$ . If job 2 is started in period  $s \in [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$ , then, since  $s \in [l_2 - p_2, u_2]$  and  $L_1 \cap U_1 = [u_2 - p_1, l_2]$ , job 1 cannot be started in  $L_1 \cap U_1$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, it follows that  $M_2 = [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$ .

Let  $j \in \{3, \dots, n\}$ . If job  $j$  is started in  $M_j$ , then it is possible to start job 1 in  $V$  before as well as after job  $j$ , which implies that  $M_j \subseteq [l, u - p_j]$ . Furthermore, it is impossible to start any job  $j' \in \{2, 3, \dots, n\} \setminus \{j\}$  in  $V$  and hence  $M_j \subseteq [u_2 - p_j, l_2]$ . We conclude that  $M_j \subseteq [u_2 - p_j, l_2] \cap [l, u - p_j]$ . If job  $j$  is started in period  $s \in [u_2 - p_j, l_2] \cap [l, u - p_j]$ , then, since  $s \in [u_2 - p_j, l_2]$ ,  $L_1 \cap U_1 = [u_2 - p_1, l_2]$ , and, by Property 6,  $l_2 < u_2$ , job 1 cannot be started in  $L_1 \cap U_1$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, it follows that  $M_j = [u_2 - p_j, l_2] \cap [l, u - p_j]$ .

Observe that by definition  $M_k \subseteq [l_k - p_k, u_k]$  for all  $k \in \{1, \dots, n\}$  and that for all but  $k = 2$  this condition is dominated by other conditions.  $\square$

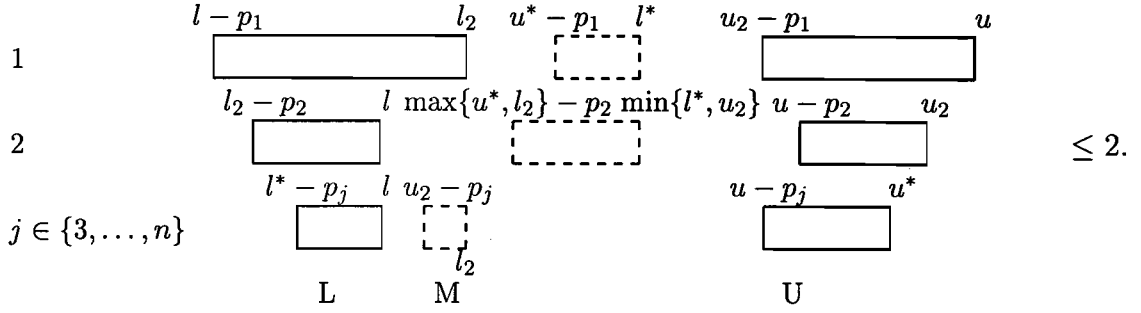
Properties 4, 5 and 7 completely determine the LMU-structure of a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$ . However, in order to emphasize the inherent structure of the intervals  $M_j$ , we prefer to use a different representation of the set  $M$ . It is easy to show that, if  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$ , then for all  $j \in \{3, \dots, n\}$  we have  $[u_2 - p_j, l] \subseteq L_j$  and  $[u - p_j, l_2] \subseteq U_j$ . We can use this observation to show that Properties 4, 5 and 7 can be combined to give the following theorem.

**Theorem 4** *A facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  has the following LMU-structure:*

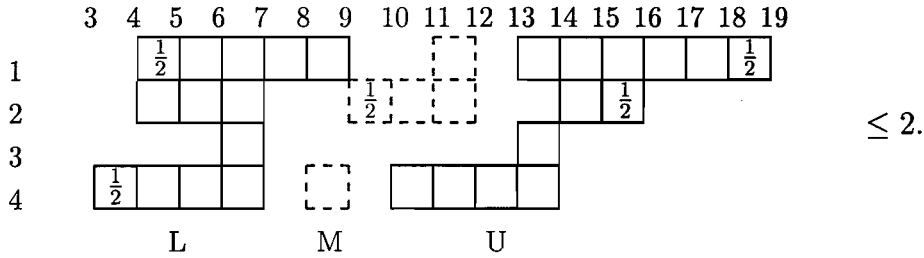
$$\begin{aligned} L_1 &= [l - p_1, l_2], & M_1 &= [u^* - p_1, l^*] \setminus (L_1 \cup U_1), \\ L_2 &= [l_2 - p_2, l], & M_2 &= [\max\{u^*, l_2\} - p_2, \min\{l^*, u_2\}] \setminus (L_2 \cup U_2), \\ L_j &= [l^* - p_j, l], & M_j &= [u_2 - p_j, l_2] \setminus (L_j \cup U_j), \\ \\ U_1 &= [u_2 - p_1, u], \\ U_2 &= [u - p_2, u_2], \\ U_j &= [u - p_j, u^*] \quad (j \in \{3, \dots, n\}), \end{aligned} \tag{4}$$

where  $[u_2 - p_j, l] \subseteq L_j$  and  $[u - p_j, l_2] \subseteq U_j$  for all  $j \in \{3, \dots, n\}$ .  $\square$

This theorem says that a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  can be represented by the following diagram:



**Example 2** Let  $n = 4$ ,  $p_1 = 3$ ,  $p_2 = 5$ ,  $p_3 = 6$ , and  $p_4 = 9$ . The inequality with LMU-structure (4) and  $l = l_1 = 7$ ,  $l_2 = 9$ ,  $l^* = 12$ ,  $u^* = 14$ ,  $u_2 = 16$  and  $u = u_1 = 19$  is given by the following diagram:



Note that the fractional solution  $x_{15} = x_{1,19} = x_{2,10} = x_{2,16} = x_{4,4} \frac{1}{2}$  violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

The following theorem shows that the given necessary conditions are also sufficient.

**Theorem 5** A valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  and LMU-structure (4) that is nondecomposable and maximal is facet inducing for  $P_{S^*}$ .

**Proof.** Let  $x(V^1) + 2x(V^2) \leq 2$  be a valid inequality with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  and LMU-structure (4) that is nondecomposable and maximal, and let  $F = \{x \in P_{S^*} \mid x(V^1) + 2x(V^2) = 2\}$ . As in the proof of Theorem 2 we show that  $\dim(F) = \dim(P_{S^*}) - 1$  by exhibiting  $\dim(P_{S^*}) - 1$  linearly independent directions in  $F$ . We give three sets of directions: unit vectors  $d_{js} = 1$  for all  $(j, s) \notin V$ ,  $d_{js} = 1, d_{1, l-p_1+1} = d_{2, u_2} = -1$  for all  $(j, s) \in V^2$ , and a set of  $|V| - |V^2| - 1$  linearly independent directions  $d_{j_1 s_1} = 1, d_{j_2, s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ . Together these give  $\dim(P_{S^*}) - 1$  linearly independent directions in  $F$ .

If  $(j, s) \notin V$ , then, since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, there is a counterexample for  $(j, s)$ , say, defined by  $x_{js} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ . Clearly this schedule is an element of  $F$ . Note that the schedule  $y_{j_1 s_1} = y_{j_2 s_2} = 1$  also is an element of  $F$  and hence  $d = x - y$  yields the direction  $d_{js} = 1$ .

Note that for  $(j, s) \in V^2$  the schedule defined by  $x_{js} = 1$  is an element of  $F$ . Since  $l < l_2$  and, by Property 6,  $l_2 < u_2$ , we have that  $y_{1, l-p_1+1} = y_{2, u_2} = 1$  defines a feasible schedule. This schedule also is an element of  $F$  and hence  $d_{js} = 1, d_{1, l-p_1+1} = d_{2, u_2} = -1$  is a direction in  $F$  for all  $(j, s) \in V^2$ .

We determine the  $|V| - |V^2| - 1$  directions  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$  in such a way that the undirected graph  $G$  whose vertices are the elements of  $V \setminus V^2$

and whose edges are given by the pairs  $\{(j_1, s_1), (j_2, s_2)\}$  corresponding to the determined directions is a spanning tree. This implies that the determined directions are linearly independent.

Observe that  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$  is a direction in  $F$  if there exists an index  $(i, t) \in V \setminus V^2$  such that  $x_{j_1 s_1} = x_{it} = 1$  and  $y_{j_2 s_2} = y_{it} = 1$  both define feasible schedules. In this case, we say that  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  is a *direction by*  $(i, t)$ .

First, we determine directions that correspond to edges in  $G$  within the sets  $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$  and  $\{(j, s) \mid s \in U_j \setminus L_j\}$ . For  $s - 1, s \in L_1 \setminus U_1$ ,  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(2, u_2)$ . If  $M_1 \neq \emptyset$ , then  $d_{1l_2} = -1, d_{1m} = 1$  is a direction by  $(2, u_2)$ , where  $m$  is the minimum of  $M_1$ , and for  $s - 1, s \in M_1$ ,  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(2, u_2)$ . Furthermore, for  $s - 1, s \in U_1 \setminus L_1$ ,  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(2, l_2 - p_2 + 1)$ . Now, let  $j \in \{2, \dots, n\}$ . For  $s - 1, s \in L_j \setminus U_j$ ,  $d_{j, s-1} = -1, d_{js} = 1$  is a direction by  $(1, u)$ . If  $M_j \neq \emptyset$ , then  $d_{jl} = -1, d_{jm} = 1$  is a direction by  $(1, u)$ , where  $m$  is the minimum of  $M_j$ , and for  $s - 1, s \in M_j$ ,  $d_{j, s-1} = -1, d_{js} = 1$  is a direction by  $(1, u)$ . Furthermore, for  $s - 1, s \in U_j \setminus L_j$ ,  $d_{j, s-1} = -1, d_{js} = 1$  is a direction by  $(1, l - p_1 + 1)$ .

Second, we determine directions that correspond to edges in  $G$  between sets  $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$  belonging to different jobs and between sets  $\{(j, s) \mid s \in U_j \setminus L_j\}$  belonging to different jobs. We define  $W = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(j, s) \mid j \in \{2, \dots, n\}, s \in V_j\}$  and  $W' = \{(1, s) \mid s \in V_j\} \cup \{(j, s) \mid j \in \{2, \dots, n\}, s \in L_j \cap U_j\}$ . Clearly  $x(W) + x(W') = x(V^1) + 2x(V^2)$  and  $x(W') \leq 1$  is a valid inequality. Since  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable, there must be a feasible schedule such that  $x(W) = 2$ , i.e.,  $\sum_{j=2}^n \sum_{s \in V_j} x_{js} = 2$ . Let  $x_{j_1 s_1} = x_{j_2 s_2} = 1$  with  $s_1 < s_2$  define such a schedule. It is easy to see that we may assume  $s_1 = l_{j_1} - p_{j_1} + 1$  and  $s_2 = u_{j_2}$ . Since  $l = l_1, y_{1, l-p_1+1} = y_{j_2 u_{j_2}} = 1$  also defines a feasible schedule and it follows that  $d_{1, l-p_1+1} = -1, d_{j_1, l_{j_1}-p_{j_1}+1} = 1$  is a direction by  $(j_2, s_2)$ . In the same way, since  $u = u_1, y_{j_1, l_{j_1}-p_{j_1}+1} = y_{1u} = 1$  defines a feasible schedule and it follows that  $d_{1u} = -1, d_{j_2 u_{j_2}} = 1$  is a direction by  $(j_1, s_1)$ . For  $j \in \{2, \dots, n\} \setminus \{j_1\}$  such that  $L_j \cup M_j \neq \emptyset$ ,  $d_{j_1, l_{j_1}-p_{j_1}+1} = -1, d_{j, l_j-p_j+1} = 1$  is a direction by  $(1, u)$ . Furthermore, for  $j \in \{2, \dots, n\} \setminus \{j_2\}$  such that  $U_j \neq \emptyset$ ,  $d_{j_2 u_{j_2}} = -1, d_{ju_j} = 1$  is a direction by  $(1, l - p_1 + 1)$ .

Finally, we determine a direction that corresponds to an edge in  $G$  between  $L \cup M$  and  $U$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable and  $x(U) \leq 1$  is a valid inequality, there exists a feasible schedule with  $x(L) + x(M) = 2$ . Let  $x_{j_1 s_1} = x_{j_2 s_2} = 1$  define such a schedule. Since  $l_1 = l$ , we may assume w.l.o.g.  $j_1 = 1$ . Since  $s_2 \in L_{j_2} \cup M_{j_2}$ ,  $y_{j_2 s_2} = y_{1u} = 1$  also is a feasible schedule. It follows that  $d_{1s_1} = -1, d_{1u} = 1$  is a direction by  $(j_2, s_2)$ .

It is easy to see that the determined directions form a spanning tree of  $G$  and we hence have determined  $|V| - |V^2| - 1$  linearly independent directions.  $\square$

The following theorem shows that the sufficient conditions given by the previous theorem are also sufficient for the original polytope if the planning horizon  $T$  is large enough.

**Theorem 6** *If  $T \geq \sum_{j=1}^n p_j + 5p_{\max}$ , then a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  and LMU-structure (4) that is nondecomposable and maximal is facet inducing for  $P_S$ .*

**Proof.** The proof is an extension of the proof of the previous theorem. Let  $F = \{x \in P_S \mid x(V^1) + 2x(V^2) = 2\}$ . Sousa and Wolsey [1992] showed that if  $T \geq \sum_{j=1}^n p_j + p_{\max}$ , then  $\dim(P_S) = \sum_{j=1}^n (T - p_j + 1) - n$ , i.e.,  $\dim(P_S) = \dim(P_{S^*}) - n$ . We show that  $\dim(F) = \dim(P_S) - 1$  by exhibiting  $\dim(P_{S^*}) - n - 1$  linearly independent directions in  $F$ . i.e., the



number of directions that we determine is  $n$  smaller than the number of directions in the previous proof.

Again, we give three sets of directions. The first set consists of directions  $d_{js} = 1, d_{js(j)} = -1$  with  $(j, s) \notin V, s \neq s(j)$ , where for each  $j$  we have that  $s(j)$  is chosen such that  $(j, s(j)) \notin V$ . The first set corresponds to the first set in the previous proof; it however contains one fewer direction for each job and hence  $n$  fewer directions. The second and third set coincide with the second and third set in the previous proof in the sense that the directions have the same value for the entries corresponding to variables in the inequality. Since the directions in the first set have zero entries for all variables in the inequality, linear independence of the directions follows in the same way as in the previous proof.

Let  $j \in \{1, \dots, n\}$ . It is not hard to see that the horizon  $T$  is so large that there exists a time period  $s(j)$  such that for any feasible schedule  $x_{j_1s_1} = x_{j_2s_2} = 1$  with  $x(V^1) + 2x(V^2) = 2$  and  $j \notin \{j_1, j_2\}$  the schedule  $x_{j_1s_1} = x_{j_2s_2} = x_{js(j)} = 1$  is also feasible. In most cases  $s(j)$  can be set equal to 1 or  $T - p_j + 1$ . Now, let  $s \neq s(j)$  be such that  $(j, s) \notin V$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, there is a counterexample for  $(j, s)$ , say defined by  $x_{js} = x_{j_1s_1} = x_{j_2s_2} = 1$ . We show that, since  $T \geq \sum_{j=1}^n p_j + 5p_{\max}$ , this schedule can be extended to a complete schedule in which the machine is idle in the periods  $s(j), s(j) + 1, \dots, s(j) + p_j - 1$ . It is not hard to see that if a schedule that contains just one job, is extended to a complete schedule by scheduling the other jobs as early as possible, then the resulting schedule has at most  $p_{\max}$  periods of idle time between the jobs. Similarly, a schedule with  $k$  jobs can be extended to a complete schedule with at most  $kp_{\max}$  periods of idle time between the jobs. In this case, we have to extend a schedule with three jobs and a given interval of idle time. As the length of this interval is at most  $p_{\max}$ , it can be viewed as a virtual job. It now follows that we can extend the schedule to a complete schedule with at most  $5p_{\max}$  periods of idle time between the jobs, where  $4p_{\max}$  is caused by three jobs plus one virtual job, and an extra  $p_{\max}$  is caused by the fact that the virtual job consists of idle time. We conclude that, since  $T \geq \sum_{j=1}^n p_j + 5p_{\max}$ , there is a complete schedule  $x^*$  with  $x_{js}^* = x_{j_1s_1}^* = x_{j_2s_2}^* = 1$  in which the machine is idle in the periods  $s(j), s(j) + 1, \dots, s(j) + p_j - 1$ . Clearly, the schedule  $y^*$  obtained from  $x^*$  by starting job  $j$  in period  $s(j)$  instead of period  $s$  is also feasible. Now,  $x^* - y^*$  yields the direction  $d_{js} = 1, d_{js(j)} = -1$ .

The directions in the second and the third set are determined like in the previous proof. For the construction of the directions we start with the same schedules as in the previous proof. Using similar arguments as in the above paragraph, it follows that the lower bound on the horizon  $T$  ensures that each of these schedules can be extended to a complete schedule. As we obtain our directions by taking the differences of these complete schedules, these directions may contain some nonzero entries corresponding to variables outside the inequality.  $\square$

Let  $x(V^1) + 2x(V^2) \leq 2$  be a valid inequality with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  and LMU-structure (4). We can show that  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable if and only if  $M_j \neq \emptyset$  for some  $j \in \{1, \dots, n\}$ , and  $l^* < u_2$  or  $l_2 < u^*$ . Observe that  $x(V^1) + 2x(V^2) \leq 2$  is maximal if and only if none of the intervals  $L_j, M_j$ , and  $U_j$  can be extended. It is not hard to show that the intervals  $M_j$  with  $j \in \{1, 3, \dots, n\}$  cannot be extended. Necessary and sufficient conditions for  $x(V^1) + 2x(V^2) \leq 2$  to be maximal can hence be derived by determining conditions under which the other intervals cannot be extended. For reasons of brevity, these conditions are omitted.

An inequality with structure (4) is determined by two jobs that w.l.o.g. are called job 1 and 2 and six time periods  $l, l_2, l^*, u^*, u_2$  and  $u$ . Recall that if the inequality is nondecomposable,

then we have  $l^* < u_2$  or  $l_2 < u^*$ . It is easy to see that  $l^* < u_2$  implies that  $U_2 \neq \emptyset$  and  $L_j \neq \emptyset$  for some  $j \in \{3, \dots, n\}$ , which implies that  $u_2 > u - p_{\max}$  and  $l^* < l + p_{\max}$ . Analogously,  $l_2 < u^*$  implies that  $l_2 < l + p_{\max}$  and  $u^* > u - p_{\max}$ . Since we also have  $l_2 \leq l^*$  and  $u_2 \geq u^*$ , it is not hard to see that the number of facet inducing inequalities with structure (4) is bounded by  $2n^2T^3p_{\max}^3$ , and is hence polynomial in the size of the formulation.

## 4.2 Case (1b)

As in case (1a), the conditions on  $l_j$  and  $u_j$  and Properties 4 and 5 completely determine the sets  $L$  and  $U$ . From these properties we can easily derive that if  $l_2 = l^*$  and  $u_3 = u^*$ , then  $L_i \neq \emptyset$  and  $U_i \neq \emptyset$  for  $i$  such that  $p_i = \max\{p_j \mid j \in \{2, \dots, n\}\}$ . But then  $l_i = l_2$  and  $u_i = u_3$  and we are in case (1a). We conclude that  $l_2 < l^*$  or  $u_3 > u^*$ . All that remains to be investigated is the structure of the set  $M$ .

**Property 8** *If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ , then  $M_1 = \emptyset$ ,  $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$ ,  $M_3 = [u^* - p_3, l_2] \cap [l, u - p_3] \cap [l^* - p_3, u_3]$ , and  $M_j = [u_3 - p_j, l_2] \cap [l, u - p_j]$  for  $j \in \{4, \dots, n\}$ .  $\square$*

The proof of this property is analogous to the proof of Property 7. Properties 4, 5, and 8 determine the LMU-structure of a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ . As in case (1a), we prefer to use a different representation of the set  $M$ , in order to emphasize the inherent structure of the intervals  $M_j$ . It turns out that a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j < l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  has the following property, which restricts the class of inequalities determined by Properties 4, 5, and 8 and leads to a simpler form of the intervals  $M_j$ .

**Property 9** *If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ , then  $l^* \leq u^*$ .*

**Proof.** Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ . To be able to prove that  $l^* \leq u^*$ , we first show that  $[u_3 - p_2, \min\{l^*, u_3\}] \subseteq V_2$  and  $[\max\{u^*, l_2\} - p_3, l_2] \subseteq V_3$ . It is easy to see that if job 2 is started in  $[u_3 - p_2, \min\{l^*, u_3\}]$  then, it is impossible to start any job  $j \in \{3, \dots, n\}$  in  $V$  and job 1 cannot be started in  $L_1 \cap U_1$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, it follows that  $[u_3 - p_2, \min\{l^*, u_3\}] \subseteq V_2$ . Analogously,  $[\max\{u^*, l_2\} - p_3, l_2] \subseteq V_3$ . Since, by assumption,  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$ , we have  $l_3 > l_2$  and  $u_2 < u_3$ . From  $[u_3 - p_2, \min\{l^*, u_3\}] \subseteq V_2$  and  $u_2 < u_3$ , we conclude that  $l^* < u_3$ . Analogously,  $u^* > l_2$ . From  $l^* < u_3$  and  $u^* > l_2$ , it follows that  $l^* < u^*$  if  $l_2 = l^*$  or  $u_3 = u^*$ . We still have to show that  $l^* \leq u^*$  if  $l_2 < l^*$  and  $u_3 > u^*$ .

Suppose  $l_2 < l^*$  and  $u_3 > u^*$ . We show that  $[u - p_j, l^*] \subseteq U_j$  for all  $j \in \{2, 4, \dots, n\}$ . Let  $j \in \{2, 4, \dots, n\}$  and let job  $j$  be started in  $[u - p_j, l^*]$ . Clearly, any job  $i \in \{3, \dots, n\} \setminus \{j\}$  cannot be started before job  $j$ . If job 2 is started before job  $j$ , then, since  $M_2 \subseteq [u_3 - p_j, l^*]$  and  $l^* < u_3$ , job 2 is not started in  $M_2$  and job 2 is hence started in  $L_2$ . It is now easy to see that at most one job can be started in  $V$  before job  $j$ . Since  $L_1 \cap U_1 = [u_3 - p_1, l_2]$  and  $l_2 < l^* < u_3$ , job 1 cannot be started in  $L_1 \cap U_1$ . Because of the maximality of  $x(V^1) + 2x(V^2) \leq 2$ , we conclude

that  $[u - p_{j_1}, l^*] \subseteq U_{j_1}$ . Observe that from  $l_2 < u^*$  and Property 8 follows that  $U_j \neq \emptyset$  for some  $j \in \{2, 4, \dots, n\}$  or  $M_2 \neq \emptyset$ . If  $U_j \neq \emptyset$  for some  $j \in \{2, 4, \dots, n\}$ , then, since  $[u - p_j, l^*] \subseteq U_j$ ,  $l^* \leq u^*$ . If  $M_2 \neq \emptyset$ , then since, by Property 8,  $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$ , we must have  $u_3 - p_2 < l^*$ . It is easy to see that if job 2 is started in  $[u_3 - p_2, l^*] \cap [l, u - p_2]$ , then job 1 is the only job that can be started before as well as after job 2 and job 1 cannot be started in  $L_1 \cap U_1$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is maximal, this implies  $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2]$  and we conclude that  $l^* \leq u^*$ .  $\square$

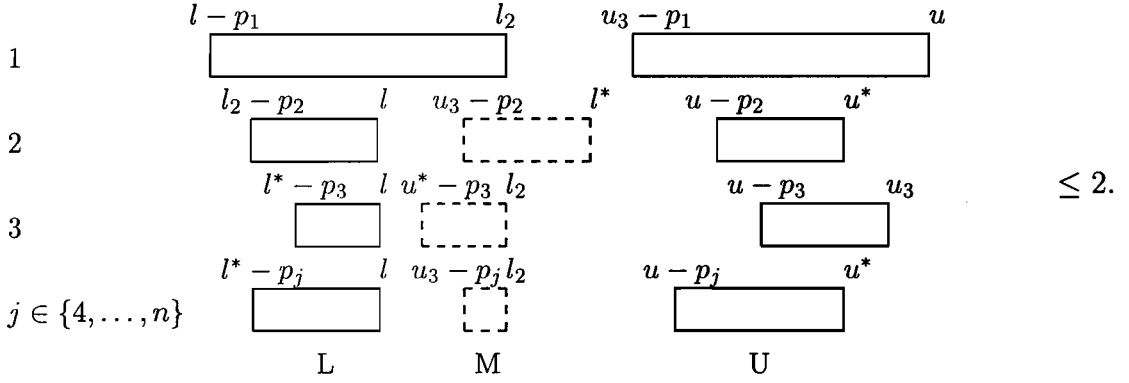
It is not hard to see that Properties 4, 5, 8, and 9 can be combined to give the following theorem.

**Theorem 7** *A facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  has the following LMU-structure:*

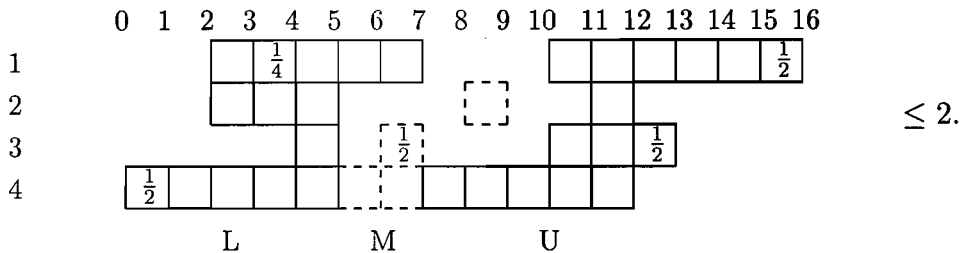
$$\begin{aligned}
 L_1 &= [l - p_1, l_2], & M_1 &= \emptyset, & U_1 &= [u_3 - p_1, u], \\
 L_2 &= [l_2 - p_2, l], & M_2 &= [u_3 - p_2, l^*] \setminus (L_2 \cup U_2), & U_2 &= [u - p_2, u^*], \\
 L_3 &= [l^* - p_3, l], & M_3 &= [u^* - p_3, l_2] \setminus (L_3 \cup U_3), & U_3 &= [u - p_3, u_3], \\
 L_j &= [l^* - p_j, l], & M_j &= [u_3 - p_j, l_2] \setminus (L_j \cup U_j), & U_j &= [u - p_j, u^*] \quad (j \in \{4, \dots, n\}),
 \end{aligned} \tag{5}$$

where  $l^* \leq u^*$ .  $\square$

This theorem says that a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  can be represented by the following diagram:



**Example 3** Let  $n = 4$ ,  $p_1 = 3$ ,  $p_2 = 5$ ,  $p_3 = 6$ , and  $p_4 = 9$ . The inequality with LMU-structure (5) and  $l = l_1 = 5$ ,  $l_2 = 7$ ,  $l^* = 9$ ,  $u^* = 12$ ,  $u_3 = 13$  and  $u = u_1 = 16$  is given by the following diagram:



Note that the fractional solution  $x_{1,16} = x_{37} = x_{3,13} = x_{41} = \frac{1}{2}$  and  $x_{14} = \frac{1}{4}$  violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

The following theorem shows that the given necessary conditions are also sufficient.

**Theorem 8** *A valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  and LMU-structure (5) that is nondecomposable and maximal is facet inducing for  $P_S$ .  $\square$*

The proof of this theorem is similar to that of Theorem 5. In the same way as in case (1a), the proof can be extended to prove that the sufficient conditions given by the previous theorem are also sufficient for the original polytope if the horizon  $T$  is large enough.

**Theorem 9** *If  $T \geq \sum_{j=1}^n p_j + 5p_{\max}$ , then a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  and LMU-structure (5) that is nondecomposable and maximal is facet inducing for  $P_S$ .  $\square$*

Furthermore, we can show that a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$ , and  $l_j > l_2$  or  $u_j < u_3$  for all  $j \in \{2, \dots, n\}$  and LMU-structure (5) is nondecomposable if and only if  $M_j \neq \emptyset$  for some  $j \in \{2, \dots, n\}$ . Necessary and sufficient conditions for such an inequality to be maximal can be derived as in case (1a).

An inequality with structure (5) is determined by three jobs and six time periods. By definition of case (1b) we have  $l_2 < u_3$ . It is easy to see that this implies that  $L_2 \neq \emptyset$  and  $U_3 \neq \emptyset$ , and it follows that  $l_2 < l + p_{\max}$  and  $u_3 > u - p_{\max}$ . We find that the number of facet inducing with structure (5) is bounded by  $n^3 T^4 p_{\max}^2$ , and hence is also polynomial in the size of the formulation.

**Remark.** It may seem more natural to define case (1a) as  $l = l_1 < l_2 < l^*$  and  $u = u_1 > u_2 > u^*$ , and case (1b) as  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_3 \geq u^*$ . Since under this definition Property 9 does not hold, we prefer the given one.

### 4.3 Case (2)

Observe that in this case the conditions on  $l_j$  and  $u_j$  and Properties 4 and 5 do not completely determine the sets  $L$  and  $U$ . It turns out to be beneficial to introduce a notion slightly different from that of  $l^*$  and  $u^*$ , namely  $l' = \min\{l_j \mid j \in \{3, \dots, n\}\}$  and  $u' = \max\{u_j \mid j \in \{3, \dots, n\}\}$ . Note that it is possible that  $l_2 > l'$  or  $u_1 < u'$ , i.e.,  $l'$  and  $u'$  do not necessarily coincide with  $l^*$  and  $u^*$  as defined in Property 5. We can however prove the following property in a way analogous to Property 5.

**Property 10** *Let  $x(V^1) + 2x(V^2) \leq 2$  be facet inducing with  $l = l_1$  and  $u = u_2$ .*

(a) *For all  $j \in \{3, \dots, n\}$  such that  $L_j \neq \emptyset$ , we have  $l_j = l'$  and for all  $j \in \{3, \dots, n\}$  such that  $L_j = \emptyset$ , we have  $l' - p_j \geq l$ , i.e.,  $L_j = [l' - p_j, l]$  for all  $j \in \{3, \dots, n\}$ .*

(b) *For all  $j \in \{3, \dots, n\}$  such that  $U_j \neq \emptyset$ , we have  $u_j = u'$  and for all  $j \in \{3, \dots, n\}$  such that  $U_j = \emptyset$ , we have  $u' \leq u - p_j$ , i.e.,  $U_j = [u - p_j, u']$  for all  $j \in \{3, \dots, n\}$ .  $\square$*

We next investigate the structure of the set  $M$ .

**Property 11** If  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1$  and  $u = u_2$ , then  $M_1 = [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$ ,  $M_2 = [u' - p_2, l'] \cap [l, \max\{u_1, u'\} - p_2] \cap [l_2 - p_2, u]$ , and  $M_j = \emptyset$  for  $j \in \{3, \dots, n\}$ .  $\square$

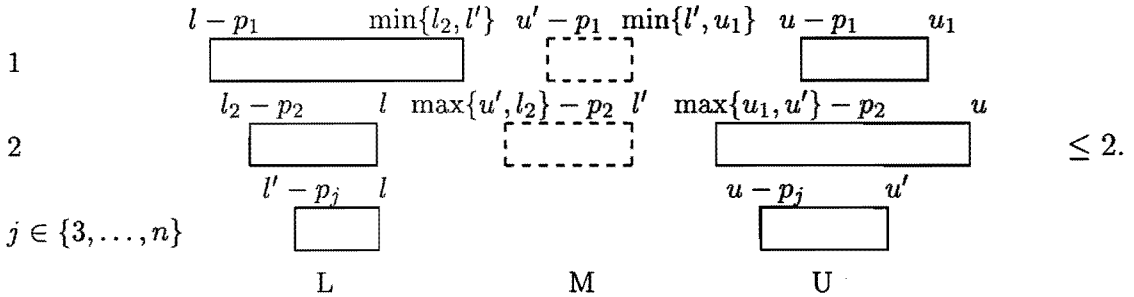
As in case (1b), the proof of this property is analogous to that of Property 7. Properties 4, 10 and 11 completely determine the LMU-structure of a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$ . As in the previous two cases, we prefer to use a different representation of the set  $M$ , in order to emphasize the inherent structure of the intervals  $M_j$ . It is easy to show that if  $x(V^1) + 2x(V^2) \leq 2$  is facet inducing with  $l = l_1$  and  $u = u_2$ , then  $[l' - p_2, l] \subseteq L_2$  and  $[u - p_1, u'] \subseteq U_1$ . It is now not hard to see that Properties 4, 10, and 11 can be combined to give the following theorem.

**Theorem 10** A facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$  has the following LMU-structure:

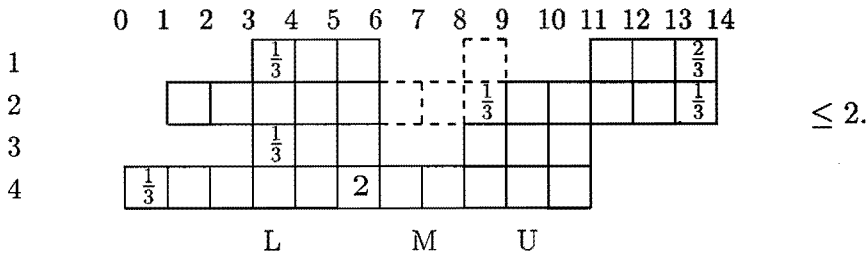
$$\begin{aligned}
 L_1 &= [l - p_1, \min\{l_2, l'\}], & M_1 &= [u' - p_1, \min\{l', u_1\}] \setminus (L_1 \cup U_1), \\
 L_2 &= [l_2 - p_2, l], & M_2 &= [\max\{u', l_2\} - p_2, l'] \setminus (L_2 \cup U_2), \\
 L_j &= [l' - p_j, l], & M_j &= \emptyset, \\
 U_1 &= [u - p_1, u_1], \\
 U_2 &= [\max\{u_1, u'\} - p_2, u], \\
 U_j &= [u - p_j, u'] \quad (j \in \{3, \dots, n\}),
 \end{aligned} \tag{6}$$

where  $[l' - p_2, l] \subseteq L_2$  and  $[u - p_1, u'] \subseteq U_1$ .  $\square$

This theorem says that a facet inducing inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$  can be represented by the following diagram:



**Example 4** Let  $n = 4$ ,  $p_1 = 3$ ,  $p_2 = 5$ ,  $p_3 = 6$ , and  $p_4 = 9$ . The inequality with LMU-structure (6) and  $l = l_1 = 6$ ,  $l_2 = 6$ ,  $l^* = 9$ ,  $u^* = 11$ ,  $u_1 = 6$ , and  $u_2 = u = 14$  is given by the following diagram:



Note that  $(4, 6) \in L \cap U$ , i.e.,  $x_{46}$  has coefficient 2. The fractional solution  $x_{14} = x_{29} = x_{2,14} = x_{34} = x_{41} = \frac{1}{3}$  and  $x_{1,14} = \frac{2}{3}$  violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

The following theorem shows that the given necessary conditions are also sufficient.

**Theorem 11** *A valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$  and LMU-structure (6) that is nondecomposable and maximal is facet inducing for  $P_{S^*}$ .*

**Proof.** The proof proceeds along the same lines as that of Theorem 5. The first and second set of directions can be determined as in the proof of Theorem 5. We consider the third set of directions, i.e., we determine the  $|V| - |V^2| - 1$  directions  $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$  with  $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$  in such a way that the undirected graph  $G$  whose vertices are the elements of  $V \setminus V^2$  and whose edges are given by the pairs  $\{(j_1, s_1), (j_2, s_2)\}$  corresponding to the determined directions is a spanning tree.

First, we determine directions that correspond to edges in  $G$  within the sets  $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$  and  $\{(j, s) \mid s \in U_j \setminus L_j\}$ . For  $s - 1, s \in L_1 \setminus U_1$ ,  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(2, u)$ . If  $M_1 \neq \emptyset$ , then  $d_{1, \min\{l_2, l'\}} = -1, d_{1m} = 1$  is a direction by  $(2, u)$ , where  $m$  is the minimum of  $M_1$ , and for  $s - 1, s \in M_1$ ,  $d_{1, s-1} = -1, d_{1s} = 1$  also is a direction by  $(2, u)$ . Let  $s - 1, s \in U_1 \setminus L_1$ . Note that  $s - 1 > \min\{l_2, l'\}$ . If  $l_2 \leq l'$ , then  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(2, l_2 - p_2 + 1)$ . If  $l_2 > l'$ , then  $d_{1, s-1} = -1, d_{1s} = 1$  is a direction by  $(j, l' - p_j + 1)$ , where  $j \in \{3, \dots, n\}$  is such that  $l_j = l'$ . In the same way we find that for  $s - 1, s \in L_2 \setminus U_2$ ,  $d_{2, s-1} = -1, d_{2s} = 1$  is a direction by  $(1, u_1)$  if  $u_1 \geq u'$ , and by  $(j, u')$  if  $u_1 < u'$ , where  $j \in \{3, \dots, n\}$  is such that  $u_j = u'$ . Observe that if job 2 is started in  $M_2$ , then job 1 is the only job that can be started before or after job 2. We find that if  $L_2 \neq \emptyset$  and  $M_2 \neq \emptyset$ , then  $d_{2l} = -1, d_{2m} = 1$  is a direction by  $(1, u_1)$ , where  $m$  is the minimum of  $M_2$ . For  $s - 1, s \in M_2$ ,  $d_{2, s-1} = -1, d_{2s} = 1$  also is a direction by  $(1, u_1)$ . Furthermore, for  $s - 1, s \in U_2 \setminus L_2$ ,  $d_{2, s-1} = -1, d_{2s} = 1$  is a direction by  $(1, l - p_1 + 1)$ . Now, let  $j \in \{3, \dots, n\}$ . Note that  $M_j = \emptyset$ . Clearly, for  $s - 1, s \in L_j \setminus U_j$ ,  $d_{j, s-1} = -1, d_{js} = 1$  is a direction by  $(2, u)$  and for  $s - 1, s \in U_j \setminus L_j$ ,  $d_{j, s-1} = -1, d_{js} = 1$  is a direction by  $(1, l - p_1 + 1)$ .

Second, we determine directions that correspond to edges in  $G$  between the sets  $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$  belonging to different jobs and between sets  $\{(j, s) \mid s \in U_j \setminus L_j\}$  belonging to different jobs. It is easy to see that for  $j \in \{3, \dots, n\}$  such that  $L_j \neq \emptyset$ ,  $d_{1, l - p_1 + 1} = -1, d_{j, l - p_1 + 1} = 1$  is a direction by  $(2, u)$ . For  $j \in \{3, \dots, n\}$  such that  $U_j \neq \emptyset$ ,  $d_{2u} = -1, d_{ju'} = 1$  is a direction by  $(1, l - p_1 + 1)$ . We still have to determine a direction that corresponds to an edge in  $G$  between  $\{(2, s) \mid s \in (L_2 \cup M_2) \setminus U_2\}$  and one of the sets  $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$  with  $j \in \{1, 3, \dots, n\}$ , and a direction that corresponds to an edge in  $G$  between  $\{(1, s) \mid s \in U_1 \setminus L_1\}$  and one of the sets  $\{(j, s) \mid s \in U_j \setminus L_j\}$  with  $j \in \{2, \dots, n\}$ . Observe that, since  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable, we must have  $(L_2 \cup M_2) \setminus U_2 \neq \emptyset$ . We define  $W = \{(1, s) \mid s \in V_1\} \cup \{(2, s) \mid s \in L_2 \cap U_2\} \cup \{(j, s) \mid j \in \{3, \dots, n\}, s \in L_j\}$  and  $W' = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(2, s) \mid s \in V_2\} \cup \{(j, s) \mid j \in \{3, \dots, n\}, s \in U_j\}$ . Note that  $x(W) + x(W') = x(V^1) + 2x(V^2)$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable, there exists a feasible schedule such that  $x(W) = 2$ , or there exists a feasible schedule such that  $x(W') = 2$ . Suppose that there exists a feasible schedule such that  $x(W) = 2$ . It is easy to see that in such a schedule some job  $j \in \{3, \dots, n\}$  is started in  $L_j$ , say job  $j_1$ , and that job 1 is started after job  $j$ . It easily follows that  $x_{j_1, l' - p_{j_1} + 1} = x_{1u_1} = 1$  defines a feasible schedule. If  $L_2 \neq \emptyset$ , then, since  $[l' - p_2, l] \subseteq L_2$ , we have  $l_2 \leq l'$ . It follows that  $y_{2, l_2 - p_2 + 1} = y_{1u_1} = 1$

defines a feasible schedule. If job 2 is started in  $M_2$ , then job 1 can be started after job 2. Hence, if  $M_2 \neq \emptyset$ , then  $y_{2,l_2-p_2+1} = y_{1u_1} = 1$  also defines a feasible schedule. We conclude that  $d_{2,l_2-p_2+1} = -1, d_{j_1,l'-p_{j_1}+1} = 1$  is a direction by  $(1, u_1)$ . As  $x_{j_1,l'-p_{j_1}+1} = x_{1u_1} = 1$  defines a feasible schedule,  $y_{j_1,l'-p_{j_1}+1} = y_{2u}$  clearly also defines a feasible schedule. We find that  $d_{1u_1} = -1, d_{2u} = 1$  is a direction by  $(j_1, l' - p_{j_1} + 1)$ .

Suppose that there is a feasible schedule such that  $x(W') = 2$ . It is now not hard to see that  $x_{2,l_2-p_2+1} = x_{ju'} = 1$  is a feasible schedule for some job  $j \in \{3, \dots, n\}$  such that  $U_j \neq \emptyset$ , say for job  $j_1$ . Similarly to the previous case, we find that  $d_{2,l_2-p_2+1} = 1, d_{1,l-p_1+1} = -1$  is a direction by  $(j_1, u')$  and that  $d_{j_1,u'} = 1, d_{1u_1} = -1$  is a direction by  $(2, l_2 - p_2 + 1)$ .

Finally, we determine a direction that corresponds to an edge in  $G$  between  $L \cup M$  and  $U$ . Since  $x(V^1) + 2x(V^2) \leq 2$  is nondecomposable and  $x(U) \leq 1$  is a valid inequality, there is a feasible schedule such that  $x(L) + x(M) = 2$ . It is easy to see that in such a schedule job 1 and job 2 are started in  $L \cup M$ . Let  $x_{1s_1} = x_{2s_2} = 1$  be such a schedule. Since  $s_1 \in L_1 \cup M_1$ ,  $y_{1s_1} = y_{2u} = 1$  also is a feasible schedule. It follows that  $d_{2s_2} = 1, d_{2u} = -1$  is a direction by  $(1, s_1)$ .

It is easy to see that the determined directions form a spanning tree of  $G$  and we have hence determined  $|V| - |V^2| - 1$  linearly independent directions.  $\square$

In the same way as in case (1a), we can extend the proof of the above theorem to show that if the time horizon is large enough, then the given sufficient conditions are also sufficient for the original polytope.

**Theorem 12** *If  $T \geq \sum_{j=1}^n p_j + 5p_{\max}$  then, a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$  and LMU-structure (6) that is nondecomposable and maximal is facet inducing for  $P_S$ .*

We can prove that a valid inequality  $x(V^1) + 2x(V^2) \leq 2$  with  $l = l_1$  and  $u = u_2$  and LMU-structure (6) is nondecomposable if and only if  $M_1 \neq \emptyset$  or  $M_2 \neq \emptyset$ , and  $l' < u_1$  or  $l_2 < u'$ . Necessary and sufficient conditions for such an inequality to be maximal can be derived as in the previous two cases.

As an inequality with structure (4), an inequality with structure (6) is determined by two jobs and six time periods. Analogously to case (1a), we can show that the number of facet inducing inequalities with structure (6) is bounded  $2n^2T^4p_{\max}^2$ , where the  $T^4$  instead of  $T^3$  stems from the fact that we do not necessarily have  $l_2 \leq l'$  and  $u_1 \geq u'$ . The number of facet inducing inequalities is clearly polynomial in the size of the formulation.

## 5 Preliminary computational results

To obtain insight in the effectiveness of the classes of facet inducing inequalities discussed above, we have developed separation algorithms that identify violated inequalities in these classes, and we have embedded these separation algorithms in MINTO.

MINTO, a Mixed INTeger Optimizer [Nemhauser, Savelsbergh, and Sigismondi 1994] is a software system that solves mixed-integer linear programs by a branch-and-bound algorithm with linear relaxations. It also provides automatic constraint classification, preprocessing, primal heuristics and constraint generation. Moreover, the user can enrich the basic algorithm by providing a variety of specialized application functions that can customize MINTO to achieve maximum efficiency for a problem class.

The performance of the resulting algorithm has been tested on the  $\mathcal{NP}$ -hard single-machine scheduling problem of minimizing the weighted sum of the completion times subject to release dates. For the instances that were tested by Sousa and Wolsey [1992], the algorithm found the optimal value without branching, as their algorithm did. We report results for 20 randomly generated instances with 30 jobs and uniformly distributed parameters, with processing times in  $[1, 5]$ , weights in  $[1, 10]$ , and release dates in  $[0, \frac{1}{2} \sum p_j]$ .

Table 1 shows the value of the initial linear program ( $Z_{LP}$ ), the value of the linear program after cuts with right-hand side 1 have been added ( $Z_{LP}^1$ ), the value of the linear program after cuts with right-hand side 1 and 2 have been added ( $Z_{LP}^2$ ), and the value of the optimal solution ( $Z_{IP}$ ). The results indicate that both classes of inequalities are effective in reducing the integrality gap.

Table 1: Preliminary computational results.

	$Z_{LP}$	$Z_{LP}^1$	$Z_{LP}^2$	$Z_{IP}$
1	6710.75	6724	6726	6727
2	6490.63	6518	6532	6533
3	5876.5	5887.5	5897	5897
4	5537	5537	5537.5	5540
5	5157.33	5165	5166.66	5185
6	4996	4996	4996	4996
7	4559.5	4589.33	4592.33	4620
8	5837	5837	5837	5837
9	5590	5590	5590	5590
10	5477	5477	5477	5477
11	5964.83	5974.83	5986	5986
12	6951	6951	6951	6951
13	5411.2	5423	5424.5	5434
14	4022.1	4029.33	4030.33	4044
15	4634	4636	4636	4636
16	4573.75	4582.125	4585.33	4597
17	6241.5	6266.5	6267.5	6279
18	4558.05	4593.52	4599	4606
19	5814	5829.8	5830	5830
20	4517.7	4536	4545.25	4558

Because of the encouraging computational results, we have started to develop and implement a full-blown branch-and-cut algorithm. We are investigating primal heuristics, branching strategies, and row management schemes. This branch-and-cut algorithm will be described in a subsequent paper, which will also include a discussion of the separation algorithms and a presentation of various computational experiments.



## 6 Related research

As mentioned in the introduction, Sousa and Wolsey [1992] and Crama and Spieksma [1993] have also studied the time-indexed formulation of single machine scheduling problems. In this section, we briefly indicate the relation between their research and our research.

Sousa and Wolsey present three classes of valid inequalities. The first class consists of inequalities with right-hand side 1, and the second and third class consist of inequalities with right-hand side  $k \in \{2, \dots, n - 1\}$ . Each class of inequalities is derived by considering a set of jobs and a certain time period. The right-hand side of the resulting inequality is equal to the cardinality of the considered set of jobs.

They show that the inequalities in the first class, which is exactly the class of inequalities with structure (3), are all facet inducing. In Section 3, we have complemented this result by showing that all facet inducing inequalities with right-hand side 1 are in this class. With respect to the other two classes of valid inequalities we make the following observations. Any inequality in the second class that has right-hand side 2 can be lifted to an inequality with LMU-structure (4) if  $p_{k_1} \neq p_{k_2}$ , and to an inequality with LMU-structure (6) if  $p_{k_1} = p_{k_2}$ , where  $\{k_1, k_2\}$  is the set of jobs considered. Any inequality in the third class that has right-hand side 2 can be written as the sum of two valid inequalities with right-hand side 1. For either of the two classes, Sousa and Wolsey give an example of a fractional solution that violates one of the inequalities in the class and for which they claim that it does not violate any valid inequality with right-hand side 1. We found that in both cases the latter claim is false.

Crama and Spieksma investigate the special case of equal processing times. They completely characterize all facet inducing inequalities with right-hand side 1 and present two other classes of facet inducing inequalities with right-hand side  $k \in \{2, \dots, n - 1\}$ .

Our characterization of all facet inducing inequalities with right-hand side 1 was found independently and generalizes their result. The inequalities in their second class that have right-hand side 2 are special cases of the inequalities with LMU-structure (6), and the inequalities in their third class that have right-hand side 2 are special cases of the inequalities with LMU-structure (4). In addition to the facet inducing inequalities reported in their paper, they have identified other classes of facet inducing inequalities with right-hand side 2 [Spieksma 1991].

### Acknowledgement

The authors wish to thank C.A.J. Hurkens for his useful remarks and suggestions.

### References

- E. BALAS (1985). On the facial structure of scheduling polyhedra. *Mathematical Programming Study* 24, 179-218.
- Y. CRAMA, F.C.R. SPIEKSMAS (1993). *Scheduling jobs of equal length: complexity and facets*. Report M 93-03, Faculty of Economics, University of Limburg, Maastricht.
- H. CROWDER, E.L. JOHNSON, M.W. PADBERG (1983). Solving large-scale zero-one linear programming problems. *Operations Research* 31, 803-834.
- M.E. DYER, L.A. WOLSEY (1990). Formulating the single machine sequencing problem with

- release dates as a mixed integer program. *Discrete Applied Mathematics* 26, 255-270.
- K.L. HOFFMAN, M.W. PADBERG (1985). LP-based combinatorial problem solving. *Annals of Operations Research* 4, 145-194.
- J.B. LASSERRE, M. QUEYRANNE (1992). Generic scheduling polyhedra and a new mixed-integer formulation for single-machine scheduling. E. BALAS, G. CORNUÉJOLS, R. KANNAN (eds.). *Integer Programming and Combinatorial Optimization*, Proceedings of the IPCO-Conference held at Carnegie Mellon University, University Printing and Publications, Carnegie Mellon University, Pittsburgh.
- G.L. NEMHAUSER, M.W.P. SAVELSBERGH (1992). A cutting plane algorithm for the single machine scheduling problem with release times. M. AKGÜL, H. HAMACHER, S. TUFECKI (eds.). *Combinatorial Optimization: New Frontiers in the Theory and Practice*, NATO ASI Series F: Computer and Systems Sciences 82, Springer, Berlin, 63-84.
- G.L. NEMHAUSER, M.W.P. SAVELSBERGH, AND G.C. SIGISMONDI (1994). MINTO, a Mixed INTegeR Optimizer. *Operations Research Letters*, to appear.
- G.L. NEMHAUSER, L.A. WOLSEY (1988). *Integer and Combinatorial Optimization*. Wiley, New York.
- M.W. PADBERG, G. RINALDI (1991). A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM Review* 33, 60-100.
- P.L. HAMMER, E.L. JOHNSON, AND U.N. PELED. (1975) Facets of regular 0-1 polytopes. *Mathematical programming* 8, 179-206.
- M. QUEYRANNE (1993). Structure of a simple scheduling polyhedron. *Mathematical Programming* 58, 89-110.
- M. QUEYRANNE, Y. WANG (1991). Single machine scheduling polyhedra with precedence constraints. *Mathematics of Operations Research* 16, 1-20.
- J.P. SOUSA, L.A. WOLSEY (1992). A time-indexed formulation of non-preemptive single-machine scheduling problems. *Mathematical Programming* 54, 353-367.
- F.C.R. SPIEKSMAN (1991). Personal communication.
- J.M. VAN DEN AKKER, C.P.M. VAN HOESEL, M.W.P. SAVELSBERGH (1993). *Facet Inducing Inequalities for Single-Machine Scheduling Problems*. COSOR-Memorandum, Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven.
- L.A. WOLSEY (1989). *Formulating single machine scheduling problems with precedence constraints*. CORE discussion paper 8924, Catholic University of Louvain, Louvain-la-Neuve.

**METEOR Research Memoranda**  
**University of Limburg , Maastricht**  
**Faculty of Economics and Business Administration**  
**Information: ++31 43 388 3830 (phone), ++31 43 325 8544 (fax)**

### Overview 1995

- RM/95/001 M. Galizzi  
Gender Discrimination and Quit Behavior: Do Future Wages Matter? An Analysis for Short-Tenure Workers
- RM/95/002 Th. Broecker  
Is Cost Sharing for R&D under the Commensurate with Income Standard Actually Arm's Length?
- RM/95/003 B. Klaus, H. Peters, T. Storcken  
Strategy-proof Division with Single-peaked Preferences and Initial Endowments
- RM/95/004 J.G. Backhaus  
Die Überführung von Produktionsmitteln in Gemeineigentum
- RM/95/005 S. Dixon  
Limit Pricing and Multi-Market Entry
- RM/95/006 M. van Wegberg  
Can R&D Alliances Facilitate the Formation of a Cartel?
- RM/95/007 A. Perea y Monsuwé, M. Jansen, H. Peters  
On The Equivalence of Weakly and Fully Consistent Beliefs in Signaling Games
- RM/95/008 C.P.M. van Hoesel, A.P.M. Wagelmans  
An  $O(T^3)$  algorithm for the economic lot-sizing problem with constant capacities
- RM/95/009 M. van Wegberg  
Cooperation between Research Companies and Manufacturing Firms: The Choice between Market Contract, Vertical Merger, and an R&D-Alliance
- RM/95/010 M. van Wegberg  
Mergers and Alliances in the Multimedia Market
- RM/95/011 M. van Wegberg  
Architectural Battles in the Multimedia Market
- RM/95/012 W. Vanhaverbeke  
De Economische Uitdagingen voor de Subregio Zuid West-Vlaanderen?
- RM/95/013 J. Derks, H. Haller  
Weighted Nucleoli
- RM/95/014 J.G. Backhaus  
The Quest for Ecological Tax Reform: a Schumpeterian Approach to Public Finance
- RM/95/015 M.C.M. Vendrik  
Dynamics of a household norm in female labour supply, non-linear dynamics
- RM/95/016 Th. Zwick, F. Bunte, M. Haan, H. Maks, J. Muysken  
The Working of a Labour Market Model with Potential Entry
- RM/95/017 A.J. Vermeulen, M.J.M. Jansen  
On the Invariance of Solutions of Finite Games
- RM/95/018 W. Vanhaverbeke  
Competing with Alliance Networks: the Case of the Risc-Microprocessor Technology
- RM/95/019 S. Dixon  
Limiting Pricing and the Mode of Entry
- RM/95/020 J.G. Backhaus  
An Economic Analysis of Constitutional Law
- RM/95/021 B. Klaus, H. Peters, T. Storcken  
Reallocation of an infinitely divisible good
- RM/95/022 S. Van Hoesel, A. Wagelmans, M. Van Vliet  
An  $O(n \log n)$  algorithm for the two-machine flow shop problem with controllable machine speeds
- RM/95/023 K. Aardal, S. Van Hoesel  
Polyhedral Techniques in Combinatorial Optimization
- RM/95/024 J.G.A. van Mierlo  
The Quest for Public Management Reform. Theory and Practice in the Case of the Netherlands 1970-1995
- RM/95/025 J.G.A. van Mierlo  
Bureaucratie en bureaucratiseren. Een Theoretische en Praktische Verkenning van Recente Ontwikkelingen in de Publieke Sector en de Particuliere Sector
- RM/95/026 P.J. Zwaneveld, S. Dauzère-Péres, S.P.M. van Hoesel, L.G. Kroon, H.E. Romeijn, M. Salomon, H.W. Ambergen  
Routing trains through railway stations: model formulation and algorithms
- RM95/027 T. Storcken  
Strategy-Proof Preference Rules
- RM/95/028 C.A. van Eijl, C.P.M. van Hoesel  
On the discrete lot-sizing and scheduling problem with Wagner-Whitin costs
- RM/95/029 J. Birner  
The surprising place of cognitive psychology in the work of F.A. Hayek
- RM/95/030 J. Birner  
Decentralization as ability to adapt
- RM/95/031 J. Birner  
Is the past really not what is used to be?
- RM/95/032 J.G.A. van Mierlo  
De Ontwikkeling van de Kosten van Beroepsziekten en Arbeidsongevallen in Nederland. Een economische inschatting van de schadelast voor verzekeraars.
- RM/95/033 Y. Crama, O.E. Flippo, J. van de Klundert, F.C.R. Spieksma  
The assembly of printed circuit boards: a case with multiple machines and multiple board types
- RM/95/034 R. Beetsma, A. Bovenberg  
Monetary Union without Fiscal Coordination May Discipline Policymakers
- RM/95/035 R. Beetsma, A. Bovenberg  
The Role of Public Debt in the Game of Double Chicken
- RM/95/036 J.G. Backhaus  
Introduction: Paracelsus (1493-1541): his relevance for modern health related policy
- RM/95/037 Y. Crama, O.E. Flippo, J. van de Klundert, F.C.R. Spieksma  
The Component Retrieval Problem in Printed Circuit Board Assembly
- RM/95/038 J. Schijns, G. Schröder  
Segment Selection by Relationship Strength

### Overview 1996

- RM/96/001 J.C. Hoekstra, J.M.C. Schijns  
There is No Need for More than One Definition of Direct Marketing
- RM/96/002 J.A.H. Maks, M. Haan

RM/96/003 Heinrich von Stackelberg's text book "Grundlagen der theoretischen Volkswirtschaftslehre" revisited  
M. Haan, J.A.H. Maks  
Stackelberg and Cournot competition under equilibrium limit pricing

RM/96/004 D. de la Croix, J.P. Urbain  
Intertemporal substitution in import demand and habit formation

RM/96/005 R. Beetsma, A.L. Bovenberg  
Designing Fiscal and Monetary Institutions for a European Monetary Union

RM/96/006 R. Beetsma, A.L. Bovenberg  
The Interaction of Fiscal and Monetary Policy in a Monetary Union: Balancing Credibility and Flexibility

RM/96/007 J.G. Backhaus  
Good Economics, Bad Economics, and European Economics

RM/96/008 T. Storcken, H. Monsuur  
Measuring Inconsistency

RM/96/009 A.J. Vermeulen, M.H.M. Jansen  
Extending Invariant Solutions

RM/96/010 J.F. Koers  
De Theorie van Overheidsinstellingen: een Netwerkbenadering

RM/96/011 K.I. Aardal, A. Hipolito, C.P.M. van Hoesel, B. Jansen  
A Branch-and-Cut Algorithm for the Frequency Assignment Problem

RM/96/012 P.K. Keizer  
A Critical Assessment of Recent Trends in Dutch Industrial Relations

RM/96/013 J.G. Backhaus  
Tausch und Geld: Ein Kommentar aufgrund von Georg Simmels Philosophie des Geldes

RM/96/014 A. Perea Y Monsuwé, H. Peters  
Limit consistent solutions in noncooperative games

RM/96/015 R.T. Baillie, W.P. Osterberg  
Central Bank Intervention and Risk in the Forward Premium

RM/96/016 B. Klaus, H. Peters, T. Storcken  
Strategy-Proof Division of a Private Good and Convex Utility

RM/96/017 J. Lemmink, J. Mattsson  
Warmth during Non-Routine Service Encounters: Impact of Ethological Events and Post-Experience Measures

RM/96/018 C. Lindenburg, A.J. van Reeken  
Elektronisch Uitgeven: een onderzoek naar de gevolgen ervan in de bedrijfskolom

RM/96/019 S. Dixon  
Trade Restrictions and Strategic Firm Behaviour

RM/96/020 S. Dixon  
Signalling for Entry: Limit Pricing and Reciprocal Entry

RM/96/021 J. Muysken, A.P. van Veen  
It does matter which side of the labour market is taxed

RM/96/022 J. Muysken, T. Zwick  
A Note on Incentive Wages with Human Capital Formation

RM/96/023 T. Storcken  
Intransitive Aggregated Preferences

RM/96/024 W. Peremans, T. Storcken  
Strategy Proof Locations of Public Bads

RM/96/025 A.J. Hoogstrate, F.C. Palm, G.A. Pfann  
The Role of the Bank for International Settlements in the

RM/96/027 De invloed van een beslissingsondersteunend hulpmiddel en ervaring op de beoordeling van AO/IC-beschrijvingen  
O.E. Flippo, A.W.J. Kolen, A.M.C.A. Koster, R.L.M.J. van de Leensel  
A Dynamic Programming Algorithm for the Local Access Network Expansion Problem

RM/96/028 D. de la Croix, F.C. Palm, J.P. Urbain  
Labor Market Dynamics When Efforts Depends on Wage Growth Comparisons

RM/96/029 H. van der Stel  
Strategy-Proofness, Pareto Optimality and Strictly Convex Norms

RM/96/030 J. Muysken, T. Zwick  
Insider Power Breeds Human Capitalists

RM/96/031 J.G. Backhaus  
The Impact of European Policies towards Science and Technology on the European Research Landscape

RM/96/032 P.P. Wakker, H. Zank  
State Dependent Expected Utility for Savage's State Space; Or: Bayesian Statistics without Prior Probabilities

RM/96/033 H. Kruiniger  
Agency Costs and the Adjustment Processes of Fixed Capital and R&D

RM/96/034 Y. Crama, J. van de Klundert  
The Approximability of Tool Management Problems

RM/96/035 S. van Hoesel, A. Wagelmans  
On the Complexity of Postoptimality Analysis of 0/1 Programs

RM/96/036 C. Allen  
An Empirical Model of Pricing, Market Share, and Market Conduct: An Application to Import Competition in US Manufacturing

RM/96/037 J.G.A. van Mierlo  
Public Entrepreneurship as Innovative Management Strategy in the Public Sector. A Public Choice-Approach

RM/96/038 J.G.A. van Mierlo  
The Experience of OECD Countries with Public Management Reform and its Relevance to Central and Eastern Europe

## Overview 1997

RM/97/001 D. Vermeulen, M. Jansen  
The Reduced form of a Game

RM/97/002 J. van den Akker, C. van Hoesel, M. Savelsbergh  
A Time-Indexed Formulation for Single-Machine Scheduling Problems: Characterization of Facets