Multibody dynamics

Multibody Dynamics
An Introduction

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Chapter 1

Introduction.

Many mechanical systems, *e.g.* uncontrolled mechanisms, manipulators and vehicles, can be modelled as multibody systems, composed of rigid or flexible bodies, coupled by connection elements. The inertia of the bodies can not be neglected while the connection elements are supposed to be massless. These elements account for the interactions between the bodies in the system and between these bodies and the environment. Connection elements can be material or immaterial. Well known examples of material elements are prismatic, revolute and spherical joints and spring-damper assemblies in the suspension of road vehicles. Immaterial elements represent interactions "on distance", *e.g.* due to gravitation or magnetic fields.

An important function of most connection elements is to restrict the relative motion of the connected bodies. Restrictions of this type are called kinematical constraints. Often the mathematical formulation of the kinematical constraints for one connection is straightforward, at least for technical connections. However, it can be quite complicated to analyse the consequences of all kinematical constraints in a system of connected (rigid) bodies, especially if the system does not have a tree structure. In fact this is the main subject in the area of kinematics of multibody systems.

Each connection element will exert forces (used in a generalized sense, *i.e.* denoting both forces and moments) on the connected bodies. These forces can be constraint forces, required to maintain the kinematical constraints of the connection. Besides, the element can exert forces that are related to energy storage or energy dissipation in the element. This will be the case, *e.g.* if the element deforms elastically or if damping occurs in the joints. Furthermore, the element can exert prescribed forces, for instance if actuators are build in in the joints. Connection elements are discussed in some detail in Chapter 3.

The laws of Newton and Euler relate the forces on each of the bodies, exerted by the connections, to the linear acceleration of the center of mass and to the angular acceleration of that body. These laws and the relevant kinematical quantities (position, orientation, velocity and acceleration) for an isolated body are considered in Chapter 2.

An essential characteristic of multibody systems is that each body can experience large rotations. These rotations can be characterized by, for instance, Cardan angles. For large rotations the common assumption from linear dynamics ($\cos(\varphi) \approx 1$ and $\sin(\varphi) \approx \varphi$, where φ is a Cardan angle) is not acceptable. This gives rise to geometrical nonlinearities and implies that the laws of Newton and Euler in general result in highly nonlinear equations.

The derivation of the equations of motion for a system with more than one rigid body and fairly arbitrary connections is discussed in Chapter 4. Two approaches are sketched. In the first approach each body is isolated from the other bodies and from the environment, all possible forces in the connections with those bodies and with the environment are introduced and the laws of Newton and Euler are applied. This approach is hampered by the fact that all constraint forces appear as unknown quantities in the obtained equations and that the elimination of these unknowns may be quite laborious. In the second approach, based on the formalism of Lagrange, these unknowns do not show up. Central quantities in this approach are the kinetic energy and the virtual work of all forces, except the constraint forces. As soon as these abstract quantities are determined as a function of the (derivative of the) degrees of freedom the rest of the derivation is very systematic and straightforward.

The final set of equations for the analysis of the dynamical behaviour of multibody systems consists of a set of second order differential equations plus a set of algebraic or first order differential equations, that represent the so-called remaining kinematical constraints. The unknowns in this set are the degrees of freedom and the constraint forces, that are required to maintain the remaining constraints. In the literature on the analysis of multibody systems many algorithms can be found to solve these equations numerically: analytical solution is hardly ever possible because the number of equations is large and, more important, because the equations are highly nonlinear. In this report no attential is given to solution procedures. More information on this subject can be found in the theoretical and the user manuals of general purpose programs like *DADS* and *ADAMS* and in the (mathematical) literature on the solution of set of differential-algebraic equations.

A trend in the design of modern mechanical systems is towards light weight constructions. Deformations of the parts of these systems becomes increasingly important and it is not realistic to model them parts as rigid bodies. Much research is going on to find feasible models for deformable parts and to analyse multibody systems with flexible bodies. It is not possible to discuss this subject in this report. An introduction can be found in *"Dynamics of Multibody Systems"* by A. Shabana, John Wiley&Sons, New York, 1989, ISBN 0-471-61494-7.

Chapter 2

Dynamics of one rigid body.

2.1 Introduction.

In this chapter some aspects of the dynamics of a rigid body are considered. First, the position, velocity and acceleration vector of an arbitrary material point of the body are introduced. Next, the relevant kinematical quantities of a rigid body, seen as a set of material points, are discussed. This results, for instance, in the introduction of the angular velocity vector and the angular variation vector. After that the equations of motion for a rigid body are considered. Finally, attention is given to the kinetic energy of a rigid body.

2.2 Position, velocity and acceleration.

The considered body $\mathcal{B}$ is seen as a set of material points. This set is invariant since $\mathcal{B}$ consists of the same material points for all $t \geq t_0$. Each material point is identified by a unique label $\underset{\sim}{\xi}$, where $\underset{\sim}{\xi}$ is a column of material coordinates. Let $\mathcal{S}$ be the invariant set of labels of all material points in $\mathcal{B}$. The material point with label $\underset{\sim}{\xi} \in \mathcal{S}$ is denoted by $P(\underset{\sim}{\xi})$.

Let $\vec{x}(\underset{\sim}{\xi}, t)$ be the current position vector of $P(\underset{\sim}{\xi})$ with respect to a fixed origin $\mathcal{O}$. It is assumed that the function $\vec{x} = \vec{x}(\underset{\sim}{\xi}, t)$ is differentiable at least twice with respect to t. Then the material rate[1] exists and is equal to the velocity $\vec{v}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$:

$$\vec{v}(\underset{\sim}{\xi}.t) = \dot{\vec{x}}(\underset{\sim}{\xi}, t) \tag{2.1}$$

Furthermore, $\dot{\vec{v}}(\underset{\sim}{\xi}, t)$ exists and is equal to the acceleration $\vec{a}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$:

$$\vec{a}(\underset{\sim}{\xi}, t) = \dot{\vec{v}}(\underset{\sim}{\xi}, t) = \ddot{\vec{x}}(\underset{\sim}{\xi}, t) \tag{2.2}$$

Often it is advantageous to consider the current position vector $\vec{x}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$ as the sum of a vector $\vec{x}_0(\underset{\sim}{\xi})$, representing the position of $P(\underset{\sim}{\xi})$ in a reference configuration (*e.g.* the

[1] Let the function $\phi = \phi(\underset{\sim}{\xi}, t)$ be differentiable with respect to t. Then the partial derivative with respect to t is called the **material rate** $\dot{\phi}$ of ϕ:

$$\dot{\phi}(\underset{\sim}{\xi}, t) = \lim_{dt \to 0} \frac{\phi(\underset{\sim}{\xi}, t + dt) - \phi(\underset{\sim}{\xi}, t)}{dt}$$

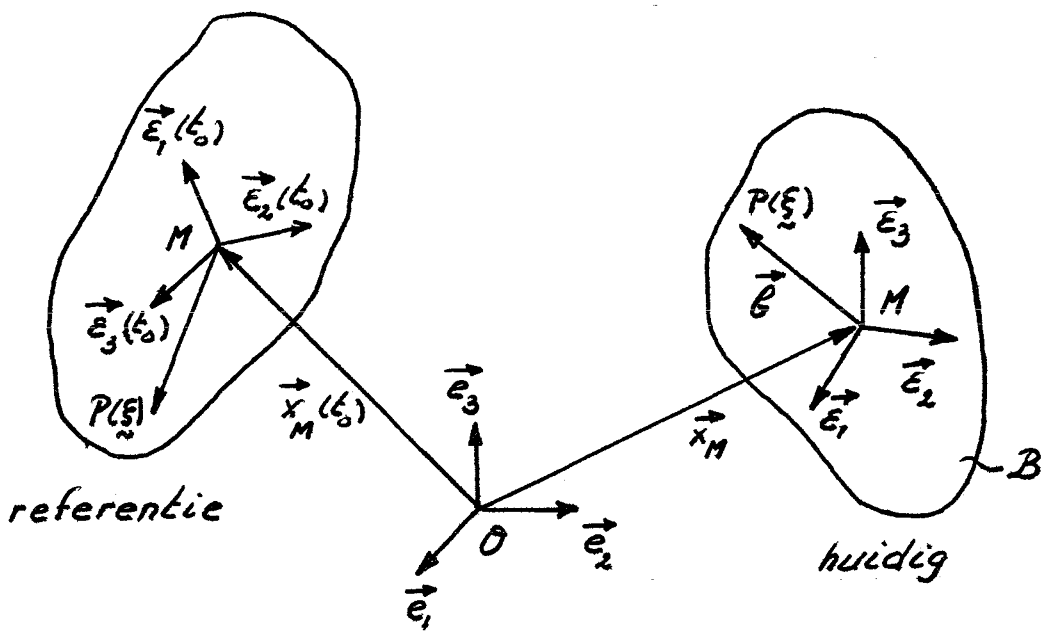

Figure 2.1: Position and orientation of a rigid body.

configuration at time t_0), plus a displacement $\vec{u}(\underset{\sim}{\xi},t)$ of $P(\underset{\sim}{\xi})$ with respect to this reference:

$$\vec{x}(\underset{\sim}{\xi},t) = \vec{x}_0(\underset{\sim}{\xi}) + \vec{u}(\underset{\sim}{\xi},t); \quad \vec{v}(\underset{\sim}{\xi},t) = \dot{\vec{u}}(\underset{\sim}{\xi},t); \quad \vec{a}(\underset{\sim}{\xi},t) = \ddot{\vec{u}}(\underset{\sim}{\xi},t)$$

2.3 Kinematics of a rigid body.

$\mathcal{B}$ is a **rigid body** if the distance between any two material points of $\mathcal{B}$ is independent of t. This is possible only if the transformation from the reference configuration to the current configuration consists of a translation as a rigid body and a rotation as a rigid body. For a rigid body $\mathcal{B}$ the position vector $\vec{x}(\underset{\sim}{\xi},t)$ of $P(\underset{\sim}{\xi})$ can be written as

$$\vec{x}(\underset{\sim}{\xi},t) = \vec{x}_M(t) + \vec{b}(\underset{\sim}{\xi},t); \quad \vec{b}(\underset{\sim}{\xi},t) = \mathbf{R}(t) \circ \vec{b}(\underset{\sim}{\xi},t_0) \tag{2.3}$$

where $\vec{x}_M(t)$ is the position vector at time t of an arbitrary point M, that is fixed to $\mathcal{B}$. This **local origin** of $\mathcal{B}$ can be a material point of $\mathcal{B}$ but that is not necessary. For the moment it is sufficient that M is moving along with $\mathcal{B}$. The displacement $\vec{u}_M(t) = \vec{x}_M(t) - \vec{x}_M(t_0)$ of M with respect to the reference configuration is called the **rigid body translation** of $\mathcal{B}$ with respect to the reference configuration[2]. The vector $\vec{b}(\underset{\sim}{\xi},t)$ in *Eq.* (2.3) is the **relative position** vector of $P(\underset{\sim}{\xi})$ with respect to the local origin M:

$$\vec{b}(\underset{\sim}{\xi},t) = \vec{x}(\underset{\sim}{\xi},t) - \vec{x}_M(t) \tag{2.4}$$

Finally, the quantity $\mathbf{R}(t)$ in *Eq.* (2.3) is a rotation tensor, so

$$\mathbf{R}^T(t) \circ \mathbf{R}(t) = \mathbf{I}; \quad det(\mathbf{R}(t)) = +1 \tag{2.5}$$

[2] This definition of the rigid body translation is not unique since the local origin M may be any point that is fixed with respect to $\mathcal{B}$. Often M is chosen in the centre of mass of $\mathcal{B}$.

for all $t \geq t_0$. This tensor characterizes the rotation of $\mathcal{B}$ and is called the **rigid body rotation tensor** of $\mathcal{B}$ with respect to the reference configuration.

Let $\vec{\underset{\sim}{\varepsilon}}(t)$ be an orthonormal vector basis with origin in M and moving along with $\mathcal{B}$. Then $\vec{\underset{\sim}{\varepsilon}}(t)$ is related to the corresponding basis $\vec{\underset{\sim}{\varepsilon}}(t_0)$ in the reference configuration by

$$\vec{\underset{\sim}{\varepsilon}}^T(t) = \mathbf{R}(t) \circ \vec{\underset{\sim}{\varepsilon}}^T(t_0)$$

These bases can be used to represent vectors and tensors quantities in matrix form. As an example the relative position vector $\vec{b}(\underset{\sim}{\xi}, t)$ is considered. This vector can be written as

$$\vec{b}(\underset{\sim}{\xi}, t) = \vec{\underset{\sim}{\varepsilon}}^T(t)\, \underset{\sim}{b}(\underset{\sim}{\xi}, t) = \mathbf{R}(t) \circ \vec{\underset{\sim}{\varepsilon}}^T(t_0)\, \underset{\sim}{b}(\underset{\sim}{\xi}, t)$$

This vector is also equal to $\mathbf{R}(t) \circ \vec{b}(\underset{\sim}{\xi}, t_0) = \mathbf{R}(t) \circ \vec{\underset{\sim}{\varepsilon}}^T(t_0)\, \underset{\sim}{b}(\underset{\sim}{\xi}, t_0)$. Hence, $\underset{\sim}{b}(\underset{\sim}{\xi}, t) = \underset{\sim}{b}(\underset{\sim}{\xi}, t_0)$ and therefore the matrix representation of the relative position vector with respect to the moving basis $\vec{\underset{\sim}{\varepsilon}}(t)$ does not depend on t. For this reason this vector is called **body fixed**[3] with respect to $\mathcal{B}$.

The velocity $\vec{v}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$ follows by differentiation of *Eq.* (2.3) with respect to t. This yields

$$\vec{v}(\underset{\sim}{\xi}, t) = \dot{\vec{x}}(\underset{\sim}{\xi}, t) = \vec{v}_M(t) + \dot{\vec{b}}(\underset{\sim}{\xi}, t); \quad \vec{v}_M(t) = \dot{\vec{x}}_M(t); \quad \dot{\vec{b}}(\underset{\sim}{\xi}, t) = \dot{\mathbf{R}}(t) \circ \vec{b}(\underset{\sim}{\xi}, t_0)$$

where $\vec{v}_M(t)$ is the velocity of M. Because $\mathbf{R}^T(t) \circ \mathbf{R}(t) = \mathbf{I}$ the relative velocity $\dot{\vec{b}}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$ with respect to M can be written as

$$\dot{\vec{b}}(\underset{\sim}{\xi}, t) = \dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) \circ \mathbf{R}(t) \circ \vec{b}(\underset{\sim}{\xi}, t_0) = \dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) \circ \vec{b}(\underset{\sim}{\xi}, t)$$

The tensor $\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t)$ is skew-symmetrical, so there exists a vector $\vec{\omega}(t)$, the axial vector of $\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t)$, such that

$$\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) \circ \vec{c} = \vec{\omega}(t) * \vec{c}, \quad \forall\, \vec{c}$$

Hence, $\dot{\vec{b}}(\underset{\sim}{\xi}, t) = \vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi}, t)$ and $\vec{v}(\underset{\sim}{\xi}, t)$ can be written as

$$\vec{v}(\underset{\sim}{\xi}, t) = \vec{v}_M(t) + \vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi}, t) \tag{2.6}$$

The vector $\vec{\omega}(t)$ is called the **rotation velocity vector** or **angular velocity vector.**

The acceleration $\vec{a}(\underset{\sim}{\xi}, t)$ of $P(\underset{\sim}{\xi})$ follows by differentiation of *Eq.* (2.6), yielding

$$\vec{a}(\underset{\sim}{\xi}, t) = \vec{a}_M(t) + \vec{\omega}(t) * [\vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi}, t)] + \dot{\vec{\omega}}(t) * \vec{b}(\underset{\sim}{\xi}, t); \quad \vec{a}_M(t) = \dot{\vec{v}}_M(t) \tag{2.7}$$

where $\vec{a}_M(t)$ is the acceleration of the local origin M on $\mathcal{B}$.

Another kinematical quantity of interest is the variation $\delta\vec{x}(\underset{\sim}{\xi}, t)$ of the position vector of

[3]This concept can be generalized: any vector or tensor, whose matrix representation with respect to the moving basis $\vec{\underset{\sim}{\varepsilon}}(t)$ is independent of t, is called **body fixed** with respect to $\mathcal{B}$.

$P(\underset{\sim}{\xi})$ if the position of $\mathcal{B}$ (*i.e.* $\vec{x}_M(t)$) and the orientation of $\mathcal{B}$ (*i.e.* $\mathbf{R}(t)$) are varied. Only infinitesimal variations are considered. Variation of $\vec{x}_M(t)$ and $\mathbf{R}(t)$ results in

$$\delta\vec{x}(\underset{\sim}{\xi},t) = \delta\vec{x}_M(t) + \delta\vec{b}(\underset{\sim}{\xi},t); \quad \delta\vec{b}(\underset{\sim}{\xi},t) = \delta\mathbf{R}(t)\circ\vec{b}(\underset{\sim}{\xi},t_0) = \delta\mathbf{R}(t)\circ\mathbf{R}^T(t)\circ\vec{b}(\underset{\sim}{\xi},t)$$

where $\delta\mathbf{R}(t)\circ\mathbf{R}^T(t)$ is skew-symmetrical. Hence, there exists a vector $\delta\vec{\pi}(t)$, the **angular variation vector**, such that

$$\delta\mathbf{R}(t)\circ\mathbf{R}^T(t)\circ\vec{c} = \delta\vec{\pi}(t) * \vec{c}, \quad \forall\, \vec{c}$$

and $\delta\vec{x}(\underset{\sim}{\xi},t)$ can be written as

$$\delta\vec{x}(\underset{\sim}{\xi},t) = \delta\vec{x}_M(t) + \delta\vec{\pi}(t) * \vec{b}(\underset{\sim}{\xi},t) \tag{2.8}$$

2.4 The equations of motion.

Each change of the motion of $\mathcal{B}$ is caused by interactions between $\mathcal{B}$ and the environment. The basic assumption in the dynamics of a rigid body is that the effect of these interactions can be taken into account by the resulting force $\vec{F}$ on the body and the resulting moment $\vec{M}$ with respect to an inertial point, i.e. a point in space with *constant* absolute velocity. Here the fixed origin $\mathcal{O}$ from the preceding sections is chosen as the reference point. To emphasize this choice the resulting moment is denoted by $\vec{M}_{\mathcal{O}}$.

Basic notions in the dynamics of a rigid body are the momentum $\vec{p}$ and the moment of momentum $\vec{L}$ with respect to a reference point. As usual, the reference point for the moment of momentum will be the same as for the resulting moment of the interactions. To emphasize this choice, the moment of momentum is denoted by $\vec{L}_{\mathcal{O}}$.

Let $\rho(\underset{\sim}{\xi})$ be the specific mass in $P(\underset{\sim}{\xi})$ of $\mathcal{B}$[4]. The **momentum** $\vec{p}(t)$ of $\mathcal{B}$ is defined by

$$\vec{p}(t) = \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{v}(\underset{\sim}{\xi},t)d\mathcal{B} \tag{2.9}$$

and with the earlier given relation for the velocity $\vec{v}(\underset{\sim}{\xi},t)$ this yields

$$\vec{p}(t) = m\vec{v}_M(t) + \vec{\omega} * \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{b}(\underset{\sim}{\xi},t)d\mathcal{B}; \quad m = \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})d\mathcal{B} \tag{2.10}$$

where m is the total mass of $\mathcal{B}$.

Until now the choice of the local origin M on $\mathcal{B}$ is completely free as long as M is fixed with respect to $\mathcal{B}$. This freedom is used to simplify the relation for $\vec{p}(t)$. For that purpose M is chosen such that

$$\int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{b}(\underset{\sim}{\xi},t)d\mathcal{B} = \vec{0} \quad \Leftrightarrow \quad \vec{x}_M(t) = \frac{1}{m}\int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{x}(\underset{\sim}{\xi},t)d\mathcal{B} \tag{2.11}$$

This point is the **center of mass** of $\mathcal{B}$. It can be shown that this point is fixed with respect to $\mathcal{B}$. However, it can be located outside $\mathcal{B}$, so it is not always a material point of $\mathcal{B}$. With this choice for M the relation for $\vec{p}(t)$ reduces to

$$\vec{p}(t) = m\vec{v}_M(t) \tag{2.12}$$

[4]The specific mass in a rigid body can depend on the position in the body, i.e. on $\underset{\sim}{\xi}$, but not on t.

The **moment of momentum** $\vec{L}_{\mathcal{O}}(t)$ with respect to $\mathcal{O}$ is defined by

$$\vec{L}_{\mathcal{O}}(t) = \int_{\mathcal{B}} \vec{x}(\underset{\sim}{\xi}, t) * \rho(\underset{\sim}{\xi})\vec{v}(\underset{\sim}{\xi}, t) d\mathcal{B} \tag{2.13}$$

and with the earlier given relation for $\vec{v}(\underset{\sim}{\xi}, t)$ this yields

$$\vec{L}_{\mathcal{O}}(t) = \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{x}(\underset{\sim}{\xi}, t) d\mathcal{B} * \vec{v}_M(t) + \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{x}(\underset{\sim}{\xi}, t) * [\vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi}, t)] d\mathcal{B}$$

With M in the center of mass the first term is equal to $m\vec{x}_M(t) * \vec{v}_M(t) = \vec{x}_M(t) * \vec{p}(t)$. Besides, with $\vec{x}(\underset{\sim}{\xi}, t) = \vec{x}_M(t) + \vec{b}(\underset{\sim}{\xi}, t)$ it follows that

$$\vec{L}_{\mathcal{O}}(t) = \vec{x}_M(t) * \vec{p}(t) + \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{b}(\underset{\sim}{\xi}, t) * [\vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi}, t)] d\mathcal{B}$$

and with $\vec{c}_1 * (\vec{c}_2 * \vec{c}_3) = (\vec{c}_1 \circ \vec{c}_3)\vec{c}_2 - (\vec{c}_1 \circ \vec{c}_2)\vec{c}_3 = \{(\vec{c}1 \circ \vec{c}_3)\mathbf{I} - \vec{c}_3\vec{c}_1\} \circ \vec{c}_2$ also that

$$\vec{L}_{\mathcal{O}}(t) = \vec{x}_M(t) * \vec{p}(t) + \vec{L}_M(t); \quad \vec{L}_M(t) = \mathbf{J}_M(t) \circ \vec{\omega}(t) \tag{2.14}$$

$\vec{L}_M(t)$ is the moment of momentum with respect to M. The tensor $\mathbf{J}_M(t)$ is called the **inertia tensor** with respect to M. This inertia tensor is defined by

$$\mathbf{J}_M(t) = \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})[\{\vec{b}(\underset{\sim}{\xi}, t) \circ \vec{b}(\underset{\sim}{\xi}, t)\}\mathbf{I} - \vec{b}(\underset{\sim}{\xi}, t)\vec{b}(\underset{\sim}{\xi}, t)] d\mathcal{B} \tag{2.15}$$

and with $\vec{b}(\underset{\sim}{\xi}, t) = \mathbf{R}(t) \circ \vec{b}(\underset{\sim}{\xi}, t_0)$ it is readily seen that

$$\mathbf{J}_M(t) = \mathbf{R}(t) \circ \mathbf{J}_{M0} \circ \mathbf{R}^T(t) \tag{2.16}$$

where the constant, symmetrical and positive definite tensor $\mathbf{J}_{M0}$ is defined by

$$\mathbf{J}_{M0}(t) = \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})[\{\vec{b}(\underset{\sim}{\xi}, t_0) \circ \vec{b}(\underset{\sim}{\xi}, t_0)\}\mathbf{I} - \vec{b}(\underset{\sim}{\xi}, t_0)\vec{b}(\underset{\sim}{\xi}, t_0)] d\mathcal{B} \tag{2.17}$$

The **laws of Newton** and **Euler** state that the derivative of the momentum $\vec{p}(t)$ and of the moment of momentum $\vec{L}_{\mathcal{O}}(t)$ with respect to the **fixed** origin $\mathcal{O}$ are related to the resulting force $\vec{F}(t)$ of the interactions with the environment and to the resulting moment $\vec{M}_{\mathcal{O}}(t)$ of those interactions with respect to $\mathcal{O}$ by

$$\dot{\vec{p}}(t) = \vec{F}(t); \quad \dot{\vec{L}}_{\mathcal{O}}(t) = \vec{M}_{\mathcal{O}}(t) \tag{2.18}$$

Substitution of the earlier derived results for $\vec{p}(t)$ and $\vec{L}_{\mathcal{O}}(t)$ yields[5]

$$\vec{F} = m\vec{a}_M; \quad \vec{M}_{\mathcal{O}} = \vec{x}_M * \vec{F} + \dot{\vec{L}}_M; \quad \dot{\vec{L}}_M = \dot{\mathbf{J}}_M \circ \vec{\omega} + \mathbf{J}_M \circ \dot{\vec{\omega}}$$

Further elaboration of the term $\dot{\mathbf{J}}_M \circ \vec{\omega}$ results in

$$\dot{\mathbf{J}}_M \circ \vec{\omega} = \dot{\mathbf{R}} \circ \mathbf{J}_{M0} \circ \mathbf{R}^T \circ \vec{\omega} + \mathbf{R} \circ \mathbf{J}_{M0} \circ \dot{\mathbf{R}}^T \circ \vec{\omega} =$$

$$= \dot{\mathbf{R}} \circ \mathbf{R}^T \circ \mathbf{J}_M \circ \vec{\omega} + \mathbf{J}_M \circ \mathbf{R} \circ \dot{\mathbf{R}}^T \circ \vec{\omega} = \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega} - \mathbf{J}_M \circ (\dot{\mathbf{R}} \circ \mathbf{R}^T) \circ \vec{\omega}$$

[5]For brevity of notation the dependence of time t is not mentioned explicitly anymore.

and because $(\dot{\mathbf{R}} \circ \mathbf{R}^T) \circ \vec{\omega} = \vec{\omega} * \vec{\omega} = \vec{0}$ it is seen that

$$\dot{\mathbf{J}}_M \circ \vec{\omega} = \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega})$$

Hence, for a rigid body the laws of Newton and Euler can be written as

$$\vec{F} = m\vec{a}_M; \quad \vec{M}_O - \vec{x}_M * \vec{F} = \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) + \mathbf{J}_M \circ \dot{\vec{\omega}}$$

The vector $\vec{M}_O - \vec{x}_M * \vec{F}$ is equal to the moment $\vec{M}_M$ of the interactions with respect to the center of mass M. This leads to the final form of the **equations of motion**

$$\vec{F} = \dot{\vec{p}} = m\vec{a}_M; \quad \vec{M}_M = \dot{\vec{L}}_M = \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) + \mathbf{J}_M \circ \dot{\vec{\omega}} \tag{2.19}$$

The first term in the last equation is quadratic in the angular velocity $\vec{\omega}$. It represents the so-called centrifugal and gyroscopic effects.

As can be seen from this result, the law of Euler also holds if the resulting moment of the interactions and the moment of momentum are determined with respect to the center of mass instead of an inertial point. In general this law is **not** valid if an arbitrary non-inertial point is chosen to determine the resulting moment and the moment of momentum!

2.5 An alternative formulation.

The equations of motion can be written in a slightly different form as

$$\vec{F} - m\vec{a}_M = \vec{0}; \quad \vec{M}_M - \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) - \mathbf{J}_M \circ \dot{\vec{\omega}} = \vec{0}$$

An alternative formulation for these equations is given by the requirement that

$$(\vec{F} - m\vec{a}_M) \circ \vec{y} + [\vec{M}_M - \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) - \mathbf{J}_M \circ \dot{\vec{\omega}}] \circ \vec{z} = 0 \tag{2.20}$$

must hold for each $\vec{y}$ and $\vec{z}$. This is the **principle of weighted residuals:** $\vec{F} - m\vec{a}_M$ and $\vec{M}_M - \vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) - \mathbf{J}_M \circ \dot{\vec{\omega}}$ are seen as residuals on the equations of motion and $\vec{y}$ and $\vec{z}$ as weighting factors for these residuals. The principle states that the weighted sum of the residuals must be zero for all weighting factors.

This mathematical formulation can be given a physical interpretation by considering $\vec{y}$ as a variation $\delta\vec{x}_M$ of the position vector of the center of mass and $\vec{z}$ as the angular variation vector $\delta\vec{\pi}$, which is a measure for the variation of the rotation of $\mathcal{B}$. The scalar relation then can be written as

$$\vec{F} \circ \delta\vec{x}_M + \vec{M}_M \circ \delta\vec{\pi} = m\vec{a}_M \circ \delta\vec{x}_M + \{\vec{\omega} * (\mathbf{J}_M \circ \vec{\omega}) + \mathbf{J}_M \circ \dot{\vec{\omega}}\} \circ \delta\vec{\pi} = 0 \tag{2.21}$$

where $\vec{F} \circ \delta\vec{x}_M$ is the work of the resulting force $\vec{F}$ if the position vector of the center of mass experiences a variation $\delta\vec{x}_M$. Besides, $\vec{M}_M \circ \delta\vec{\pi}$ is the work of the resulting moment $\vec{M}_M$ if the rotation of $\mathcal{B}$ experiences a variation $\delta\vec{\pi}$. Hence, the left hand side of *Eq.*(2.21) is equal to the work of the interactions if the position and orientation of $\mathcal{B}$ are varied. Usually this work is called the **virtual work** of those interactions.

The alternative formulation with the given interpretation for the weighting functions is called the **principle of d'Alembert** or the **principle of virtual work**. It plays a major role in procedures to determine approximate solutions of the equations of motion.

2.6 The kinetic energy T.

The kinetic energy $T(t)$ plays an important role in some energy principles to derive the equations of motion for multibody systems. This energy is defined by

$$T(t) = \frac{1}{2}\int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{v}(\underset{\sim}{\xi},t) \circ \vec{v}(\underset{\sim}{\xi},t)d\mathcal{B} \tag{2.22}$$

Substitution of $\vec{v}(\underset{\sim}{\xi},t) = \vec{v}_M(t) + \vec{\omega}(t) * \vec{b}(\underset{\sim}{\xi},t)$ yields after some calculations that

$$T(t) = \frac{1}{2}m\vec{v}_M(t) \circ \vec{v}_M(t) + \vec{v}_M(t) * \vec{\omega}(t) \circ \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})\vec{b}(\underset{\sim}{\xi},t)d\mathcal{B} +$$

$$+\frac{1}{2}\vec{\omega}(t) \circ \int_{\mathcal{B}} \rho(\underset{\sim}{\xi})[\{\vec{b}(\underset{\sim}{\xi},t) \circ \vec{b}(\underset{\sim}{\xi},t)\}\mathbf{I} - \vec{b}(\underset{\sim}{\xi},t)\vec{b}(\underset{\sim}{\xi},t)]d\mathcal{B} \circ \vec{\omega}(t)$$

The second term on the right hand side disappears since M is in the center of mass. Besides, according to *Eq.* (2.16), the integral in the last term is equal to the inertia tensor $\mathbf{J}_M(t)$ of $\mathcal{B}$ with respect to M. Hence, the kinetic energy is given by

$$T(t) = \frac{1}{2}m\vec{v}_M(t) \circ \vec{v}_M(t) + \frac{1}{2}\vec{\omega}(t) \circ \mathbf{J}_M(t) \circ \vec{\omega}(t) \tag{2.23}$$

The term $\frac{1}{2}m\vec{v}_M(t) \circ \vec{v}_M(t)$ is the kinetic energy if the mass of $\mathcal{B}$ is concentrated in the center of mass. This term, the kinetic energy due to the translation of $\mathcal{B}$, is often called the kinetic energy of the center of mass. The term $\frac{1}{2}\vec{\omega}(t) \circ \mathbf{J}_M(t) \circ \vec{\omega}(t)$ is the kinetic energy due to the rotation of $\mathcal{B}$. This term is commonly called the kinetic energy around the center of mass. In this terminology the total kinetic energy of a rigid body is said to be the sum of the kinetic energy of the center of mass and the kinetic energy around this center.

Chapter 3

Connection elements.

3.1 Introduction.

Each mechanical interaction between two or more bodies in a multibody system[1] can be modeled in terms of kinematical, dynamical and external quantities. Examples of kinematical quantities are the position vectors of the centers of gravity and the rotation tensors of the interacting bodies. Dynamical quantities are, for instance, the forces and torques between these bodies. External quantities or inputs are prescribed displacements, prescribed electrical currents to actuators, etc.

Two bodies $\mathcal{B}^i$ and $\mathcal{B}^j$, that interact directly upon each other, are called **contiguous**. An interaction or coupling can occur via a material element, *e.g.* a bar, a hinge or a spring, or can be immaterial, *e.g.* via a magnetic field. It is assumed that all interactions between two contiguous bodies can be combined in one **connection element** between those bodies. The description of the interactions then boils down to the characterization of the behaviour of that element. Furthermore, it is assumed that the connection elements are massless, *i.e.* that the momentum and the moment of momentum of these elements are negligible compared to the momentum and the moment of momentum of the (rigid) bodies in the system.

Each connection element will represent the interactions between exactly two bodies. If more than two bodies interact directly it is often (but not always!) possible to take the interactions into account by the introduction of a connection element between any pair of those bodies. It turns out that this restriction is not very severe for mechanisms and mechanical manipulators.

It is assumed that the behaviour of each connection element can be described in terms of kinematical, dynamical and external quantities in a finite number of points of that element, the so-called **nodal points** or **points of attachment**. Let n_e be the number of points of attachment of element e. Then $n_e \geq 2$ because there will be at least one point of attachment on each of the connected bodies. Mostly $n_e = 2$ but sometimes it is necessary or advantageous to introduce more than one point of attachment on one or both bodies. This is the case, for instance, if the bodies are coupled by a four bar mechanism.

[1]The fixed world is considered as one of the bodies of the system. This makes it possible to consider the interactions with the environment as interactions between bodies of the system.

Often the function (or one of the functions) of a connection element between two bodies $\mathcal{B}^i$ and $\mathcal{B}^j$ is to restrict the possible motions of one of these bodies, say $\mathcal{B}^j$, with respect to the other, $\mathcal{B}^i$. A restriction of this type is called a **kinematical constraint**. To formulate the constraints in a mathematical form it is noted that a free, rigid body $\mathcal{B}^k$ $(k = i, j)$, moving in the three-dimensional space $\mathcal{E}$, has six degrees of freedom, *i.e.* that six quantities are required to specify the position of each material point of $\mathcal{B}^k$. A possible choice for these quantities are the Cartesian coordinates of the center of mass M^k of $\mathcal{B}^k$ plus the three Cardan angles in the column $\underset{\sim}{\varphi}^k$, which characterize the rotation tensor $\mathbf{R}^k$ of $\mathcal{B}^k$.

It turns out that each kinematical constraint of a connection between $\mathcal{B}^i$ and $\mathcal{B}^j$ can be represented by an implicit relation in the columns $\underset{\sim}{x}^i$ and $\underset{\sim}{\varphi}^i$ of $\mathcal{B}^i$, the columns $\underset{\sim}{x}^j$ and $\underset{\sim}{\varphi}^j$ of $\mathcal{B}^j$ and, eventually, time derivatives of these quantities. More specific, it turns out that kinematical constraints for a connection between $\mathcal{B}^i$ and $\mathcal{B}^j$ can always be formulated in terms of the relative position vector $\vec{c} = \vec{x}^j - \vec{x}^i$ (from the center of mass M^i of $\mathcal{B}^i$ to the center of mass M^j of $\mathcal{B}^j$) and of the relative rotation tensor $\mathbf{C}$ (which transforms the local basis $\underset{\sim}{\vec{e}}^i$ on $\mathcal{B}^i$ into the local basis $\underset{\sim}{\vec{e}}^j$ on $\mathcal{B}^j$) or, as an often attractive alternative, in terms of the matrix representations $\underset{\sim}{c}$ and $\underline{C}$ of $\vec{c}$ and $\mathbf{C}$ in the local basis $\underset{\sim}{\vec{e}}^i$.

One of the main topics in the theory of multibody systems is the formulation of the kinematical constraints, that are induced by the connections in the system and the analysis of the consequences of these constraints for the degrees of freedom of the bodies in the system. This is discussed in some detail in the following section, where first some attention is given to the kinematics of one rigid body and to the description of the relative motion of one rigid body with respect to another. After that, kinematical constraints are introduced in a more formal manner and notions like relative velocity, kinematically admissable variations etc. are discussed.

Another topic in the theory of multibody systems is the description of the forces and moments that can be exerted by a connection on the connected bodies. Some remarks on this subject are put forward in the last section, where also the notion of the virtual work of these forces and moments for kinematically admissable variations is introduced. Besides, some attention is given to external quantities, *i.e.* to inputs to the system.

It is noted in advance that there is no aim at generality in the following sections. Only some aspects of interactions between contiguous bodies are highlighted and some aids and appliances to take these interactions into account in modeling a system of interacting (rigid) bodies are given.

3.2 Kinematical aspects.

3.2.1 Kinematics of one rigid body.

Let $\vec{x}^k$ be the position vector of the center of mass M^k of $\mathcal{B}^k$ and let $\mathbf{R}^k$ be the rotation tensor of $\mathcal{B}^k$, *i.e.* the tensor that transforms the fixed or inertial, orthonormal vector basis $\underset{\sim}{\vec{e}}$ into the local, orthonormal vector basis $\underset{\sim}{\vec{e}}^k$, that is fixed to $\mathcal{B}^k$:

$$(\underset{\sim}{\vec{e}}^k)^T = \mathbf{R}^k \circ \underset{\sim}{\vec{e}}^T \tag{3.1}$$

The elements of the matrixrepresentations $\underset{\sim}{x}^k$ of $\vec{x}^{\,k}$ and $\underline{R}^k$ of $\mathbf{R}^k$ with respect to the inertial basis follow from

$$\underset{\sim}{x}^k = \underset{\sim}{\vec{e}} \circ \vec{x}^{\,k}; \quad \underline{R}^k = \underset{\sim}{\vec{e}} \circ \mathbf{R}^k \circ \underset{\sim}{\vec{e}}^{\,T} \tag{3.2}$$

$\underline{R}^k$ is an orthonormal matrix, so $\underline{R}^k(\underline{R}^k)^T = \underline{I}$, and the nine entries of $\underline{R}^k$ can be written as functions of three independent variables (for instance Euler or Cardan angles) or as functions of four variables (for instance Euler parameters) that have to satisfy one constraint condition[2]. Here only Cardan angles are considered. For $\mathcal{B}^j$ these angles are denoted by φ_1^k, φ_2^k and φ_3^k and seen as the components of the column $\underset{\sim}{\varphi}^k$. Then $\underline{R}^k = \underline{R}(\underset{\sim}{\varphi}^k)$ where (see the appendix on vectors and tensors)

$$\underline{R}(\underset{\sim}{\varphi}) = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}; \quad c_s = \cos(\varphi_s); \quad s_s = \sin(\varphi_s) \tag{3.3}$$

The three elements of $\underset{\sim}{x}^k$ plus the three Cardan angles in $\underset{\sim}{\varphi}^k$ completely determine the position and orientation of $\mathcal{B}^k$ with respect to the fixed basis $\underset{\sim}{\vec{e}}$. They are called the attitude coordinates $\underset{\sim}{z}^k$ of $\mathcal{B}^k$:

$$\underset{\sim}{z}^k = \begin{bmatrix} \underset{\sim}{x}^k \\ \underset{\sim}{\varphi}^k \end{bmatrix} \tag{3.4}$$

The vector basis $\underset{\sim}{\vec{e}}^{\,k}$ is fixed with respect to the moving body $\mathcal{B}^k$ and will therefore depend on time t. The time derivative follows from

$$(\dot{\underset{\sim}{\vec{e}}}^{k})^T = \dot{\mathbf{R}}^k \circ (\underset{\sim}{\vec{e}})^T = \dot{\mathbf{R}}^k \circ (\mathbf{R}^k)^T \circ (\underset{\sim}{\vec{e}}^{\,k})^T$$

where $\dot{\mathbf{R}}^k \circ (\mathbf{R}^k)^T$ is skew-symmetrical. The axial vector of this tensor is the **angular** (or rotational) velocity $\vec{\omega}^{\,k}$ of $\mathcal{B}^k$. With this vector it is seen that

$$(\dot{\underset{\sim}{\vec{e}}}^{k})^T = \vec{\omega}^{\,k} * (\underset{\sim}{\vec{e}}^{\,k})^T \tag{3.5}$$

The representation $\underset{\sim}{\omega}^k$ of $\vec{\omega}^{\,k}$ in the fixed basis $\underset{\sim}{\vec{e}}$ follows from

$$\underset{\sim}{\omega}^k = \underset{\sim}{\vec{e}} \circ \vec{\omega}^{\,k}; \quad \underset{\sim}{\omega}^k = \underline{W}^k \dot{\underset{\sim}{\varphi}}^k \tag{3.6}$$

where $\underline{W}^k$ is given by (see, for instance, the appendix on vectors and tensors or the earlier mentioned book of Wittenburg)

$$\underline{W}^k = \underline{W}(\underset{\sim}{\varphi}^k); \quad \underline{W}(\underset{\sim}{\varphi}) = \begin{bmatrix} 1 & 0 & s_2 \\ 0 & c_1 & -s_1c_2 \\ 0 & s_1 & +c_1c_2 \end{bmatrix}, \quad s_i = \sin(\varphi_i), \quad c_i = \cos(\varphi_i) \tag{3.7}$$

The (translational) velocity $\vec{v}^{\,k}$ of $\mathcal{B}^k$, *i.e.* the velocity of the center of mass M^k, is given by

$$\vec{v}^{\,k} = \dot{\vec{x}}^{\,k} = \underset{\sim}{\vec{e}}^{\,T}\underset{\sim}{v}^k; \quad \underset{\sim}{v}^k = \dot{\underset{\sim}{x}}^k \tag{3.8}$$

[2]More information on this subject can be found in the appendix on vectors and tensors or in "*Dynamics of Systems of Rigid Bodies*" by J. Wittenburg (B.G. Teubner, Stuttgart, 1977, ISBN 3-519-12337-7).

Finally, the **angular variation vector** $\delta\vec{\pi}^k$, which characterizes the variation of the orientation of $\mathcal{B}^k$ if the Cardan angles in $\underset{\sim}{\varphi}^k$ are varied, can be determined from

$$\delta\vec{\pi}^k = (\underset{\sim}{\vec{e}})^T \delta\underset{\sim}{\pi}^k; \quad \delta\underset{\sim}{\pi}^k = \underline{W}^k \, \delta\underset{\sim}{\varphi}^k \tag{3.9}$$

Let P be a material point of $\mathcal{B}^k$ and let $\vec{b}$ be the vector from M^k to P. Then $\vec{b}$ is fixed with respect to $\mathcal{B}^k$, *i.e.* the elements of the representation $\underset{\sim}{b} = \underset{\sim}{\vec{e}}^k \circ \vec{b}$ of $\vec{b}$ in the moving basis $\underset{\sim}{\vec{e}}^k$ are constant. Hence, the time derivative $\dot{\vec{b}}$ of $\vec{b}$ and the variation $\delta\vec{b}$ of $\vec{b}$ for variations of the orientation of $\mathcal{B}^k$ are given by

$$\dot{\vec{b}} = (\dot{\underset{\sim}{\vec{e}}}^k)^T \underset{\sim}{b} = \vec{\omega}^k * (\underset{\sim}{\vec{e}}^k)^T \underset{\sim}{b} = \vec{\omega}^k * \vec{b} \tag{3.10}$$

$$\delta\vec{b} = \delta(\underset{\sim}{\vec{e}}^k)^T \underset{\sim}{b} = \delta\vec{\pi}^k * (\underset{\sim}{\vec{e}}^k)^T \underset{\sim}{b} = \delta\vec{\pi}^k * \vec{b} \tag{3.11}$$

3.2.2 Relative motion of contiguous bodies.

Let $\mathcal{B}^i$ and $\mathcal{B}^j$ be contiguous bodies. The orientation of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ can be described with the **relative rotation tensor** $\mathbf{C}$, which transforms the local basis $\underset{\sim}{\vec{e}}^i$ on $\mathcal{B}^i$ into the local basis $\underset{\sim}{\vec{e}}^j$ on $\mathcal{B}^j$:

$$(\underset{\sim}{\vec{e}}^j)^T = \mathbf{C} \circ (\underset{\sim}{\vec{e}}^i)^T; \quad \mathbf{C} = (\underset{\sim}{\vec{e}}^j)^T \underset{\sim}{\vec{e}}^i \tag{3.12}$$

From $(\underset{\sim}{\vec{e}}^j)^T = \mathbf{R}^j \circ (\underset{\sim}{\vec{e}})^T$ and $(\underset{\sim}{\vec{e}}^i)^T = \mathbf{R}^i \circ (\underset{\sim}{\vec{e}})^T$ it is easily seen that[3]

$$\mathbf{R}^j = \mathbf{C} \circ \mathbf{R}^i \tag{3.13}$$

The representation $\underline{C}$ of $\mathbf{C}$ with respect to the *moving* basis $\underset{\sim}{\vec{e}}^i$ follows from

$$\underline{C} = \underset{\sim}{\vec{e}}^i \circ \mathbf{C} \circ (\underset{\sim}{\vec{e}}^i)^T = \underset{\sim}{\vec{e}}^i \circ (\underset{\sim}{\vec{e}}^j)^T \tag{3.14}$$

Hence, the representation $\underline{R}^j$ of $\mathbf{R}^j$ with respect to the *fixed* basis $\underset{\sim}{\vec{e}}$ can be determined from

$$\underline{R}^j = \underset{\sim}{\vec{e}} \circ \mathbf{R}^j \circ (\underset{\sim}{\vec{e}})^T = \underset{\sim}{\vec{e}} \circ \mathbf{C} \circ \mathbf{R}^i \circ (\underset{\sim}{\vec{e}})^T = \underset{\sim}{\vec{e}} \circ (\underset{\sim}{\vec{e}}^i)^T \underline{C} = \underline{R}^i \, \underline{C}$$

$\underline{C}$ is an orthonormal matrix, so the elements of $\underline{C}$ can be written as functions of three Cardan angles ψ_1, ψ_2 and ψ_3, *i.e.*

$$\underline{C} = \underline{R}(\underset{\sim}{\psi}); \quad \underset{\sim}{\psi} = [\psi_1 \quad \psi_2 \quad \psi_3]^T \tag{3.15}$$

Therefore, with $\underline{R}^j = \underline{R}(\underset{\sim}{\varphi}^j)$, $\underline{R}^i = \underline{R}(\underset{\sim}{\varphi}^i)$ and $\underline{C} = \underline{R}(\underset{\sim}{\psi})$, it is seen that

$$\underline{R}(\underset{\sim}{\varphi}^j) = \underline{R}(\underset{\sim}{\varphi}^i) \, \underline{R}(\underset{\sim}{\psi}) \tag{3.16}$$

This matrix equation is equivalent to a set of three independent algebraic equations for the Cardan angles $\underset{\sim}{\varphi}^j$ of $\mathcal{B}^j$. It can be used to determine $\underset{\sim}{\varphi}^j$ if $\underset{\sim}{\psi}$ (the Cardan angles of the connection element) and $\underset{\sim}{\varphi}^i$ (the Cardan angles of $\mathcal{B}^i$) are given.

[3] Usually the relative position vector and the relative rotation tensor of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ are denoted by $\vec{c}^{ij}$ (or $\vec{c}^{ji}$) and $\mathbf{C}^{ij}$ (or $\mathbf{C}^{ji}$.). For the sake of readability the upper indices are omitted here.

The position of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ can be described with the **relative position vector** $\vec{c}$ from the center of mass M^i of $\mathcal{B}^i$ to the center of mass M^j of $\mathcal{B}^j$:

$$\vec{x}^{\,j} = \vec{x}^{\,i} + \vec{c} \tag{3.17}$$

With the matrix representation $\underset{\sim}{c}$ of $\vec{c}$ with respect to the *moving* basis $\underset{\sim}{\vec{e}}^{\,i}$ of $\mathcal{B}^i$, *i.e.* with

$$\underset{\sim}{c} = \underset{\sim}{\vec{e}}^{\,i} \circ \vec{c}, \tag{3.18}$$

the matrix representation $\underset{\sim}{x}^j$ of $\vec{x}^{\,j}$ with respect to the *fixed* basis $\underset{\sim}{\vec{e}}$ can be written as

$$\underset{\sim}{x}^j = \underset{\sim}{x}^i + \underline{R}^i \underset{\sim}{c} = \underset{\sim}{x}^i + \underline{R}(\underset{\sim}{\varphi}^i)\underset{\sim}{c} \tag{3.19}$$

This equation relates the coordinates $\underset{\sim}{x}^j$ of the center of mass of $\mathcal{B}^j$ to the coordinates $\underset{\sim}{x}^i$ and $\underset{\sim}{\varphi}^i$ of $\mathcal{B}^i$ and to the elements of the column $\underset{\sim}{c}$ of the connection element. As noted earlier, the Cardan angles $\underset{\sim}{\varphi}^j$ are determined by the Cardan angles $\underset{\sim}{\varphi}^i$ of $\mathcal{B}^i$ and the Cardan angles $\underset{\sim}{\psi}$ of the connection. Often the elements of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$ are called the coordinates of the connection element.

To relate the angular velocity $\vec{\omega}^{\,j}$ of $\mathcal{B}^j$ to the angular velocity $\vec{\omega}^{\,i}$ of $\mathcal{B}^i$ and to (derivatives of) $\underset{\sim}{\psi}$ it is noted that $(\underset{\sim}{\vec{e}}^{\,j})^T = \mathbf{C} \circ (\underset{\sim}{\vec{e}}^{\,i})^T = (\underset{\sim}{\vec{e}}^{\,i})^T \underline{C}$ and therefore

$$(\dot{\underset{\sim}{\vec{e}}}^{\,j})^T = \vec{\omega}^{\,j} * (\underset{\sim}{\vec{e}}^{\,j})^T = \vec{\omega}^{\,i} * (\underset{\sim}{\vec{e}}^{\,j})^T + (\underset{\sim}{\vec{e}}^{\,i})^T \dot{\underline{C}} = \vec{\omega}^{\,i} * (\underset{\sim}{\vec{e}}^{\,j})^T + (\underset{\sim}{\vec{e}}^{\,i})^T (\dot{\underline{C}}\,\underline{C}^T)\,\underline{C}$$

With $\underline{C} = \underset{\sim}{\vec{e}}^{\,i} \circ (\underset{\sim}{\vec{e}}^{\,j})^T$ and $\dot{\underline{C}}\,\underline{C} = -(\dot{\underline{C}}\,\underline{C})^T$ it is readily seen that

$$\vec{\omega}^{\,j} = \vec{\omega}^{\,i} + \vec{\omega}_{rel} \tag{3.20}$$

where $\vec{\omega}_{rel}$ is the axial vector of the skew-symmetrical tensor $(\underset{\sim}{\vec{e}}^{\,i})^T(\dot{\underline{C}}\,\underline{C}^T)\underset{\sim}{\vec{e}}^{\,i}$. It is the **relative angular velocity** vector of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ as measured by an observer, which is fixed to $\mathcal{B}^i$. It is noted emphatically that $\vec{\omega}_{rel}$ is defined, using the matrix representation $\underline{C}$ of $\mathbf{C}$ with respect to the basis $\underset{\sim}{\vec{e}}^{\,i}$ that is fixed to $\mathcal{B}^i$. Therefore, this vector is not related directly to the skew-symmetrical tensor $\dot{\mathbf{C}} \circ \mathbf{C}^T$. However, it can be shown that

$$\underset{\sim}{\omega}_{rel} = \underline{W}(\underset{\sim}{\psi})\dot{\underset{\sim}{\psi}} \tag{3.21}$$

The absolute velocity $\vec{v}^{\,j}$ of M^j is equal to $\vec{v}^{\,i} + \dot{\vec{c}}$, where $\dot{\vec{c}}$ follows from $\vec{c} = (\underset{\sim}{\vec{e}}^{\,i})^T \underset{\sim}{c}$:

$$\dot{\vec{c}} = (\dot{\underset{\sim}{\vec{e}}}^{\,i})^T \underset{\sim}{c} + (\underset{\sim}{\vec{e}}^{\,i})^T \dot{\underset{\sim}{c}} = \vec{\omega}^{\,i} * \vec{c} + \vec{v}_{rel}; \quad \vec{v}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \dot{\underset{\sim}{c}} \tag{3.22}$$

The vector $\vec{v}_{rel}$ is the **relative velocity** of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$. It is the velocity of the center of mass of $\mathcal{B}^j$ as measured by an observer, located in the center of mass of $\mathcal{B}^i$ and moving along with $\mathcal{B}^i$.

For the absolute velocity $\vec{v}^{\,j}$ of the center of mass of $\mathcal{B}^j$ it is now seen that

$$\vec{v}^{\,j} = \vec{v}^{\,i} + \dot{\vec{c}}; \quad \dot{\vec{c}} = \vec{\omega}^{\,i} * \vec{c} + \vec{v}_{rel} \tag{3.23}$$

or, written in terms of matrix representations with respect to the inertial basis, that

$$\underset{\sim}{v}^j = \underset{\sim}{v}^i + \underline{R}^i \dot{\underset{\sim}{c}} + \underset{\sim}{\omega}^i * (\underline{R}^i \underset{\sim}{c}) \tag{3.24}$$

Apart from the position, the orientation and the translational and angular velocity of $\mathcal{B}^j$ also the variation of the position and the orientation of $\mathcal{B}^j$ in terms of the variation of the position and of the orientation of $\mathcal{B}^i$ and the variation of quantities of the connection element play a role in further analyses. The variation of $\mathbf{R}^j = \mathbf{R}(\underset{\sim}{\varphi}^j)$ can be characterized by the angular variation vector $\delta\vec{\pi}^{\,j}$,

$$\delta\vec{\pi}^{\,j} * \vec{w} = \delta\mathbf{R}^j \circ (\mathbf{R}^j)^T \circ \vec{w}; \quad \delta\vec{\pi}^{\,j} = \underset{\sim}{\vec{e}}^{\,T} \delta\underset{\sim}{\pi}^j; \quad \delta\underset{\sim}{\pi}^j = \underline{W}(\underset{\sim}{\varphi}^j)\delta\underset{\sim}{\varphi}^j \tag{3.25}$$

In a simular way as for the angular velocity $\vec{\omega}^{\,j}$ it can be shown that the angular variation vector $\delta\vec{\pi}^{\,j}$ is given by

$$\delta\vec{\pi}^{\,j} = \delta\vec{\pi}^{\,i} + \delta\vec{\pi}_{rel}; \quad \delta\vec{\pi}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{\pi}_{rel}; \quad \delta\underset{\sim}{\pi}_{rel} = \underline{W}(\underset{\sim}{\psi})\delta\underset{\sim}{\psi} \tag{3.26}$$

where $\delta\underset{\sim}{\psi}$ is a variation of the Cardan angles $\underset{\sim}{\psi}$ of the connection element. Furthermore, for the variation $\delta\vec{x}^{\,j}$ of the position vector of M^j it can be shown that

$$\delta\vec{x}^{\,j} = \delta\vec{x}^{\,i} + \delta\vec{c}_{rel} + \delta\vec{\pi}^{\,i} * \vec{c}; \quad \delta\vec{c}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{c} \tag{3.27}$$

where $\delta\underset{\sim}{c}$ is a variation of the matrix representation $\underset{\sim}{c}$ of $\vec{c}$ with respect to $\underset{\sim}{\vec{e}}^{\,i}$.

3.2.3 Kinematical constraints.

From the given analysis it follows that the motion of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ can be characterized completely in terms of the relative position vector $\vec{c}$ and the relative rotation tensor $\mathbf{C}$ or, alternatively, in terms of the elements of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$.

Usually the connection element between $\mathcal{B}^i$ and $\mathcal{B}^j$ restricts the motion of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ or, put in a more mathematical form, enforces one or more constraint relations in the elements of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$ and, eventually, their time derivatives. A relation of this type is a **kinematical constraint**. For technical connection elements only $\underset{\sim}{c}$, $\underset{\sim}{\psi}$ and sometimes their first derivative occur in the kinematical constraints. Furthermore, if derivatives occur then the constraints always are linear in those derivatives.

Kinematical constraints can be inequalities. A simple example is given by a connection element in the form of an inextensible string of length ℓ between the centers of mass of the contiguous bodies. The constraint then takes the form $||\vec{c}|| \leq \ell$.

Although inequality constraints are of great importance they are not discussed here in more detail. In the sequel only constraints of the form

$$\underline{K}_c(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\dot{\underset{\sim}{c}} + \underline{K}_\psi(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\dot{\underset{\sim}{\psi}} + \underset{\sim}{h}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0}, \tag{3.28}$$

are considered. If a constraint explicitly depends on time t then that constraint is **rheonomic**, otherwise it is **scleronomic**.

Constraints of the given form can be integrable. For these so-called **holonomic** constraints there exists a function $\underset{\sim}{k}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0}$, such that

$$\dot{\underset{\sim}{k}}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underline{K}_c(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\dot{\underset{\sim}{c}} + \underline{K}_\psi(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\dot{\underset{\sim}{\psi}} + \underset{\sim}{h}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0}, \tag{3.29}$$

Holonomic constraints can always be written in the form of *Eq.* (3.28), where $\underline{K}_c(\underset{\sim}{c}, \underset{\sim}{\psi}, t)$, $\underline{K}_\psi(\underset{\sim}{c}, \underset{\sim}{\psi}, t)$ and $\underset{\sim}{h}(\underset{\sim}{c}, \underset{\sim}{\psi}, t)$ then represent the derivatives of $\underset{\sim}{k} = \underset{\sim}{k}(\underset{\sim}{c}, \underset{\sim}{\psi}, t)$ with respect to $\underset{\sim}{c}$, $\underset{\sim}{\psi}$ and t.

Non-integrable constraints of the form of *Eq.* (3.28) are called **non-holonomic** and can *not* be formulated in terms of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$ only. Examples of systems with non-holonomic constraints are mechanisms on castors. Although non-holonomic constraints often occur in models for technical systems only holonomic constraints are considered in the sequel[4].

Let $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{\psi}$ be variations of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$. These variations are called **kinematically admissable** if they satisfy the condition

$$\underline{K}_c(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\delta\underset{\sim}{c} + \underline{K}_\psi(\underset{\sim}{c}, \underset{\sim}{\psi}, t)\delta\underset{\sim}{\psi} = \underset{\sim}{0} \tag{3.30}$$

For holonomic constraints this has a simple interpretation: $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{\psi}$ are kinematically admissable if $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$ as well as $\underset{\sim}{c} + \delta\underset{\sim}{c}$ and $\underset{\sim}{\psi} + \delta\underset{\sim}{\psi}$ satisfy the constraints, *i.e.* if

$$\underset{\sim}{k}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0} \quad \wedge \quad \underset{\sim}{k}(\underset{\sim}{c} + \delta\underset{\sim}{c}, \underset{\sim}{\psi} + \delta\underset{\sim}{\psi}, t) = \underset{\sim}{0} \tag{3.31}$$

Further elaboration yields the given condition for kinematically admissable variations.

Holonomic constraints $\underset{\sim}{k}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0}$ are (implicit) equations in $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$. Let n ($n < 6$) be the number of holonomic constraints. It is assumed that they are independent for all relevant values of $\underset{\sim}{c}$, $\underset{\sim}{\psi}$ and t. Then, using $\underset{\sim}{k}(\underset{\sim}{c}, \underset{\sim}{\psi}, t) = \underset{\sim}{0}$, n elements of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$ can be written as functions of the other elements of $\underset{\sim}{c}$ and $\underset{\sim}{\psi}$. An attractive alternative is to introduce a column $\underset{\sim}{q}$ with $6 - n$ elements and to determine functions $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t)$ such that $\underset{\sim}{k}(\underset{\sim}{c}(\underset{\sim}{q}, t), \underset{\sim}{\psi}(\underset{\sim}{q}, t), t) = \underset{\sim}{0}$ for all relevant $\underset{\sim}{q}$ and t. For the position vector $\vec{x}^{\,j}$ and the rotation tensor $\mathbf{R}^j$ of body this results in

$$\vec{x}^{\,j} = \vec{x}^{\,i} + \vec{c}; \quad \vec{c} = (\vec{\underset{\sim}{e}}^{\,i})^T \underset{\sim}{c}; \quad \underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t) \tag{3.32}$$

$$\mathbf{R}^j = \mathbf{C} \circ \mathbf{R}^i; \quad \mathbf{C} = (\vec{\underset{\sim}{e}}^{\,i})^T \underline{C}\, \vec{\underset{\sim}{e}}^{\,i}; \quad \underline{C} = \underline{R}(\underset{\sim}{\psi}); \quad \underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t) \tag{3.33}$$

The elements of $\underset{\sim}{q}$ are called the **generalized coordinates** of the connection.

This given result for $\mathbf{R}^j$ can be reformulated in terms of the Cardan angles $\underset{\sim}{\varphi}^i$ of $\mathcal{B}^i$ and $\underset{\sim}{\varphi}^j$ of $\mathcal{B}^i$ and the variables $\underset{\sim}{q}$ of the connection. With $\underline{R}^j = \underline{R}^i\underline{C}$, $\underline{R}^j = \underline{R}(\underset{\sim}{\varphi}^j)$, $\underline{R}^i = \underline{R}(\underset{\sim}{\varphi}^i)$ and $\underline{C} = \underline{R}(\underset{\sim}{\psi})$ it follows that

$$\underline{R}(\underset{\sim}{\varphi}^j) = \underline{R}(\underset{\sim}{\varphi}^i)\underline{R}(\underset{\sim}{\psi}); \quad \underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t) \tag{3.34}$$

[4] More information on non-holonomic constraints can be found, *e.g.*, in the earlier mentioned book of J. Wittenburg or in "*Dynamics of Multibody Systems*" by A.A. Shabana (John Wiley & Sons, Inc., New York, 1989, ISBN 0-471-61494-7).

According to the previous section, the velocity $\vec{v}^{\,j}$ and the angular velocity $\vec{\omega}^{\,j}$ are related to $\vec{v}^{\,i}$ and $\vec{\omega}^{\,j}$ by

$$\vec{v}^{\,j} = \vec{v}^{\,i} + \vec{\omega}^{\,i} * \vec{c} + \vec{v}_{rel}; \quad \vec{\omega}^{\,j} = \vec{\omega}^{\,i} + \vec{\omega}_{rel} \tag{3.35}$$

where $\vec{v}_{rel}$ and $\vec{\omega}_{rel}$ are functions of the generalized coordinates $\underset{\sim}{q}$ and the generalized velocities $\dot{\underset{\sim}{q}}$:

$$\vec{v}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \underset{\sim}{v}_{rel}; \quad \underset{\sim}{v}_{rel} = \underline{c}_q(\underset{\sim}{q},t)\dot{\underset{\sim}{q}} + \frac{\partial}{\partial t}(\underset{\sim}{c}(\underset{\sim}{q},t)) \tag{3.36}$$

$$\vec{\omega}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \underset{\sim}{\omega}_{rel}; \quad \underset{\sim}{\omega}_{rel} = \underline{W}(\underset{\sim}{\psi})\dot{\underset{\sim}{\psi}}; \quad \dot{\underset{\sim}{\psi}} = \underline{\psi}_q(\underset{\sim}{q},t)\dot{\underset{\sim}{q}} + \frac{\partial}{\partial t}(\underset{\sim}{\psi}(\underset{\sim}{q},t)) \tag{3.37}$$

Here $\underline{c}_q = \underline{c}_q(\underset{\sim}{q},t)$ and $\underline{\psi}_q = \underline{\psi}_q(\underset{\sim}{q},t)$ are the derivatives of $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q},t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q},t)$ with respect to $\underset{\sim}{q}$.

Simular relations hold for the variations $\delta\vec{x}^{\,j}$ and $\delta\vec{\pi}^{\,j}$. They are related to the corresponding variations of $\mathcal{B}^i$ by

$$\delta\vec{x}^{\,j} = \delta\vec{x}^{\,i} + \delta\vec{\pi}^{\,i} * \vec{c} + \delta\vec{c}_{rel}; \quad \delta\vec{\pi}^{\,j} = \delta\vec{\pi}^{\,i} + \delta\vec{\pi}_{rel} \tag{3.38}$$

where $\delta\vec{c}_{rel}$ and $\delta\vec{\pi}_{rel}$ are linear functions of the variation $\delta\underset{\sim}{q}$ of $\underset{\sim}{q}$:

$$\delta\vec{c}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{c}_{rel}; \quad \delta\underset{\sim}{c}_{rel} = \underline{c}_q(\underset{\sim}{q},t)\delta\underset{\sim}{q} \tag{3.39}$$

$$\delta\vec{\pi}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{\pi}_{rel}; \quad \delta\underset{\sim}{\pi}_{rel} = \underline{W}(\underset{\sim}{\psi})\delta\underset{\sim}{\psi}; \quad \delta\underset{\sim}{\psi} = \underline{\psi}_q(\underset{\sim}{q},t)\delta\underset{\sim}{q} \tag{3.40}$$

As a simple example the hinged bodies $\mathcal{B}^i$ and $\mathcal{B}^j$ in *Fig.* 3.1 are considered. Only motions in the plane of this figure are possible. The fixed basis $\underset{\sim}{\vec{e}}$ and the local bases $\underset{\sim}{\vec{e}}^{\,i}$ and $\underset{\sim}{\vec{e}}^{\,j}$ are chosen, such that $\vec{e}_3$, $\vec{e}^{\,i}_3$ and $\vec{e}^{\,j}_3$ are perpendicular to the plane of *Fig.* 3.1. The requirement that all motions out of the plane of *Fig.* 3.1 are suppressed can be represented in mathematical form by $\mathbf{C} \circ \vec{e}^{\,i}_3 = \vec{e}^{\,j}_3$. Hence, the rotation of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ is a rotation around the carrier of $\vec{e}^{\,i}_3$, so the Cardan angles ψ_1 and ψ_2 are zero and Cardan angle ψ_3 is equal to the angle β between the carriers of $\vec{e}^{\,i}_1$ and $\vec{e}^{\,j}_1$, *i.e.*

$$\underset{\sim}{\psi} = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}; \quad \underline{C} = \underline{R}(\underset{\sim}{\psi}) = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = (\underset{\sim}{\vec{e}}^{\,i})^T \underline{C}\, \underset{\sim}{\vec{e}}^{\,i} = \cos(\beta)[\vec{e}^{\,i}_1\vec{e}^{\,i}_1 + \vec{e}^{\,i}_2\vec{e}^{\,i}_2] + \sin(\beta)[\vec{e}^{\,i}_2\vec{e}^{\,i}_1 - \vec{e}^{\,i}_1\vec{e}^{\,i}_2] + \vec{e}^{\,i}_3\vec{e}^{\,i}_3$$

To relate $\vec{\omega}^{\,j}$ to $\vec{\omega}^{\,i}$ and (derivatives) of β it is noted that

$$\vec{\omega}^{\,j} = \vec{\omega}^{\,i} + \vec{\omega}_{rel}; \quad \vec{\omega}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \underset{\sim}{\omega}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \underline{W}(\underset{\sim}{\psi})\dot{\underset{\sim}{\psi}}$$

Substitution of $\underset{\sim}{\psi} = [0 \;\; 0 \;\; \beta]^T$ in *Eq.* (3.7) yields $\underline{W}(\underset{\sim}{\psi}) = \underline{I}$ and therefore

$$\vec{\omega}_{rel} = \dot{\beta}\vec{e}_3$$

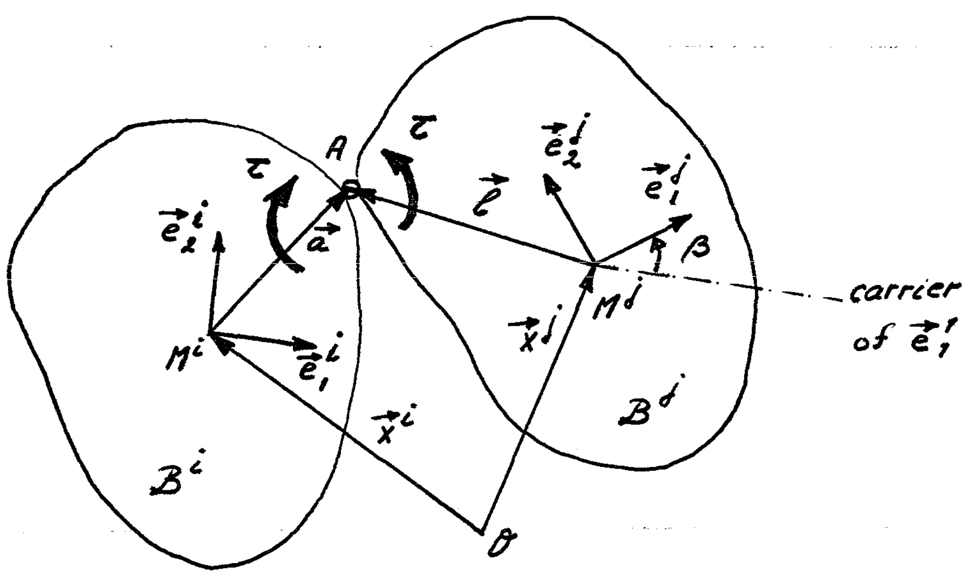

Figure 3.1: $\mathcal{B}^i$ and $\mathcal{B}^j$, connected by a revolute hinge.

The second kinematical condition of this connection follows from the requirement that the center A of the hinge, seen as a point of $\mathcal{B}^j$, must coincide with A, seen as a point of $\mathcal{B}^i$, *i.e.* that $\vec{x}^{\,i} + \vec{a} = \vec{x}^{\,j} + \vec{b}$. Hence,

$$\vec{x}^{\,j} = \vec{x}^{\,i} + \vec{c}; \quad \vec{c} = \vec{a} - \vec{b}; \quad \underset{\sim}{c} = \underset{\sim}{\vec{e}}^{\,i} \circ \vec{c} = \underset{\sim}{a} - \underset{\sim}{\vec{e}}^{\,i} \circ (\underset{\sim}{\vec{e}}^{\,2})^T \underset{\sim}{b} = \underset{\sim}{a} - \underline{C}\,\underset{\sim}{b}$$

The vector $\vec{a}$ from M^i to A is fixed with respect to $\mathcal{B}^i$, so the representation $\underset{\sim}{a}$ of $\vec{a}$ in the basis $\underset{\sim}{\vec{e}}^{\,i}$ is constant. Simularly, the vector $\vec{b}$ from M^j to A is fixed with respect to $\mathcal{B}^j$ and the representation $\underset{\sim}{b}$ in the basis $\underset{\sim}{\vec{e}}^{\,j}$ is constant. This means that the representation $\underset{\sim}{c}$ of the relative position vector $\vec{c}$ in the basis $\underset{\sim}{\vec{e}}^{\,i}$ is completely determined by the angle β. Hence, this connection allows one relative motion of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ and the column $\underset{\sim}{q}$ has only one component, $\underset{\sim}{q} = [\beta]$.

The velocity $\vec{v}^{\,j}$ of M^j follows by differentiation of $\vec{x}^{\,j} = \vec{x}^{\,i} + \vec{c}$ and $\vec{c} = \vec{a} - \vec{b}$. Because $\vec{a}$ is fixed to $\mathcal{B}^i$ and $\vec{b}$ is fixed to $\mathcal{B}^j$ their time derivatives satisfy

$$\dot{\vec{a}} = \vec{\omega}^{\,i} * \vec{a}; \quad \dot{\vec{b}} = \vec{\omega}^{\,j} * \vec{b} = (\vec{\omega}^{\,i} + \vec{\omega}_{rel}) * \vec{b} = (\vec{\omega}^{\,i} + \dot{\beta}\,\vec{e}_3) * \vec{b}$$

and therefore the velocity of M^j follows from

$$\vec{v}^{\,j} = \vec{v}^{\,i} + \dot{\vec{c}}; \quad \dot{\vec{c}} = \dot{\vec{a}} - \dot{\vec{b}} = \vec{\omega}^{\,i} * \vec{c} + \vec{v}_{rel}; \quad \vec{v}_{rel} = -\vec{\omega}_{rel} * \vec{b} = -\dot{\beta}\,\vec{e}_3 * \vec{b}$$

As a second example the system in *Fig.* 3.2 is considered. The rigid bodies $\mathcal{B}^i$ and $\mathcal{B}^j$ are connected by a rigid rod of length ℓ. The hinges between the rod and the bodies are kinematically ideal (no play). Only motions in the plane of *Fig.* 3.2 are allowed. The vector $\vec{a}$ from M^i to the point of attachment A^1 of the rod on $\mathcal{B}^i$ is fixed to $\mathcal{B}^i$. A simular remark holds

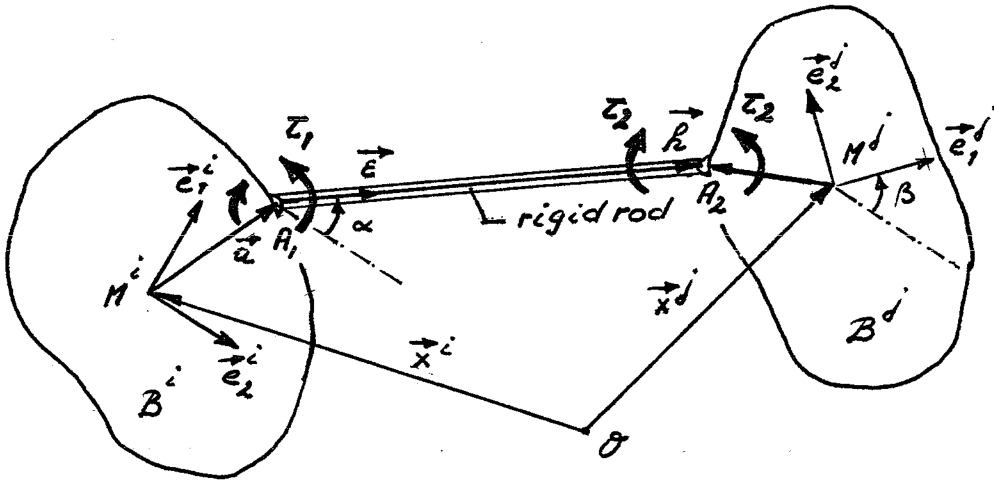

Figure 3.2: $\mathcal{B}^i$ and $\mathcal{B}^j$, connected by a hinged, rigid rod.

for the vector $\vec{b}$ from M^j to the point of attachment A^2 of the rod on $\mathcal{B}^j$: this vector is fixed to $\mathcal{B}^j$. As in the first example, the inertial basis $\underset{\sim}{\vec{e}}$ and the local bases $\underset{\sim}{\vec{e}}^i$ and $\underset{\sim}{\vec{e}}^j$ are chosen, such that $\vec{e}_3$, $\vec{e}^{\,i}_3$ and $\vec{e}^{\,j}_3$ are perpendicular to the plane of *Fig.* 3.2.

The kinematical constraints for this system are holonomic and scleronomic and follow from the conditions that there is no motion out of the plane of *Fig.* 3.2 and that the length of the rod is constant and equal to ℓ. The first condition is given in mathematical form by

$$\mathbf{C} \circ \vec{e}^{\,i}_3 = \vec{e}^{\,j}_3$$

As in the first example, this implies that the rotation of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ must be a rotation around the axis perpendicular to the plane of *Fig.* 3.2, so again the Cardan angles ψ_1 and ψ_2 are zero and Cardan angle ψ_3 is the angle β between the carriers of $\vec{e}^{\,i}_1$ and $\vec{e}^{\,j}_1$. This results in the same rotation matrix $\underline{C}$ and the same rotation tensor $\mathbf{C}$ as in the first example.

To give a mathematical formulation for the second constraint it is noted that the relative position vector $\vec{c}$ is given by $\vec{c} = \vec{a} + \vec{h} - \vec{b}$, where $\vec{h}$ is the vector from A^1 to A^2 along the axis of the rod. Then this is constraint can be written as

$$\|\vec{h}\| = \ell \quad \Leftrightarrow \quad \|\vec{c} - \vec{a} + \vec{b}\| = \ell$$

and to satisfy it the angle α between the carriers of $\vec{h}$ and $\vec{e}^{\,1}_1$ is introduced. Then

$$\vec{h} = \ell\vec{\varepsilon}; \quad \vec{\varepsilon} = \cos(\alpha)\vec{e}^{\,i}_1 + \sin(\alpha)\vec{e}^{\,i}_2; \quad \vec{c} = \vec{a} + \ell\vec{\varepsilon} - \vec{b}$$

and $\|\vec{h}\| = \ell$ is satisfied for all values of α.

In this example two quantities (α and β) are required the describe the position and orientation of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$. Therefore, the number of elements of $\underset{\sim}{q}$ is two and $\underset{\sim}{q}$ can be defined by

$$\underset{\sim}{q} \equiv \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The rotation tensor $\mathbf{R}^j$ of $\mathcal{B}^j$ follows from $\mathbf{R}^j = \mathbf{C} \circ \mathbf{R}^i$ where $\mathbf{C}$ is the relative rotation tensor from the first example. Hence, the relation between $\vec{\omega}^{\,j}$, $\vec{\omega}^{\,i}$ and β again is given by

$$\vec{\omega}^{\,j} = \vec{\omega}^{\,i} + \vec{\omega}_{rel}; \quad \vec{\omega}_{rel} = \dot{\beta}\vec{e}_3$$

The position vector $\vec{x}^{\,j}$ of M^j follows from

$$\vec{x}^{\,j} = \vec{x}^{\,i} + \vec{c}; \quad \vec{c} = \vec{a} + \vec{h} - \vec{b}; \quad \vec{h} = \ell\,\vec{\varepsilon}; \quad \vec{\varepsilon} = \cos(\alpha)\vec{e}^{\,i}_1 + \sin(\alpha)\vec{e}^{\,i}_2$$

Differentiation yields a relation for the velocity $\vec{v}^{\,j}$ of M^j. With $\dot{\vec{\varepsilon}} = \vec{\omega}^{\,i} * \vec{\varepsilon} + \dot{\alpha}\,\vec{e}_3 * \vec{\varepsilon}$ it is seen that

$$\vec{v}^{\,j} = \vec{v}^{\,i} + \vec{\omega}^{\,i} * (\vec{a} + \ell\,\vec{\varepsilon}) + \dot{\alpha}\,\ell\,\vec{e}_3 * \vec{\varepsilon} - \vec{\omega}^{\,j} * \vec{b} = \vec{v}^{\,i} + \vec{\omega}^{\,i} * \vec{c} + \vec{e}_3 * (\dot{\alpha}\,\ell\,\vec{\varepsilon} - \dot{\beta}\,\vec{b})$$

If required a relation for $\vec{v}_{rel}$ can be determined from $\vec{v}^{\,j} = \vec{v}^{\,i} + \vec{\omega}^{\,i} * \vec{c} + \vec{v}_{rel}$. This yields

$$\vec{v}_{rel} = \vec{e}_3 * (\dot{\alpha}\,\ell\,\vec{\varepsilon} - \dot{\beta}\,\vec{b})$$

Finally, relations for the variations $\delta\vec{x}^{\,j}$ and $\delta\vec{\pi}^{\,j}$ as functions of $\delta\vec{x}^{\,i}$, $\delta\vec{\pi}^{\,i}$ and $\delta\underset{\sim}{q}$ are given. It follows that

$$\delta\vec{x}^{\,j} = \delta\vec{x}^{\,i} + \delta\vec{\pi}^{\,i} * \vec{c} + \vec{e}_3 * (\delta\alpha\,\ell\,\vec{\varepsilon} - \delta\beta\,\vec{b}); \quad \delta\vec{\pi}^{\,j} = \delta\vec{\pi}^{\,i} + \delta\beta\vec{e}_3$$

Often it is not feasible to determine generalized coordinates $\underset{\sim}{q}$ and functions $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t)$ such that all holonomic constraints are satisfied for all values of $\underset{\sim}{q}$. Then it is common practice to introduce coordinates $\underset{\sim}{q}$, such that as many constraints as possible are taken into account with $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t)$. The remaining constraints then take the form

$$\underset{\sim}{k}_{rem}(\underset{\sim}{q}, t) = \underset{\sim}{0}$$

and can be seen as (implicit) relations for $\underset{\sim}{q}$. The elements of $\underset{\sim}{q}$ are called the **generalized coordinates** of the connection.

As an example the system in *Fig.* 3.3 is considered. The rods between $\mathcal{B}^i$ and $\mathcal{B}^j$ are rigid and the hinges are ideal (no play). Only motions in the plane of *Fig.* 3.3 are allowed, again resulting in the constraint $\mathbf{C} \circ \vec{e}^{\,i}_3 = \vec{e}^{\,j}_3$. The fact that the rods are rigid leads to the constraints $\|\vec{h}_1\| = \ell_1$ and $\|\vec{h}_2\| = \ell_2$. These constraints, *i.e.* $\mathbf{C} \circ \vec{e}^{\,i}_3 = \vec{e}^{\,j}_3$, $\|\vec{h}_1\| = \ell_1$ and $\|\vec{h}_2\| = \ell_2$, can be taken into account by the introduction of the new coordinates $s_1 \equiv \alpha_1$, $s_2 \equiv \alpha_2$ and $s_3 \equiv \beta$ (see *Fig.* 3.3). Then

$$\underset{\sim}{\psi} = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}; \quad (\underset{\sim}{\vec{e}}^{\,2})^T = (\underset{\sim}{\vec{e}}^{\,1})^T \underline{C}; \quad \underline{C} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{h}_1 = \ell_1\{\cos(\alpha_1)\vec{e}^{\,1}_1 + \sin(\alpha_1)\vec{e}^{\,1}_2\}; \quad \vec{h}_2 = \ell_2\{\cos(\alpha_2)\vec{e}^{\,1}_1 + \sin(\alpha_2)\vec{e}^{\,1}_2\}$$

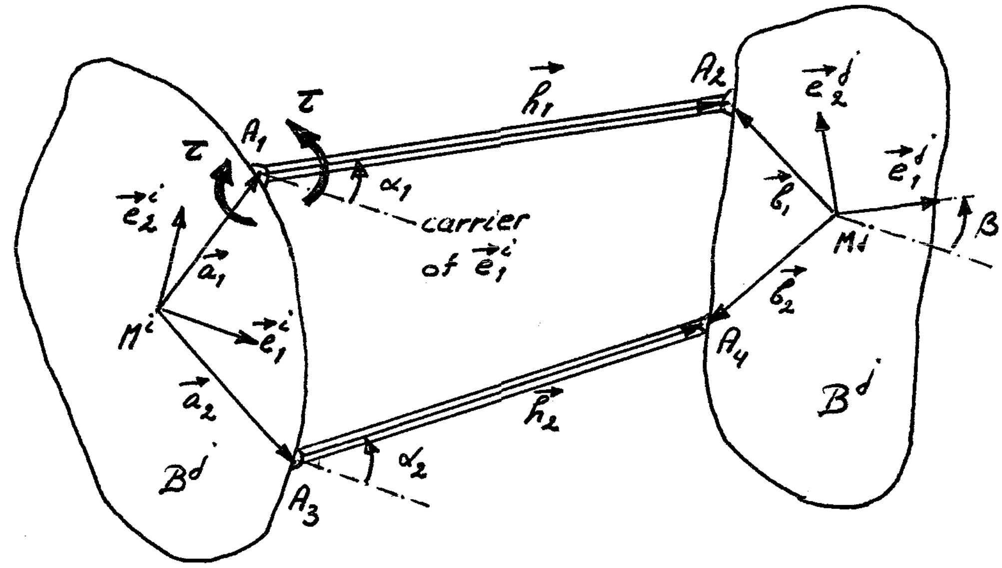

Figure 3.3: $\mathcal{B}^i$ and $\mathcal{B}^j$, connected by two hinged, rigid rods.

However, in this example there is another kinematical constraint: at each moment the position vector $\vec{x}^{\,i} + \vec{a}_1 + \vec{h}_1 - \vec{b}_1 + \vec{b}_2$ of the point of attachment A_4, seen as a point of $\mathcal{B}^j$, must be equal to the position vector $\vec{x}^{\,i} + \vec{a}_2 + \vec{h}_2$ of A_4, seen as the end of rod 2, *i.e.*

$$\vec{a}_1 + \vec{h}_1 - \vec{b}_1 = \vec{a}_2 + \vec{h}_2 - \vec{b}_2$$

This constraint results in two highly nonlinear equations in α_1, α_2 and β. It is not feasible to use these equations to determine two of these quantities, say α_1 and α_2, as a function of the third, β. As a consequence the column $\underset{\sim}{q}$ has three elements $q_1 \equiv \alpha_1$, $q_2 \equiv \alpha_2$ and $q_3 \equiv \beta$, which have to satisfy two constraint equations.

3.2.4 Dynamical quantities.

Characterization of the connection between $\mathcal{B}^i$ and $\mathcal{B}^j$ requires a description of the forces and torques that can be exerted by that connection on $\mathcal{B}^i$ and $\mathcal{B}^j$. The basic idea is to remove the connection and to replace it by the forces and moments it exerts on the connected bodies.

Let $\vec{F}^{\,k}$ be the force and $\vec{M}_M^k$ $(k = i, j)$ be the moment with respect to M^k, exerted by the connection on $\mathcal{B}^k$. The third law of Newton ("*action=−reaction*") states that $\mathcal{B}^k$ exerts a force $-\vec{F}^{\,k}$ and a moment $-\vec{M}_M^k$ on the connection element. Because this element is massless the equations of motion degenerate to the equilibrium equations

$$\vec{F}^{\,i} + \vec{F}^{\,j} = \vec{0}; \quad \vec{M}_M^i + \vec{M}_M^j + \vec{c} * \vec{F}^{\,j} = \vec{0} \tag{3.41}$$

The virtual work δA of the forces and moments on the connection element is given by

$$\delta A = -\vec{F}^{\,i} \circ \delta\vec{x}^{\,i} - \vec{F}^{\,j} \circ \delta\vec{x}^{\,j} - \vec{M}_M^i \circ \delta\vec{\pi}^{\,i} - \vec{M}_M^j \circ \delta\vec{\pi}^{\,j} \tag{3.42}$$

With $\delta\vec{x}^{\,j} = \delta\vec{x}^{\,i} + \delta\vec{\pi}^{\,i} * \vec{c} + \delta\vec{c}_{rel}$, $\delta\vec{\pi}^{\,j} = \delta\vec{\pi}^{\,i} + \delta\vec{\pi}_{rel}$ and the equilibrium equations this can be written as

$$\delta A = -\vec{F}^{\,j} \circ \delta\vec{c}_{rel} - \vec{M}^{j} \circ \delta\vec{\pi}_{rel} \tag{3.43}$$

As discussed in the previous section it is possible to eliminate all or some of the holonomic constraints: introduce a set $\underset{\sim}{q}$ of generalized coordinates and determine functions $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t)$ such that those constraints are satisfied for all values of $\underset{\sim}{q}$. The variations $\delta\vec{c}_{rel}$ and $\delta\vec{\pi}_{rel}$ then are linear combinations[5] of $\delta\underset{\sim}{q}$:

$$\delta\vec{c}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{c}; \quad \delta\underset{\sim}{c} = \underset{\sim}{c}_q \delta\underset{\sim}{q}$$

$$\delta\vec{\pi}_{rel} = (\underset{\sim}{\vec{e}}^{\,i})^T \delta\underset{\sim}{\pi}_{rel}; \quad \delta\underset{\sim}{\pi}_{rel} = \underline{W}(\underset{\sim}{\psi})\delta\underset{\sim}{\psi}; \quad \delta\underset{\sim}{\psi} = \underline{\psi}_q \delta\underset{\sim}{q}$$

With these results for $\delta\vec{c}_{rel}$ and $\delta\vec{\pi}_{rel}$ the virtual work δA can be written as

$$\delta A = \delta\underset{\sim}{q}^T \underset{\sim}{F} \tag{3.44}$$

where the column $\underset{\sim}{F}$ of connection forces is defined by

$$\underset{\sim}{F} = -\underset{\sim}{c}_q^T \underset{\sim}{\vec{e}}^{\,i} \circ \vec{F}^{\,j} - \underline{\psi}_q^T \underline{W}^T \underset{\sim}{\vec{e}}^{\,i} \circ \vec{M}_M^j \tag{3.45}$$

Hardly any general rule can be given for the forces and torques, that can be exerted by a connection element: they strongly depend on the connection at hand. For that reason only some examples are given.

The first example is the connection of *Fig.* 3.1, *i.e.* a hinge between $\mathcal{B}^i$ and $\mathcal{B}^j$. Suppose that $\mathcal{B}^j$ is driven by an actuator in this hinge. Let $\vec{M} = \tau\,\vec{e}_3$ be the moment, that is exerted by this actuator on $\mathcal{B}^j$. The hinge can also transmit a force between $\mathcal{B}^i$ and $\mathcal{B}^j$. Let $\vec{F}$ be the force that is exerted in the hinge on $\mathcal{B}^j$. According to the third law of Newton, the actuator also exerts a moment $-\vec{M}$ on $\mathcal{B}^i$ and $\mathcal{B}^j$ exerts a force $-\vec{F}$ on $\mathcal{B}^i$. Hence, the resulting force $\vec{F}^i$ on $\mathcal{B}^i$, the resulting moment $\vec{M}_M^i$ on $\mathcal{B}^i$ with respect to M^i, the resulting force $\vec{F}^{\,j}$ on $\mathcal{B}^j$ and the resulting moment $\vec{M}_M^j$ on $\mathcal{B}^j$ with respect to M^j are given by

$$\vec{F}^{\,i} = -\vec{F}; \quad \vec{M}_M^i = -\tau\vec{e}_3 - \vec{a} * \vec{F}; \quad \vec{F}^{\,j} = +\vec{F}; \quad \vec{M}_M^j = +\tau\vec{e}_3 + \vec{b} * \vec{F}$$

where $\vec{a}$ and $\vec{b}$ are the vectors from M^i, respectively M^j to the hinge.

The virtual work δA of these forces and moments can easily be determined, yielding

$$\delta A = \tau\,\delta\beta$$

where β is the rotation of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$. Hence, the contribution of the force $\vec{F}$, transmitted in the hinge between $\mathcal{B}^i$ and $\mathcal{B}^j$, is equal to zero. This force is required to maintain the kinematical constraint of this connection and is called the **constraint force** of the connection. The fundamental theorem of Lagrange mechanics states that constraint forces never contribute to the virtual work if the variations of the position and orientation of

[5] As in the previous section $\underset{\sim}{c}_q = \underset{\sim}{c}_q(\underset{\sim}{q}, t)$ and $\underline{\psi}_q = \underline{\psi}_q(\underset{\sim}{q}, t)$ are the derivatives of $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{q}, t)$ with respect to $\underset{\sim}{q}$.

the connected bodies are kinematically admissable.

As a second example the system of *Fig.* 3.2 is considered again. In each of the hinges an actuator is build in. Let $\vec{M}_1 = \tau_1 \vec{e}_3$ be the moment, exerted on the rod by the actuator in A_1 and let $\vec{M}_2 = \tau_2 \vec{e}_3$ be the moment, exerted on $\mathcal{B}^j$ by the actuator in A_2. Then, by the third law of Newton, there is a moment $-\vec{M}_1$ in A_1 on $\mathcal{B}^i$ and a moment $-\vec{M}_2$ in A_2 on the rod.

Apart from these moments also forces can be exerted by the rod on $\mathcal{B}^i$ and $\mathcal{B}^j$ and, as a consequence of the third law of Newton, by these bodies on the rod. Let $\vec{F}_1$ be the force, exerted in A_1 by $\mathcal{B}^i$ on the rod and let $\vec{F}_2$ be the force, exerted in A_2 by $\mathcal{B}^j$ on the rod. Then the equilibrium equations of the connection element are given by

$$\vec{F}_1 + \vec{F}_2 = \vec{0}; \quad \tau_1 \vec{e}_3 - \tau_2 \vec{e}_3 + \vec{h} * \vec{F}_2 = \vec{0}$$

where $\vec{h} = \ell \vec{\varepsilon}$ is the vector from A_1 to A_2.

The resulting force $\vec{F}^{\,i}$ on $\mathcal{B}^i$, the resulting moment $\vec{M}^i_M$ on $\mathcal{B}^i$ with respect to M^i and the corresponding quantities for $\mathcal{B}^j$ follow from

$$\vec{F}^{\,i} = -\vec{F}_1; \quad \vec{M}^i_M = -\tau_1 \vec{e}_3 - \vec{a} * \vec{F}_1$$

$$\vec{F}^{\,j} = -\vec{F}_2; \quad \vec{M}^j_M = +\tau_2 \vec{e}_3 - \vec{b} * \vec{F}_2$$

The virtual work δA of the forces and moments on the connection element follows from

$$\delta A = \delta \vec{x}^{\,i} \circ \vec{F}^{\,i} + \delta \vec{\pi}^{\,i} \circ \vec{M}^i + \delta \vec{x}^{\,j} \circ \vec{F}^{\,j} + \delta \vec{\pi}^{\,j} \circ \vec{M}^j$$

With $\delta \vec{x}^{\,j} = \delta \vec{x}^{\,i} + \delta \vec{a} + \delta \vec{h} - \delta \vec{b} = \delta \vec{x}^{\,i} + \delta \vec{\pi}^{\,i} * \vec{a} + \delta \vec{h} - \delta \vec{\pi}^{\,j} * \vec{b}$ this relation can be simplified. After some elementary calculations it is seen that

$$\delta A = \vec{F}_2 \circ [-\delta \vec{h} + \delta \vec{\pi}^{\,i} * \vec{h}] + \tau_2 \vec{e}_3 \circ (\delta \vec{\pi}^{\,j} - \delta \vec{\pi}^{\,i})$$

From the previous section it is known that $\delta \vec{h} - \delta \vec{\pi}^{\,i} * \vec{h} = \delta \alpha\, \vec{e}_3 * \vec{h}$ and $\delta \vec{\pi}^{\,j} - \delta \vec{\pi}^{\,i} = \delta \beta \vec{e}_3$ and therefore

$$\delta A = \tau_1 \delta \alpha + \tau_2 \delta \beta$$

It is seen again that the constraint forces $\vec{F}_1$ and $\vec{F}_2$, required to maintain the kinematical constraints of this connection, do not contribute to the virtual work.

As a final example the system of *Fig.* 3.3 is considered. The motion of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ is controlled by an actuator in A_1 between rod 1 and $\mathcal{B}^i$. The moment of this actuator on the rod is given by $\vec{M} = \tau \vec{e}_3$. $\mathcal{B}^i$ exerts a force $\vec{F}_1$ on rod 1 and a force $\vec{F}_3$ on rod 2 while $\mathcal{B}^j$ exerts a force $\vec{F}_2$ on rod 1 and a force $\vec{F}_4$ on rod 2. According to the third law of Newton rod 1 exerts a moment $-\vec{M}$ and a force $-\vec{F}_1$ on $\mathcal{B}^i$ and a force $-\vec{F}_2$ on $\mathcal{B}^j$. Besides, rod 2 exerts a force $-\vec{F}_3$ on $\mathcal{B}^i$ and a force $-\vec{F}_4$ on $\mathcal{B}^j$.

The rods are massless, so the resulting forces and moments on each of the rods have to cancel. This yields the equilibrium equations for the connection:

$$\vec{F}_1 + \vec{F}_2 = \vec{0}; \quad \tau \vec{e}_3 + \vec{h}_1 * \vec{F}_2 = \vec{0}; \quad \vec{F}_3 + \vec{F}_4 = \vec{0}; \quad \vec{h}_2 * \vec{F}_4 = \vec{0}$$

where $\vec{h}_1$ and $\vec{h}_2$ are the vectors along the rods.

The resulting force $\vec{F}^i$ and resulting moment $\vec{M}^i_M$ with respect to M^i, exerted by the connection on $\mathcal{B}^i$, and the resulting force $\vec{F}^j$ and resulting moment $\vec{M}^j_M$ with respect to M^j, exerted by the connection on $\mathcal{B}^j$, are given by

$$\vec{F}^i = -\vec{F}_1 - \vec{F}_3; \quad \vec{M}^i_M = -\tau\vec{e}_3 - \vec{a}_1 * \vec{F}_1 - \vec{a}_2 * \vec{F}_3$$

$$\vec{F}^j = -\vec{F}_2 - \vec{F}_4; \quad \vec{M}^j_M = -\vec{b}_1 * \vec{F}_2 - \vec{b}_2 * \vec{F}_4$$

To arrive at a relation for the virtual work δA it is noted that $\delta\vec{\pi}^j$ and $\delta\vec{x}^j$ satisfy

$$\delta\vec{\pi}^j = \delta\vec{\pi}^i + \delta\beta\vec{e}_3; \quad \delta\vec{x}^j = \delta\vec{x}^i + \delta\vec{\pi}^i * \vec{a}_1 + \delta\vec{\pi}^i * \vec{h}_1 + \delta\alpha_1\vec{e}_3 * \vec{h}_1 - \delta\vec{\pi}^j * \vec{b}_1$$

Substitution in $\delta A = \vec{F}^i \circ \delta\vec{x}^i + \vec{F}^j \circ \delta\vec{x}^j + \vec{M}^i \circ \delta\vec{\pi}^i + \vec{M}^j \circ \delta\vec{\pi}^j$ yields, after a fairly prolix calculation, that

$$\delta A = \tau\delta\alpha_1 + \vec{F}_4 \circ (\delta\vec{g}^* - \delta\vec{g})$$

where $\vec{g}$ is the position vector of the center of hinge A_4, seen as a point of body $\mathcal{B}^j$, and $\vec{g}^*$ is the position vector of this center, seen as the end point of rod 2:

$$\vec{g} = \vec{x}^j + \vec{b}_2; \quad \vec{g}^* = \vec{x}^i + \vec{a}_2 + \vec{h}_2$$

As stated in the previous section the constraint $\vec{g} = \vec{g}^*$ is not taken into account in the definition of the generalized coordinates $q_1 \equiv \alpha_1$, $q_2 \equiv \alpha_2$ and $q_3 \equiv \beta$ of the connection, so it is not required a priori that $\delta\vec{g} = \delta\vec{g}^*$ and a term $\vec{F}_4 \circ (\delta\vec{g}^* - \delta\vec{g})$, representing the virtual work of the constraint force in hinge A_4, shows up in the relation for δA. However, it is seen $\delta A = \tau\delta\alpha_1$ as soon as the constraint $\vec{g} = \vec{g}^*$ is also taken into account: the variations $\delta\underset{\sim}{q}$ then must satisfy $\delta\vec{g}^* = \delta\vec{g}$.

The connection elements in all examples discussed until now can not store or dissipate energy. However, often connections do not or not only restrict the relative motion of the connected bodies but also store or dissipate energy. Again hardly any general rules for modelling connections of this type can be given since the variety in possible connections is too large. For that reason only some simple examples are given.

In the first example again the system of *Fig.* 3.1 is considered but now it is assumed that damping occurs in the hinge. This damping is modelled by a linear, viscous damper (damping coefficient b), parallel to the actuator. Because the angular velocity of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$ is equal to $\dot{\beta}$ this damping results in a moment $-b\dot{\beta}\vec{e}_3$ on $\mathcal{B}^j$ and, according to the third law of Newton, a moment $+b\dot{\beta}\vec{e}_3$ on $\mathcal{B}^i$. Hence, in the hinge a resulting moment $(\tau - b\dot{\beta})\vec{e}_3$ is exerted on $\mathcal{B}^j$ and a resulting moment $-(\tau - b\dot{\beta})\vec{e}_3$ on $\mathcal{B}^i$. All results of the previous section for the system in *Fig.* 3.1 remain valid if τ is replaced by $\tau - b\dot{\beta}$. This means, for instance, that the virtual work δA for this system with viscous damping is given by

$$\delta A = (\tau - b\dot{\beta})\delta\beta$$

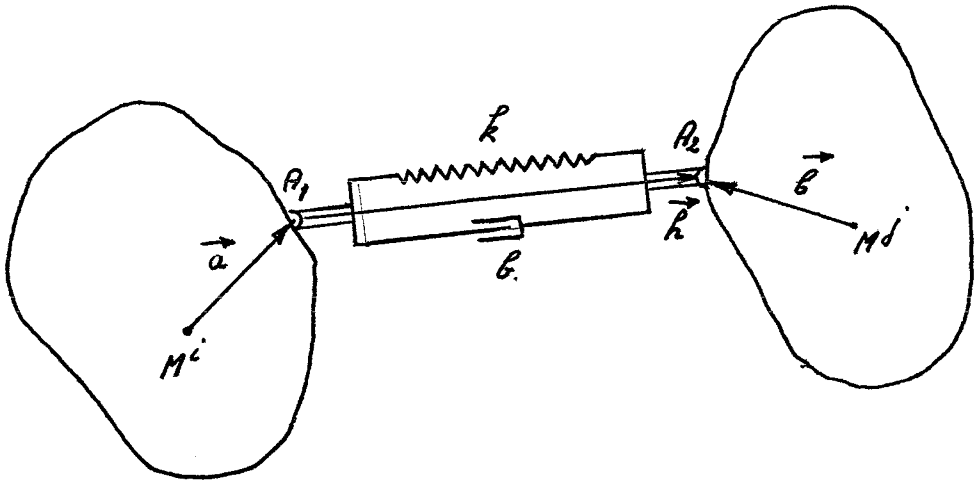

Figure 3.4: $\mathcal{B}^i$ and $\mathcal{B}^j$, connected by a spring and a damper.

In a final example the system in *Fig.* 3.2 is considered again but now the actuator is removed and the rigid rod is replaced by a parallel assembly of a linear, elastic spring (stiffness k, unstressed length ℓ_0) and a linear, viscous damper (damping coefficient b). The kinematical constraint in this case follows from the requirement that the connected bodies only move in the plane of *Fig.* 3.2 and can be taken into account with the angle β between the $\vec{e}^{\,j}_1$ and $\vec{e}^{\,i}_1$. The distance ℓ between A_1 to A_2 is *not* constant now but follows from

$$\ell = ||\vec{h}||; \quad \vec{h} = \vec{x}^{\,j} + \vec{b} - (\vec{x}^{\,i} + \vec{a})$$

The connection can only transmit forces along its axis, *i.e.* along the carrier of $\vec{h}$. Hence, the force $\vec{F}_2$, that is exerted in A_2 on the spring-damper assembly by $\mathcal{B}^j$, must be a vector in the direction of the unit vector $\vec{\varepsilon}$ along this axis, *i.e.*

$$\vec{F}_2 = F\vec{\varepsilon}; \quad \vec{\varepsilon} = \frac{1}{\ell}\vec{h}$$

Here F is the sum of the force in the spring and the force in the damper, *i.e.*

$$F = k(\ell - \ell_0) + b\,\dot{\ell}$$

Equilibrium of the massless connection requires that the force $\vec{F}_1$, exerted in A_1 by $\mathcal{B}^i$ on the connection, is equal to $-\vec{F}_2$, so

$$\vec{F}_1 = -\vec{F}_2 = -F\vec{\varepsilon}; \quad F = k(\ell - \ell_0) + b\,\dot{\ell}$$

Since the connection can not transmit torques the moments in A_1 and A_2 are zero. This means that the resulting force $\vec{F}^{\,i}$ and the resulting moment $\vec{M}^{\,i}$ with respect to M^i, exerted on $\mathcal{B}^i$ by the connection, and the resulting force $\vec{F}^{\,j}$ and the resulting moment $\vec{M}^{\,j}$ with respect to M^j, exerted on $\mathcal{B}^j$ by the connection, are given by

$$\vec{F}^{\,i} = -\vec{F}_1 = +F\vec{\varepsilon}; \quad \vec{M}^{\,i} = -\vec{a} * \vec{F}_1 = +F\vec{a} * \vec{\varepsilon}$$

$$\vec{F}^{\,j} = -\vec{F}_2 = -F\vec{\varepsilon}; \quad \vec{M}^{\,j} = -\vec{b} * \vec{F}_2 = -F\vec{b} * \vec{\varepsilon}$$

The relation for the virtual work δA of the system with this connection is quite different from the relation for the system with a connection in the form of a hinged rigid rod. The reason is that the rigid rod results in the constraint $\|\vec{h}\| = \ell = constant$ and therefore restricts the motion of $\mathcal{B}^j$ with respect to $\mathcal{B}^i$: the position vector $\vec{x}^j$ of M^j has to satisfy $\vec{x}^j = \vec{x}^i + \vec{a} + \ell\vec{\varepsilon} - \vec{b}$, where ℓ is constant and $\vec{\varepsilon} = \cos(\alpha)\vec{e}^{\,i}_1 + \sin(\alpha)\vec{e}^{\,i}_2$. Hence, ℓ is not a degree of freedom for the connection with the rigid rod: for that connection only α and β are degrees of freedom. However, for the connection with the spring and the damper also ℓ is a degree of freedom. This has consequences for the virtual work. After some straight-forward calculations it is seen that

$$\delta A = F\delta\ell = [k(\ell - \ell_0) + b\,\dot{\ell}]\delta\ell = \delta[\frac{1}{2}k(\ell - \ell_0)^2] + b\,\dot{\ell}\,\delta\ell$$

Chapter 4

Equations of motion for multibody systems.

4.1 Introduction.

An essential assumption in the "classical" theory for multibody systems is that the system can be divided in rigid parts with mass, the rigid bodies, and parts without mass, the connections. All parts with mass must be rigid and all flexible parts must be massless. This assumption is very restrictive. It excludes, for instance, flexible manipulators but is nevertheless adopted in this chapter. Unless stated otherwise it is assumed that the bodies move in three-dimensional Euclidian space $\mathcal{E}$.

The equations of motion of a multibody system relate dynamical quantities (resulting forces and moments on the rigid bodies) to kinematical quantities (linear and angular acceleration). The derivation of these equations requires a specification of the relevant dynamical and kinematical quantities. In the next section first kinematical quantities are considered. Attention is focussed on the attitude coordinates of the bodies, on kinematical constraints and on the introduction of generalized coordinates to take these constraints into account. As soon as the attitude coordinates are written as functions of the generalized coordinates the linear and angular velocity and acceleration of all bodies can be determined by differentiation.

Two different approaches can be distinguished to arrive at the equations of motion for a multibody system. The first approach is based on the laws of Newton and Euler, the second on some energy principle. Examples of these approaches are given in the last sections of this chapter. There also some results of general validity for the final set of equations of motion for multibody systems are presented.

It is noted that the discussion in this chapter is fairly ad hoc: much of the theory is not presented in a general context but only sketched and illustrated for some simple examples.

4.2 Kinematics of multibody systems.

Let n_b be the number of rigid bodies in the considered system. These bodies are numbered from 1 to n_b. The body with number i is denoted by $\mathcal{B}^i$. In $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$ an orthonormal vector basis $\vec{\underset{\sim}{e}}^i$, the **local basis**, with origin in the center of mass M^i of $\mathcal{B}^i$ is defined. This basis is fixed with respect to $\mathcal{B}^i$, *i.e.* $\vec{e}_1^{\,i}$, $\vec{e}_2^{\,i}$ and $\vec{e}_3^{\,i}$ are moving along with $\mathcal{B}^i$.

The environment of the system, the fixed world, is denoted by $\mathcal{B}^0$. It is used as the reference for the motion of the bodies in the system. The orthonormal vector basis $\underset{\sim}{\vec{e}}^{\,0}$, that is fixed with respect to $\mathcal{B}^0$, is called the **fixed** or **global basis.** For simplicity it is denoted by $\underset{\sim}{\vec{e}}$ instead of $\underset{\sim}{\vec{e}}^{\,0}$.

The position and orientation of $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$ is determined completely by the position vector $\vec{x}^{\,i}$ from the origin $\mathcal{O}$ of the global basis to M^i and by the rotation tensor $\mathbf{R}^i$, which maps the global basis $\underset{\sim}{\vec{e}}$ on the local basis $\underset{\sim}{\vec{e}}^{\,i}$, *i.e.*

$$(\underset{\sim}{\vec{e}}^{\,i})^T = \mathbf{R}^i \circ (\underset{\sim}{\vec{e}})^T \tag{4.1}$$

In practice it is very clumsy to describe the orientation of $\mathcal{B}^i$ with a rotation tensor, because this tensor has to satisfy the orthogonality condition $(\mathbf{R}^i)^T \circ \mathbf{R}^i = \mathbf{I}$. As a consequence, the matrix representation $\underline{R}^i$ of this tensor,

$$\underline{R}^i = \underset{\sim}{\vec{e}} \circ \mathbf{R}^i \circ (\underset{\sim}{\vec{e}})^T \quad \Leftrightarrow \quad \mathbf{R}^i = (\underset{\sim}{\vec{e}})^T \underline{R}^i \, \underset{\sim}{\vec{e}},$$

has to satisfy the condition $(\underline{R}^i)^T \underline{R} = \underline{I}$ and the entries of $\underline{R}^i$ can be written as functions of three independent quantities, *e.g.* Euler or Cardan angles[1]. Only Cardan angles are considered here. For $\mathcal{B}^i$ they are denoted by φ_1^i, φ_2^i and φ_3^i and are seen as the elements of a column $\underset{\sim}{\varphi}^i$. Then the relation between $\underline{R}^i$ and $\underset{\sim}{\varphi}^i$ is given by

$$\underline{R}^i = \underline{R}(\underset{\sim}{\varphi}^i); \quad \underline{R}(\underset{\sim}{\varphi}) = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}$$

where $c_s = \cos(\varphi_s)$ and $s_s = \sin(\varphi_s)$. The Cardan angles $\underset{\sim}{\varphi}^i$ determine the rotation matrix $\underline{R}^i$ and also the rotation tensor $\mathbf{R}^i = (\underset{\sim}{\vec{e}})^T \underline{R}^i \, \underset{\sim}{\vec{e}}$. Therefore, the position and orientation of $\mathcal{B}^i$ can be described with $\vec{x}^{\,i}$ and $\underset{\sim}{\varphi}^i$ or with the **attitude coordinates** $\underset{\sim}{z}^i$ of $\mathcal{B}^i$:

$$\underset{\sim}{z}^i = \begin{bmatrix} \underset{\sim}{x}^i \\ \underset{\sim}{\varphi}^i \end{bmatrix}$$

where $\underset{\sim}{x}^i$ is the matrix representation of $\vec{x}^{\,i}$ with respect to the global vector basis $\underset{\sim}{\vec{e}}$, *i.e.*

$$\vec{x}^{\,i} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{x}^i \quad \Leftrightarrow \quad \underset{\sim}{x}^i = \underset{\sim}{\vec{e}} \circ \vec{x}^{\,i}$$

The total number of attitude coordinates for n_b rigid bodies in three-dimensional space $\mathcal{E}$ is equal to $n_z = 6.n_b$. These coordinates are the elements a column $\underset{\sim}{z} \in \mathcal{R}^{n_z}$,

$$\underset{\sim}{z} = \begin{bmatrix} \underset{\sim}{z}^1 \\ \underset{\sim}{z}^2 \\ \vdots \\ \underset{\sim}{z}^{n_b} \end{bmatrix}$$

The connections between the bodies and between the bodies and the fixed world can induce

[1] See, for instance, the appendix on vectors and tensors or "*Dynamics of Systems of Rigid Bodies*" by J. Wittenburg (B.G. Teubner, Stuttgart, 1977, ISBN 3-519-12337-7).

kinematical constraints on the possible motions of the bodies. Only holonomic constraints are considered here. They are discussed in some detail in the previous chapter.

Let n_t be the number of independent kinematical constraints, given by

$$\underset{\sim}{k}_t(\underset{\sim}{z},t) = \underset{\sim}{0}, \quad \underset{\sim}{k}_t \in \mathcal{R}^{n_t}$$

Then n_t attitude coordinates can be eliminated and written as functions of the other $n_f = n_z - n_t$ coordinates. An alternative is to introduce a set $\underset{\sim}{s}$ of n_f **degrees of freedom** and to write the attitude coordinates as functions of $\underset{\sim}{s}$, *i.e.* $\underset{\sim}{z} = \underset{\sim}{z}(\underset{\sim}{s},t)$, such that the constraints are satisfied for all $\underset{\sim}{s}$ and t:

$$\underset{\sim}{k}_t(\underset{\sim}{z}(\underset{\sim}{s},t),t) = \underset{\sim}{0}, \quad \forall \underset{\sim}{s} \in \mathcal{R}^{n_f}, \quad \forall t \in \mathcal{R}$$

As argued in the previous chapter it is not always feasible nor advantageous to try to find a set $\underset{\sim}{s}$ and functions $\underset{\sim}{z} = \underset{\sim}{z}(\underset{\sim}{s},t)$, such that *all* constraints are satisfied for all values of $\underset{\sim}{s}$. In that case the set $\underset{\sim}{k}_t(\underset{\sim}{z},t) = \underset{\sim}{0}$ can, eventually after renumbering, be partitioned in

$$\underset{\sim}{k}_t(\underset{\sim}{z},t) = \begin{bmatrix} \underset{\sim}{k}_r(\underset{\sim}{z},t) \\ \underset{\sim}{k}_e(\underset{\sim}{z},t) \end{bmatrix} = \underset{\sim}{0}, \quad \underset{\sim}{k}_r \in \mathcal{R}^{n_c}, \quad \underset{\sim}{k}_e \in \mathcal{R}^{n_t - n_c}$$

where $\underset{\sim}{k}_e(\underset{\sim}{z},t) = \underset{\sim}{0}$ represents the $n_t - n_c$ constraints that are taken into account in the choice of the new coordinates while $\underset{\sim}{k}_r(\underset{\sim}{z},t) = \underset{\sim}{0}$ represents the n_c constraints that are left out of consideration in this choice. To emphasize that these new coordinates are not based on *all* constraints they are called **generalized coordinates** instead of degrees of freedom and the column with these coordinates as elements is denoted by $\underset{\sim}{q}$ instead of $\underset{\sim}{s}$. Then

$$\underset{\sim}{z} = \underset{\sim}{z}(\underset{\sim}{q},t); \quad \underset{\sim}{k}_e(\underset{\sim}{z}(\underset{\sim}{q},t),t) = \underset{\sim}{0}, \quad \forall \underset{\sim}{q} \in \mathcal{R}^{n_q}, \quad \forall t \in \mathcal{R}$$

where n_q is the number of elements of $\underset{\sim}{q}$:

$$n_q = 6.n_b - (n_t - n_c) = n_z + n_c - n_t$$

The remaining constraints $\underset{\sim}{k}_r(\underset{\sim}{z},t) = \underset{\sim}{0}$ can be rewritten in terms of $\underset{\sim}{q}$ and t, resulting in

$$\underset{\sim}{k}_{rem}(\underset{\sim}{q},t) = \underset{\sim}{0}, \quad \underset{\sim}{k}_{rem} \in \mathcal{R}^{n_c}$$

Important topics in a discussion on the kinematics of multibody systems are

1. formulation of the kinematical constraints
2. choice of the generalized coordinates $\underset{\sim}{q}$, based on all or part of the kinematical constraints
3. determination of the relation between the attitude coordinates $\underset{\sim}{z}$ on the one side and $\underset{\sim}{q}$ and t on the other hand
4. formulation of eventual remaining constraints in terms of $\underset{\sim}{q}$ and t

In general, none of these topics will cause essential problems for technical systems: technical connections like prismatic, revolute or spherical joints are fairly easy to characterize, as was shown in the examples in the previous chapter. This means that the choice of the generalized coordinates usually is straightforward. It turns out that, as a rule, the determination of the relation between $\underset{\sim}{z}$, $\underset{\sim}{q}$ and t is merely a problem of correct book-keeping. Also the last item mentioned above generally does not give rise to large difficulties since a formulation of the kinematical constraints in terms of $\underset{\sim}{z}$ and t usually is fairly simple and reformulation in terms of $\underset{\sim}{q}$ and t is merely a book-keeping problem.

From $\underset{\sim}{z} = \underset{\sim}{z}(\underset{\sim}{q}, t)$ the attitude coordinates $\underset{\sim}{z}^i$ of $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$ can be extracted. Written in terms of its partitions, *i.e.* in terms of the Cartesian coordinates $\underset{\sim}{x}^i$ of M^i and the Cardan angles $\underset{\sim}{\varphi}^i$, this results in

$$\underset{\sim}{z}^i = \begin{bmatrix} \underset{\sim}{x}^i \\ \underset{\sim}{\varphi}^i \end{bmatrix}; \quad \underset{\sim}{x}^i = \underset{\sim}{x}^i(\underset{\sim}{q}, t); \quad \underset{\sim}{\varphi}^i = \underset{\sim}{\varphi}^i(\underset{\sim}{q}, t)$$

For the position vector $\vec{x}^{\,i}$ of M^i and the rotation tensor $\mathbf{R}^i$ of $\mathcal{B}^i$ it is seen that

$$\vec{x}^{\,i} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{x}^i; \quad \underset{\sim}{x}^i = \underset{\sim}{x}^i(\underset{\sim}{q}, t)$$

$$\mathbf{R}^i = (\underset{\sim}{\vec{e}})^T \, \underline{R}^i \, \underset{\sim}{\vec{e}}; \quad \underline{R}^i = \underline{R}^i(\underset{\sim}{\varphi}^i); \cdot \quad \underset{\sim}{\varphi}^i = \underset{\sim}{\varphi}^i(\underset{\sim}{q}, t)$$

Differentiation of these relations with respect to t yields relations for the velocity $\vec{v}^{\,i}$ of M^i and for the angular velocity $\vec{\omega}^{\,i}$ of $\mathcal{B}^i$. From $\underset{\sim}{x}^i = \underset{\sim}{x}^i(\underset{\sim}{q}, t)$ and $\underset{\sim}{\varphi}^i = \underset{\sim}{\varphi}^i(\underset{\sim}{q}, t)$ it is seen that

$$\dot{\underset{\sim}{x}}^i = \underline{X}^i\,(\underset{\sim}{q}, t)\dot{\underset{\sim}{q}} + \underset{\sim}{v}^i_{part}(\underset{\sim}{q}, t)$$

$$\dot{\underset{\sim}{\varphi}}^i = \underline{\Phi}^i\,(\underset{\sim}{q}, t)\dot{\underset{\sim}{q}} + \underset{\sim}{\omega}^i_{part}(\underset{\sim}{q}, t)$$

where the matrix functions $\underline{X}^i = \underline{X}^i(\underset{\sim}{q}, t)$ and $\underline{\Phi}^i = \underline{\Phi}^i(\underset{\sim}{q}, t)$ represent the derivative of $\underset{\sim}{x}^i = \underset{\sim}{x}^i(\underset{\sim}{q}, t)$, respectively $\underset{\sim}{\varphi}^i = \underset{\sim}{\varphi}^i(\underset{\sim}{q}, t)$ with respect to $\underset{\sim}{q}$ while $\underset{\sim}{v}^i_{part} = \underset{\sim}{v}^i_{part}(\underset{\sim}{q}, t)$ and $\underset{\sim}{\omega}^i_{part} = \underset{\sim}{\omega}^i_{part}(\underset{\sim}{q}, t)$ represent the derivative of $\underset{\sim}{x}^i = \underset{\sim}{x}^i(\underset{\sim}{q}, t)$, respectively $\underset{\sim}{\varphi}^i = \underset{\sim}{\varphi}^i(\underset{\sim}{q}, t)$ with respect to t. With these results $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$ can be written as[2]

$$\vec{v}^{\,i} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{v}^i; \quad \underset{\sim}{v}^i = \dot{\underset{\sim}{x}}^i$$

$$\vec{\omega}^{\,i} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{\omega}^i; \quad \underset{\sim}{\omega}^i = \underline{W}^i \, \dot{\underset{\sim}{\varphi}}^i$$

where $\underline{W}^i$ depends on $\underset{\sim}{\varphi}^i$ only. Because the elements of $\underset{\sim}{\varphi}^i$ are Cardan angles this matrix is given by

$$\underline{W}^i = \underline{W}(\underset{\sim}{\varphi}^i); \quad \underline{W}(\underset{\sim}{\varphi}) = \begin{bmatrix} 1 & 0 & \sin(\varphi_2) \\ 0 & \cos(\varphi_1) & -\sin(\varphi_1)\cos(\varphi_2) \\ 0 & \sin(\varphi_1) & \cos(\varphi_1)\cos(\varphi_2) \end{bmatrix}$$

[2] Further information on this subject can be found in, *e.g.*, the appendix on vectors and tensors and in the earlier mentioned book of Wittenburg

Relations for the acceleration of M^i and for the angular acceleration of $\mathcal{B}^i$ can be found be differentiation of these relations for $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$. This straightforward process results in fairly bulky equations, that are not given here.

Analogous to the results for $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$ also relations for the variation $\delta\vec{x}^{\,i}$ of the position vector of M^i and for the angular variation $\delta\vec{\pi}^{\,i}$ of $\mathcal{B}^i$ can be derived. This results in

$$\delta\vec{x}^{\,i} = (\underset{\sim}{\vec{e}})^T \delta\underset{\sim}{x}^i; \quad \delta\underset{\sim}{x}^i = \underline{X}^i(\underset{\sim}{q},t)\delta\underset{\sim}{q}$$

$$\delta\vec{\pi}^{\,i} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{\pi}^i; \quad \delta\underset{\sim}{\pi}^i = \underline{W}^i\delta\underset{\sim}{\varphi}^i; \quad \delta\underset{\sim}{\varphi}^i = \underline{\Phi}^i(\underset{\sim}{q},t)\delta\underset{\sim}{q}$$

The generalized coordinates $\underset{\sim}{q}$ must satisfy the remaining constraints $\underset{\sim}{k}_{rem}(\underset{\sim}{q},t) = \underset{\sim}{0}$. Variations of $\underset{\sim}{q}$, such that both $\underset{\sim}{k}_{rem}(\underset{\sim}{q},t) = \underset{\sim}{0}$ and $\underset{\sim}{k}_{rem}(\underset{\sim}{q}+\delta\underset{\sim}{q},t) = \underset{\sim}{0}$ are satisfied, are called **kinematically admissable.** If the derivative of $\underset{\sim}{k}_{rem} = \underset{\sim}{k}_{rem}(\underset{\sim}{q},t)$ with respect to $\underset{\sim}{q}$ is denoted by $\underline{K} = \underline{K}(\underset{\sim}{q},t)$, *i.e.* if

$$\underset{\sim}{k}_{rem}(\underset{\sim}{q}+\delta\underset{\sim}{q},t) = \underset{\sim}{k}_{rem}(\underset{\sim}{q},t) + \underline{K}(\underset{\sim}{q},t)\delta\underset{\sim}{q}$$

for all infinitesimal $\delta\underset{\sim}{q}$, then $\delta\underset{\sim}{q}$ is kinematically admissable if

$$\underline{K}(\underset{\sim}{q},t)\delta\underset{\sim}{q} = \underset{\sim}{0}$$

Kinematically admissable variations play an important role if the equations of motion of the multibody system are derived, using an energy principle.

To illustrate the previous description of the kinematics of a multibody system the simple crank-shaft mechanism of *Fig.* 3.1 is considered. The bodies $\mathcal{B}^1$ and $\mathcal{B}^2$ are rigid and all motions of $\mathcal{B}^1$ and $\mathcal{B}^2$ out of the plane of this figure are suppressed. The revolute joints (hinges) A_1 and A_2 allow relative rotations only. The combination of a revolute and a prismatic joint A_3 permits both a rotation of $\mathcal{B}^2$ with respect to the fixed world and a translation in horizontal direction. The center of mass M^1 of $\mathcal{B}^1$ is located on the line from A_1 to A_2 on a distance c_1 from A_1 and the center of mass M^2 of $\mathcal{B}^2$ is located on the line from A_2 to A_3 on a distance c_2 from A_2.

The global basis vectors $\vec{e}_1$ and $\vec{e}_3$ are horizontal, respectively perpendicular to the plane of *Fig.* 3.1. The origin $\mathcal{O}$ is on the line between A_1 and A_3 at a distance a_1 from A_1. The local basis vector $\vec{e}^{\,i}_3$ on $\mathcal{B}^i$ ($i = 1,2$) is perpendicular to the plane of *Fig.* 3.1. Furthermore, the local basis vector $\vec{e}^{\,1}_1$ on $\mathcal{B}^1$ is directed from A_1 to A_2 and the local basis vector $\vec{e}^{\,2}_1$ on $\mathcal{B}^2$ is directed from A_2 to A_3.

Because $\mathcal{B}^i$ ($i = 1,2$) can only rotate in the plane of *Fig.* 3.1 the Cardan angles φ^i_1 and φ^i_2 of $\mathcal{B}^i$ are zero and the orientation of $\mathcal{B}^i$ can be characterized by the angle $\varphi^i_3 \equiv \beta_i$ between the carriers of global basis vector $\vec{e}_1$ and the local basis vector $\vec{e}^{\,i}_1$:

$$(\underset{\sim}{\varphi}^i)^T = [0 \;\; 0 \;\; \beta_i]$$

With this angle β_i the local basis vectors $\vec{e}^{\,i}_1$ and $\vec{e}^{\,i}_2$ are given by

$$\vec{e}^{\,i}_1 = \cos(\beta_i)\vec{e}_1 + \sin(\beta_i)\vec{e}_2; \quad \vec{e}^{\,i}_2 = \vec{e}_3 * \vec{e}^{\,i}_1 = -\sin(\beta_i)\vec{e}_1 + \cos(\beta_i)\vec{e}_2$$

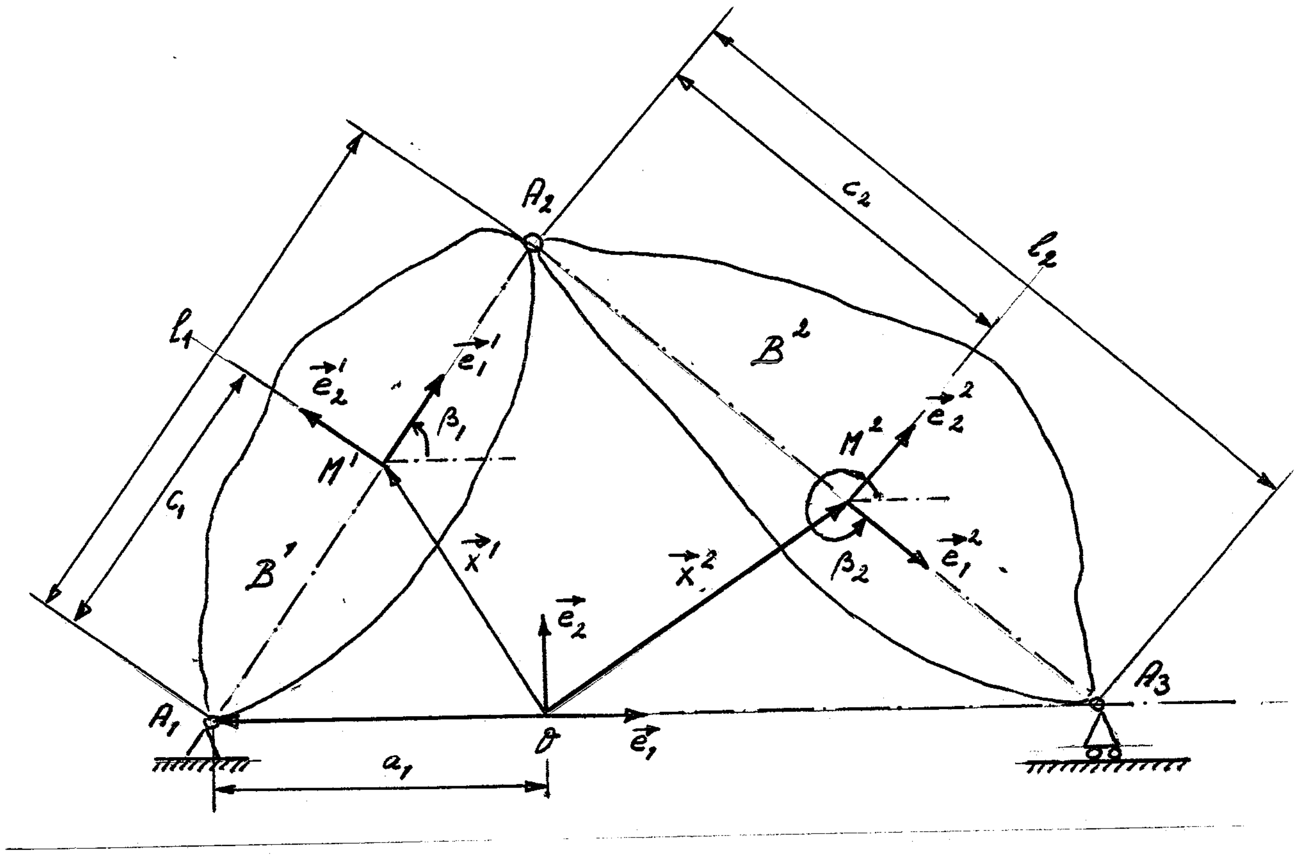

Figure 4.1: Crank-shaft mechanism.

Two Cartesian coordinates are required to describe the position of M^i $(i = 1, 2)$ since the coordinate of M^i in $\vec{e}_3$–direction is equal to zero, *i.e.* $\vec{x}^{\,i} \circ \vec{e}_3 = 0$.

The hinge A_1 suppresses displacements of $\mathcal{B}^1$ in A_1 with respect to the fixed world: the position vector of A_1, seen as a point of $\mathcal{B}^1$, must be equal to $\vec{a}_1 = -a_1\vec{e}_1$. Therefore, the position vector $\vec{x}^{\,1}$ of M^1 can be written as

$$\vec{x}^{\,1} = -a_1\vec{e}_1 + c_1\vec{e}_1^{\,1} = -a_1\vec{e}_1 + c_1\cos(\beta_i)\vec{e}_1 + c_1\sin(\beta_i)\vec{e}_2)$$

and it is seen that $\vec{x}^{\,1}$ is determined completely by β_1. The hinge in A_2 suppresses all relative displacements of $\mathcal{B}^1$ and $\mathcal{B}^2$ in this point: the position vector of A_2, seen as a point of $\mathcal{B}^1$, must be equal to the position vector of A_2, seen as a point of $\mathcal{B}^2$. This means that $\vec{x}^{\,1} + (\ell_1 - c_1)\vec{e}_1^{\,1} = \vec{x}^{\,2} - c_2\vec{e}_1^{\,2}$ and therefore

$$\vec{x}^2 = -a_1\vec{e}_1 + \ell_1\vec{e}_1^{\,1} + c_2\vec{e}_1^{\,2}$$

With the earlier given relations for $\vec{x}^{\,1}$, $\vec{e}_1^{\,1}$ and $\vec{e}_1^{\,2}$ it follows that $\vec{x}^{\,2}$ is determined completely by β_1 and β_2.

The kinematical constraint, induced by the connection A_3, is that the vertical coordinate of A_3, seen as a point of $\mathcal{B}^2$, must be zero, *i.e.* that

$$\vec{e}_2 \circ [\vec{x}^{\,2} + (\ell_2 - c_2)\vec{e}_1^{\,2}] = \ell_1\sin(\beta_1) + \ell_2\sin(\beta_2) = 0$$

It is possible to solve β_2 from this nonlinear relation as a function of β_1. Then $\vec{x}^{\,1}$, $\vec{x}^{\,2}$, $\underset{\sim}{\varphi}^1$ and $\underset{\sim}{\varphi}^2$ are functions of β_1 only and β_1 can be considered as the only degree of freedom of this system, *i.e.* $\underset{\sim}{s} = [\beta_1]$. However, the resulting functions for $\vec{x}^{\,1}$, $\vec{x}^{\,2}$ and $\underset{\sim}{\varphi}^2$ are highly nonlinear and very cumbersome to handle. For that reason it is attractive to introduce two generalized coordinates $q_1 \equiv \beta_1$ and $q_2 \equiv \beta_2$, *i.e.*

$$\underset{\sim}{q} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

and accept that these coordinates have to satisfy the remaining constraint

$$\ell_1\sin(\beta_1) + \ell_2\sin(\beta_2) \equiv \ell_1\sin(q_1) + \ell_2\sin(q_2) = 0$$

The angular velocity $\vec{\omega}^{\,i}$ and the angular variation $\delta\vec{\pi}^{\,i}$ $(i = 1, 2)$ are given by

$$\vec{\omega}^{\,i} = \dot{\beta}_i\vec{e}_3 \equiv \dot{q}_i\vec{e}_3; \quad \delta\vec{\pi}^{\,i} = \delta\beta_i\vec{e}_3 \equiv \delta q_i\vec{e}_3$$

Finally, the velocity $\vec{v}^{\,i}$ of M^i $(i = 1, 2)$ can be determined by differentiation of the given relations for the position vectors $\vec{x}^{\,i}$ M^i. This yields

$$\vec{v}^{\,1} = c_1\dot{\vec{e}}_1^{\,1} = c_1\vec{\omega}^{\,1} * \vec{e}_1^{\,1} = c_1\dot{\beta}_1\vec{e}_3 * \vec{e}_1^{\,1} = c_1\dot{\beta}_1\vec{e}_2^{\,1} \equiv c_1\dot{q}_1\vec{e}_2^{\,1}$$

$$\vec{v}^{\,2} = \ell_1\dot{\vec{e}}_1^{\,1} + c_2\dot{\vec{e}}_1^{\,2} = \ell_1\vec{\omega}^{\,1} * \vec{e}_1^{\,1} + \ell_2\vec{\omega}^{\,2} * \vec{e}_1^{\,2} = \ell_1\dot{\beta}_1\vec{e}_2^{\,1} + c_2\dot{\beta}_2\vec{e}_2^{\,2} \equiv \ell_1\dot{q}_1\vec{e}_2^{\,1} + c_2\dot{q}_2\vec{e}_2^{\,2}$$

If required the linear and angular accelerations can be found by differentiation of these relations for $\vec{v}^{\,1}$, $\vec{v}^{\,2}$, $\vec{\omega}^{\,1}$ and $\vec{\omega}^{\,2}$.

4.3 Dynamics of multibody systems.

4.3.1 Introduction.

The theory of multibody dynamics focusses on the derivation and solution of the equations of motion in terms of (derivatives) of the generalized coordinates. Two quite different approaches to arrive at these equations can be distinguished. The first approach is based on an imaginary experiment in which all connections in the system are removed and replaced by the forces and moments they exert on the bodies. After that, the laws of Newton and Euler are applied to each of the "isolated" bodies to arrive at a set of equations of motion for the original system. In this set the constraint forces and moments, *i.e.* the forces and moments to maintain the kinematical constraints, play an important role. Their elimination usually is very laborious. This is an important drawback of this otherwise very generally applicable approach.

The second approach is based on some energy principle, for instance the principle of virtual work or the principle of d'Alembert. Energy principles offer a basis for some very powerful (approximation) methods like the finite element method.

For dynamical, mechanical systems there are many specialized energy principles that can facilitate the derivation of the equations of motion. Examples are the principle of Hamilton, the principle of Jourdan and the principle of Lagrange. The last mentioned principle is of special importance. It results in the so-called equations of Lagrange. The application of these equations is the subject of the last subsection. No attention is given here to the backgrounds of these equations since these can be found in, for instance, *"Dynamics of Multibody Systems"* by A. Shabana (John Wiley & Sons, New York, 1989, ISBN 0471-61494-7) or in any advanced text-book on dynamics of mechanical systems.

4.3.2 Application of the laws of Newton and Euler.

A connection between two bodies (or between a body and the environment) can exert forces and moments on the connected bodies. These forces and moments can be distinguished in

- constraint forces and moments, required to maintain the kinematical constraints. There is one constraint force or moment for each kinematical constraint, so there will be n_t constraint forces or moments.

- other forces and moments, *e.g.* due to actuators in the connections, due to the gravitation field or due to dissipation or storage of energy in connection element. This is the case if the connection consists of springs, dampers, flexible beams, etc.

The third law of Newton (*"action=−reaction"*) states that, if a connection exerts a force and moment on a body, then that body exerts an equal but opposite force and moment on the connection. Because the connection elements are assumed to be massless the resulting forces and moments on each of these elements have to cancel, *i.e.* each element has to be in equilibrium.

The momentum $\vec{p}^{\,i}$ of $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$ and the moment of momentum $\vec{L}^{\,i}$ of $\mathcal{B}^i$ with respect to M^i are related to the velocity $\vec{v}^{\,i}$ of M^i and to the angular velocity $\vec{\omega}^{\,i}$ of $\mathcal{B}^i$ by

$$\vec{p}^{\,i} = m_i\,\vec{v}^{\,i}; \quad \vec{L}^{\,i} = \mathbf{J}_i \circ \vec{\omega}^{\,i} \tag{4.2}$$

where m_i is the mass of $\mathcal{B}^i$ and $\mathbf{J}_i$ is the inertia tensor of $\mathcal{B}^i$ with respect to M^i. As discussed in the previous section, $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$ can be written as functions of n_q generalized coordinates $\underset{\sim}{q}$ and their first derivative $\dot{\underset{\sim}{q}}$, the generalized velocities. Both $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$ are linear in $\dot{\underset{\sim}{q}}$, so $\vec{p}^{\,i}$ and $\vec{L}^{\,i}$ are also linear in $\dot{\underset{\sim}{q}}$.

The second law of Newton states that the derivative of the momentum $\vec{p}^{\,i}$ is equal to the resulting force $\vec{F}^{\,i}$ on $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$. Furthermore, the law of Euler (in fact a special form of the second law of Newton for rotating rigid bodies) states that the derivative of the moment of momentum $\vec{L}^{\,i}$ *with respect to the center of mass* of $\mathcal{B}^i$ is equal to the resulting moment $\vec{M}^{\,i}$ on $\mathcal{B}^i$ *with respect to this center of mass.* Application of these laws results in

$$\vec{F}^{\,i} = \dot{\vec{p}}^{\,i} = m_i \dot{\vec{v}}^{\,i}; \quad \vec{M}^{\,i} = \dot{\vec{L}}^{\,i} = \dot{\mathbf{J}}_i \circ \vec{\omega}^{\,i} + \mathbf{J}_i \circ \dot{\vec{\omega}}^{\,i} \tag{4.3}$$

where $\dot{\mathbf{J}}_i \circ \vec{\omega}^{\,i} = \vec{\omega}^{\,i} * (\mathbf{J}_1 \circ \vec{\omega}^{\,i})$ (see the chapter on the dynamics of one rigid body). Therefore, the equations of motion for $\mathcal{B}^i$ $(i = 1, 2, \cdots, n_b)$ can be written as

$$\vec{F}^{\,i} = m_i \dot{\vec{v}}^{\,i}; \quad \vec{M}^{\,i} = \mathbf{J}_i \circ \dot{\vec{\omega}}^{\,i} + \vec{\omega}^{\,i} * (\mathbf{J}_i \circ \vec{\omega}^{\,i}) \tag{4.4}$$

and because both $\vec{v}^{\,i}$ and $\vec{\omega}^{\,i}$ are linear in the generalized velocities $\dot{\underset{\sim}{q}}$ it follows that the right hand sides of these equations are linear in $\ddot{\underset{\sim}{q}}$.

For motions in three dimensional space these equations of motion for $\mathcal{B}^1$, $\mathcal{B}^2$, $\cdots$, $\mathcal{B}^{n_b}$ represent a set of $6.n_b$ second order differential equations. The unknown quantities in these equations are n_t constraint forces and moments and n_q generalized coordinates. The number of generalized coordinates is given by $n_q = 6.n_b - (n_t - n_c)$, where n_c is the number of the remaining constraints $\underset{\sim}{k}_{rem}(\underset{\sim}{q}, t) = \underset{\sim}{0}$. Hence, the number of unknowns $n_t + n_q = 6.n_b + n_c$ is equal to the number of equations ($6.n_b$ second order differential equations and n_c remaining constraints).

As an example again the system of *Fig.* 3.1 is considered. It is assumed that the global basis vector $\vec{e}_3$ is an eigenvector of the inertia tensors $\mathbf{J}_1$ of $\mathcal{B}^1$ and $\mathbf{J}_2$ of $\mathcal{B}^2$. The corresponding eigenvalues are denoted by J_1 and J_2, *i.e.*

$$\mathbf{J}_1 \circ \vec{e}_3 = J_1 \vec{e}_3; \quad \mathbf{J}_2 \circ \vec{e}_3 = J_2 \vec{e}_3$$

An actuator in the revolute joint A_1 can exert a moment $\tau \vec{e}_3$ on $\mathcal{B}^1$ and, as a consequence of *"action=−reaction"*, also a moment $-\tau \vec{e}_3$ on the fixed world. The gravitation field results in vertical forces $-m_1 g \vec{e}_2$ in M^1 and $-m_2 g \vec{e}_2$ in M^2. Furthermore there is an external, horizontal force $-F\vec{e}_1$ in A_3 on $\mathcal{B}^2$. The constraint forces in the revolute joints A_1 and A_2, required to suppress relative displacements of the connected bodies, are given by the force $\vec{F}_1$ in A_1 on $\mathcal{B}^1$, the force $\vec{F}_2$ in A_2 on $\mathcal{B}^2$ and (*"action=−reaction"*!) a force $-\vec{F}_1$ in A_1 on the fixed world and a force $-\vec{F}_2$ in A_2 on $\mathcal{B}^1$. Finally, there is a constraint force in the connection A_3. Here only the vertical displacement of $\mathcal{B}^2$ is suppressed and therefore the constraint force is a vertical force $N\vec{e}_2$ on $\mathcal{B}^i$ and a vertical force $-N\vec{e}_2$ on the fixed world.

The other forces on $\mathcal{B}^1$ and $\mathcal{B}^2$ are caused by damping in the connections. For simplicity only linear, viscous damping is considered. The rotational damper in the hinge A_1 (damper coefficient b_1) exerts a moment $-b_1\dot{\beta}_1\vec{e}_3$ on $\mathcal{B}^1$ while the rotational damper in the hinge A_2 (damper coefficient b_2) exerts a moment $-b_2(\dot{\beta}_2 - \dot{\beta}_1)\vec{e}_3$ on $\mathcal{B}^2$ and (*"action=−reaction"*) a

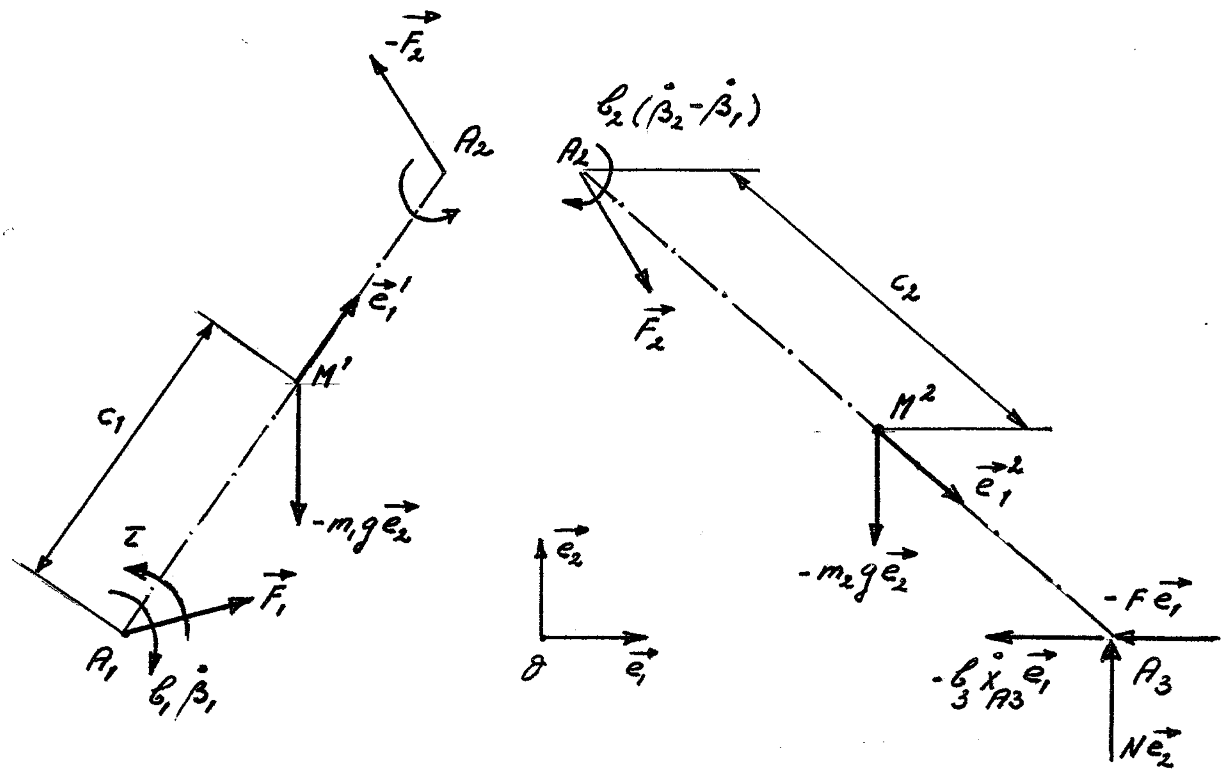

Figure 4.2: Forces and moments on the "isolated" bodies.

moment $+b_2(\dot\beta_2 - \dot\beta_1)\vec e_3$ on $\mathcal{B}^1$. The linear damper in A_3 (damper coefficient b_3) results in a force $-b_3\dot x_{A3}\vec e_1$ on $\mathcal{B}^2$, where $\dot x_{A3}$ is the horizontal velocity of A_3. Because

$$\vec x_{A3} = -a_1\vec e_1 + \ell_1\vec e_1^{\,1} + \ell_2\vec e_1^{\,2}; \quad x_{A3} = \vec x_{A3} \circ \vec e_1 = -a_1 + \ell_1 \cos(\beta_1) + \ell_2\cos(\beta_2)$$

this velocity is given by

$$\dot x_{A3} = -\ell_1\,\dot\beta_1 \sin(\beta_1) - \ell_2\,\dot\beta_2\sin(\beta_2)$$

Application of the laws of Newton and Euler to $\mathcal{B}^1$ and $\mathcal{B}^2$ results in

$$\dot{\vec p}^{\,1} \equiv m_1\,\dot{\vec v}^{\,1} = \vec F_1 - m_1\,g\,\vec e_2 - \vec F_2$$

$$\dot{\vec L}^{\,1} \equiv \mathbf{J}_1 \circ \dot{\vec\omega}^{\,1} = [\tau - b_1\dot\beta_1 + b_2(\dot\beta_2 - \dot\beta_1)]\vec e_3 - c_1\vec e_1^{\,1} * \vec F_1 - (\ell_1 - c_1)\vec e_1^{\,1} * \vec F_2$$

$$\dot{\vec p}^{\,2} \equiv m_2\,\dot{\vec v}^{\,2} = \vec F_2 - m_2\,g\,\vec e_2 - (F + b_3\dot x_{A3})\vec e_1$$

$$\dot{\vec L}^{\,2} \equiv \mathbf{J}_2 \circ \dot{\vec\omega}^{\,2} = -b_2(\dot\beta_2 - \dot\beta_1)\vec e_3 - c_2\vec e_1^{\,2} * \vec F_2 + (\ell_2 - c_2)\vec e_1^{\,2} * [N\vec e_2 - (F + b_3\dot x_{A3})\vec e_1]$$

where $\vec v^{\,1}$, $\vec\omega^{\,1}$, $\vec v^{\,2}$, $\vec\omega^{\,2}$ and the local basis vectors $\vec e_1^{\,1}$, $\vec e_2^{\,1}$ on $\mathcal{B}^1$ and $\vec e_1^{\,2}$, $\vec e_2^{\,2}$ on $\mathcal{B}^2$ follow from

$$\vec v^{\,1} = \dot{\vec x}^{\,1}; \quad \vec x^{\,1} = -a_1\vec e_1 + c_1\vec e_1^{\,1}; \quad \vec\omega^{\,1} = \dot\beta_1\vec e_3$$

$$\vec v^{\,2} = \dot{\vec x}^{\,2}; \quad \vec x^{\,2} = \ell_1\vec e_1^{\,1} + c_2\vec e_1^{\,2}; \quad \vec\omega^{\,2} = \dot\beta_2\vec e_3$$

$$\vec e_1^{\,1} = \cos(\beta_1)\vec e_1 + \sin(\beta_1)\vec e_2; \quad \vec e_2^{\,1} = -\sin(\beta_1)\vec e_1 + \cos(\beta_1)\vec e_2$$

$$\vec e_1^{\,2} = \cos(\beta_2)\vec e_1 + \sin(\beta_2)\vec e_2; \quad \vec e_2^{\,2} = -\sin(\beta_2)\vec e_1 + \cos(\beta_2)\vec e_2$$

The constraint forces $\vec F_1$ and $\vec F_2$ can be solved easily and eliminated. Then, using $\mathbf{J}_1 \circ \vec e_3 = J_1\vec e_3$ and $\mathbf{J}_2 \circ \vec e_3 = J_2\vec e_3$, finally two second order differential equations for β_1 and β_2 are found. These equations are very bulky and are not given here. However, the unknown constraint force N in the connection A_3 shows up in these equations and to arrive at a complete set the differential equations must be supplemented with the remaining constraint

$$\ell_1\sin(\beta_1) + \ell_2\sin(\beta_2) = 0$$

4.3.3 Application of the equations of Lagrange.

The formalism of Lagrange offers a very systematic approach to find the equations of motion for mechanical systems. The derivation of this formalism from the laws of Newton and Euler is omitted here but can be found in any advanced text-book on the dynamics of mechanical systems.

Central notions in this approach are the kinetic energy of the system and the virtual work, done by the forces[3] in and on the system. First attention is given to the kinetic energy and the virtual work. After that the formalism of Lagrange is given and applied in an example.

[3] In this subsection the word "forces" will be used in a generalized sense and will denote both forces and moments.

The kinetic energy of body $\mathcal{B}^i$, denoted by T_i, is given by (see, for instance, the chapter on the dynamics of one rigid body)

$$T_i = \frac{1}{2} m_i \vec{v}^{\,i} \circ \vec{v}^{\,i} + \frac{1}{2} \vec{\omega}^{\,i} \circ \mathbf{J}_i \circ \vec{\omega}^{\,i} \tag{4.5}$$

where the velocity $\vec{v}^{\,i}$ of the center of mass M^i and the angular velocity $\vec{\omega}^{\,i}$ of $\mathcal{B}^i$ are linear functions of the generalized velocities $\dot{\underset{\sim}{q}}$ and, in general, nonlinear functions of the generalized coordinates $\underset{\sim}{q}$. As a consequence, T_i is a quadratic form in the generalized velocities, *i.e.*

$$T_i = \frac{1}{2} \dot{\underset{\sim}{q}}^T \underline{M}_i \dot{\underset{\sim}{q}} \tag{4.6}$$

where $\underline{M}_i$ is the **mass matrix** of $\mathcal{B}^i$. The entries of this matrix of dimension $(n_q * n_q)$ can depend on $\underset{\sim}{q}$ but not on $\dot{\underset{\sim}{q}}$.

An eventual skew-symmetrical part of $\underline{M}_i$ does not contribute to T_i. Therefore it may be assumed, without any restriction, that $\underline{M}_i$ is symmetrical. Furthermore, $m_i \neq 0$ and $\mathbf{J}_i$ is a positive definite tensor, so $T_i > 0$ if $\vec{v}^{\,i} \neq \vec{0}$ or $\vec{\omega}^{\,i} \neq \vec{0}$. Hence, $\underline{M}_i$ is at least semi-positive definite[4]:

$$\underline{M}_i = \underline{M}_i^T; \quad \underline{M}_i \geq 0; \quad \underline{M}_i = \underline{M}_i(\underset{\sim}{q}) \tag{4.7}$$

The total kinetic energy T of the system is the sum of the kinetic energies T_1, T_2, $\cdots$, T_{n_b} of the individual bodies, *i.e.*

$$T = \sum_{k=1}^{n_b} T_k = \frac{1}{2} \dot{\underset{\sim}{q}}^T \underline{M} \dot{\underset{\sim}{q}}; \quad \underline{M} = \sum_{k=1}^{n_b} \underline{M}_i \tag{4.8}$$

where $\underline{M}$ is the **total** or **system mass matrix**. Of course, $\underline{M}$ is symmetrical and its entries can be functions of $\underset{\sim}{q}$ but not of $\dot{\underset{\sim}{q}}$. Besides, $\underline{M}$ is at least semi-positive definite. Since each $\underset{\sim}{q} \neq \underset{\sim}{0}$ results for at least one body, say body $\mathcal{B}^i$, in a velocity $\vec{v}^{\,i} \neq \vec{0}$ or $\vec{\omega}^{\,i} \neq \vec{0}$ and therefore in a kinetic energy $T_i > 0$ it is seen that $T > 0$ for each $\underset{\sim}{q} \neq \underset{\sim}{0}$. Hence, the total mass matrix $\underline{M}$ is positive definite, so

$$\underline{M} = \underline{M}^T; \quad \underline{M} > 0; \quad \underline{M} = \underline{M}(\underset{\sim}{q}) \tag{4.9}$$

The forces in the system can be divided in external forces (for instance due to gravitation or due to an actuator between a body and the environment), internal forces (for instance due to actuators between bodies, or due to friction and damping in the connections) and constraint forces. The constraint forces can be split in so-called remaining constraint forces, required to maintain remaining constraints and constraint forces, required to maintain constraints that are already taken into account in the choice of the generalized coordinates. The fundamental law of Lagrange mechanics (in fact a special version of the law of *"action=−reaction"*) states that the virtual work, done by the constraint forces of the last kind, is equal to zero for all variations $\delta \underset{\sim}{q}$ of the generalized coordinates. Furthermore, this law states that the virtual work of the remaining constraint forces is equal to zero for all kinematically admissable variations,

[4] It can not be concluded that $\underline{M}_i$ is positive definite because there may exist columns $\underset{\sim}{q} \neq \underset{\sim}{0}$ for which $\vec{v}^{\,i} = \vec{0}$ and also $\vec{\omega}^{\,i} = \vec{0}$ and therefore $T_i = 0$.

i.e. for all variations $\delta \underset{\sim}{q}$ of $\underset{\sim}{q}$, such that $\underline{K}\delta\underset{\sim}{q} = \underset{\sim}{0}$, where $\underline{K} = \underline{K}(\underset{\sim}{q}, t)$ is the derivative of $\underset{\sim}{k}_{rem} = \underset{\sim}{k}_{rem}(\underset{\sim}{q}, t)$ with respect to $\underset{\sim}{q}$. The virtual work δA_r of the remaining constraint forces for a variation $\delta\underset{\sim}{q}$ is proportional to $\delta\underset{\sim}{q}$ and can be written as

$$\delta A_r = (\delta\underset{\sim}{q})^T \underset{\sim}{Q}_r$$

The earlier mentioned fundamental law of Lagrange mechanics implies that $\delta A_r = 0$ for all kinematically admissable $\delta\underset{\sim}{q}$, *i.e.* if $\underline{K}\delta\underset{\sim}{q} = \underset{\sim}{0}$. Hence, the n_q elements of $\underset{\sim}{Q}_r$ can be written as

$$\underset{\sim}{Q}_r = \underline{K}^T(\underset{\sim}{q}, t)\underset{\sim}{\lambda} \tag{4.10}$$

where $\underset{\sim}{\lambda}$ is a column with only n_c elements.

Let δA be the work, done by all internal and external forces (*i.e.* by all forces except the constraint forces). Then δA is proportional to $\delta\underset{\sim}{q}$ and can be written as

$$\delta A = (\delta\underset{\sim}{q})^T \underset{\sim}{Q} \tag{4.11}$$

The elements of $\underset{\sim}{Q}$ are called the **generalized forces**. They can be considered as the relevant resultants of the internal and external forces.

The total virtual work δA_t of all forces is the sum of δA_r and δA, *i.e.*

$$\delta A_t = (\delta\underset{\sim}{q})^T \underset{\sim}{Q}_t; \quad \underset{\sim}{Q}_t = \underset{\sim}{Q} + \underset{\sim}{Q}_r = \underset{\sim}{Q} + \underline{K}^T \underset{\sim}{\lambda} \tag{4.12}$$

The heart of the formalism of Lagrange is given by the so-called equations of Lagrange. These equations relate partial derivatives of the kinetic energy with the generalized forces:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_{tk}, \quad k = 1, 2, \cdots, n_q \tag{4.13}$$

The left hand side of this equation, the so-called inertia term, deserves further attention. From $T = \frac{1}{2}\dot{\underset{\sim}{q}}^T \underline{M}\dot{\underset{\sim}{q}}$ and the fact that the entries of $\underline{M}$ can depend on $\underset{\sim}{q}$ but not on $\dot{\underset{\sim}{q}}$ it follows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{d}{dt}\sum_{i=1}^{n_q}(M_{ki}\dot{q}_i) - \frac{1}{2}\sum_{i=1}^{n_q}\sum_{j=1}^{n_q}\left(\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\right) =$$

$$= \frac{d}{dt}(\underline{M}\dot{\underset{\sim}{q}})_i - \frac{1}{2}\sum_{i=1}^{n_q}\left[\left(\frac{\partial \underline{M}}{\partial q_k}\dot{\underset{\sim}{q}}\right)_i \dot{q}_i\right] = (\underline{M}\ddot{\underset{\sim}{q}})_i + (\dot{\underline{M}}\dot{\underset{\sim}{q}})_i - \frac{1}{2}\sum_{i=1}^{n_q}\left[\left(\frac{\partial \underline{M}}{\partial q_k}\dot{\underset{\sim}{q}}\right)_i \dot{q}_i\right]$$

For brevity a matrix $\underline{G}$ of dimension $n_q * n_q$ is introduced, such that column i of $\underline{G}$ is equal to the partial derivative of $\underline{M}\dot{\underset{\sim}{q}}$ with respect to q_i

$$\underline{G} = [\underset{\sim}{g}_1 \;\; \underset{\sim}{g}_2 \;\; \cdots \;\; \underset{\sim}{g}_{n_q}]; \quad \underset{\sim}{g}_i = \frac{\partial \underline{M}}{\partial q_i}\dot{\underset{\sim}{q}}, \quad i = 1, 2, \cdots, n_q \tag{4.14}$$

Then it is readily seen that[5]

$$\dot{\underline{M}}\dot{\underset{\sim}{q}} = \sum_{i=1}^{n_q}\left[\left(\frac{\partial \underline{M}}{\partial q_i}\dot{\underset{\sim}{q}}\right)\dot{q}_i\right] = \underline{G}\dot{\underset{\sim}{q}}; \quad \sum_{i=1}^{n_q}\left[\left(\frac{\partial \underline{M}}{\partial q_k}\dot{\underset{\sim}{q}}\right)\right]_i \dot{q}_i = (\underline{G}^T\dot{\underset{\sim}{q}})_k \tag{4.15}$$

[5] This result does *not* mean that $\dot{\underline{M}} = \underline{G}$ because both $\dot{\underline{M}}$ and $\underline{G}$ depend on $\dot{\underset{\sim}{q}}$.

and that the inertia term can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = (\underline{M}\ddot{\underline{q}})_k + (\dot{\underline{M}}\dot{\underline{q}})_k - \frac{1}{2}(\underline{G}^T\dot{\underline{q}})_k$$

Substitution in the equations of Lagrange yields $(\underline{M}\ddot{\underline{q}})_k + (\dot{\underline{M}}\dot{\underline{q}})_k - \frac{1}{2}(\underline{G}^T\dot{\underline{q}})_k = Q_{tk}$ for $k = 1, 2, \cdots, n_q$ or, written in matrix form

$$\underline{M}\ddot{\underline{q}} + \dot{\underline{M}}\dot{\underline{q}} - \frac{1}{2}\underline{G}^T\dot{\underline{q}} = \underline{Q}_t = \underline{Q} + \underline{K}^T\underline{\lambda}$$

Because $\dot{\underline{M}}\dot{\underline{q}} = \underline{G}\dot{\underline{q}}$ it follows that $\dot{\underline{M}}\dot{\underline{q}} = \frac{1}{2}\dot{\underline{M}}\dot{\underline{q}} + \frac{1}{2}\underline{G}\dot{\underline{q}}$. This leads to the final form of the equations of motion as derived with the Lagrange formalism:

$$\underline{M}\ddot{\underline{q}} + \underline{C}\dot{\underline{q}} = \underline{q} + \underline{K}^T\underline{\lambda}; \quad \underline{C} = \frac{1}{2}\dot{\underline{M}} + \frac{1}{2}(\underline{G} - \underline{G}^T) \tag{4.16}$$

The mass matrix $\underline{M}$ is symmetrical and positive definite and therefore special procedures can be used if these differential equations have to be solved numerically. Another remarkable property of these equations concerns the matrix $\underline{C}$. It is readily seen that $\underline{C} - \frac{1}{2}\dot{\underline{M}}$ is equal to $\frac{1}{2}(\underline{G} - \underline{G}^T)$ and therefore is skew-symmetrical. This property is of eminent importance, for instance for stability proofs of computed torque and adaptive controllers.

The equations of motion form a set of n_q second order differential equations for n_q generalized coordinates and n_c constraint quantities $\underline{\lambda}$. To arrive at a complete set of equations the equations of motion must be completed with the n_c remaining constraints, *i.e.* with

$$\underline{k}_{rem}(\underline{q}, t) = \underline{0} \tag{4.17}$$

To illustrate the Lagrangian approach to derive the equations of motion again the crank-shaft mechanism of the previous subsection is considered. With the results of that subsection it is easily seen that

$$T_1 = \frac{1}{2}(c_1^2 m_1 + J_1)\dot{\beta}_1^2$$

$$T_2 = \frac{1}{2}\ell_1^2 m_2 \dot{\beta}_1^2 + \ell_1 c_2 m_2 \cos(\beta_2 - \beta_1)\dot{\beta}_1\dot{\beta}_2 + \frac{1}{2}(c_2^2 m_2 + J_2)\dot{\beta}_2^2$$

Substitution in the relation $T = T_1 + T_2$ for the total kinetic energy T yields

$$T = \frac{1}{2}\dot{\underline{q}}^T\underline{M}\,\dot{\underline{q}}; \quad \underline{q} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}; \quad \underline{M} = \begin{bmatrix} J_1 + c_1^2 m_1 + \ell_1^2 m_2 & \ell_1 c_2 m_2 \cos(\beta_2 - \beta_1) \\ \ell_1 c_2 m_2 \cos(\beta_2 - \beta_1) & J_2 + c_2^2 m_2 \end{bmatrix}$$

With this symmetrical and positive definite mass matrix as the starting point the matrices $\underline{G}$ and $\underline{C}$ can easily be determined:

$$\underline{G} = \ell_1 c_2 m_2 \sin(\beta_2 - \beta_1)\begin{bmatrix} \dot{\beta}_2 & -\dot{\beta}_2 \\ \dot{\beta}_1 & -\dot{\beta}_1 \end{bmatrix}; \quad \underline{C} = \ell_1 c_2 m_2 \sin(\beta_2 - \beta_1)\begin{bmatrix} 0 & +\dot{\beta}_2 \\ -\dot{\beta}_1 & 0 \end{bmatrix}$$

The work δA_r of by the remaining constraint force N in the connection A_3 follows from

$$\delta A_r \equiv (\delta \underline{q})^T \underline{Q}_r = \delta\vec{x}_{A3} \circ (N\vec{e}_2)$$

where $\vec{x}_{A3} = \ell_1\vec{e}_1^{\,1} + \ell_2\vec{e}_1^{\,2}$ and therefore $\delta\vec{x}_{A3} = \ell_1\delta\beta_1\vec{e}_2^{\,1} + \ell_2\delta\beta_2\vec{e}_2^{\,2}$. Hence,

$$\delta A_r \equiv (\delta \underset{\sim}{q})^T \underset{\sim}{Q}_r = \ell_1\cos(\beta_1)N\delta\beta_1 + \ell_2\cos(\beta_2)N\delta\beta_2; \quad \underset{\sim}{Q}_r = \begin{bmatrix} \ell_1\cos(\beta_1) \\ \ell_2\cos(\beta_2) \end{bmatrix} N$$

and indeed $\underset{\sim}{Q}_r = \underline{K}^T\underset{\sim}{\lambda}$ because in this example the remaining constraint is given by $\ell_1\sin(\beta_1)+\ell_2\sin(\beta_2) = 0$ and therefore $\underline{K} = [\ell_1\cos(\beta_1) \;\; \ell_2\cos(\beta_2)]$. Furthermore, $\underset{\sim}{\lambda}$ has only one element and that element can be interpreted as the normal force N, required to maintain the remaining constraint. The virtual work δA of the internal and external forces is given by

$$\begin{aligned} \delta A \equiv (\delta \underset{\sim}{q})^T \underset{\sim}{Q} &= [\tau - b_1\dot{\beta}_1 + b_2(\dot{\beta}_2 - \dot{\beta}_1)]\delta\beta_1 - m_1 g\vec{e}_2 \circ \delta\vec{x}^{\,1} + \\ &- b_2(\dot{\beta}_2 - \dot{\beta}_1)\delta\beta_2 + m_2 g\vec{e}_2 \circ \delta\vec{x}^{\,2} - (F + b_3\dot{x}_{A3})\vec{e}_1 \circ \delta\vec{x}_{A3} \end{aligned}$$

or, written in matrix form, by

$$\delta A = (\delta \underset{\sim}{q})^T \underset{\sim}{Q}$$

where the column $\underset{\sim}{Q}$ of generalized forces is defined by

$$\begin{aligned} \underset{\sim}{Q} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} - \begin{bmatrix} (b_1+b_2) & -b_2 \\ -b_2 & b_2 \end{bmatrix} \dot{\underset{\sim}{q}} - \begin{bmatrix} (c_1 m_1 + \ell_1 m_2) g\cos(\beta_1) \\ c_2 m_2 g\cos(\beta_2) \end{bmatrix} g + \\ + (F + b_3\dot{x}_{A3}) \begin{bmatrix} \ell_1\sin(\beta_1) \\ \ell_2\sin(\beta_2) \end{bmatrix} \end{aligned}$$

The final set of equations for the unknown generalized coordinates $q_1 \equiv \beta_1$ and $q_2 \equiv \beta_2$ and the unknown remaining constraint force N therefore is given by two second order differential equations (the equations of motion) plus one algebraic equation (the remaining constraint):

$$\begin{aligned} &\underline{M}\,\ddot{\underset{\sim}{q}} + \underline{C}\,\dot{\underset{\sim}{q}} = \underset{\sim}{Q} + \underset{\sim}{Q}_r \\ &\ell_1\sin(\beta_1) + \ell_2\sin(\beta_2) \equiv \ell_1\sin(q_1) + \ell_2\sin(q_2) = 0 \end{aligned}$$

4.3.4 Some concluding remarks.

The discussion in this chapter concerned the derivation of analytical expressions for the equations of motion for a multibody system with massless connections and rigid bodies. Two approaches are sketched. The approach, based on isolation of each of the rigid bodies and application of the laws of Newton and Euler, is very generally applicable. However, elimination of the constraint forces from the obtained equations often is very laborious. The approach, based on the equations of Lagrange, requires the introduction of abstract notions like the kinetic energy and the virtual work. However, as soon as the quantities are determined as functions of the generalized coordinates the rest of the derivation is very systematic and can be done by hand or by a symbolic manipulation program, like *MATHEMATICA*, *MAPLE* or *REDUCE*.

For simulation purposes often software packages like *DADS* or *ADAMS* are used. In these packages the generalized coordinates are calculated by numerical integration of the equations of motion. Several methods are developed to obtain these equations. It is not possible to discuss any of these methods here. For more information the reader is referred to the theoretical manuals and the user manuals of these software packages.

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Appendix A

Vectors and tensors.

In this appendix some essentials of the calculus of vectors and tensors are mentioned. First vectors, operations on vectors, matrix representation of vectors and vector functions are considered. Next a simular treatment of tensors is given. Finally, the last section gives a more detailed discussion on a special kind of tensors, the so-called rotation tensors.

It is noted in advance that the words "vector" and "column" have a completely different meaning: as usual, the word "column" is just an abbreviation for "matrix with one column".

For more information on the use of vectors and tensors in multibody dynamics see, for instance, Wittenburg[1].

A.1 Vectors.

A.1.1 Vectors in the Euclidian space.

A **vector** $\vec{a}$ in the three dimensional Euclidian space $\mathcal{E}$ is characterized by its direction, sense and length and can be seen as an arrow from a point $\mathcal{O}$ to a point $\mathcal{A}$. The line through $\mathcal{O}$ and $\mathcal{A}$ is the **carrier** of $\vec{a}$. The **length** of $\vec{a}$, denoted by $||\vec{a}||$, is the distance between $\mathcal{O}$ and $\mathcal{A}$ and the **direction** of $\vec{a}$ is the direction of the carrier of $\vec{a}$. Two vectors are **parallel** if their carriers are parallel.

A vector with length 0 is a **null vector**, denoted by $\vec{0}$. A vector with length 1 is a **unit vector**. Often used symbols for unit vectors are $\vec{e}$, $\vec{m}$ and $\vec{n}$.

A.1.2 Some operations on vectors.

Let $\vec{a}$ and $\vec{c}$ be two non-parallel vectors with the same origin $\mathcal{O}$. The carriers of $\vec{a}$ and $\vec{c}$ span a plane in $\mathcal{E}$. The **sum** $\vec{a}+\vec{c}$ of $\vec{a}$ and $\vec{c}$ is a vector in this plane, determined by the parallelogram-rule as depicted in *Fig.* A.1. The operation "sum of two vectors" is both commutative and associative, *i.e.* for all $\vec{a}$, $\vec{c}$ and $\vec{v}$ holds

$$\vec{a}+\vec{c}=\vec{c}+\vec{a}; \quad \vec{a}+(\vec{c}+\vec{v})=(\vec{a}+\vec{c})+\vec{v} \tag{A.1}$$

[1] *"Dynamics of Systems of Rigid Bodies"* by J. Wittenburg, B.G Teubner, Stuttgart, 1977, ISBN 3-519-12337-7

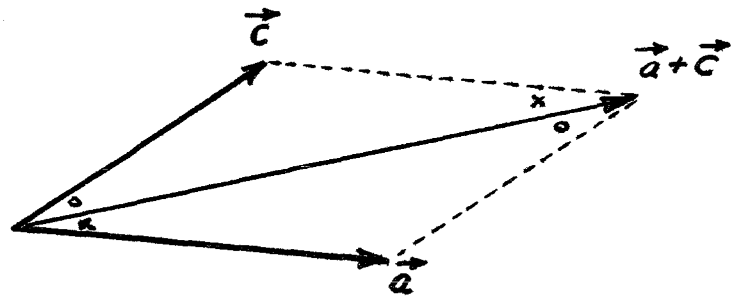

Figure A.1: Sum $\vec{a}+\vec{c}$ of two vectors $\vec{a}$ and $\vec{c}$

The **algebraic product** of $\vec{a}$ with a real number α is a vector $\alpha\vec{a}$ with the same direction as $\vec{a}$, length $\|\alpha\vec{a}\| = |\alpha|\|\vec{a}\|$ and the same (opposite) sense as $\vec{a}$ if α is positive (negative). Each vector $\vec{a}$ is the algebraic product of its length and a unit vector $\vec{e}$ with the same direction and sense as $\vec{a}$:

$$\vec{a} = \|\vec{a}\|\vec{e}; \quad \|\vec{e}\| = 1 \tag{A.2}$$

The **scalar product** or **inproduct** $\vec{a}\circ\vec{c}$ of $\vec{a}$ and $\vec{c}$ is the product of $\|\vec{a}\|$, $\|\vec{c}\|$ and $\cos(\phi(\vec{a},\vec{c}))$, where $\phi(\vec{a},\vec{c})$ is the smallest angle between $\vec{a}$ and $\vec{c}$, *i.e.*

$$\vec{a}\circ\vec{c} = \|\vec{a}\|\,\|\vec{c}\|.\cos(\phi(\vec{a},\vec{c})); \quad \phi(\vec{a},\vec{c}) \in [0,\pi) \tag{A.3}$$

For all $\vec{a}$, $\vec{c}$ and $\vec{v}$ and all real numbers α and β holds

$$\vec{a}\circ\vec{a} \geq 0; \quad \vec{a}\circ\vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}; \quad \|\vec{a}\| = (\vec{a}\circ\vec{a})^{\frac{1}{2}} \tag{A.4}$$

$$\vec{a}\circ\vec{c} = \vec{c}\circ\vec{a} \tag{A.5}$$

$$(\alpha\vec{a}+\beta\vec{c})\circ\vec{v} = \alpha(\vec{a}\circ\vec{v}) + \beta(\vec{c}\circ\vec{v}) \tag{A.6}$$

If $\vec{a}\circ\vec{c} = 0$ then $\vec{a}$ and $\vec{c}$ are said to be **orthogonal** or perpendicular.

The **vector product** of $\vec{a}$ and $\vec{c}$ is a vector, denoted by $\vec{a}*\vec{c}$ and given by

$$\vec{a}*\vec{c} = \|\vec{a}\|\,\|\vec{c}\|\sin(\phi(\vec{a},\vec{c}))\,\vec{n}(\vec{a},\vec{c}) \tag{A.7}$$

where $\vec{n}(\vec{a},\vec{c})$ is the unit vector, orthogonal to both $\vec{a}$ and $\vec{c}$ and such that $\vec{a}$, $\vec{c}$ and $\vec{n}(\vec{a},\vec{c})$ form a right hand system. The length $\|\vec{a}*\vec{c}\|$ of $\vec{a}*\vec{c}$ is equal to the area of the parallelogram, spanned by $\vec{a}$ and $\vec{c}$ (see *Fig.* A.1).

The **triple product** of $\vec{a}$, $\vec{c}$ and $\vec{v}$ is a scalar, denoted by $\vec{a}\circ\vec{c}*\vec{v}$ and given by

$$\vec{a}\circ\vec{c}*\vec{v} = (\vec{a}*\vec{c})\circ\vec{v} = \|\vec{a}\|\,\|\vec{c}\|\sin(\phi(\vec{a},\vec{c}))\vec{v}\circ\vec{n}(\vec{a},\vec{c}) \tag{A.8}$$

The absolute value $| \vec{a} \circ \vec{c} * \vec{v} |$ of this triple product is equal to the volume V of the parallelopipedum, spanned by $\vec{a}$, $\vec{c}$ and $\vec{v}$, where $\vec{a} \circ \vec{c} * \vec{v} = +V$ if $\vec{a}$, $\vec{c}$ and $\vec{v}$ form a right hand system and $\vec{a} \circ \vec{c} * \vec{v} = -V$ otherwise.

Another operation involving three vectors is the **combined vector product** $\vec{v} * (\vec{a} * \vec{c})$. This product is a vector, given by

$$\vec{v} * (\vec{a} * \vec{c}) = (\vec{v} \circ \vec{c})\vec{a} - (\vec{v} \circ \vec{a})\vec{c} \tag{A.9}$$

and therefore $\vec{v} * (\vec{a} * \vec{c})$ is a linear combination of $\vec{a}$ and $\vec{c}$.

The **dyadic product** or **dyad** of $\vec{a}$ and $\vec{c}$ is a linear map of $\mathcal{E}$ in $\mathcal{E}$, denoted by $\vec{a}\vec{c}$. The image of $\vec{v}$ under this map is a vector, denoted by $(\vec{a}\vec{c}) \circ \vec{v}$ and given by

$$(\vec{a}\vec{c}) \circ \vec{v} = (\vec{c} \circ \vec{v})\vec{a} \tag{A.10}$$

Hence, the dyad $\vec{a}\vec{c}$ maps $\vec{v}$ on a vector in the direction of $\vec{a}$.

The **sum** of two dyads $\vec{a}_1\vec{c}_1$ and $\vec{a}_2\vec{c}_2$ is a linear map $\vec{a}_1\vec{c}_1 + \vec{a}_2\vec{c}_2$, defined by

$$(\vec{a}_1\vec{c}_1) \circ \vec{v} + (\vec{a}_2\vec{c}_2) \circ \vec{v} = (\vec{a}_1\vec{c}_1 + \vec{a}_2\vec{c}_2) \circ \vec{v} \tag{A.11}$$

for all $\vec{v}$. In a simular way other operations on dyadic products can be introduced. For instance, the product $((\vec{a}_1\vec{c}_1) \circ (\vec{a}_2\vec{c}_2))$ of $\vec{a}_1\vec{c}_1$ and $\vec{a}_2\vec{c}_2$ is defined by

$$\{(\vec{a}_1\vec{c}_1) \circ (\vec{a}_2\vec{c}_2)\} \circ \vec{v} = (\vec{a}_1\vec{c}_1) \circ \{(\vec{a}_2\vec{c}_2) \circ \vec{v}\} = \vec{a}_1(\vec{c}_1 \circ \vec{a}_2)(\vec{c}_2 \circ \vec{v}) \tag{A.12}$$

for each $\vec{v}$ and it is easily seen that

$$(\vec{a}_1\vec{c}_1) \circ (\vec{a}_2\vec{c}_2) = (\vec{c}_1 \circ \vec{a}_2)(\vec{a}_1\vec{c}_2) \tag{A.13}$$

Many operations on vectors can be written in terms of dyadic products. As an example the product $\vec{v} * (\vec{a} * \vec{c})$ is considered. From

$$\vec{v} * (\vec{a} * \vec{c}) = (\vec{v} \circ \vec{c})\vec{a} - (\vec{v} \circ \vec{a})\vec{c} = (\vec{a}\vec{c} - \vec{c}\vec{a}) \circ \vec{v} \tag{A.14}$$

it is seen that $\vec{v} * (\vec{a} * \vec{c})$ is the image of $\vec{v}$ under the linear map $(\vec{a}\vec{c} - \vec{c}\vec{a})$.

Three vectors $\vec{a}$, $\vec{c}$ and $\vec{v}$ are **linear independent** if $\alpha\vec{a} + \gamma\vec{c} + \nu\vec{v} = \vec{0}$ implies that $\alpha = \gamma = \nu = 0$. This is the case if and only if these vectors are not co-planar, *i.e.* if the volume of the parallelopipedum, spanned by these vectors, is unequal 0. Hence, $\vec{a}$, $\vec{c}$ and $\vec{v}$ are linear independent if and only if $\vec{a} \circ (\vec{c} * \vec{v}) \neq 0$.

Each set $\{\vec{a}, \vec{c}, \vec{v}\}$ of three linearly independent vectors $\vec{a}$, $\vec{c}$ and $\vec{v}$ is a **vector basis** in $\mathcal{E}$. This basis is orthonormal of $\vec{a}$, $\vec{c}$ and $\vec{w}$ are orthogonal unit vectors.

A.1.3 Matrixrepresentation.

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be an orthonormal basis in $\mathcal{E}$. Then each vector $\vec{a}$ can be written as a linear combination of the basis vectors, *i.e.*

$$\vec{a} = \sum_{i=1}^{3}(a_i\vec{e}_i) \tag{A.15}$$

where $a_i = \vec{a} \cdot \vec{e}_i$ $(i = 1, 2, 3)$ is the component of $\vec{a}$ in the direction of $\vec{e}_i$. These components are the elements of a column $\underset{\sim}{a}$, the **matrix representation** of $\vec{a}$ with respect to the basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$:

$$\underset{\sim}{a} = [a_1 \;\; a_2 \;\; a_3]^T = [\vec{e}_1 \circ \vec{a} \;\; \vec{e}_2 \circ \vec{a} \;\; \vec{e}_3 \circ \vec{a}]^T \tag{A.16}$$

The representation $\underset{\sim}{a}$ of $\vec{a}$ is *not* a characteristic of $\vec{a}$ since $\underset{\sim}{a}$ changes if another basis is used. The use of vectors instead of their matrix representations is especially advantageous if more than one vector basis has to be used. This is often the case in multibody dynamics.

For compactness of notation it is quite attractive to adopt the bookkeeping system of Wittenburg. Usually, the elements of a matrix or column are integer, real or complex numbers. Wittenburg generalized this concept. In his approach the elements of a matrix or column can, for instance, be vectors or numbers. As an example, the basis vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ can be seen as the elements of a column $\underset{\sim}{\vec{e}}$:

$$\underset{\sim}{\vec{e}} = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \tag{A.17}$$

The usual rules for matrix operations, like sum and product, apply (see the book of Wittenburg). This implies that *Eq.*(A.15) for $\vec{a}$ and *Eq.*(A.16) for $\underset{\sim}{a}$ can be written in a very compact form as

$$\vec{a} = [\vec{e}_1 \; \vec{e}_2 \; \vec{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \underset{\sim}{a}^T \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}; \quad \underset{\sim}{a} = \begin{bmatrix} \vec{e}_1 \circ \vec{a} \\ \vec{e}_2 \circ \vec{a} \\ \vec{e}_3 \circ \vec{a} \end{bmatrix} = \underset{\sim}{\vec{e}} \circ \vec{a} \tag{A.18}$$

Substitution of $\underset{\sim}{a} = \underset{\sim}{\vec{e}} \circ \vec{a}$ in $\vec{a} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$ yields $\vec{a} = \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} \circ \vec{a}$, where $\underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}}$ is just a very short notation for a sum of dyads:

$$\underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} = [\vec{e}_1 \; \vec{e}_2 \; \vec{e}_3] \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3$$

A.1.4 Vectors as a function of one variable.

Let $\vec{v} : \mathcal{R} \rightarrow \mathcal{E}$ be a **vector function** of $\mathcal{R}$ in $\mathcal{E}$ and let $\underset{\sim}{v} = \underset{\sim}{\vec{e}} \circ \vec{v}$ be the matrix representation of $\vec{v}$ with respect to a *constant* vector basis $\underset{\sim}{\vec{e}} = [\vec{e}_1 \; \vec{e}_2 \; \vec{e}_3]^T$ in $\mathcal{E}$ [2]. Then the function $\vec{v}$ is

[2] In the sequel a vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ will be denoted by $\underset{\sim}{\vec{e}} = [\vec{e}_1 \; \vec{e}_2 \; \vec{e}_3]^T$, *i.e.* by the column with the basis vectors as elements.

continuous if the functions v_1, v_2 and v_3, *i.e.* the elements of the matrix representation $\underset{\sim}{v}$, are continuous. The vector function is **differentiable** if each of the elements of $\underset{\sim}{v}$ is differentiable. The derivative of $\vec{v}$ is denoted by $\frac{d\vec{v}}{dx}$ or $\vec{v}\,'$ or by $\dot{\vec{v}}$ if the independent variable is time t. It is easily seen that

$$(\alpha\vec{v})' = \alpha'\vec{v} + \alpha\vec{v}\,' \tag{A.19}$$

$$(\vec{v} \circ \vec{w})' = \vec{v}\,' \circ \vec{w} + \vec{v} \circ \vec{w}\,' \tag{A.20}$$

$$(\vec{v} * \vec{w})' = \vec{v}\,' * \vec{w} + \vec{v} * \vec{w}\,' \tag{A.21}$$

$$(\vec{v}\vec{w})' = \vec{v}\,'\vec{w} + \vec{v}\vec{w}\,' \tag{A.22}$$

for differentiable scalar functions α and differentiable vector functions $\vec{v}$ and $\vec{w}$. Other relations, for instance for triple product and sum of vector functions, can easily be derived.

If the length $\|\vec{v}(x)\|$ of $\vec{v}(x)$ is constant for all x in the neighbourhood of $x_0 \in \mathcal{R}$ then

$$\vec{v}(x_0) \circ \vec{v}\,'(x_0) = 0 \tag{A.23}$$

and it is seen that $\vec{v}(x_0)$ and the derivative $\vec{v}\,'(x_0)$ are orthogonal.

A.2 Second order tensors.

A.2.1 Tensors as a linear map.

A **second order tensor** (for short: a tensor) $\mathbf{A}$ is a **linear** map of $\mathcal{E}$ in $\mathcal{E}$. The image of $\vec{a}$ under the map $\mathbf{A}$ is denoted by $\mathbf{A} \cdot \vec{a}$ and is called the **dot product** or **inproduct** of $\mathbf{A}$ and $\vec{a}$. Since $\mathbf{A}$ is linear the inproduct $\mathbf{A} \circ (\alpha\vec{a} + \gamma\vec{c})$ will satisfy

$$\mathbf{A} \circ (\alpha\vec{a} + \gamma\vec{c}) = \alpha(\mathbf{A} \circ \vec{a}) + \gamma(\mathbf{A} \circ \vec{c}) \tag{A.24}$$

for all $\vec{a}$ and $\vec{c}$ and all real numbers α and γ.

The unit tensor $\mathbf{I}$ maps each vector on itself and the null tensor $\mathbf{0}$ maps each vector on the null vector, *i.e.*

$$\mathbf{I} \circ \vec{a} = \vec{a}; \quad \mathbf{0} \circ \vec{a} = \vec{0} \tag{A.25}$$

The dyad $\vec{a}\vec{c}$ is an example of a second order tensor because $\vec{a}\vec{c}$, operating on $\vec{v} \in \mathcal{E}$, results in the vector $(\vec{c} \circ \vec{v})\vec{a} \in \mathcal{E}$ and the map, specified by $\vec{a}\vec{c}$, is linear. For this reason the dyadic product of two vectors is often called the **tensor product** of those vectors.

A.2.2 Some operations on tensors.

The **sum** of two tensors $\mathbf{A}$ and $\mathbf{B}$ is a tensor, denoted by $\mathbf{A} + \mathbf{B}$ and such that

$$(\mathbf{A} + \mathbf{B}) \circ \vec{a} = \mathbf{A} \circ \vec{a} + \mathbf{B} \circ \vec{a} \tag{A.26}$$

for each $\vec{a}$. From this definition it follows that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}; \quad (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{A.27}$$

Let $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ be an orthonormal vector basis in $\mathcal{E}$. Then each $\vec{a} \in \mathcal{E}$ can be written as a linear combination of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, *i.e.*

$$\vec{a} = (\vec{e}_1 \circ \vec{a})\vec{e}_1 + (\vec{e}_2 \circ \vec{a})\vec{e}_2 + (\vec{e}_3 \circ \vec{a})\vec{e}_3 = (\vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \circ \vec{a} \tag{A.28}$$

Since $\vec{a} = \mathbf{I} \circ \vec{a}$ it is seen that the unit tensor $\mathbf{I}$ can be written as the sum of three dyads:

$$\mathbf{I} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3 = \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} \tag{A.29}$$

The **algebraic product** of a tensor $\mathbf{A}$ and a real number α is a tensor, denoted by $\alpha\mathbf{A}$ and such that

$$(\alpha\mathbf{A}) \circ \vec{a} = \alpha(\mathbf{A} \circ \vec{a}) \tag{A.30}$$

for each $\vec{a}$.

The **product** of two tensors $\mathbf{A}$ and $\mathbf{B}$ is a tensor, denoted by $\mathbf{A} \circ \mathbf{B}$ and such that

$$(\mathbf{A} \circ \mathbf{B}) \circ \vec{a} = \mathbf{A} \circ (\mathbf{B} \circ \vec{a}) \tag{A.31}$$

for each $\vec{a}$. Hence, $\mathbf{A} \circ \mathbf{I} = \mathbf{A}$ and $\mathbf{A} \circ \mathbf{0} = \mathbf{0}$. Furthermore,

$$\alpha(\mathbf{A} \circ \mathbf{B}) = (\alpha\mathbf{A}) \circ \mathbf{B} = \mathbf{A} \circ (\alpha\mathbf{B}) \tag{A.32}$$

$$\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) = (\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ \mathbf{B} \circ \mathbf{C} \tag{A.33}$$

for all $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and all real numbers α. In general $\mathbf{A} \circ \mathbf{B}$ is **not** equal to $\mathbf{B} \circ \mathbf{A}$!

The **transpose** of a tensor $\mathbf{A}$ is a tensor, denoted by $\mathbf{A}^T$ and such that

$$\vec{c} \circ (\mathbf{A} \circ \vec{a}) = \vec{a} \circ (\mathbf{A}^T \circ \vec{c}) \tag{A.34}$$

for each $\vec{a}$ and $\vec{c}$. Each tensor has a unique transpose and

$$(\mathbf{A}^T)^T = \mathbf{A}; \quad (\mathbf{A} \circ \mathbf{B})^T = \mathbf{B}^T \circ \mathbf{A}^T; \quad (\alpha\mathbf{A} + \beta\mathbf{B})^T = \alpha\mathbf{A}^T + \beta\mathbf{B}^T \tag{A.35}$$

for each $\mathbf{A}$ and $\mathbf{B}$ and all real numbers α and β.

The **adjoint** of a tensor $\mathbf{A}$ is a tensor, denoted by $\mathbf{A}^a$ and such that

$$\mathbf{A}^a \circ (\vec{a} * \vec{c}) = (\mathbf{A} \circ \vec{a}) * (\mathbf{A} \circ \vec{c}) \tag{A.36}$$

for each $\vec{a}$ and $\vec{c}$. Without prove it is noted that each tensor has a unique adjoint.

The **Euclidian norm** of a tensor $\mathbf{A}$ is a non-negative real number, denoted by $||\mathbf{A}||$ and determined by

$$||\mathbf{A}|| = \max_{\vec{a} \neq \vec{0}} \frac{||\mathbf{A} \circ \vec{a}||}{||\vec{a}||} \tag{A.37}$$

Other definitions of the norm are possible, but will not be considered here. Without proof it is noted that

$$||\mathbf{A}|| \geq 0; \quad ||\mathbf{A}|| = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}; \quad ||\alpha\mathbf{A}|| = | \alpha | \, ||\mathbf{A}|| \tag{A.38}$$

$$||\mathbf{A} \circ \mathbf{B}|| \leq ||\mathbf{A}|| \, ||\mathbf{B}||; \quad ||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}|| \tag{A.39}$$

for all $\mathbf{A}$ and $\mathbf{B}$ and all real numbers α. The last relation is called the triangular or Schwartz inequality.

A.2.3 Regularity, symmetry, orthogonality etc.

A tensor $\mathbf{A}$ is called **regular, invertible** or **non-singular** if $\mathbf{A} \circ \vec{a} = \vec{0}$ is satisfied only for the trivial solution $\vec{a} = \vec{0}$. If $\mathbf{A} \circ \vec{a} = \vec{0}$ has a solution $\vec{a} \neq \vec{0}$ then $\mathbf{A}$ is **singular**. For each regular tensor $\mathbf{A}$ there exists a tensor $\mathbf{B}$, such that $\mathbf{A} \circ \mathbf{B} = \mathbf{I}$. This tensor, the **inverse** of $\mathbf{A}$, is denoted by $\mathbf{A}^{-1}$. If the inverse exists then $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$. Furthermore, for regular tensors $\mathbf{A}$ and $\mathbf{B}$ the product $\mathbf{A} \circ \mathbf{B}$ is regular and

$$(\mathbf{A} \circ \mathbf{B})^{-1} = \mathbf{B}^{-1} \circ \mathbf{A}^{-1} \tag{A.40}$$

If $\mathbf{A}$ or $\mathbf{B}$ is singular then $\mathbf{A} \circ \mathbf{B}$ is also singular.

A tensor $\mathbf{A}$ is **symmetrical** if $\mathbf{A} = \mathbf{A}^T$. If $\mathbf{A} = -\mathbf{A}^T$ then $\mathbf{A}$ is **skew-symmetrical** and $\vec{a} \circ (\mathbf{A} \circ \vec{a}) = 0$ for each $\vec{a}$. Because $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ each tensor $\mathbf{A}$ is the sum of a symmetrical tensor $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and a skew-symmetrical tensor $\frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$. This decomposition is unique.

A tensor $\mathbf{A}$ is **orthogonal** if

$$(\mathbf{A} \circ \vec{a}) \circ (\mathbf{A} \circ \vec{b}) = \vec{a} \circ \vec{b} \tag{A.41}$$

for all $\vec{a}$ and $\vec{b}$. Hence, $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T \circ \mathbf{A} = \mathbf{I}$. Each orthogonal tensor $\mathbf{A}$ is regular and the inverse $\mathbf{A}^{-1}$ is equal to the transpose $\mathbf{A}^T$.

A tensor $\mathbf{A}$ is **positive definite**, respectively **semi- positive definite** if $\vec{a} \circ (\mathbf{A} \circ \vec{a}) > 0$, respectively $\vec{a} \circ (\mathbf{A} \circ \vec{a}) \geq 0$ for each $\vec{a} \neq \vec{0}$. In a simular way negative definite and semi-negative definite tensors can be defined: $\mathbf{A}$ is **negative definite**, respectively **semi- negative definite** if $\vec{a} \circ (\mathbf{A} \circ \vec{a}) < 0$, respectively $\vec{a} \circ (\mathbf{A} \circ \vec{a}) \leq 0$ for each $\vec{a} \neq \vec{0}$.

A.2.4 Invariants.

Let $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ be linear independent vectors, so $a = \vec{a}_1 \circ (\vec{a}_2 * \vec{a}_3) \neq 0$. Furthermore, let $\vec{\beta}_i$ be defined by

$$\vec{\beta}_i = \mathbf{A} \circ \vec{a}_i, \quad i = 1, 2, 3$$

The scalars $J_1(\mathbf{A})$, $J_2(\mathbf{A})$ and $J_3(\mathbf{A})$, defined by

$$J_1(\mathbf{A}) = \frac{1}{a}[\vec{\beta}_1 \circ (\vec{a}_2 * \vec{a}_3) + \vec{\beta}_2 \circ (\vec{a}_3 * \vec{a}_1) + \vec{\beta}_3 \circ (\vec{a}_1 * \vec{a}_2)] \tag{A.42}$$

$$J_2(\mathbf{A}) = \frac{1}{a}[\vec{a}_1 \circ (\vec{\beta}_2 * \circ\vec{\beta}_3) + \vec{a}_2 \circ (\vec{\beta}_3 * \circ\vec{\beta}_1) + \vec{a}_3 \circ (\vec{\beta}_1 * \circ\vec{\beta}_2)] \tag{A.43}$$

$$J_3(\mathbf{A}) = \frac{1}{a}\vec{\beta}_1 \circ (\vec{\beta}_2 * \vec{\beta}_3) \tag{A.44}$$

depend on $\mathbf{A}$ but *not* on $\vec{a}_1$, $\vec{a}_2$ or $\vec{a}_3$. Hence, $J_1(\mathbf{A})$, $J_2(\mathbf{A})$ and $J_3(\mathbf{A})$ are invariants of $\mathbf{A}$. The **first invariant** $J_1(\mathbf{A})$ is the **trace** of $\mathbf{A}$, denoted by $tr(\mathbf{A})$, the **third invariant** $J_3(\mathbf{A})$ is the **determinant** of $\mathbf{A}$, denoted by $det(\mathbf{A})$:

$$tr(\mathbf{A}) \equiv J_1(\mathbf{A}); \quad det(\mathbf{A}) \equiv J_3(\mathbf{A}) \tag{A.45}$$

With an orthonormal, right hand vector basis $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ it is easily seen that

$$J_1(\mathbf{A}) = tr(\mathbf{A}) = \sum_{i=1}^{3}\{\vec{e}_i \circ (\mathbf{A} \circ \vec{e}_i)\} \tag{A.46}$$

From the definition of the invariants a number of properties can be derived. Some of them, valid for all $\mathbf{A}$ and $\mathbf{B}$ and all real numbers α and β, are

$$tr(\mathbf{I}) = 3; \quad J_2(\mathbf{I}) = 3; \quad det(\mathbf{I}) = 1 \tag{A.47}$$

$$tr(\mathbf{A}^T) = tr(\mathbf{A}); \quad tr(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha tr(\mathbf{A}) + \beta tr(\mathbf{B}) \tag{A.48}$$

$$J_2(\mathbf{A}^T) = J_2(\mathbf{A}); \quad J_2(\alpha\mathbf{A}) = \alpha^2 J_2(\mathbf{A}) \tag{A.49}$$

$$det(\mathbf{A}^T) = det(\mathbf{A}); \quad det(\alpha\mathbf{A}) = \alpha^3 det(\mathbf{A}); \quad det(\mathbf{A} \circ \mathbf{B}) = det(\mathbf{A})det(\mathbf{B}) \tag{A.50}$$

Besides, it can be shown that for all $\vec{a}$ and $\vec{c}$ will hold

$$tr(\mathbf{A})\vec{a} * \vec{c} = \mathbf{A}^T \circ (\vec{a} * \vec{c}) + \vec{a} * (\mathbf{A} \circ \vec{c}) + (\mathbf{A} \circ \vec{a}) * \vec{c} \tag{A.51}$$

$$J_2(\mathbf{A})\vec{a} * \vec{c} = (\mathbf{A} \circ \vec{a}) * (\mathbf{A} \circ \vec{c}) + \mathbf{A}^T \circ [\vec{a} * (\mathbf{A} \circ \vec{c}) + (\mathbf{A} \circ \vec{a}) * \vec{c}] \tag{A.52}$$

$$det(\mathbf{A})\vec{a} * \vec{c} = \mathbf{A}^T \circ [(\mathbf{A} \circ \vec{a}) * (\mathbf{A} \circ \vec{c})] = \mathbf{A}^T \circ \mathbf{A}^a \circ (\vec{a} * \vec{c}) \tag{A.53}$$

From the last equation it is seen that the determinant, the transpose and the adjoint of a tensor $\mathbf{A}$ are related by

$$\mathbf{A}^T \circ \mathbf{A}^a = \mathbf{A}^a \circ \mathbf{A}^T = det(\mathbf{A})\mathbf{I} \tag{A.54}$$

Further elaboration on the above results yields the Cayley-Hamilton lemma, *i.e.*

$$\mathbf{A}^3 - tr(\mathbf{A})\mathbf{A}^2 + J_2(\mathbf{A})\mathbf{A} - det(\mathbf{A})\mathbf{I} = \mathbf{0} \tag{A.55}$$

If $det(\mathbf{A}) \neq 0$ then $(\mathbf{A} \circ \vec{a}) * (\mathbf{A} \circ \vec{c}) \neq \vec{0}$ for all $\vec{a} * \vec{c} \neq \vec{0}$. This is possible if and only if $\mathbf{A} \circ \vec{a} \neq \vec{0}$ for each $\vec{a} \neq \vec{0}$ and therefore if and only if $\mathbf{A}$ is a regular tensor. Hence, $\mathbf{A}$ is regular if and only if $det(\mathbf{A}) \neq 0$.

A.2.5 Eigenvalues and eigenvectors.

A vector $\vec{n} \neq \vec{0}$ is an **eigenvector** of $\mathbf{A}$ if $\vec{n}$ and $\mathbf{A} \circ \vec{n}$ have the same direction, *i.e.* if

$$\mathbf{A} \circ \vec{n} = \lambda \vec{n} \tag{A.56}$$

where λ is a real or complex number. The length and sense of $\vec{n}$ are not determined by this requirement: if $\vec{n}$ is an eigenvector of $\mathbf{A}$ then $\alpha\vec{n}$ (with real or complex α) is also an eigenvector of $\mathbf{A}$.

From $\mathbf{A} \circ \vec{n} = \lambda\vec{n}$ it is seen that $(\mathbf{A} - \lambda\mathbf{I}) \circ \vec{n} = \vec{0}$ and this vector equation has a non-trivial solution $\vec{n} \neq \vec{0}$ if and only if $\mathbf{A} - \lambda\mathbf{I}$ is singular, *i.e.* if and only if

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{A.57}$$

With the definition of the determinant this **characteristic equation** can be written as

$$[(\mathbf{A} - \lambda\mathbf{I}) \circ \vec{a}_1] \circ [(\mathbf{A} - \lambda\mathbf{I}) \circ \vec{a}_2] * [(\mathbf{A} - \lambda\mathbf{I}) \circ \vec{a}_3] = 0 \tag{A.58}$$

where $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are linear independent vectors. Elaboration results in a polynomial in λ with the invariants of $\mathbf{A}$ as coefficients:

$$\lambda^3 - tr(\mathbf{A})\lambda^2 + J_2(\mathbf{A})\lambda - det(\mathbf{A}) = 0 \tag{A.59}$$

The three solutions for λ are the **eigenvalues** of $\mathbf{A}$. Since the coefficients of this equation are real eventual complex eigenvalues come in conjugate pairs: if $\alpha + j\beta$ (with real α and β) is an eigenvalue then $\alpha - j\beta$ is also an eigenvalue. Hence, the number of complex eigenvalues is even and each second order tensor has at least one real eigenvalue.

Let λ_1, λ_2 and λ_3 the eigenvalues of $\mathbf{A}$. Because they are the roots of the characteristic equation this equation can be written as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0 \tag{A.60}$$

Comparison with the earlier given characteristic equation yields relations between the eigenvalues and the invariants of a tensor:

$$tr(\mathbf{A}) \equiv J_1(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 \tag{A.61}$$

$$J_2(\mathbf{A}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \tag{A.62}$$

$$det(\mathbf{A}) \equiv J_3(\mathbf{A}) = \lambda_1\lambda_2\lambda_3 \tag{A.63}$$

If $\mathbf{A}$ is singular, i.e. if $det(\mathbf{A}) = 0$, at least one of the eigenvalues of $\mathbf{A}$ is equal to 0. All eigenvalues of a regular tensor are unequal to 0.

A.2.6 Symmetrical tensors.

The eigenvalues λ_1, λ_2 and λ_3 of a symmetrical tensor $\mathbf{A} = \mathbf{A}^T$ are real. Besides, each symmetrical tensor has three real, linear independent eigenvectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$.

The eigenvectors are orthogonal if the eigenvalues are different. The proof is simple: if $\mathbf{A}$ is symmetrical then $\vec{n}_j \circ (\mathbf{A} \circ \vec{n}_i) - \vec{n}_i \circ (\mathbf{A} \circ \vec{n}_j) = (\lambda_i - \lambda_j)\vec{n}_j \circ \vec{n}_i = 0$. Therefore, $\vec{n}_i \circ \vec{n}_j = 0$, *i.e.* $\vec{n}_i$ and $\vec{n}_j$ are orthogonal, if $\lambda_i \neq \lambda_j$. If $\lambda_i = \lambda_j$ $(i \neq j)$ then $\vec{n}_i$ and $\vec{n}_j$ span a plane in $\mathcal{E}$ and any two orthogonal vectors in this plane can be taken as the eigenvectors, corresponding to the eigenvalue $\lambda_i = \lambda_j$. Hence, for a symmetrical tensor it is always possible to choose the eigenvectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ such that they are orthogonal, form a right hand system and have unit length, *i.e.* such that $\underset{\sim}{\vec{n}} = [\vec{n}_1\ \vec{n}_2\ \vec{n}_3]^T$ is an orthonormal, right hand vector basis in $\mathcal{E}$. With this basis and using $\mathbf{A} \circ \vec{n}_i = \lambda_i \vec{n}_i$ $(i = 1,2,3)$ and $\mathbf{I} = \underset{\sim}{\vec{n}}^T \underset{\sim}{\vec{n}}$ it is seen that

$$\mathbf{A} = \sum_{i=1}^{3}(\lambda_i \vec{n}_i \vec{n}_i) = \underset{\sim}{\vec{n}}^T \underline{\Lambda}\, \underset{\sim}{\vec{n}} \tag{A.64}$$

where $\underline{\Lambda}$ is a diagonal matrix with the eigenvalues as the main diagonal elements:

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{A.65}$$

This representation of $\mathbf{A}$ in terms of eigenvalues and eigenvectors is called the **spectral** or **normal representation** of $\mathbf{A}$.

If $\mathbf{A} = \mathbf{A}^T$ is regular, *i.e.* if $det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3 \neq 0$, then the inverse $\mathbf{A}^{-1}$ can be determined easily if the spectral representation of $\mathbf{A}$ is given. It can be shown that

$$\mathbf{A}^{-1} = \underset{\sim}{\vec{n}}^T \underline{\Lambda}^{-1} \underset{\sim}{\vec{n}} = \sum_{i=1}^{3}(\frac{1}{\lambda_i}\vec{n}_i \vec{n}_i) \tag{A.66}$$

Hence, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^{-1}$ are the same but the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocal of the eigenvalues of $\mathbf{A}$.

A.2.7 Skew-symmetrical tensors.

Let $\mathbf{A}$ be a skew-symmetrical tensor, *i.e.* $\mathbf{A} = -\mathbf{A}^T$. Then $\vec{a} \circ (\mathbf{A} \circ \vec{a}) = 0$ for each $\vec{a}$. Let $\vec{n}$ be a real eigenvector[3] of $\mathbf{A}$ and let λ be the associated eigenvalue, *i.e.* $\mathbf{A} \circ \vec{n} = \lambda \vec{n}$. From $\lambda \vec{n} \circ \vec{n} = \vec{n} \circ (\mathbf{A} \circ \vec{n}) = 0$ it then follows that $\lambda = 0$ and therefore that $det(\mathbf{A}) = 0$ since $det(\mathbf{A})$ is the product of the eigenvalues. Hence, each skew-symmetrical tensor is singular. Furthermore, from the definition of the invariants $J_1(\mathbf{A}) = tr(\mathbf{A})$ and $J_2(\mathbf{A})$ it is seen that $tr(\mathbf{A}) = 0$ and $J_2(\mathbf{A}) > 0$ if $\mathbf{A} = -\mathbf{A}^T$. Therefore, the characteristic equation is given by

$$\lambda[\lambda^2 + J_2(\mathbf{A})] = 0 \tag{A.67}$$

so two of the eigenvalues of a skew-symmetrical tensor are purely imaginairy.

Let $\vec{n}$ be the unit eigenvector, corresponding to the real eigenvalue $\lambda = 0$, and let $\vec{e}$ be a unit vector, orthogonal to $\vec{n}$. The vectors $\{\vec{n}$, $\vec{e}$ and $\vec{n} * \vec{e}\}$ then form an orthonormal basis in

[3] As stated earlier each second order tensor has at least one real eigenvector and corresponding real eigenvalue.

$\mathcal{E}$ and each vector $\vec{a}$ and its image $\mathbf{A} \circ \vec{a}$ can be written as a linear combination of $\vec{n}$, $\vec{e}$ and $\vec{n} * \vec{e}$. With this representation of $\vec{a}$ and $\mathbf{A} \circ \vec{a}$ it can be shown that

$$\mathbf{A} \circ \vec{a} = [(\vec{n} * \vec{e}) \circ (\mathbf{A} \circ \vec{e})] * \vec{a} \tag{A.68}$$

and that $(\vec{n} * \vec{e}) \circ (\mathbf{A} \circ \vec{e})$ does not depend on $\vec{e}$ as long as $\vec{e}$ is a unit vector orthogonal to $\vec{n}$. Hence there exists a vector $\vec{\omega}(\mathbf{A})$, the **axial vector** of $\mathbf{A} = -\mathbf{A}^T$, such that

$$\mathbf{A} \circ \vec{a} = \vec{\omega}(\mathbf{A}) * \vec{a} \tag{A.69}$$

for each $\vec{a}$. Using any right hand, orthonormal basis $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ it can be shown that

$$\vec{\omega}(\mathbf{A}) = \frac{1}{2} \sum_{i=1}^{3} [\vec{e}_i * (\mathbf{A} \circ \vec{e}_i)] \tag{A.70}$$

or, written in terms of matrixrepresentations, as

$$\vec{\omega}(\mathbf{A}) = \underset{\sim}{\vec{e}}^T \underset{\sim}{\omega}(\mathbf{A}); \quad \underset{\sim}{\omega} = \begin{bmatrix} \vec{e}_3 \circ (\mathbf{A} \circ \vec{e}_2) \\ \vec{e}_1 \circ (\mathbf{A} \circ \vec{e}_3) \\ \vec{e}_2 \circ (\mathbf{A} \circ \vec{e}_1) \end{bmatrix} \tag{A.71}$$

A.2.8 Matrixrepresentation of tensors.

Let $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ be an orthonormal basis in $\mathcal{E}$ and let $\underset{\sim}{a}$ be the matrix representation of a vector $\vec{a}$ with respect to this basis, *i.e* $\underset{\sim}{a} = \underset{\sim}{\vec{e}} \circ \vec{a}$ and $\vec{a} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$. The matrix representation $\underset{\sim}{v}$ of the image $\vec{v} = \mathbf{A} \circ \vec{a}$ then follows from

$$\underset{\sim}{v} = \underset{\sim}{\vec{e}} \circ \vec{v} = \underset{\sim}{\vec{e}} \circ \mathbf{A} \circ \underset{\sim}{\vec{e}}^T \underset{\sim}{a} = \underline{A}\, \underset{\sim}{a}$$

where the (3*3) matrix $\underline{A}$, the **matrix representation** of $\mathbf{A}$ with respect to the chosen basis, is given by

$$\underline{A} = \underset{\sim}{\vec{e}} \circ \mathbf{A} \circ \underset{\sim}{\vec{e}}^T \tag{A.72}$$

The individual entries A_{ij} $(i, j = 1, 2, 3)$ of this matrix can be determined from $A_{ij} = \vec{e}_i \circ \mathbf{A} \circ \vec{e}_j$. With $\underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} = \mathbf{I}$ it can easily be shown that

$$\mathbf{A} = \underset{\sim}{\vec{e}}^T \underline{A}\, \underset{\sim}{\vec{e}} = \sum_{i=1}^{3} \sum_{j=1}^{3} (A_{ij} \vec{e}_i \vec{e}_j) \tag{A.73}$$

This is the so-called dyadic representation of $\mathbf{A}$. It is noted that, in general, the matrix representation of $\mathbf{A}$ will change if another basis is chosen, *i.e.* that the matrix $\underline{A}$ is **not** a characteristic of the tensor $\mathbf{A}$.

A.2.9 Tensors as a function of one variable t.

Let $\mathbf{A}$ be a function of the independent variable $x \in \mathcal{R}$ and let $\vec{a}$ be a constant vector. Then the tensor function $\mathbf{A}$ is **continuous** if the vector function $\mathbf{A} \circ \vec{a}$ is continuous for each $\vec{a}$. Furthermore, this tensor function is **differentiable** if $\mathbf{A} \circ \vec{a}$ is a differentiable vector function for each $\vec{a}$.

If $\mathbf{A}$ is a differentiable tensor function then there exists another tensor function, the derivative $\mathbf{A}'$ of $\mathbf{A}$, such that

$$\mathbf{A}' \circ \vec{a} = (\mathbf{A} \circ \vec{a})' \tag{A.74}$$

For the derivative of tensor functions simular relations hold as for the derivatives of scalar functions. For example,

$$(\mathbf{A} \circ \mathbf{B})' = \mathbf{A}' \circ \mathbf{B} + \mathbf{A} \circ \mathbf{B}' \tag{A.75}$$

All tensor functions used in the sequel are assumed to be differentiable.

Let $\mathbf{A}(x)$ be regular for all $x \in \mathcal{R}$. From $\mathbf{A}(x) \circ \mathbf{A}^{-1}(x) = \mathbf{I}$ it then follows that

$$(\mathbf{A}^{-1})' = -\mathbf{A}^{-1} \circ \mathbf{A}' \circ \mathbf{A}^{-1} \tag{A.76}$$

If $\mathbf{A}(x)$ is orthogonal, *i.e.* if $\mathbf{A}(x) \circ \mathbf{A}^T(x) = \mathbf{I}$ for all $x \in \mathcal{R}$, it is seen that

$$\mathbf{A}' \circ \mathbf{A}^T = -\mathbf{A} \circ (\mathbf{A}^T)' = -\mathbf{A} \circ (\mathbf{A}')^T = -(\mathbf{A}' \circ \mathbf{A}^T)^T \tag{A.77}$$

Hence, $\mathbf{A}' \circ \mathbf{A}^T$ is skew-symmetrical and there exists a vector $\vec{\omega}$, the axial vector of $\mathbf{A}' \circ \mathbf{A}^T$, such that

$$\mathbf{A}' \circ \mathbf{A}^T \circ \vec{a} = \vec{\omega} * \vec{a} \tag{A.78}$$

for each $\vec{a}$.

Relations for the derivatives of the invariants of a tensor follow from the definitions of those invariants. Some elementary calculations result in

$$(tr(\mathbf{A}))' = tr(\mathbf{A}') \tag{A.79}$$

$$(J_2(\mathbf{A}))' = tr(\mathbf{A})tr(\mathbf{A}') - tr(\mathbf{A} \circ \mathbf{A}') \tag{A.80}$$

$$(det(\mathbf{A}))' = tr((\mathbf{A}')^T \circ \mathbf{A}^a) \tag{A.81}$$

If $\mathbf{A}$ is regular then, using the definition of the adjoint $\mathbf{A}^a$, it is seen that

$$(det(\mathbf{A}))' = det(\mathbf{A})tr(\mathbf{A}^{-1} \circ \mathbf{A}') \tag{A.82}$$

A.3 Rotation tensors.

A.3.1 Introduction.

Let $\mathbf{R}$ be an orthogonal tensor, *i.e.* $\mathbf{R}^T \circ \mathbf{R} = \mathbf{I}$ and $det(\mathbf{R})det(\mathbf{R}^T) = det^2(\mathbf{R}) = 1$. Orthogonal tensors $\mathbf{R}$ with $det(\mathbf{R}) = +1$ are called **rotation tensors**. They play a very important role in, for instance, multibody dynamics[4] since the current orientation of a moving rigid body can be described with tensors of this kind.

Rotation tensors $\mathbf{R}$ and their matrix representations $\underline{R}$ with respect to an orthonormal vector basis have some special properties. One of these properties is that the nine entries of

[4] Orthogonal tensors $\mathbf{R}$ with $det(\mathbf{R}) = -1$ are called reflection tensors. They are not considered here since they do not occur in multibody dynamics.

$\underline{R}$ can be written as functions of three independent variables or as functions of four variables, that have to satisfy one constraint condition.

In the next section first some general remarks on rotation tensors and their matrix representations are put forward. After that Euler parameters are discussed as one of the possibilities to write a rotation tensor as a function of four variables. Finally attention is focussed on Cardan angles. These angles (or simular alternatives like Euler angles) can be used if the rotation tensor has to be written as a function of three independent variables. Attention is given to the time derivative of rotation tensors and to the rotation velocity vector that is associated with this derivative.

A.3.2 Rotation tensors: some general remarks.

Many possibilities are known to describe the current orientation of a moving rigid body $\mathcal{B}$. Common in most of them is the introduction of a fixed basis $\underset{\sim}{\vec{e}}$ and a moving basis $\underset{\sim}{\vec{\varepsilon}}$, that is rigidly connected to $\mathcal{B}$. Only orthonormal, right hand bases are considered, so

$$\underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} = \mathbf{I}; \quad \underset{\sim}{\vec{\varepsilon}}^T \underset{\sim}{\vec{\varepsilon}} = \mathbf{I}$$

The current orientation of $\mathcal{B}$ can be described by the time dependent tensor $\mathbf{R}(t)$ that maps the fixed basis $\underset{\sim}{\vec{e}}$ on the moving basis $\underset{\sim}{\vec{\varepsilon}}(t)$:

$$\underset{\sim}{\vec{\varepsilon}}^T(t) = \mathbf{R}(t) \circ \underset{\sim}{\vec{e}}^T \tag{A.83}$$

Since $\underset{\sim}{\vec{e}}$ and $\underset{\sim}{\vec{\varepsilon}}$ are orthonormal, right hand vector bases $\mathbf{R}(t)$ must be a rotation tensor, so

$$\mathbf{R}(t) \circ \mathbf{R}^T(t) = \mathbf{R}^T(t) \circ \mathbf{R}(t) = \mathbf{I}; \quad det(\mathbf{R}(t)) = 1 \tag{A.84}$$

Differentiation of $\underset{\sim}{\vec{\varepsilon}}^T(t) = \mathbf{R}(t) \circ \underset{\sim}{\vec{e}}^T$ with respect to t results in

$$\dot{\underset{\sim}{\vec{\varepsilon}}}^T(t) = \dot{\mathbf{R}}(t) \circ \underset{\sim}{\vec{e}}^T = \dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) \circ \underset{\sim}{\vec{\varepsilon}}^T(t)$$

where $\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t)$ is skew-symmetrical, as can be seen by differentiation of *Eq.*(A.84):

$$\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) + \mathbf{R}(t) \circ \dot{\mathbf{R}}^T(t) = \dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) + \{\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t)\}^T = \mathbf{0}$$

Hence, there exists a vector $\vec{\omega}(t)$, the **rotation velocity vector**, such that

$$\dot{\mathbf{R}}(t) \circ \mathbf{R}^T(t) \circ \vec{c} = \vec{\omega}(t) * \vec{c} \tag{A.85}$$

for each $\vec{c}$. Therefore, the time derivative of the moving basis can be written as[5]

$$\dot{\underset{\sim}{\vec{\varepsilon}}}^T = \vec{\omega} * \underset{\sim}{\vec{\varepsilon}}^T \tag{A.86}$$

The matrix representation $\underline{R}$ of $\mathbf{R}$ with respect to the fixed basis $\underset{\sim}{\vec{e}}$ follows from

$$\underline{R} = \underset{\sim}{\vec{e}} \circ \mathbf{R} \circ \underset{\sim}{\vec{e}}^T \tag{A.87}$$

[5]It would be clean to write $\mathbf{R}(t)$, $\underset{\sim}{\vec{\varepsilon}}(t)$ etc. instead of $\mathbf{R}$, $\underset{\sim}{\vec{\varepsilon}}$ etc. Because this complicates the notation too much the dependence on t is not mentioned explicitly in the sequel unless it is required to avoid confusion.

and using $Eq.$(A.84) it is seen that

$$\underline{R}\,\underline{R}^T = \underline{R}^T\underline{R} = \underline{I} \tag{A.88}$$

This represents a set of nine scalar equations. Due to the symmetry of $\underline{R}\,\underline{R}^T$ only six of these equations are independent, so the nine entries of $\underline{R}$ have to satisfy six constraints. Because it is very inconvenient to work with nine quantities and six constraints it is common practice to write the entries of $\underline{R}$ as functions of just three independent quantities φ_1, φ_2 and φ_3 or as functions of four quantities q_0, q_1, q_2 and q_3 that have to satisfy one constraint. In literature various choices for φ_1, φ_2 and φ_3 and for q_0, q_1, q_2 and q_3 are given. Here only Euler parameters and Cardan angles are discussed in some detail. Other possibilities can be found in, for instance, the book of Wittenburg.

A.3.3 Euler parameters.

With $\mathbf{R} \circ (\mathbf{R}^T - \mathbf{I}) = \mathbf{I} - \mathbf{R}$ it is seen that $det(\mathbf{R})det(\mathbf{R} - \mathbf{I}) = -det(\mathbf{R} - \mathbf{I})$ and because $det(\mathbf{R}) = 1$ also $det(\mathbf{R} - \mathbf{I}) = 0$. From the definition of the eigenvalues it follows that $\lambda = 1$ is an eigenvalue of $\mathbf{R}$. Let $\vec{n}$ be such that

$$\mathbf{R} \circ \vec{n} = \vec{n}; \quad \vec{n} \circ \vec{n} = 1 \tag{A.89}$$

Then $\vec{n}$ is an eigenvector of $\mathbf{R}$ and since $\vec{n} = \mathbf{R}^T \circ \mathbf{R} \circ \vec{n} = \mathbf{R}^T \circ \vec{n}$ also of $\mathbf{R}^T$. The carrier of $\vec{n}$ is called the **helical axis** of $\mathbf{R}$.

Let $\vec{e}$ be a unit vector perpendicular to the helical axis, *i.e.*

$$\vec{e} \circ \vec{e} = 1; \quad \vec{e} \circ \vec{n} = 0 \tag{A.90}$$

The vector $\vec{n} * \vec{e}$ then is a unit vector perpendicular to $\vec{n}$ and $\vec{e}$, so $\{\vec{n}, \vec{e}, \vec{n} * \vec{e}\}$ is an orthonormal vector basis. Hence, the image $\mathbf{R} \circ \vec{e}$ of $\vec{e}$ can be written as

$$\mathbf{R} \circ \vec{e} = \mathbf{I} \circ (\mathbf{R} \circ \vec{e}) = \vec{n}(\vec{n} \circ \mathbf{R} \circ \vec{e}) + \vec{e}(\vec{e} \circ \mathbf{R} \circ \vec{e}) + (\vec{n} * \vec{e})[(\vec{n} * \vec{e}) \circ \mathbf{R} \circ \vec{e}]$$

and, using $\vec{n} \circ \mathbf{R} \circ \vec{e} = (\mathbf{R}^T \circ \vec{n}) \circ \vec{e} = \vec{n} \circ \vec{e} = 0$, also as

$$\mathbf{R} \circ \vec{e} = \alpha\vec{e} + \beta\vec{n} * \vec{e}; \quad \alpha = \vec{e} \circ \mathbf{R} \circ \vec{e}; \quad \beta = (\vec{n} * \vec{e}) \circ \mathbf{R} \circ \vec{e}$$

A simular treatment of $\mathbf{R} \circ (\vec{n} * \vec{e})$ results in

$$\mathbf{R} \circ (\vec{n} * \vec{e}) = \gamma\vec{e} + \nu\vec{n} * \vec{e}; \quad \gamma = \vec{e} \circ \mathbf{R} \circ (\vec{n} * \vec{e}); \quad \nu = (\vec{n} * \vec{e}) \circ \mathbf{R} \circ (\vec{n} * \vec{e})$$

The factors α, β, γ and ν have to satisfy some constraints that follow from the requirement that $\mathbf{R} \circ \vec{e}$ and $\mathbf{R} \circ (\vec{n} * \vec{e})$ are orthogonal unit vectors and that $det(\mathbf{R}) = 1$:

$$\begin{aligned}
(\mathbf{R} \circ \vec{e}) \circ (\mathbf{R} \circ \vec{e}) = \vec{e} \circ \vec{e} = 1 \quad &\Rightarrow \quad \alpha^2 + \beta^2 = 1 \\
\{\mathbf{R} \circ (\vec{n} * \vec{e})\} \circ \{\mathbf{R} \circ (\vec{n} * \vec{e})\} = (\vec{n} * \vec{e}) \circ (\vec{n} * \vec{e}) = 1 \quad &\Rightarrow \quad \gamma^2 + \nu^2 = 1 \\
(\mathbf{R} \circ \vec{e}) \circ [\mathbf{R} \circ (\vec{n} * \vec{e})] = \vec{e} \circ (\vec{n} * \vec{e}) = 0 \quad &\Rightarrow \quad \alpha\gamma + \beta\nu = 0 \\
det(\mathbf{R}) = (\mathbf{R} \circ \vec{n}) \circ [(\mathbf{R} \circ \vec{e}) * \{\mathbf{R} \circ (\vec{n} * \vec{e})\}] = 1 \quad &\Rightarrow \quad \alpha\nu - \beta\gamma = 1
\end{aligned}$$

The solution is given by $\alpha = \nu = \cos(\varphi)$ and $\beta = -\gamma = \sin(\varphi)$ and therefore

$$\mathbf{R} \circ \vec{e} = \cos(\varphi)\vec{e} + \sin(\varphi)\vec{n} * \vec{e}; \quad \mathbf{R} \circ (\vec{n} * \vec{e}) = -\sin(\varphi)\vec{e} + \cos(\varphi)\vec{n} * \vec{e}$$

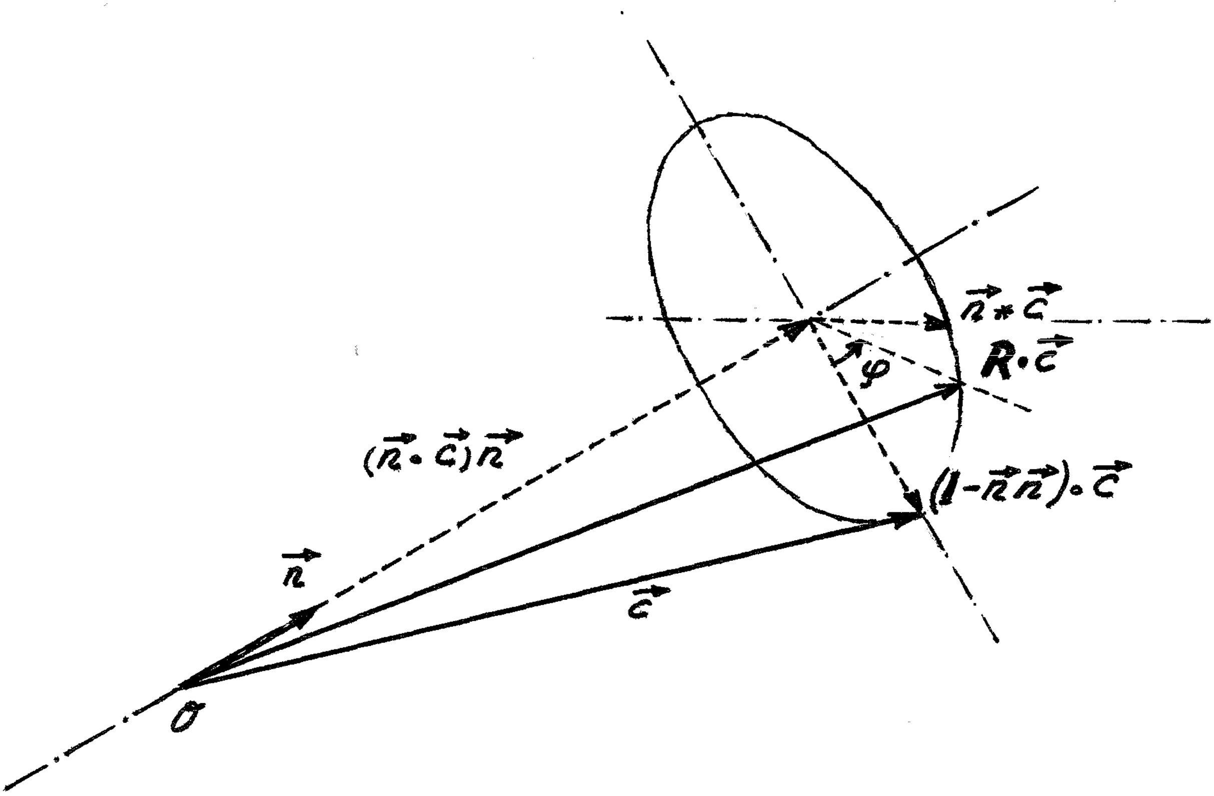

Figure A.2: Geometrical interpretation of $\mathbf{R} \circ \vec{c}$

These results can be used to determine the image $\mathbf{R} \circ \vec{c}$ of an arbitrary vector $\vec{c}$. With

$$\mathbf{R} \circ \vec{c} = \vec{n}(\vec{n} \circ \mathbf{R} \circ \vec{c}) + \vec{e}(\vec{e} \circ \mathbf{R} \circ \vec{c}) + (\vec{n} * \vec{e})[(\vec{n} * \vec{e}) \circ \mathbf{R} \circ \vec{c}]$$

this yields after some calculations

$$\mathbf{R} \circ \vec{c} = \vec{n}\vec{n} \circ \vec{c} + \cos(\varphi)(\mathbf{I} - \vec{n}\vec{n}) \circ \vec{c} + \sin(\varphi)\vec{n} * \vec{c} \tag{A.91}$$

The component $(\vec{n} \circ \vec{c})\vec{n}$ of $\vec{c}$ in the direction of $\vec{n}$, *i.e.* the component of $\vec{c}$ along the helical axis, remains unchanged while the component $\vec{c} - (\vec{n} \circ \vec{c})\vec{n} = -\vec{n} * (\vec{n} * \vec{c})$ of $\vec{c}$ perpendicular to the helical axis experiences a rotation φ in the plane perpendicular to that axis. Hence, the effect of $\mathbf{R}$ on any vector $\vec{c}$ is a rotation φ of that vector around the helical axis. This geometrical interpretation is depicted in the following figure.

From the preceding analysis it can be concluded that each rotation tensor $\mathbf{R}$ can be written as a function of the unit vector $\vec{n}$ along the helical axis and a rotation φ around that axis. In a more formal notation this can be written as

$$\mathbf{R} = \mathbf{R}(\vec{n}, \varphi), \quad \vec{n} \circ \vec{n} = 1 \tag{A.92}$$

The matrixrepresentation $\underline{R}$ of $\mathbf{R}$ with respect to the fixed basis $\underset{\sim}{\vec{e}}$ can be determined from $\underline{R} = \underset{\sim}{\vec{e}} \circ (\mathbf{R} \circ \underset{\sim}{\vec{e}}^T)$ and *Eq.*(A.91). The result is not very interesting and is omitted here.

For a given rotation tensor $\mathbf{R}$ the vector $\vec{n}$ must be determined from

$$\mathbf{R} \circ \vec{n} = \vec{n} \ \wedge \ \vec{n} \circ \vec{n} = 1$$

To determine φ it is noted that the first invariant of $\mathbf{R}$, *i.e.* $tr(\mathbf{R})$, is equal to

$$tr(\mathbf{R}) = J_1(\mathbf{R}) = 1 + 2\cos(\varphi)$$

and, as can be seen from *Eq.*(A.91) that

$$\sin(\varphi) = (\vec{n} * \vec{e}) \circ (\mathbf{R} \circ \vec{e})$$

for each unit vector $\vec{e}$ perpendicular to $\vec{n}$. These equations completely determine φ if $\vec{n}$ is given and if only values of φ in the interval $[0, 2\pi)$ are accepted.

The given analysis only proves that for every rotation tensor $\mathbf{R}$ there exist at least one unit vector $\vec{n}$ and one angle φ such that (A.91) holds for each vector $\vec{c}$. It is easily seen that neither $\vec{n}$ nor φ are unique: if (A.91) is satisfied for a given $\vec{n}$ and φ it is also satisfied if $\vec{n}$ and $\sin(\varphi)$ are replaced by -$\vec{n}$ and -$\sin(\varphi)$ or, equivalently, if $\vec{n}$ and φ are replaced by -$\vec{n}$ and $\varphi \pm \pi$. It turns out that $\vec{n}$ and φ may be replaced by $\vec{n}^*$ and φ^* as long as

$$\vec{n}^*\vec{n}^* = \vec{n}\vec{n} \quad \wedge \quad \sin(\varphi^*)\vec{n}^* = \sin(\varphi)\vec{n}$$

An important aspect of the representation *Eq.*(A.91) of $\mathbf{R}$ concerns the question whether or not this representation is unique. For an investigation all eigenvalues of $\mathbf{R}$ are considered. They follow from the characteristic equation

$$\lambda^3 - tr(\mathbf{R})\lambda^2 + J_2(\mathbf{R})\lambda - det(\mathbf{R}) = 0,$$

where $tr(\mathbf{R}) = 1 + 2\cos(\varphi)$, $det(\mathbf{R}) = 1$ and $J_2(\mathbf{R})$ is the second invariant of $\mathbf{R}$. With the definition of this invariant it can easily be shown that

$$J_2(\mathbf{R}) = tr(\mathbf{R}) = 1 + 2\cos(\varphi)$$

The characteristic equation therefore becomes $(\lambda - 1)[\lambda^2 - 2\lambda\cos(\varphi) + 1] = 0$ or, after a minor recasting,

$$(\lambda - 1)[\{\lambda - \cos(\varphi)\}^2 + \sin^2(\varphi)] = 0$$

If $\varphi \neq 2k\pi$ (integer k) then two of the eigenvalues are complex and the eigenvalue $\lambda = 1$ and the direction of the associated eigenvector $\vec{n}$ are unique. However, if $\varphi = 2k\pi$ then all eigenvalues are equal to 1 and $\mathbf{R} = \mathbf{I}$. In this case there is no rotation at all and each vector is an eigenvector of $\mathbf{R}$. This can cause problems in general purpose computer programs. That is the main reason to use the quantities $\vec{\eta}$ and q_0, defined by

$$\vec{\eta} = \sin(\frac{1}{2}\varphi)\vec{n}; \quad q_0 = \cos(\frac{1}{2}\varphi) \tag{A.93}$$

and not $\vec{n}$ and φ as parameters to characterize $\mathbf{R}$. In terms of $\vec{\eta}$ and q_0 the representation (A.91) of $\mathbf{R}$ can be written as

$$\mathbf{R} \circ \vec{c} = [2\vec{\eta}\vec{\eta} + (2q_0^2 - 1)\mathbf{I}] \circ \vec{c} + 2q_0\vec{\eta} * \vec{c} \tag{A.94}$$

The components of $\vec{\eta}$ with respect to the fixed basis $\underset{\sim}{\vec{e}}$ are denoted by q_1, q_2 and q_3, *i.e.* $q_i = \vec{\eta} \circ \vec{e}_i = \sin(\varphi/2)\vec{n} \circ \vec{e}_i$ for $i = 1, 2, 3$. Together with q_0 they are seen as the elements of a column $\underset{\sim}{q}$, *i.e.*

$$\underset{\sim}{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2) \\ \vec{\eta} \circ \vec{e}_1 \\ \vec{\eta} \circ \vec{e}_2 \\ \vec{\eta} \circ \vec{e}_3 \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2) \\ \sin(\varphi/2)\vec{n} \circ \vec{e}_1 \\ \sin(\varphi/2)\vec{n} \circ \vec{e}_2 \\ \sin(\varphi/2)\vec{n} \circ \vec{e}_3 \end{bmatrix} \tag{A.95}$$

The components of $\underset{\sim}{q}$ are called **Euler parameters**. They completely determine the matrixrepresentation $\underline{R}$ of $\mathbf{R}$ with respect to the basis $\underset{\sim}{\vec{e}}$. However, if the components of $\underline{R}$ are considered as functions of these parameters it must be realized that they are **not** independent, so

$$\underline{R} = \underline{R}(\underset{\sim}{q}); \quad \underset{\sim}{q}^T \underset{\sim}{q} = 1 \tag{A.96}$$

The rotation velocity vector $\vec{\omega}$ can be determined by differentiation of *Eq.*(A.91). With $\vec{z} = \mathbf{R} \circ \vec{c}$ and constant $\vec{c}$ this results in

$$\begin{aligned} \dot{\vec{z}} &= \dot{\mathbf{R}} \circ \vec{c} = \dot{\mathbf{R}} \circ \mathbf{R}^T \circ \vec{z} = \vec{\omega} * \vec{z} = \\ &= \dot{\varphi}[-\sin(\varphi)(\mathbf{R}^T - \vec{n}\vec{n}) \circ \vec{z} + \cos(\varphi)\mathbf{R}^T \circ (\vec{n} * \vec{z})] + \\ &\quad + [1 - \cos(\varphi)][\vec{n}(\vec{z} \circ \mathbf{R} \circ \dot{\vec{n}} + \dot{\vec{n}}(\vec{n} \circ \vec{z})] + \sin(\varphi)\dot{\vec{n}} * (\mathbf{R}^T \circ \vec{z}) \end{aligned}$$

After a fairly prolix derivation this finally yields

$$\vec{\omega} = \dot{\varphi}\vec{n} + [1 - \cos(\varphi)]\vec{n} * \dot{\vec{n}} + \sin(\varphi)\dot{\vec{n}} \tag{A.97}$$

or, written in terms of $\vec{\eta} = \sin(\varphi/2)\vec{n}$ and $q_0 = \cos(\varphi/2)$,

$$\vec{\omega} = -2\dot{q}_0\vec{\eta} + 2\vec{\eta} * \dot{\vec{\eta}} + 2q_0\dot{\vec{\eta}} \tag{A.98}$$

A.3.4 Cardan angles.

In the previous section the rotation tensor $\mathbf{R}$ is written as a function of four parameters, being the angle φ and the components of the eigenvector $\vec{n}$, where $\vec{n}$ has to satisfy the constraint $\vec{n} \circ \vec{n} = 1$. It is possible to avoid the constraint if $\mathbf{R}$ is written as a function of three suitable chosen elements φ_1, φ_2 and φ_3 of a column $\underset{\sim}{\varphi}$. Often used choices for these elements are Euler angles and Bryant or Cardan angles. Then the total rotation is seen as the result of three subsequent rotations around specified axes. The differences arise from the choice of the axes for these partial rotations.

Here only Cardan angles are considered. Then the total rotation from the fixed basis $\underset{\sim}{\vec{e}}$ to the moving basis $\underset{\sim}{\vec{\epsilon}}$, is the result of

1. a rotation φ_1 around the carrier of $\vec{e}_1$. This transforms $\underset{\sim}{\vec{e}}$ into $\underset{\sim}{\vec{e}}' = [\vec{e}_1{}' \; \vec{e}_2{}' \; \vec{e}_3{}']^T$
2. a rotation φ_2 around the carrier of $\vec{e}_2{}'$. This transforms $\underset{\sim}{\vec{e}}'$ into $\underset{\sim}{\vec{e}}^* = [\vec{e}_1^{\,*} \; \vec{e}_2^{\,*} \; \vec{e}_3^{\,*}]^T$

3. a rotation φ_3 around the carrier of $\vec{e}_3^*$. This transforms $\underset{\sim}{\vec{e}}^*$ into $\underset{\sim}{\vec{\varepsilon}}$.

Each of these partial rotations can be described with the results of the previous section. This results in

$$(\underset{\sim}{\vec{e}}')^T = \mathbf{R}(\vec{e}_1, \varphi_1) \circ (\underset{\sim}{\vec{e}})^T$$

$$(\underset{\sim}{\vec{e}}^*)^T = \mathbf{R}(\vec{e}'_2, \varphi_2) \circ (\underset{\sim}{\vec{e}}')^T$$

$$\underset{\sim}{\vec{\varepsilon}}^T = \mathbf{R}(\vec{e}_3^*, \varphi_3) \circ (\underset{\sim}{\vec{e}}^*)^T$$

or, in more detail for the individual basis vectors, in

$$\vec{e}'_i = \mathbf{R}(\vec{e}_1, \varphi_1) \circ \vec{e}_i = [1 - \cos(\varphi_1)]\vec{e}_1(\vec{e}_1 \circ \vec{e}_i) + \cos(\varphi_1)\vec{e}_i + \sin(\varphi_1)\vec{e}_1 * \vec{e}_i$$

$$\vec{e}_i^* = \mathbf{R}(\vec{e}'_2, \varphi_2) \circ \vec{e}'_i = [1 - \cos(\varphi_2)]\vec{e}'_2(\vec{e}'_2 \circ \vec{e}'_i) + \cos(\varphi_2)\vec{e}'_i + \sin(\varphi_2)\vec{e}'_1 * \vec{e}'_i$$

$$\vec{\varepsilon}_i = \mathbf{R}(\vec{e}_3^*, \varphi_3) \circ \vec{e}_i^* = [1 - \cos(\varphi_3)]\vec{e}_3^*(\vec{e}_3^* \circ \vec{e}_i^*) + \cos(\varphi_3)\vec{e}_i^* + \sin(\varphi_3)\vec{e}_3^* * \vec{e}_i^*$$

These results can also be written in matrix form as

$$(\underset{\sim}{\vec{e}}')^T = \underset{\sim}{\vec{e}}^T \underline{R}_1(\varphi_1); \quad \underline{R}_1(\varphi_1) = \underset{\sim}{\vec{e}} \circ \mathbf{R}(\vec{e}_1, \varphi_1) \circ (\underset{\sim}{\vec{e}})^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}$$

$$(\underset{\sim}{\vec{e}}^*)^T = (\underset{\sim}{\vec{e}}')^T \underline{R}_2(\varphi_2); \quad \underline{R}_2(\varphi_2) = \underset{\sim}{\vec{e}}' \circ \mathbf{R}(\vec{e}'_2, \varphi_2) \circ (\underset{\sim}{\vec{e}}')^T = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}$$

$$\underset{\sim}{\vec{\varepsilon}}^T = (\underset{\sim}{\vec{e}}^*)^T \underline{R}_3(\varphi_3); \quad \underline{R}_3(\varphi_3) = \underset{\sim}{\vec{e}}^* \circ \mathbf{R}(\vec{e}_3^*, \varphi_3) \circ (\underset{\sim}{\vec{e}}^*)^T = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $c_i = \cos(\varphi_i)$ and $s_i = \sin(\varphi_i)$ for $i = 1, 2$ and 3.

Of course it is also possible to relate $\underset{\sim}{\vec{\varepsilon}}$ directly to $\underset{\sim}{\vec{e}}$. This finally yields

$$\underset{\sim}{\vec{\varepsilon}}^T = \mathbf{R} \circ \underset{\sim}{\vec{e}}^T = (\underset{\sim}{\vec{e}})^T \underline{R}(\underset{\sim}{\varphi}); \quad \underline{R} = \underset{\sim}{\vec{e}} \circ \mathbf{R} \circ \underset{\sim}{\vec{e}}^T$$

$$\underline{R}(\underset{\sim}{\varphi}) = \underline{R}_3(\varphi_3)\underline{R}_2(\varphi_2)\underline{R}_1(\varphi_1) = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}$$

The rotation velocity vector $\vec{\omega}$ can be determined from

$$\dot{\vec{e}}'_i = \dot{\mathbf{R}}(\vec{e}_1, \varphi_1) \circ \mathbf{R}^T(\vec{e}_1, \varphi_1) \circ \vec{e}'_i + \mathbf{R}(\vec{e}_1, \varphi_1) \circ \dot{\vec{e}}_i = \dot{\varphi}_1 \vec{e}_i * \vec{e}'_i$$

$$\dot{\vec{e}}_i^* = \dot{\mathbf{R}}(\vec{e}'_2, \varphi_2) \circ \mathbf{R}^T(\vec{e}'_2, \varphi_2) \circ \vec{e}_i^* + \mathbf{R}(\vec{e}'_2, \varphi_2) \circ \dot{\vec{e}}'_i = \{\dot{\varphi}_1\vec{e}_1 + \dot{\varphi}_2\vec{e}'_2\} * \vec{e}_i^*$$

$$\dot{\vec{\varepsilon}}_i = \dot{\mathbf{R}}(\vec{e}_3^*, \varphi_3) \circ \mathbf{R}^T(\vec{e}_3^*, \varphi_3) \circ \vec{\varepsilon}_i + \mathbf{R}(\vec{e}_3^*, \varphi_3) \circ \dot{\vec{e}}_i^* = \{\dot{\varphi}_1\vec{e}_1 + \dot{\varphi}_2\vec{e}'_2 + \dot{\varphi}_3\vec{e}_3^*\} * \vec{\varepsilon}_i$$

With $\dot{\vec{\varepsilon}}_i = \vec{\omega} * \vec{\varepsilon}_i$ this yields a relation for $\vec{\omega}$ in terms of $\underset{\sim}{\varphi}$ and $\dot{\underset{\sim}{\varphi}}$:

$$\vec{\omega} = \dot{\varphi}_1 \vec{e}_1 + \dot{\varphi}_2 \vec{e}\,'_2 + \dot{\varphi}_3 \vec{e}\,^*_i = \underset{\sim}{\vec{w}}^T \dot{\underset{\sim}{\varphi}} \tag{A.99}$$

where the column $\underset{\sim}{\vec{w}}$ is defined by

$$\underset{\sim}{\vec{w}} = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vec{w}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \\ \vec{e}\,'_2 \\ \vec{e}\,^*_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \\ c_1 \vec{e}_2 + s_1 \vec{e}_3 \\ s_2 \vec{e}_1 - s_1 c_2 \vec{e}_2 + c_1 c_2 \vec{e}_3 \end{bmatrix} \tag{A.100}$$

An important problem when using Cardan angles is that it is not always possible to determine $\dot{\underset{\sim}{\varphi}}$ for a given $\vec{\omega}$. Let $\underset{\sim}{\omega} = \underset{\sim}{\vec{e}} \circ \vec{\omega}$ be the matrix representation of $\vec{\omega}$ with respect to the basis $\underset{\sim}{\vec{e}}$. Then the earlier derived vector equation for $\vec{\omega}$,

$$\vec{\omega} = (\dot{\varphi}_1 + s_2 \dot{\varphi}_3)\vec{e}_1 + (c_1 \dot{\varphi}_2 - s_1 c_2 \dot{\varphi}_3)\vec{e}_2 + (s_1 \dot{\varphi}_2 + c_1 c_2 \dot{\varphi}_3)\vec{e}_3,$$

can be transformed into three algebraic equations for $\dot{\underset{\sim}{\varphi}}$:

$$\underset{\sim}{\omega} = \underset{\sim}{\vec{e}} \circ \vec{\omega} = \underset{\sim}{\vec{e}} \circ \underset{\sim}{\vec{w}}^T \dot{\underset{\sim}{\varphi}} = \underline{W}\, \dot{\underset{\sim}{\varphi}}$$

where the matrix $\underline{W}$ is given by

$$\underline{W} = \underset{\sim}{\vec{e}} \circ \underset{\sim}{\vec{w}}^T = \begin{bmatrix} 1 & 0 & s_2 \\ 0 & c_1 & -s_1 c_2 \\ 0 & s_1 & c_1 c_2 \end{bmatrix} \tag{A.101}$$

$\underline{W}$ is singular if $det(\underline{W}) = c_2 = \cos(\varphi_2) = 0$, *i.e.* if $\varphi_2 = \pi/2 + k\pi$ (integer k). Then it is not possible to solve $\dot{\underset{\sim}{\varphi}}$ from $\underset{\sim}{\omega} = \underline{W}\dot{\underset{\sim}{\varphi}}$. This can cause problems in general purpose programs for the analysis of multibody systems and, as a consequence, Cardan angles are hardly ever used in those programs. Because simular problems occur with other sets of three independent parameters nearly all general purpose programs use Euler parameters or a variant like Rodrigues parameters.