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MEASUREMENT ERROR INFLUENCE ON HELICAL AXIS ACCURACY IN THE DESCRIPTION OF 3-D, FINITE JOINT MOVEMENT IN BIOMECHANICS

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The recent availability of a number of stereometric technologies such as ultrasonic digitization, cinephotogrammetry, Röntgenphotogrammetry, and electrogoniometry has fostered a considerable amount of research in 3-D kinematics of biomechanical joint movement. Particularly the helical (or screw-) axis concept for representing finite, spatial, rigid-body movement has acquired increased popularity. The purpose of this paper is to evaluate the accuracy of the helical axis parameters (position and direction, rotation angle and shift) if these are derived from experimentally determined, stereometric data. For this purpose, a statistical model is presented, following the work by Panjabi *et al.* (1982) which demonstrated a high sensitivity of the centroid to measurement errors in the planar case. The present analysis generalizes their results to spatial movements; it demonstrates that position and direction of the helical axis are highly error-prone, and that the landmark distribution should preferably surround the helical axis.

We assume a rigid body (a bone) applied with a right-handed, cartesian coordinate system E_x, to undergo a finite displacement from a position 1 to a position 2 (see Fig. 1). The bone has a given distribution of landmarks with known coordinates $\underline{x}_k$ in E_x ($k=1 \ldots m \geq 3$), at least 3 of which should be non-collinear for determinacy to obtain. The landmarks' centre of gravity coincides with the origin of E_x, and its position in the global (fixed) coordinate system E_y is denoted by $\underline{d}_i$ ($i=1,2$). The attitude of E_x with respect to E_y is described by the rotation matrix R_i. The experimental procedure implies the determination of each landmark position $\underline{y}_{ik}$ in the global system E_y. For errorfree $\underline{y}_{ik}$ and $\underline{x}_k$, the $\underline{d}_i$ and R_i can subsequently be determined from the model

$$\underline{y}_{ik} = \underline{d}_i + R_i\underline{x}_k \quad (i=1,2;\ k=1 \ldots m \geq 3) \tag{1}$$

using a technique such as described by Spoor & Veldpaus (1980). The helical axis representing the displacement ($\underline{d}_1 \rightarrow \underline{d}_2$, $R_1 \rightarrow R_2$) is characterized in E_y (see Fig. 2) by a unit direction vector $\underline{n}$ and by the position vector $\underline{s}$ of some point on the helical axis; the rotation angle about the axis is denoted by θ, and the shift along the axis by t. Following Spoor & Veldpaus (1980), $\underline{n}$ and θ can be derived from

$$\sin\theta \cdot S(\underline{n}) = \tfrac{1}{2}(R - R'), \text{ with } R \triangleq R_2R_1', \ \underline{n}'\underline{n} = 1, \ 0 < \theta < \pi \text{ rad} \tag{2}$$

and $S(\underline{a})$ a *skew-symmetric* matrix uniquely defined by the property $S(\underline{a})\cdot\underline{b} \triangleq \underline{a} * \underline{b}$ for arbitrary $\underline{a}$ and $\underline{b}$, and $*$ denoting the vector product operator (NB: ' denotes transposition). Contrary to Spoor & Veldpaus (1980), $\underline{s}$ is chosen to be the projection onto the helical axis of the midpoint $\underline{p}$ on the differential translation $\underline{d}$ between $\underline{d}_1$ and $\underline{d}_2$. It appears that $\underline{s} - \underline{p}$ is the shortest line between $\underline{d}$ and the helical axis, so $\underline{s}$ is presumably minimally sensitive to measurement errors, for all points on the helical axis. From the condition $\underline{n}'(\underline{s} - \underline{p}) = 0$, $\underline{s}$ and t now follow as

$$\underline{s} = \underline{p} + \{2\tan(\tfrac{1}{2}\theta)\}^{-1}\cdot \underline{n} * \underline{d}, \quad t = \underline{n}'\underline{d} \tag{3}$$

where

$$\underline{p} \triangleq \tfrac{1}{2}(\underline{d}_1 + \underline{d}_2), \quad \underline{d} \triangleq \underline{d}_2 - \underline{d}_1 \tag{4}$$

The *observed* landmark coordinates $\underline{y}^o_{ik}$ are modelled as subject to measurement errors $\underline{n}_{ik}$ (i.e., $\underline{y}^o_{ik} = \underline{y}_{ik} + \underline{n}_{ik}$), and the $\underline{n}_{ik}$ are modelled as zero-mean ('unbiased'), uncorrelated, and isotropic, with covariance matrix $\sigma^2 I$. The question is now how these errors propagate in statistically optimal estimates $\hat{\underline{d}}_i$ and $\hat{R}_i$ for $\underline{d}_i$ and R_i, respectively, and subsequently in the helical axis parameters.

The effect on $\underline{d}_i$ is denoted by an additive term $\Delta\underline{d}_i$, and the effect on R_i may be modelled by means of a rotation matrix $R(\Delta\underline{\phi}_i)$ which premultiplies the true matrix R_i. For reasonably small errors $\underline{n}_{ik}$, the errors $\Delta\underline{d}_i$ and $\Delta\underline{\phi}_i$ will be unbiased: $E(\Delta\underline{d}_i) = \underline{0}$ and $E(\Delta\underline{\phi}_i) \cong \underline{0}$. To a first order approximation, $\Delta\underline{\phi}_i$ may be viewed as a vector describing small rotations about the axes of E_y. Using a minimum variance (*Markov*) estimator, the covariance matrices for $\Delta\underline{d}_i$ and $\Delta\underline{\phi}_i$ may be shown to be

$$\Sigma_{\Delta\underline{d}_i} = \sigma^2/m \cdot I, \quad \Sigma_{\Delta\underline{\phi}_i} = \sigma^2\left(\sum_{k=1}^{m}\{R_i(\underline{x}_k'\underline{x}_k \cdot I - \underline{x}_k\underline{x}_k')R_i'\}\right)^{-1} \tag{5}$$

For the present noise model, the algorithm of Spoor & Veldpaus (1980) is suitable, but some form of iterative adjustment may be required if the noise distribution is anisotropic. For certain landmark distributions, the rotation error covariance matrix in (5) assumes diagonal form, with equal diagonal elements. A case in point is the situation that the landmarks are at the vertices of a regular polyhedron; it is contended without proof that the rotation error covariance matrix for such a landmark distribution becomes

$$\Sigma_{\Delta\underline{\phi}_i} = \sigma^2/(\rho^2 m)\cdot I, \text{ with } \rho^2 = \tfrac{2}{3}\cdot r^2 \tag{6}$$

and where r is the radius of the sphere circumscribing the polyhedron, e.g., a pyramid ($m=4$), a cube ($m=6$), etc. For more general distributions meeting (6), ρ is the effective landmark distribution radius.

The variances and covariance matrices of the helical parameter estimates follow from (5 - 6), and from the partial derivatives of θ, t, $\underline{n}$, and $\underline{s}$ to the $\Delta\underline{d}_i$ and $\Delta\underline{\phi}_i$. Defining χ as the angle between $\underline{n}$ and $\underline{d}$, this yields after some matrix calculus

$$\sigma^2_\theta = 2\sigma^2/(m\rho^2), \quad \Sigma_{\underline{n}} = \sigma^2_\theta\cdot\{2\sin(\tfrac{1}{2}\theta)\}^{-2}\cdot(I - \underline{n}\,\underline{n}') \tag{7}$$

$$\sigma^2_t = 2\sigma^2/m\cdot\left(1 + \{\cos(\tfrac{1}{2}\theta)\}^{-2}\cdot|(\underline{s}-\underline{p})/\rho|^2\right) \tag{8}$$

$$\begin{aligned}\Sigma_{\underline{s}} = {} & \tfrac{1}{2}\sigma^2/m\cdot I + 2\sigma^2/m\cdot\{2\tan(\tfrac{1}{2}\theta)\}^{-2}\cdot(I - \underline{n}\,\underline{n}') + \\ & \sigma^2/(m\rho^2)\cdot\left(2\{2\sin(\tfrac{1}{2}\theta)\}^{-2}\cdot(\sin\chi)^{-2}\cdot(\underline{s}-\underline{p})'(\underline{s}-\underline{p})\cdot(I - \underline{d}\,\underline{d}'/\underline{d}'\underline{d}) + \right. \\ & \left. \tfrac{1}{2}\{\cos(\tfrac{1}{2}\theta)\}^{-2}\cdot(\underline{s}-\underline{p})\cdot(\underline{s}-\underline{p})'\right)\end{aligned} \tag{9}$$

For small rotation angles $|\theta| \ll 1$ rad, (3) and (7 - 9) may be simplified by replacing the circular functions with their small-angle approximations. Apparently, the measurement errors have a profound effect on most of the helical parameters if θ is small. Other things being equal, the helical errors are minimal if the landmarks' centre of gravity is on the helical axis, but even then, some of them may be inordinately large. For $|\theta| \ll 1$ rad and $|\underline{s}-\underline{p}| = 0$, (7 - 9) result in

$$\sigma_\theta = \sqrt{2/m}\,\sigma/\rho, \quad \sigma_{\underline{n}} \triangleq \sqrt{\mathrm{Tr}(\Sigma_{\underline{n}})} \simeq 2\sigma/(\rho\theta\sqrt{m}) \tag{10}$$

$$\sigma_t \simeq \sqrt{2/m}\,\sigma, \quad \sigma_{\underline{s}} \triangleq \sqrt{\mathrm{Tr}(\Sigma_{\underline{s}})} \simeq 2\sigma/(\theta\sqrt{m})\cdot\{1 + t^2/(\theta\rho)^2\}^{1/2} \tag{11}$$

As clarified in the Figures 3 and 4, $\sigma_{\underline{n}}$ denotes the direction error standard deviation of $\underline{n}$, and $\sigma_{\underline{s}}$ the mean standard deviation of $\underline{s}$. Taking typical values in wrist Röntgenphotogrammetry as an example, one might choose $\sigma = 50\,\mu m$, $m=4$, $\rho = 5$ mm, $t = |\underline{d}| = 0$ mm, and $\theta = 0.1$ rad $= 5.7°$. This results in $\sigma_\theta = 0.41°$, $\sigma_{\underline{n}} = 5.7°$, $\sigma_t = 35\,\mu m$, and $\sigma_{\underline{s}} = 0.5$ mm. If $|\underline{s}-\underline{p}| \gg |\rho\sin\chi|$, the errors affecting t and $\underline{s}$ are much higher. Thus, ρ and $\sqrt{m}/\sigma$ should be sufficiently large, and the landmark distribution should be sufficiently close to the helical axis. Furthermore, the large errors incurred for small θ demonstrate the limited utility of the finite helical axis as an approximate sample for the instantaneous helical axis.

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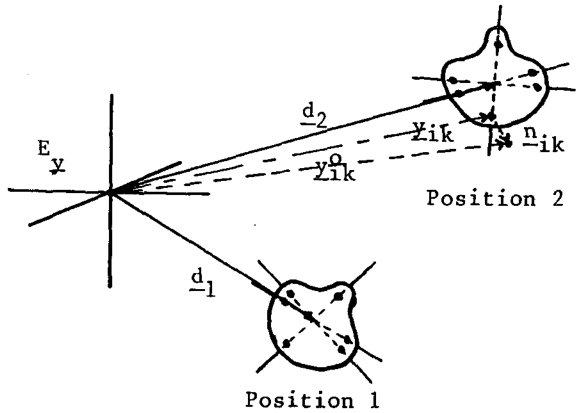

Figure 1. Global (fixed) coordinate system $E_{\underline{y}}$ in which a rigid body with attached coordinate system $E_{\underline{x}}$ moves from position 1 to position 2. The body has a given landmark distribution with known coordinates in $E_{\underline{x}}$, and true coordinates $\underline{y}_{ik}$ in $E_{\underline{y}}$, with $i = 1,2$, $k = 1 \ldots m \geq 3$. Due to measurement errors $\underline{n}_{ik}$, only *observed* coordinates $\underline{y}^o_{ik}$ are available. See text for definition of other symbols.

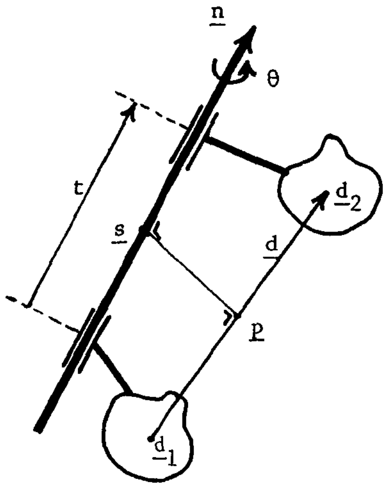

Figure 2. Helical axis in the finite displacement case. By virtue of the vector-product nature of (3), $\underline{s} - \underline{p}$ is normal to both $\underline{n}$ and $\underline{d}$. By consequence, $\underline{s} - \underline{p}$ is the shortest distance vector interconnecting $\underline{d}$ and the helical axis. See text for the definition of other symbols.

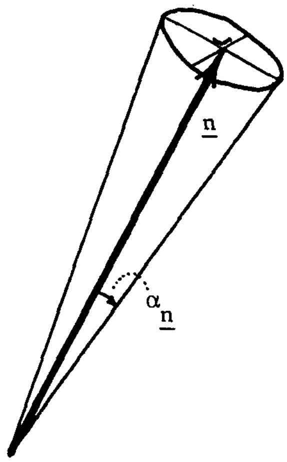

Figure 3. Angular uncertainty in the direction vector $\underline{n}$. The errors in $\underline{n}$ are constrained to the normal plane of $\underline{n}$. For small errors, the direction uncertainty angle $\alpha_{\underline{n}}$ has mean value zero, and standard deviation $\sigma_{\underline{n}}$ as defined in (10).

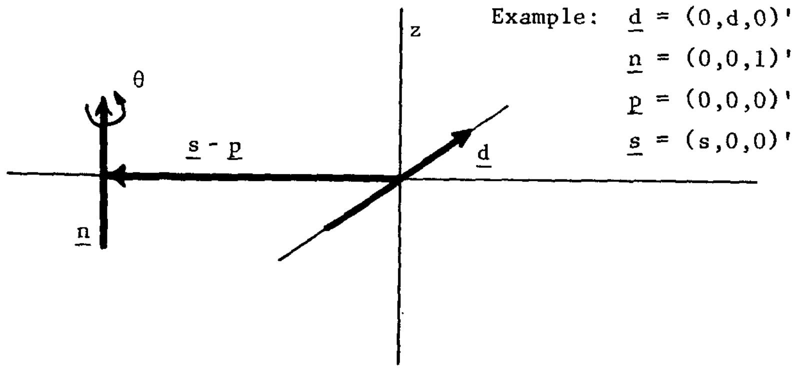

Figure 4. Covariance contributions in $\Sigma_{\underline{s}} \triangleq \Sigma_{\underline{s}|\underline{p}} + \Sigma_{\underline{s}|\underline{d}} + \Sigma_{\underline{s}|R}$, for the given assumptions on noise and landmark distributions.

1) $\Sigma_{\underline{s}|\underline{p}} = \frac{1}{2}\sigma^2/m \cdot I$, spatially isotropic, due to errors in $p \triangleq \frac{1}{2}(\underline{d}_1 + \underline{d}_2)$.

2) $\Sigma_{\underline{s}|\underline{d}} = 2\sigma^2/m \cdot \{2\tan(\frac{1}{2}\theta)\}^{-2} \cdot (I - \underline{n}\,\underline{n}')$, isotropic in the normal plane of $\underline{n}$, due to errors in $\underline{d} \triangleq \underline{d}_2 - \underline{d}_1$.

3) $\Sigma_{\underline{s}|R} = \sigma^2/(\rho^2 m) \cdot \big(2\{2\sin(\frac{1}{2}\theta)\}^{-2}(\sin\chi)^{-2} \cdot (\underline{s} - \underline{p}) \cdot (\underline{s} - \underline{p}) \cdot (I - \underline{d}\,\underline{d}'/\underline{d}'\underline{d}) + \frac{1}{2}\{\cos(\frac{1}{2}\theta)\}^{-2} \cdot (\underline{s} - \underline{p}) \cdot (\underline{s} - \underline{p})' \big)$, anisotropic in the normal plane of $\underline{d}$, due to errors in the rotation matrix $R \triangleq R_2 R_1'$.

For sufficiently small measurement errors, the errors in $\underline{s}$ are zero-mean, with mean standard deviation $\sigma_{\underline{s}}$ averaged over all directions as defined in (11). For the 'optimal' case $\underline{s} = \underline{p}$, $\Sigma_{\underline{s}|R} = 0$, and the errors in $\underline{s}$ for small θ are mainly located in the normal plane of $\underline{n}$, since $Tr(\Sigma_{\underline{s}|\underline{d}}) \gg Tr(\Sigma_{\underline{s}|\underline{p}})$ if $|\theta| \ll 1$.