Approximation numbers

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Approximation Numbers
Extended Notes of a Lecture by A.A. Melkman

by

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1. Summary.

The problem is to approximate, in the p-norm, the unit matrix I_m of size m by rank n matrices, $n \le m$, that is, to determine the numbers

$$a_n = a_n(I_m;\ \ell_1^m,\ \ell_p^m) := \min_{A \in \mathcal{A}_n} \max_{\|\underline{x}\|_1 = 1} \| (I_m - A)\underline{x} \|_p ,$$

where $\mathcal{A}_n$ denotes the set of all $m \times m$ matrices of rank $\le n$.
Since only the extremes of the octahedron $\sum_{i=1}^{m} |x_i| \le 1$ are of importance this is equivalent to

$$a_n = \min_{A \in \mathcal{A}_n} \max_{1 \le k \le m} \| \underline{e}_k - A\underline{e}_k \|_p ,$$

where $\underline{e}_1, \ldots, \underline{e}_m$ denote the columns of I_m. In terms of subspaces $X_n \subset \mathbb{R}^m$ this is equivalent to

$$a_n = \min_{X_n \subset \mathbb{R}^m} \max_{1 \le k \le m} \min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_p .$$

It follows from a theorem by Sofman [5] that

$$a_n (I_m;\ \ell_1^m,\ \ell_2^m) = \sqrt{1 - \frac{n}{m}} .$$

Melkman [1] proves that for $p \geqslant 2$

$$a_n(I_m;\ \ell_1^m,\ \ell_p^m) \geqslant \left[1 + (m-1) \left[\frac{(m-1)(m-n)}{n}\right]^{-\frac{p}{2(p-1)}} \right]^{-1 + \frac{1}{p}}$$

For the case $p = \infty$ this amounts to

$$a_n (I_m ; \ell_1^m , \ell_\infty^m) \geqslant \left(1 + \sqrt{\frac{(m - 1)n}{m - n}}\right)^{-1}$$

Moreover, Melkman shows that for $p \neq 2$ equality holds if and only if there exists a regular two-graph on m vertices with the multiplicities n and m - n.

2. Eutactic stars ([4] ,[3])

Let X_m denote a real inner product space with an orthonormal basis $\underline{e}_1,\ldots,\underline{e}_m$. Let $P : X_m \rightarrow X_m$ be a projection operator (linear, symmetric, idempotent), and let $PX_m = X_n$. Then, for $k,\ell = 1,\ldots,m$,

$$(\underline{e}_k, P\underline{e}_\ell) = (P\underline{e}_k, P\underline{e}_\ell) = (P\underline{e}_k, \underline{e}_\ell),$$

hence for the vectors $P\underline{e}_1,\ldots,P\underline{e}_m \in X_n$ the Gram matrix equals the coordinate matrix with respect to $\underline{e}_1,\ldots \underline{e}_m$. This matrix is symmetric and idempotent, has trace n hence

$$\sum_{k=1}^{m} (P\underline{e}_k, P\underline{e}_k) = n .$$

By definition, the vectors $P\underline{e}_1,\ldots,P\underline{e}_m$ constitute a eutactic star. This star is spherical whenever all $(P\underline{e}_k,P\underline{e}_k)$ are equal. If, in addition, $|(P\underline{e}_k, P\underline{e}_\ell)|$ is constant for all $k \neq \ell$, then the lines spanned by $P\underline{e}_1,\ldots,P\underline{e}_m$ constitute a set of m equiangular lines in X_n at $\cos\varphi = \sqrt{\frac{m - n}{n(m - 1)}}$. Such a set is extremal in the following sense. For any set of m equiangular lines in $\mathbb{R}^n$ at angle ψ, let G denote the Gram matrix of a set of m unit vectors, one along each of the lines. Then $G = I + C \cos\psi$, where C is a symmetric matrix of size m with diagonal zero and entries ± 1 elsewhere. Since G is positive semidefinite of rank n, its nonzero eigenvalues $\lambda_1,\ldots,\lambda_n$ satisfy

$$m = \mathrm{tr}\, G = \lambda_1 + \ldots + \lambda_n \; , \; m + m(m - 1) \cos^2 \psi = \mathrm{tr}\, G^2 = \lambda_1^2 + \ldots + \lambda_n^2 \; ,$$

hence $\frac{m^2}{n} \leqslant m + m(m - 1) \cos^2\psi$, $\frac{m - n}{n(m - 1)} \leqslant \cos^2\psi$.

Equality holds iff $\lambda_1 = \ldots = \lambda_n$, that is, iff C has just two eigenvalues, of multiplicities n and m - n.

A triple of equiangular lines is of acute or obtuse type, according as the lines are spanned by a triple of equiangular vectors at acute or at obtuse angle. An extremal set of m equiangular lines in $\mathbb{R}^n$ is characterized by the property that each pair of lines is in a constant number of triples of obtuse type. This is equivalent to the existence of a regular two-graph on n vertices, whose eigenvalues have the multiplicities n and m - n. For the definition and a survey cf. [3] .

3. Sofman's theorem ([5], see also [1]).

Theorem. Let $\underline{e}_1,\ldots,\underline{e}_m$ be an orthonormal basis of X_m.

The conditions $\sum_{k=1}^{m} \xi_k^2 = n$, $0 \le \xi_k \le 1$ are necessary and sufficient for the existence of a subspace X_n of X_m such that, for $k = 1,\ldots,m$, the projection of $\underline{e}_k$ onto X_n has length ξ_k.

Proof. The necessity of the condition has been observed in 2. For the sufficiency we use induction on n.

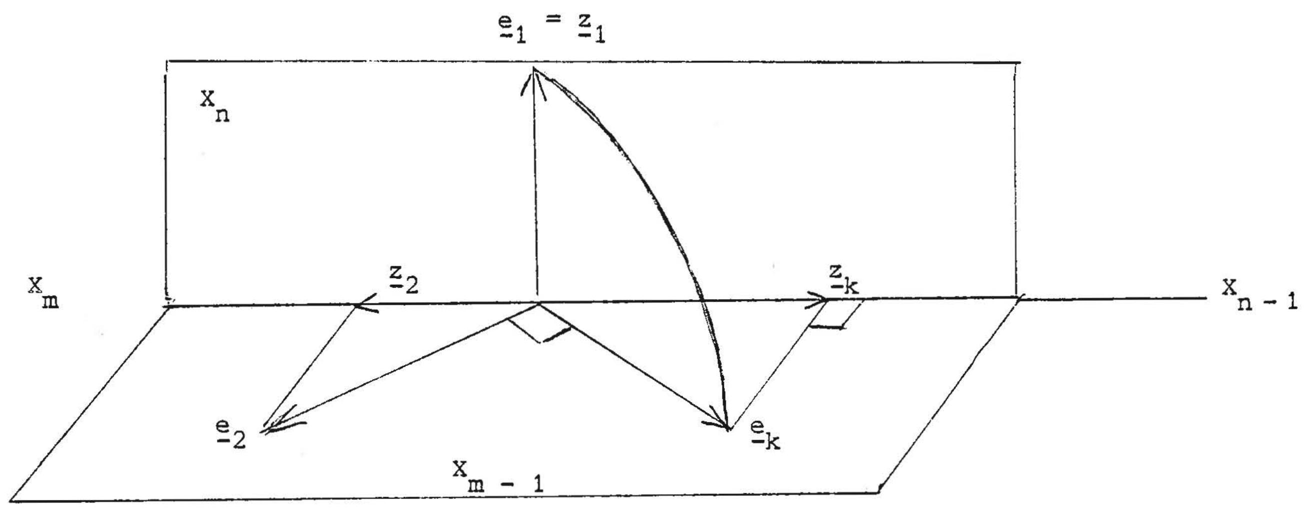
Suppose, for $k = 1,\ldots,m$.

$$0 \le \xi_k \le 1, \quad \max_k \ \xi_k \le \xi_1, \quad \xi_1^2 + \xi_2^2 + \ldots + \xi_m^2 = n .$$

Then $\sum_{k=2}^{m} \xi_k^2 = n - \xi_1^2 \ge n - 1$, so we may choose $\eta_1,\ldots,\eta_m$ such that $\eta_1 = 0$, $0 \le \eta_2 \le \xi_2,\ldots,0 \le \eta_m \le \xi_m$, $\sum_{k=1}^{m} \eta_k^2 = n - 1$.

By the induction hypothesis there exists $X_{n-1} \subset X_m$ such that, for $k = 1,\ldots,m$, the projection of $\underline{e}_k$ onto X_{n-1} has length η_k . Then $X_{n-1} \perp \underline{e}_1$. We define $X_n := \langle \underline{e}_1 \rangle \oplus X_{n-1}$. Then, for $k = 1, \ldots, m$, the projections $\underline{z}_k$ of $\underline{e}_k$ onto X_n have lengths

$$\zeta_1 = 1,\ \zeta_2 = \eta_2,\ldots,\zeta_m = \eta_m, \text{ and } \sum_{k=1}^{m} \zeta_k^2 = n .$$



We wish to rotate $\underline{e}_1,\dots,\underline{e}_m$ so as to move step by step from $\zeta_1,\dots\zeta_m$ to $\xi_1,\dots,\xi_m$. Take any $k = 2,\dots,m$, and consider a rotation about α in the plane $\langle \underline{e}_1, \underline{e}_k \rangle$, leaving all other $\underline{e}_i$ fixed:

$$\underline{e}_1(\alpha) = \underline{e}_1 \cos\alpha - \underline{e}_k \sin\alpha,\ \underline{e}_k(\alpha) = \underline{e}_1 \sin\alpha + \underline{e}_k \cos\alpha,\ \underline{e}_i(\alpha) = \underline{e}_i \ .$$

Then $\zeta_j(\alpha) := \| P(\underline{e}_j(\alpha)) \|$ satisfy $\zeta_1^2(\alpha) + \zeta_k^2(\alpha) = \zeta_1^2 + \zeta_k^2$, and $\zeta_i(\alpha) = \zeta_i$ for $i \neq 1,k$. Since $\zeta_k(\alpha)$ is an increasing function of α from $\zeta_k(0) = \zeta_k$ to $\zeta_k(\frac{\pi}{2}) = \zeta_1 = 1$, and since $1 = \zeta_1 \geqslant \xi_1 \geqslant \xi_k \geqslant \zeta_k$, we may choose α such that $\zeta_k(\alpha) = \xi_k$. Then $\zeta_i(\alpha) = \zeta_i \leq \xi_i$ implies

$$\sum_{i=1}^{m} \zeta_i^2(\alpha) = n = \sum_{i=1}^{m} \xi_i^2 \ \text{ and } \zeta_i \leq \xi_1 \leq \zeta_1(\alpha) \ .$$

Thus we have made ζ_k into ξ_k , and $\zeta_1(\alpha)$ is again the largest number . Now repeat the process with each of the indices $\neq 1$, so as to arrive at an orthonormal basis whose projections onto X_n have given lengths $\xi_2,\dots,\xi_m$. Then also the first length fits with ξ_1. This finishes the induction step from $n - 1$ to n. For $n = 1$ the theorem is true, since then the line spanned by the vector $(\xi_1,\dots,\xi_m)$ applies. Thus the proof is completed.
The theorem may be rephrased in the following ways.

Corollary. Necessary and sufficient for the existence of a eutactic star in X_n consisting of m vectors at lengths $\xi_1,\dots,\xi_m$ are $\xi_1^2 + \dots + \xi_m^2 = n,\ 0 \leqslant \xi_k \leqslant 1$.

Corollary. Necessary and sufficient for the existence of a symmetric idempotent matrix C are:

$$\text{trace } C = \text{rank } C \ , \ \text{diag } C \geqslant \underline{0}$$

In particular, taking $\xi_1 = \ldots = \xi_m = \sqrt{\frac{n}{m}}$ we have :

Corollary. X_n contains spherical eutactic stars of any cardinality m.

Corollary. Given $m, n \in \mathbb{N}$, $m \geq n$, there exists a symmetric zero-diagonal matrix of size m whose only eigenvalues are n and n - m.

Corollary. $a_n(I_m \ ; \ \ell_1^m \ , \ \ell_2^m) = \sqrt{1 - \frac{n}{m}}$.

Proof. For the ℓ_2-norm we know

$$\min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \| = \| \underline{e}_k - P\underline{e}_k \| \ .$$

Now
$$\| P\underline{e}_1 \|^2 + \ldots + \| P\underline{e}_m \|^2 = n \ ,$$

$$\| \underline{e}_1 - P\underline{e}_1 \|^2 + \ldots + \| \underline{e}_m - P\underline{e}_m \|^2 = m - n$$

implies

$$\max_{1 \leq k \leq m} \| \underline{e}_k - P(\underline{e}_k) \| \geqslant \sqrt{1 - \frac{n}{m}} \ .$$

By Sofman's theorem, an X_n with $\| P\underline{e}_1 \| = \ldots = \| P\underline{e}_m \| = \sqrt{\frac{n}{m}}$ really exists, hence for such X_n :

$$\max_{1 \leq k \leq m} \| \underline{e}_k - P(\underline{e}_k) \| = \sqrt{1 - \frac{n}{m}} \ . \qquad \square$$

Remark. Any X_n, for which

$$\max_{1 \leq k \leq m} \| \underline{e}_k - P\underline{e}_k \| = \sqrt{1 - \frac{n}{m}} \ ,$$

satisfies $\| P\underline{e}_1 \| = \ldots = \| P\underline{e}_m \| = \sqrt{\frac{n}{m}}$. Indeed,

$$\| P\underline{e}_k \|^2 + \| \underline{e}_k - P\underline{e}_k \|^2 = 1 ,$$

$$n + \sum_{k=1}^{m} \| \underline{e}_k - P\underline{e}_k \|^2 = m = n + m \max_{1 \le k \le m} \| \underline{e}_k - P\underline{e}_k \|^2 ,$$

$$\forall_{1 \le k \le m} \| \underline{e}_k - P\underline{e}_k \|^2 = \max_{1 \le k \le m} \| \underline{e}_k - P\underline{e}_k \|^2 ,$$

hence all $\| P\underline{e}_k \|$ are equal.

4. The theorem of Hahn - Banach.

Let V be a Banach space with norm $\| \ \|$, and let W be a closed subspace. The quotient space V/W is a Banach space with norm

$$\| \underline{v} + W \| := \inf_{w \in W} \| \underline{v} - \underline{w} \| .$$

The dual space V^* of all continuous linear functionals f on V is a Banach space with norm

$$\| f \|_* := \sup_{\substack{\| \underline{x} \| = 1 \\ \underline{x} \in V}} | f(\underline{x}) | = \sup_{0 \ne \underline{x} \in V} \frac{| f(\underline{x}) |}{\| \underline{x} \|} .$$

The following is a consequence of the Hahn - Banach theorem.

Theorem. $$\| \underline{x} \| = \sup_{\substack{\| f \|_* = 1 \\ f \in V^*}} | f(\underline{x}) | = \sup_{0 \ne f \in V^*} \frac{| f(\underline{x}) |}{\| f \|_*}$$

Furthermore, we need the following

Theorem. There exists an isometric isomorphism of the spaces

$$(V/W)^* \text{ and } W^\perp := \{ f \in V^* \mid f(W) = 0 \} .$$

We apply the above to the linear space X_m provided with the ℓ_p-norm

$$\| \underline{x} \|_p := \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} .$$

<u>Hölder</u> : $\sum_{i=1}^{m} | x_i y_i | \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{m} |y_i|^{p'}\right)^{1/p'}$

with $\frac{1}{p} + \frac{1}{p'} = 1,\ 1 < p < \infty$.

As a consequence, X_m^* has $\ell_{p'}$-norm

$$\|y\|_{p'} = \left(\sum_{i=1}^{m} |y_i|^{p'}\right)^{\frac{1}{p'}}, \ \frac{1}{p} + \frac{1}{p'} = 1 \ , \ y \in X_m^* \ .$$

For a subspace X_n of X_m we now have

$$X_n^{\perp} = \{y \in X_m^* \mid y_1 x_1 + \ldots + y_m x_m = 0, \ \forall_{\underline{x} \in X_n}\} \ ,$$

where $y_i = y(\underline{e}_i)$. Apply Hahn-Banach to the quotient $V/W = X_m/X_n$, then we obtain (since the dimension is finite) :

<u>Theorem</u> . $\| \underline{e}_k + X_n \|_p := \inf_{x \in X_n} \| \underline{e}_k - \underline{x} \|_p = \max_{0 \neq y \in X_n^{\perp}} \frac{|y_k|}{\|y\|_{p'}}$.

In 6. we will use the following consequence of Hölder's inequality .

<u>Theorem</u> . $\left(\sum_{\substack{j=1 \\ j \neq k}}^{m} | x_j|^p\right)^{\frac{1}{p}} \geqslant \left(\sum_{\substack{j=1 \\ j \neq k}}^{m} |x_j|^r\right)^{\frac{1}{r}} (m-1)^{\frac{1}{p} - \frac{1}{r}}$, for $p \geqslant r$.

Indeed, apply Hölder with $q \geqslant 1$ to the $(m-1)$-vectors

$$|x_1|^r, \ldots \diagdown\!\!\!\!\!|x_k|^r\!\diagup \ldots, |x_m|^r \text{ and } 1, \ldots \diagdown\!\!\!1\!\diagup \ldots 1 \ , \text{ then}$$

$$\sum_{j \neq k} |x_j|^r \leqslant (m-1)^{1-\frac{1}{q}} \left(\sum_{j \neq k} |x_j|^{rq}\right)^{\frac{1}{q}} ,$$

$$\left(\sum_{j \neq k} |x_j|^r\right)^{\frac{1}{r}} (m-1)^{\frac{1}{qr} - \frac{1}{r}} \leqslant \left(\sum_{j \neq k} |x_j|^{qr}\right)^{\frac{1}{qr}} .$$

For $p = r$, that is $q = 1$, our inequality is an equality.

For $p > r$, equality holds iff the $(m-1)$-vectors are proportional.

5. Melkman's theorem for $p = \infty$.

In the following proof we will use back and forth the consequence of Hahn-Banach exposed in 4. In addition, we will use the Cauchy-Schwarz inequality in the following form:

$$\sum_{\substack{i=1\\ i\neq k}}^{m} |y_i| . 1 \leq \sqrt{\sum_{\substack{i=1\\ i\neq k}}^{m} |y_i|^2} \sqrt{m - 1} .$$

We first prove the $p = \infty$ case, since this case is representative.

Theorem. $a_n (I_m ; \ell_1^m , \ell_\infty^m) \geq \left(1 + \sqrt{\frac{(m - 1)n}{m - n}}\right)^{-1}$,

and equality holds iff there exists a regular two-graph on m vertices with multiplicities n, m - n .

Proof. For $k \in \{1,2,\ldots,m\}$ we have

$$\min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_\infty = \max_{\underline{0} \neq \underline{y} \perp X_n} \frac{|(\underline{y}, \underline{e}_k)|}{\| \underline{y} \|_1} =$$

$$= \max_{\underline{0} \neq \underline{y} \perp X_n} \frac{|y_k|}{|y_k| + \sum_{i\neq k} |y_i|} \geq$$

$$\geq \max_{\underline{0} \neq \underline{y} \perp X_n} \frac{|y_k|}{|y_k| + \sqrt{m - 1} \sqrt{\sum_{i=1}^{m} |y_i|^2 - |y_k|^2}} =$$

$$= \max_{\underline{0} \neq \underline{y} \perp X_n} \left[1 + \sqrt{m - 1} \sqrt{\left(\frac{|y_k|}{\| y \|_2}\right)^{-2} - 1}\, \right]^{-1} =$$

$$= \left[1 + \sqrt{m - 1} \sqrt{\left(\max_{\underline{0} \neq \underline{y} \in X_n^\perp} \frac{|y_k|}{\| \underline{y} \|_2} \right)^{-2} - 1}\, \right]^{-1} =$$

$$= \left[1 + \sqrt{m-1} \sqrt{\left(\min_{\underline{x} \in X_n} \| \underline{e}_k - x \|_2 \right)^{-2} - 1} \right]^{-1} .$$

Since $f(z) := \left[1 + \sqrt{m-1} \sqrt{z^{-2} - 1} \right]^{-1}$ is a monotone increasing function of z, this implies

$$\min_{X_n \subset X_m} \max_{1 \le k \le m} \min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_\infty \geq$$

$$\geq \left[1 + \sqrt{m-1} \sqrt{\left(\min_{X_n \subset X_m} \max_{1 \le k \le m} \min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_2 \right)^{-2} - 1} \right]^{-1} .$$

Thus we have expressed $a_n(I_m ; \ell_1^m , \ell_\infty^m)$ in terms of $a_n(I_m; \ell_1^m , \ell_2^m)$, which equals $\sqrt{1 - \frac{n}{m}}$ by 3. Substitution yields the inequality of the theorem. Now suppose we have equality :

$$a_n(I_m ; \ell_1^m , \ell_\infty^m) = \left[1 + \sqrt{\frac{n(m-1)}{m-n}} \right]^{-1} .$$

We analyse the various steps performed in the proof of the inequality. First, for an optimal X_n we have

$$\max_{1 \le k \le m} \min_{x \in X_n} \| \underline{e}_k - \underline{x} \|_2 = \sqrt{1 - \frac{n}{m}}$$

hence, by the remark at the end of 3,

$$\min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_2 = \sqrt{1 - \frac{n}{m}} \text{ , for all } k = 1,\dots,m \text{ .}$$

In addition, for any $1 \le k \le m$, the $\underline{x}$ which achieves $\min \| \underline{e}_k - \underline{x} \|_\infty$, also minimizes $\| \underline{e}_k - \underline{x} \|_2$. Hence the matrix A which approximates I_m in the ℓ_∞-norm also works for the ℓ_2-norm.

Secondly, if equality holds in Cauchy-Schwarz, then the corresponding vectors of length m - 1 :

$$(|y_1|, |y_2|, \dots, \widehat{|y_k|}, \dots, |y_m|) \text{ and } (1,1,\dots,\widehat{1},\dots,1)$$

are proportional. So we may take $|y_1| = \dots = |y_m| = 1$ except for $|y_k|$ which equals

$$|y_k| = \sqrt{\frac{(m-1)(m-n)}{n}}, \text{ since } \frac{|y_k|}{|y_k| + m - 1} = \frac{1}{1 + \sqrt{\frac{n(m-1)}{m-n}}} .$$

For each k = 1,...,m we find such a vector $\underline{y}$, which is proportional to the projection of $\underline{e}_k$ onto $X_n^{\perp}$. These vectors are taken as the columns of the following coordinate matrix B (= Gram matrix, cf. 2) of rank m - n :

$$B = \begin{bmatrix} \gamma & \varepsilon_{12} & . & . & . & \varepsilon_{1m} \\ \varepsilon_{21} & \gamma & . & . & . & \varepsilon_{2m} \\ . & . & . & & & . \\ . & . & & . & & . \\ \varepsilon_{m1} & \varepsilon_{m2} & . & . & & \gamma \end{bmatrix}, \quad |\varepsilon_{ij}| = 1, \ \varepsilon_{ij} = \varepsilon_{ji}, \quad \gamma = |y_k| = \sqrt{\frac{(m-1)(m-n)}{n}} .$$

It follows that the lines spanned by these vectors are equiangular, at $\cos^2\chi = \frac{n}{(m-1)(m-n)}$. This set of equiangular lines in $X_n^{\perp}$ is extremal in the sense of 2 (just interchange n and m - n). Equivalently, X_n contains an extremal set of equiangular lines at $\cos^2\varphi = \frac{m-n}{(m-1)n}$. Equivalently, there exists a regular two-graph on m vertices with the multiplicities n and m - n.

6. Melkman's theorem for p > 2 .

For $p \geq r$, a lower bound for $a_n(I_m ; \ell_1^m, \ell_p^m)$ in terms of $a_n(I_m ; \ell_1^m, \ell_r^m)$ is obtained, and specialized to the case r = 2, since $a_n(I_m ; \ell_1^m, \ell_2^m) = \sqrt{1 - \frac{n}{m}}$ is known. Instead of Cauchy-Schwarz we use the following consequence of Hölder, cf. 4 :

$$\Big(\sum_{j\neq k} |y_j|^{p'}\Big)^{\frac{1}{p'}} \leqslant \Big(\sum_{j\neq k} |y_j|^{r'}\Big)^{\frac{1}{r'}} (m-1)^{\frac{1}{p'}-\frac{1}{r'}} ,$$

for $p > r$; for $p = r$ this degenerates into an equality.

For a fixed k, we abbreviate as follows:

$$\eta_p := \min_{\underline{x} \in X_n} \| \underline{e}_k - \underline{x} \|_p = \max_{\underline{0} \neq \underline{y} \perp X_n} \frac{|y_k|}{\|\underline{y}\|_{p'}} .$$

Then for $p \geqslant r$,

$$(\eta_p)^{p'} = \max_{\underline{y} \perp X_n} \frac{|y_k|^{p'}}{|y_k|^{p'} + \sum_{j\neq k} |y_j|^{p'}} \geqslant$$

$$\geqslant \max_{\underline{y} \perp X_n} \frac{|y_k|^{p'}}{|y_k|^{p'} + (m-1)^{1-\frac{p'}{r'}} \Big(\sum_{j\neq k} |y_j|^{r'}\Big)^{\frac{p'}{r'}}} =$$

$$= \max_{\underline{y} \perp X_n} \left[1 + (m-1)^{1-\frac{p'}{r'}} \left[\left(\frac{|y_k|}{\|\underline{y}\|_{r'}} \right)^{-r'} - 1 \right]^{\frac{p'}{r'}} \right]^{-1} =$$

$$= \left[1 + (m-1)^{1-\frac{p'}{r'}} \left[\left(\max_{\underline{y} \perp X_n} \frac{|y_k|}{\|\underline{y}\|_{r'}} \right)^{-r'} - 1 \right]^{\frac{p'}{r'}} \right]^{-1} .$$

It follows that

$$\eta_p \geqslant \left[1 + (m-1)^{1-\frac{p'}{r'}} \left((\eta_r)^{-r'} - 1 \right)^{\frac{p'}{r'}} \right]^{-\frac{1}{p'}} .$$

<u>Theorem</u>. For $p > 2$

$$a_n(I_m ; \ell_1^m , \ell_p^m) \geqslant \left| 1 + (m-1) \left(\frac{(m-1)(m-n)}{n} \right)^{\frac{-p}{2(p-1)}} \right|^{-1+\frac{1}{p}} ,$$

and equality holds iff there exists a regular two-graph on m vertices with multiplicities n, m - n.

Proof. Put r = 2 in the formula for η_p, proceed as in 5, and substitute the value of a_n for r = 2. This yields the inequality. For the case of equality we must have equality in the consequence of Hölder's inequality. Since p > 2, again the (m - 1)-vectors are proportional, and the reasoning of 5 works.

Remark. The second part of the reasoning above does not work for p = 2. Indeed, then the consequance of Hölder's inequality is an equality, and yields nothing new. In fact, for p = 2 we do have

$$a_n(I_m ; \ell_1^m , \ell_2^m) = \sqrt{1 - \frac{n}{m}} ,$$

but the extremal sets need not be extremal sets of equiangular lines. Any spherical eutactic star provides an extremal set. As an example we mention the root systems [4].

[1] A.A. Melkman, The distance of a subspace of R^m from its axes and n-widths of octahedra, to be published.

[2] A. Pinkus, Matrices and n-widths, to appear, Lin. Alg.and Appl.

[3] J.J. Seidel, A survey of two-graphs, Proc.Intern. Coll.Teorie Combinatorie (Roma, 1973), Accad. Naz. Lincei, Roma 1976, Vol I, 481 - 511.

[4] J.J. Seidel, Eutactic stars, Coll. Math. Soc.J. Bolyai 18, Combinatorics , Keszthely 1976, 983 - 999.

[5] L. B. Sofman, Diameters of octahedra, Mat. Notes 5 (1969) 258 - 262.