EINDHOVEN UNIVERSITY OF TECHNOLOGY

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Problem 60-4

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PROBLEMS AND SOLUTIONS

EDITED BY MURRAY S. KLAMKIN

All problems and solutions should be sent to Murray S. Klamkin, Physics T-434, AVCO Research and Advanced Development Division, Wilmington, Massachusetts, and should be submitted in accordance with the instructions given on the inside front cover. An asterisk placed beside a problem number indicates that the problem was submitted without solution.

Problem 61-8, *A Coin Tossing Problem*, By D. J. NEWMAN (Yeshiva University and WALTER WEISSBLUM (Sylvania Electronic Systems).

Two persons are gambling by tossing a fair coin. If at the end of $2n$ tosses they break even, what is the probability that the first man was ahead k/n of the time.

Equivalently: How many sequences can be formed from n plus ones and n minus ones such that exactly k of its odd partial sums are positive.

Problem 61-9*, *A Definite Integral*, By W. L. BADE (Avco Research and Advanced Development Division).

Evaluate the integral

$$Q = \int_0^\infty xe^{-2x}\{\psi(x)\}^2\,dx,$$

where

$$\psi(x) = \int_0^{\pi/2} \{1 - e^{x(1-\csc\theta)}\}\sec^2\theta\,d\theta.$$

Physically, Q is proportional to the cross-section corresponding to the exponential repulsive potential $\phi = Ae^{-r/r_0}$ in the limit of high relative velocity.

A numerical integration for Q gives a value of 0.3333. Consequently, it is conjectured that $Q = \frac{1}{3}$.

Problem 61-10, *The Expected Value of a Product*, by LAWRENCE SHEPP (University of California, Berkeley).

Let E_n be the expected value of the product $x_1x_2x_3 \cdots x_n$, where x_1 is chosen at random (with a uniform distribution) in (0, 1) and x_k is chosen at random (with a uniform distribution) in $(x_{k-1}, 1)$, $k = 2, 3, \cdots, n$. Show that

$$\lim_{n\to\infty} E_n = \frac{1}{e}.$$

SOLUTIONS

Problem 60-4*, *Vorticity Interaction*, by SIN-I CHENG (Princeton University)

The shock wave over the blunt nose of a slender body in a hypersonic stream leaves a large vorticity in the downstream flow field. This vorticity interacts with the displacement flow of the boundary layer over the body to produce an induced pressure gradient. For hypersonic flow at very high altitudes, this self-induced pressure gradient along the body surface is so large as to govern the development of the boundary layer itself. This is distinctly different from ordinary boundary layers for which the pressure gradient acting on the boundary layer is known and essentially is independent of the downstream development of the boundary layer.

The mathematical analysis of one such an interacting boundary layer leads to the following proposed problem:

Determine α and L such that

$$\lim_{x\to\infty} F'(x) = L,$$

where $F(x)$ satisfies the differential equation

$$\{D^3 + x^2D^2 - xD + 1\}F(x) = \alpha, \tag{1}$$

and boundary conditions

$$F(0) = 0,$$
$$F'(0) = 0,$$
$$F''(0) = 1.$$

Solution by J. Ernest Wilkins, Jr. (Nuclear Development Corporation of America, White Plains, N. Y.).

Introduce new independent and dependent variables as follows:

$$s = -x^3/3, \qquad F = \alpha + xw, \qquad v = dw/dx.$$

Then v satisfies the confluent hypergeometric equation

$$s\frac{d^2v}{ds^2} + (c - s)\frac{dv}{ds} - av = 0$$

if $a = \frac{1}{3}$, $c = \frac{5}{3}$. Therefore the general solution of the original differential equation may be written in the form [p. 252, Eqs. 1, 2, 3. All references are to Erdelyi et al. *Higher Transcendental Functions, vol. 1*].

$$F = \alpha + \mu x\int_0^x \phi(\tfrac{1}{3}, \tfrac{5}{3}; -t^3/3)\,dt + \lambda x\int_x^X t^{-2}\phi(-\tfrac{1}{3}, \tfrac{1}{3}; -t^3/3)\,dt$$

in which μ, λ and X are arbitrary constants, and $\phi(a, c; s)$ is the standard confluent hypergeometric function regular at the origin. Determining the constants in such a manner that F satisfies the boundary conditions when $x = 0$, we find that

$$F = \alpha x\int_0^x t^{-2}[\phi(-\tfrac{1}{3}, \tfrac{1}{3}; -t^3/3) - 1]\,dt + (x/2)\int_0^x \phi(\tfrac{1}{3}, \tfrac{5}{3}; -t^3/3)\,dt$$

Since it is known [p. 278] that $\phi(a, c; s) \sim \Gamma(c)(-s)^{-a}/\Gamma(c - a)$ as $s \to -\infty$, it follows that

$$F \sim \left[\left\{\frac{\alpha\Gamma(\frac{1}{3})}{3^{1/3}\Gamma(\frac{2}{3})}\right\} + \left\{\frac{3^{1/3}\Gamma(\frac{5}{3})}{2\Gamma(\frac{4}{3})}\right\}\right] x \ln x,$$

and hence

$$\alpha = -\frac{3^{2/3}\Gamma(\frac{2}{3})\Gamma(\frac{5}{3})}{2\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})} = -0.5339$$

if $F'(x)$ has a limit at ∞. If α is selected in this manner, then

$$L = F'(\infty) = \tfrac{1}{2}\int_0^\infty \Big[\phi(\tfrac{1}{3}, \tfrac{4}{3}; -t^3/3) - \frac{3^{2/3}\Gamma(\frac{2}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})} t^{-2}\{\phi(-\tfrac{1}{3}, \tfrac{1}{3}; -t^3/3) - 1\}\Big] dt$$

Let $u = t^3/3$ and make use of the Kummer relation [p. 253, Eq. 7]

$$\phi(a, c; x) = e^x\phi(c - a, c; -x),$$

and the definition of the irregular solution of the confluent hypergeometric equation [p. 257, Eq. 7]

$$\psi(a, c; x) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)}\phi(a, c; x) + \frac{\Gamma(c - 1)}{\Gamma(a)}x^{1-c}\phi(a - c + 1, 2 - c; x).$$

Then we see that

$$L = \frac{\Gamma(\frac{2}{3})}{3^{5/3}\Gamma(\frac{1}{3})}\int_0^\infty \left[-e^{-u}\psi(\tfrac{4}{3}, \tfrac{5}{3}; u) + \frac{3\Gamma(\frac{2}{3})u^{-2/3}}{\Gamma(\frac{1}{3})}\right] u^{-2/3}\, du.$$

Since $d[x^a\psi(a, c; x)]/dx = a(a - c + 1)x^{a-1}\psi(a + 1, c; x)$ [p. 258, Eq. 13], this can be written as

$$L = \frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\int_0^\infty \left\{e^{-u}\frac{d}{du}[u^{1/3}\psi(\tfrac{1}{3}, \tfrac{5}{3}; u)] - \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\frac{d}{du}u^{-1/3}\right\} du$$

$$= \frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\int_0^\infty e^{-u}u^{1/3}\psi(\tfrac{1}{3}, \tfrac{5}{3}; u)\, du$$

after an integration by parts, since $e^{-u}u^{1/3}\psi(\frac{1}{3}, \frac{5}{3}; u) - \Gamma(\frac{2}{3})/u^{1/3}\Gamma(\frac{1}{3})$ vanishes when $u = 0$ [p. 257, Eq. 7] and when $u = \infty$ [p. 278, Eq. 1] It is known [p. 270, Eq. 7] that

$$\int_0^\infty e^{-u}u^{b-1}\psi(a, c; u)\, du = \Gamma(b)\Gamma(b - c + 1)/\Gamma(a + b - c + 1)$$

if $b > 0$, $c < b + 1$, and hence

$$L = \frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\Gamma(\tfrac{4}{3})\Gamma(\tfrac{2}{3}) = \{\Gamma(\tfrac{2}{3})\}^2/3^{2/3} = 0.8815.$$

These results are consistent with those asserted by Lu Ting, *Boundary Layer* *over a Flat Plate in Presence of Shear Flow.* The Physics of Fluids, vol 3 (1960), pp 78–81, who claims that, for the solution $f(\eta)$ of the system

$$3f''' + \eta^2 f'' - \eta f' + f = -\beta, f(0) = f'(0) = 0,$$

$f'(\infty) = 1, f''(0) = 0.7866, \beta = 0.8695.$

If we set $f(\eta) = 3^{1/3}L^{-1}F(x)$, $x = 3^{-1/3}\eta$, $\beta = -3^{1/3}L^{-1}\alpha$, then $F(x)$ is the solution of the system discussed above and so we would get $\beta = 3^{5/3}/\Gamma^2(\frac{1}{3}) = 0.8695, f''(0) = 3^{-1/3}L = 3^{1/3}/\Gamma^2(\frac{2}{3}) = 0.7866.$

Solution by Yudell L. Luke (Midwest Research Institute, Kansas City, Missouri*).

It is easy to see that $F(x) = \alpha$ is the particular solution of (1) and that $F(x) = x$ is a complementary solution. By the usual power series approach, the other two complementary solutions of (1) may be expressed in hypergeometric form [1] as

$$F_1(x) = {}_2F_2(-\tfrac{1}{3}, -\tfrac{1}{3}; \tfrac{1}{3}, \tfrac{2}{3}; -\xi), \tag{2}$$

$$F_2(x) = \tfrac{1}{2}x^2{}_2F_2(\tfrac{1}{3}, \tfrac{1}{3}; \tfrac{4}{3}, \tfrac{5}{3}; -\xi). \xi = \frac{x^3}{3}. \tag{3}$$

Using the boundary conditions at the origin, it follows that

$$F(x) = \alpha - \alpha F_1(x) + F_2(x). \tag{4}$$

To determine the behavior of $F(x)$ for x large, we appeal to the asymptotic theory of hypergeometric functions. For the case at hand, we follow Meijer [2] who shows that

$$\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(1+b_1)\Gamma(1+b_2)}\, {}_2F_2\left(\begin{matrix} a_1, a_2 \\ 1+b_1, 1+b_2 \end{matrix}; -z\right) \backsim L_{2,2}(z), \quad |z| \to \infty, |\arg z| < \frac{\pi}{2} \tag{5}$$

where

$$L_{2,2}(z) = \frac{z^{-a_1}\Gamma(a_1)\Gamma(a_2-a_1)}{\Gamma(1+b_1-a_1)\Gamma(1+b_2-a_1)}\, {}_3F_1\left(\begin{matrix} a_1, a_1-b_1, a_1-b_2 \\ 1+a_1-a_2 \end{matrix}; z^{-1}\right) \tag{6}$$

+ a like expression with a_1 and a_2 interchanged.

Note that for both (2) and (3) $a_1 = a_2 = b_1$ and the expression for $L_{2,2}(z)$ must be found by a limiting process. By L'Hospital's rule,

$$\begin{aligned} L_{2,2}(z) = {} & \frac{z^{-1-a_1}\Gamma(a_1+1)}{\Gamma(b_2-a_1)}\, {}_4F_2\left(\begin{matrix} 1, 1, a_1+1, a_1+1-b_2 \\ 2, 2 \end{matrix}; z^{-1}\right) \\ & + \frac{z^{-a_1}\Gamma(a_1)}{\Gamma(1+b_2-a_1)}\{\ln z - \psi(a_1) - \psi(1+b_2-a_1) + \psi(1)\}, \end{aligned} \tag{7}$$

* This solution was obtained using results of work supported by the Applied Mathematics Laboratory of the David W. Taylor Model Basin.

where $\psi(z)$ is the logarithmic derivative of the gamma function.

As a remark aside, (7) is not given by Meijer. In some unpublished notes, we study a more general function notated $L_{p,q}(z)$ for the case where two numerator parameters of a given ${}_pF_q$ differ by an integer or zero.

Collecting (2–7), we get

$$\begin{aligned} F(x) \backsim{} & \alpha + \frac{\alpha}{27}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\xi^{-2/3}G(x) + \frac{3^{-1/3}}{9}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\xi^{-2/3}G(x) \\ & + \frac{\alpha}{3}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\xi^{1/3}\left[\ln\xi - \psi\left(-\frac{1}{3}\right) - \psi\left(\frac{2}{3}\right) + \psi(1)\right] \\ & + 3^{-1/3}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\xi^{1/3}\left[\ln\xi - \psi\left(\frac{1}{3}\right) - \psi\left(\frac{4}{3}\right) + \psi(1)\right], \end{aligned} \tag{8}$$

$$|\xi| \to \infty, \qquad |\arg\xi| < \frac{\pi}{2},$$

where

$$G(x) = {}_4F_2\left(\begin{matrix}1, 1, \frac{2}{3}, \frac{4}{3} \\ 2, 2\end{matrix}; \xi^{-1}\right). \tag{9}$$

If $\lim_{x\to\infty} F'(x) = L$, a constant, then the coefficient of $\ln\xi$ in (8) must vanish. This determines

$$\alpha = -\left\{\frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}^2 \tag{10}$$

A straightforward calculation gives

$$L = \frac{3^{-2/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\left[3 + 2\psi\left(\frac{5}{3}\right) - 2\psi\left(\frac{4}{3}\right)\right] = 0.88152. \tag{11}$$

REFERENCES

1. A. Erdélyi, et al., *Higher Transcendental Functions*, v. 1, McGraw-Hill Book Company, Inc., 1953.
2. C. S. Meijer, *"On the G-Function, VIII,"* Nederlandsche Akademie Van Wetenschappen, Proceedings, pp. 1165–1175, 1946.

Solution by Jan Petersson, Chalmers University of Technology, Gothenburg, Sweden.

The problem is solved by Laplace transforms. Since F' is bounded on the interval $(0, \infty)$, the Laplace integrals $\int_0^\infty e^{-sx}F^{(k)}(x)dx$ $(k = 0, 1, 2, \cdots)$ are convergent, at least in the half-plane $Re\ s > 0$. Putting

$$L(F'') = y(s) = \int_0^\infty e^{-sx}F''(x)\,dx$$

and observing the initial values $F(0) = F'(0) = 0$, $F''(0) = 1$, we obtain $L(F''') = sy - 1$, $L(x^2F'') = y''$, $L(-xF') = (s^{-1}y)'$ and $L(F) = s^{-2}y$. Hence

$y(s)$ satisfies the differential equation

$$y'' + s^{-1}y' + sy = 1 + \alpha s^{-1}. \tag{2}$$

Since $\lim_{x\to\infty} F'(x) = L$, we have ([1], p. 455)

$$y(s) = L + o(1), \quad (s \to 0, |\arg s| \leqq \psi < \pi/2). \tag{3}$$

Furthermore, $F''(0) = 1$ implies ([1], p. 473) that

$$y(s) = s^{-1}(1 + o(1)), \quad (s \to \infty, |\arg s| \leqq \psi < \pi/2). \tag{4}$$

Making use of the principal branch of the transformation $t = \frac{2}{3}s^{3/2}$, it turns out that $v(t) = y(s)$ satisfies Bessels equation

$$tv'' + v' + tv = (\tfrac{2}{3})^{2/3}t^{1/3} + \alpha(\tfrac{2}{3})^{4/3}t^{-1/3}. \tag{2'}$$

The asymptotic properties of $v(t)$ are described by

$$v(t) = L + o(1), \quad (t \to 0, |\arg t| \leqq \psi < 3\pi/4) \tag{3'}$$

$$v(t) = (\tfrac{3}{2}t)^{-2/3}(1 + o(1), \quad (t \to \infty, |\arg t| \leqq \psi < 3\pi/4). \tag{4'}$$

From (2′) and (3′) we conclude that

$$v(t) = LJ_0(t) + (\tfrac{2}{3})^{2/3}s_{1/3,0}(t) + \alpha(\tfrac{2}{3})^{4/3}s_{-1/3,0}(t), \tag{5}$$

where

$$s_{\mu,0}(t) = \frac{\pi}{2}\left[Y_0(t)\int_0^t \tau^\mu J_0(\tau)\,d\tau - J_0(t)\int_0^t \tau^\mu Y_0(\tau)\,d\tau\right]$$

are Lommel functions. (See [2], p. 346).

Introducing the Lommel functions $S_{\mu,0}(t)$, ([2], p. 347)

$$S_{\mu,0}(t) = s_{\mu,0}(t) + 2^{\mu-1}\Gamma^2\left(\frac{\mu+1}{2}\right)\left[\sin\frac{\mu\pi}{2}J_0(t) - \cos\frac{\mu\pi}{2}Y_0(t)\right]$$

we find that $v(t)$ has the representation

$$\begin{aligned} v(t) = {} & \left(\frac{2}{3}\right)^{2/3} S_{1/3,0}(t) + \alpha\left(\frac{2}{3}\right)^{4/3} S_{-1/3,0}(t) \\ & + \left[L - \frac{1}{2}3^{-2/3}\Gamma^2\left(\frac{2}{3}\right) + \frac{\alpha}{2}3^{-4/3}\Gamma^2\left(\frac{1}{3}\right)\right]J_0(t) \\ & + \frac{\sqrt{3}}{2}\left[3^{-2/3}\Gamma^2\left(\frac{2}{3}\right) + \alpha 3^{-4/3}\Gamma^2\left(\frac{1}{3}\right)\right]Y_0(t). \end{aligned} \tag{6}$$

Next we observe the following estimates

$$\left.\begin{aligned} J_0(t) &= \left(\frac{2}{\pi t}\right)^{1/2}\left[\cos\left(t - \frac{\pi}{4}\right) + O(t^{-1})\right] \\ Y_0(t) &= \left(\frac{2}{\pi t}\right)^{1/2}\left[\sin\left(t - \frac{\pi}{4}\right) + O(t^{-1})\right] \\ S_{1/3,0}(t) &= t^{-2/3}[1 + O(t^{-2})] \\ S_{-1/3,0}(t) &= t^{-4/3}[1 + O(t^{-2})] \end{aligned}\right\} \quad \left(t \to \infty, |\arg t| \leq \psi < \frac{3\pi}{4}\right)$$

Combining these estimates with (6), we find that the coefficients in (6) belonging to $J_0(t)$ and $Y_0(t)$ must vanish. Hence

$$\alpha = -3^{2/3}\left\{\Gamma\left(\frac{2}{3}\right)\Big/\Gamma\left(\frac{1}{3}\right)\right\}^2 = -\frac{3^{5/3}\Gamma^4(\frac{2}{3})}{4\pi^2} \tag{7}$$

$$L = 3^{-2/3}\Gamma^2(\tfrac{2}{3}). \tag{8}$$

REFERENCES

1. G. Doetsch, *Handbuch der Laplacetransformation I*, Basel 1950.
2. G. N. Watson, *Theory of Bessel functions*, 2nd ed., Cambridge 1944.

The next solution by P. J. de Doelder and J. H. van Lint, jointly (Technische Hogeschool, Te Eindhoven, Nederland) *and M. S. Klamkin* (AVCO) are essentially the same.

Differentiating Eq. (1);

$$\{D^2 + X^2D + X\}F'' = 0.$$

Multiplying thru by $e^{x^3/6}$ and using the exponential shift theorem;

$$\left\{D^2 - \frac{x^4}{4}\right\}F''e^{x^3/6} = 0.$$

This is a modified Bessel equation whose solution is given by

$$F''(x) = \sqrt{x}e^{-x^3/6}\left\{AI_{1/6}\left(\frac{x^3}{6}\right) + BI_{-1/6}\left(\frac{x^3}{6}\right)\right\}. \tag{2}$$

Here,

$$I_n(x) = \frac{(x/2)^n}{\Gamma(1+n)}\left\{1 + \frac{(x/2)^2}{1\cdot(n+1)} + \frac{(x/2)^4}{1\cdot2\cdot(n+1)(n+2)} + \cdots\right\}.$$

Since $F''(0) = 1$,

$$B = \frac{\Gamma(\frac{5}{6})}{12^{1/6}}.$$

From Eq. (1) it follows that $F'''(0) = \alpha$. Consequently, by differentiating Eq. (2),

$$A = 12^{1/6}\Gamma(\tfrac{7}{6})\alpha.$$

It now follows that

$$F'(x) = \int_0^x \sqrt{x}e^{-x^3/6}\left\{12^{1/6}\Gamma\left(\frac{7}{6}\right)\alpha I_{1/6}\left(\frac{x^3}{6}\right) + \frac{\Gamma(\frac{5}{6})}{12^{1/6}}I_{-1/6}\left(\frac{x^3}{6}\right)\right\}dx. \tag{3}$$

Since

$$I_n(x) \underset{x\to\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}},$$

the integrand (of Eq. 3) is asymptotic to

$$\frac{\sqrt{3/\pi}}{x}\left\{12^{1/6}\Gamma\left(\frac{7}{6}\right)\alpha + \Gamma\left(\frac{5}{6}\right)12^{-1/6}\right\}.$$

Thus for arbitrary α,

$$F'(x) \sim c \log x.$$

In order for $\lim_{x\to\infty} F'(x)$ to exist,

$$\alpha = -\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{7}{6})}\cdot 12^{-1/3},$$

and then

$$F'(\infty) = \frac{2\Gamma(\frac{5}{6})}{\sqrt[6]{2^5\cdot 3^4}}\int_0^\infty \{I_{-1/6}(y) - I_{1/6}(y)\}\frac{e^{-y}\,dy}{\sqrt{y}}.$$

From a table of Laplace transforms,

$$t^{\mu-1/2}K_{\nu+1/2}(at) \supset \sqrt{\frac{\pi}{2a}}\,\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)s^{-\mu}P_\nu^{-\mu}(p/a).$$

where $s = \sqrt{p^2 - a^2}$.
Since

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\{I_{-\nu}(x) - I_\nu(x)\},$$

$$F'(\infty) = \left(\frac{2}{3}\right)^{2/3}\frac{\sqrt{\pi}\Gamma(\frac{5}{6})}{2\cos\pi/6}P_{-1/3}(1) = \left(\frac{2}{3}\right)^{2/3}\sqrt{\frac{\pi}{3}}\,\Gamma\left(\frac{5}{6}\right).$$

Also solved by the proposer and incompletely by William Squire (Southwest Research Institute, San Antonio, Texas).

Editorial note: Even though the expressions for α and L in the different solutions are dissimilar, it is easy to show that they are equivalent.

Problem 60-8, *Another Sorting Problem*, by J. H. van Lint (Technische Hogeschool, Te Eindhoven, Nederland).

Consider all sequences of length M consisting of p_1 1's, p_2 2's, $\cdots$ p_k k's $(k \geqq 2)$ whose maximal monotonic non-decreasing subsequences of contiguous numbers are of length n where n satisfies the inequalities

$$(1) \qquad n > M - p_i \qquad i = 1, 2, \cdots, k.$$

Determine the number $N(p_1, p_2, \cdots, p_k; n)$ of sequences with this property.

Editorial note: This problem is similar to that of Brock's Optimum Sorting Procedure, Problem 59-3. However, in the latter problem, contiguity is not required.

Solution by the proposer. As a consequence of (1) the subsequence must contain every number 1, 2, $\cdots$, k at least once and hence it starts with the number 1 and ends with the number k.