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FOR REPAIRABLE SYSTEMS

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ON SOME POINT PROCESS MODELS FOR REPAIRABLE SYSTEMS

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SUMMARY

The use of a model driven approach to the analysis of repairable is considered and shown to be useful as a way of understanding the characteristics of a system. However, the statistical problems that arise from the use of a set of standard model building elements are shown to be considerable. In particular identification problems arise in many of the models. The conclusion is that in many cases the exploratory data analysis approach is as effective as the use of more sophisticated models.

Keywords: Reliability modelling; modulated renewal process; identifiability; estimation.

1. Introduction

The successful use of stochastic models and the application of statistical techniques in the study of complex repairable systems seems to remain rather limited. There seems to be a gap between a model driven approach and the statistical (data analysis) approach. Models can be used to give insight into system behaviour but are difficult to analyse statistically in terms of finding estimators of their parameters. Simpler approaches based on what is intuitively attractive in statistical terms also seem to lose the essential features of the systems studied. In other words, given knowledge of the structure of the system and the behaviour of the components we can model it successfully, but when we must estimate model parameters in order to extrapolate the behaviour of a system we meet problems.

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In this paper I outline some of the models and indicate theoretically why estimation through likelihood methods is not always successful. Secondly on the basis of a few examples the difficulties with likelihood are also illustrated. The conclusions to be drawn are that in most cases the exploratory data analysis approach outlined by Bendell and Walls (1985) and Ansell and Phillips (1989) yield as much information as the data is capable of giving. In particular the lack of identifiability in some of the model based approaches means that observation of explanatory variables adds nothing to the knowledge obtained from the event data.

2. Dependency in system modelling

Some of the difficulties seem to lie with our feelings about what a model should describe and with a lack of clarity about what we want to know. The problems mainly seem to arise from the desire for simplification so that statistical analysis is possible. The difficulty forms a sub-theme in Ascher and Feingold (1984) where they discuss extensively the problems of aggregation. In a complex system subsystems and components follow different processes, but the data are often recorded without reference to these subprocesses. Thus we try to analyse the data with one of the simpler or better known point processes. The history of the subject also plays a part. Engineers have always been more concerned with models which deal with structural dependence, until relatively recently mostly in a deterministic sense, but also through the use of Markov models, for example (Billinton and Allan, 1983 & 1984). These models capture the essence of system behaviour, allow different scenarios to be explored but are usually difficult from the point of view of statistical analysis. On the other hand, many of the methods of dealing with dependence on explanatory variables have been borrowed from other areas of statistics, medical statistics in particular (Gore *et al.*, 1984; Gail *et al.*, 1980; Prentice *et al.*, 1981). The questions asked in medical statistics are frequently about the effect of a particular treatment on survival, they deal rarely with the times between reoccurrences of an illness, and pay little regard to the structure of the system. Recent

developments in econometrics have lead to techniques for dealing with the statistical analysis of stochastic processes that more closely resemble the problems faced in reliability modelling (Heckman and Singer, 1984; Lancaster, 1990).

Below some natural approaches to dealing with dependency on covariates for repairable systems are outlined. The common feature of this approach is the assumption that the intervals between failures depend only on the state at the beginning of an interval. The development through time is reflected in the parameters of the distribution of the interval, and these parameters are often in their turn modelled as functions of explanatory covariates. The covariates may be as simple as the index number of the failure interval or a measure of maintenance effort. Most of the models fall in the class of semi-Markov processes, but in only one is the state transition matrix explicitly required (Brown and Proschan, 1983).

3. What do we want from the models?

The requirements can be divided into three statistical aspects, and into a number of probabilistic or statistical representations of the characteristics of the physical process. The statistical properties are: the model should fit observed data; it should be possible to discriminate between models; and on the basis of a chosen model we should be able to make predictions. We shall see that models can be fitted to many data sets, but the the problem of discrimination and prediction is much more difficult (Lawless, 1983). A model which fits the data well allows us to determine trends and to find indications of dependency, but inability to discriminate between models means that there is no effective way to choose between the predictions made by different models (Lawless, 1983). In such a situation judgement must be made on non-statistical grounds. When viewed in this way, the non-homogeneous process with a suitable rate function frequently appears good enough to characterise the behaviour of the process through time. The discrimination problem appears as identifiability problems in the models described below.

When we consider the physical basis of models for repairable it is easiest to think in terms of the hazard rate for the current interval (Ansell and Phillips, 1989). We want to model three aspects of the development of the system, the rate of aging between events, the effects of repair, and the superimposition of processes or properties of components (in short, heterogeneity). Some of these aspects can be captured in the behaviour of the hazard rate between failures, and the values of the hazard rate at the beginning of each interval. Since a hazard rate describes the rate of aging a change of time scale or a simple scaling up of the hazard rate can capture the effects of explanatory factors between failures. The effects of repair can be captured in the initial value of the hazard rate. A hazard rate that is zero at time zero indicates that a repair has removed all age effects. In the following paragraphs the behaviour within an interval and at a repair is discussed. The comments and observations apply equally well to the analysis of data from non-repairable systems where a common underlying model is assumed to be modified by field or experimental conditions.

The knowledge of component behaviour obtained from survival analysis can be used as a building block in the study of complex systems. A renewal process describes a system where after a failure it is simply replaced by a new system with the same characteristics so that the life distribution of the system is enough to deduce all the properties of the system. In more complex cases if a system has regeneration points and the distribution of time between regeneration points is known a great deal can be learnt about the system's behaviour. The simplest interpretation of a regeneration point is an event at which the state of the system is, probabilistically, identical to the initial state. Many models for non-repairable systems can reappear in what Cox (1972) calls modulated renewal processes. Proportional hazards extensions of modulated renewal process idea are discussed in detail by Prentice *et al.* (1981).

As well as studying the effects of the measurable covariates on the life of the component we may also have to deal with heterogeneity in the observed systems. The heterogeneity may be intrinsic, for example different suppliers may deliver systems of differing qualities, or there may be errors of measurement in the duration data and the covariates (Lancaster, 1990; Vaupel, Manton and Stallard, 1979). In the following paragraphs the basic models are outlined and illustrated, they can be used alone or in combination as building blocks to model system behaviour.

4. Accelerated Failure Time Models

Perhaps the most intuitively attractive model is the accelerated failure time model. In this model the effect of the covariates is assumed to be seen as changes in the time scale for the system. Suppose that the duration is a random variable T and that the covariates are summarised in a vector $z=(\zeta_1, \zeta_2, \zeta_3 \dots \zeta_k)$. The assumption is that the duration of a system under standard conditions is T_0 and its duration under conditions described by z is

$$T = \psi(z)T_0 . \quad (1)$$

The life is increased or decreased according to whether $\psi>1$ or $\psi<1$. The behaviour of the model is most easily understood by considering the effect of the transformation (1) on a one-parameter family of distributions. Suppose that $F_0(x;\alpha)$ is a one parameter distribution function with density function $f_0(x;\alpha)$ and hazard rate $h_0(x;\alpha)$. The system operating with covariate $z_j=(\zeta_{j1}, \zeta_{j2}, \zeta_{j3}, \dots \zeta_{js})$ has

$$F_j(x;z_j,\alpha) = F_0\left[\frac{x}{\psi(z_j)};\alpha\right]$$

Estimation and inference for this model are particularly straightforward if the acceleration factor ψ is taken to be a quasi-linear in the sense that ψ can be written as

$$\psi(z,\beta) = \psi(\beta'z)$$

where β is a vector of parameters. A detailed treatment of estimation and interpretation for such models can be found in (Newby & Winterton, 1983; Newby, 1985 & 1988).

Models with a location parameter offer the possibility of including more of the features of a process as it unfolds through time. These models can be written in terms of the baseline distribution and densities as

$$F_j(x; z_j, \alpha) = F_0 \left[\frac{x + \theta(z_j)}{\psi(z_j)} ; \alpha \right]$$

The presence of the location parameter allows the effects of accumulated aging or imperfect repair to be more effectively represented. The likelihoods for these models are not much more complicated than those for the two parameter models although care has to be taken with the estimation of location parameters (Newby, 1988; Smith and Naylor, 1987). Generally a grouped likelihood performs better than the likelihood itself (Cheng and Amin, 1983; Cheng and Iles, 1987).

5. Proportional Hazards

One of the most widely used methods for the study of the effects of covariates is the proportional hazards model. The basis of the model is the simple assumption that the hazard rate is affected in a multiplicative way by a relative risk factor. Since the hazard rate is a measure of aging an increase in relative risk is an indication of more rapid aging. The model is simply expressed in terms of a *baseline hazard* rate $h_0(x)$ and a relative risk factor $\psi(z)$ dependent as above on a covariate vector z . The system operating with covariate $z_j = (\zeta_{j1}, \zeta_{j2}, \zeta_{j3}, \dots, \zeta_{js})$ has a hazard rate

$$h_j(x) = \psi(z_j; \beta) h_0(x)$$

with the assumption that for *standard conditions* $\psi(z) = 1$. Naturally, ψ must be non-negative and the commonest choice is the *Cox-Model* in which

$$\psi(z; \beta) = \exp(\beta' z) \quad .$$

The basic properties of the model are easy to deduce. However, the most valuable property is that if we are prepared to leave the baseline hazard rate unspecified the analysis of the effects of covariates can proceed on the basis of a *partial likelihood* that is independent of h_0 . The problem can be studied through the use of the partial likelihood alone. With such a non-parametric approach the importance of covariates can be explored, and predictions can be made about the effect of covariates on the relative risk, however, to make predictions about the future behaviour of systems a parametric model generally has to be employed. The details of these models are readily accessible in the literature (Ansell and Phillips, 1989 & 1990; Cox and Oakes, 1984; Prentice *et al.*, 1981) and are also available in statistical packages such as BMDP.

6. Additive Hazards

In this case the baseline hazard h_0 is assumed to be modified in an additive way by the effect of covariates (Pijnenburg, 1991; Sander, 1991),

$$h_j(x) = \psi(z_j) + h_0(x) \quad .$$

The cumulative hazard functions are

$$H_j(x) = \psi(z_j)x + H_0(x) \quad ,$$

the survivor functions become

$$R_j(x) = \exp[-H_j(x)] = \exp\{-[\psi(z_j)x + H_0(x)]\} = \exp[-\psi(z_j)x]R_0(x)$$

and the densities

$$f_j(x) = [\psi(z_j) + h_0]\exp\{-[\psi(z_j)x + H_0]\} \quad .$$

On considering the competing risks model in the following section it is clear that the additive hazards model is a form of competing risks model with two, possibly fictional, components in series. However, the failing component, or cause of failure, can not be determined. This remark is useful for simulating data from an additive model, since the failure time T_a is simply the minimum of a random variable T_e sampled from the exponential

distribution with parameter ψ_j and a random variable T_0 sampled from the baseline distribution. Mercer (1961) showed how an additive form arose naturally in a wear dependent renewal process in which a stochastic wear process contributed additively to the failure rate of a component.

If observations are made at n values of the covariate, $z_1, z_2, z_3, \dots, z_n$, $\psi_j = \psi(\beta'z_j)$, $L^{(j)}(\beta, \theta)$, $\beta \in \mathbb{R}^p$ and $\theta \in \mathbb{R}^q$, denotes the contribution to the log-likelihood for the observations (actual and censored) at z_j , $\psi_j = \psi_j(z_j; \beta)$, x_{ji} is the i -th observation at level z_j , C_j is the index set of censored observations, and D_j the index set for the observed failures, then the required log-likelihoods are

$$\begin{aligned} L^{(j)}(\beta, \theta) &= \sum_{i \in D_j} \ell n[f_j(x_{ji}; \psi_j)] + \sum_{i \in C_j \cup D_j} \ell n[R_j(x_{ji}; \psi_j)] \\ &= \sum_{i \in D_j} \ell n[\psi_j + h_0(x_{ji})] - \psi_j \sum_{i \in C_j \cup D_j} x_{ji} - \sum_{i \in C_j \cup D_j} H_0(x_{ji}) \end{aligned}$$

where θ is the parameter vector for f_0 . The overall log-likelihood is

$$\mathcal{L} = \sum_{j=1}^{j=n} L^{(j)}(\beta, \theta)$$

The likelihood equations fall into two linked sets, one for the model parameters β_k and the other for the parameters of the baseline hazard θ_r . For the baseline hazard rate

$$\frac{\partial \mathcal{L}}{\partial \theta_r} = \sum_j \frac{\partial L^{(j)}}{\partial \theta_r} = \sum_j \left\{ \sum_{i \in D_j} \frac{h_{0r}(x_{ji})}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} H_{0r}(x_{ji}) \right\} = 0$$

and for the model

$$\frac{\partial \mathcal{L}}{\partial \beta_k} = \sum_j \frac{\partial L^{(j)}}{\partial \beta_k} = \sum_j \psi_{jk}(\beta) \left\{ \sum_{i \in D_j} \frac{1}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} x_{ji} \right\} = 0 \quad (2)$$

Where h_{0r} and H_{0r} denote the derivatives of h_0 and H_0 with respect to θ_r . Equation (2) shows that for a fixed θ there is always the solution

$$\sum_{i \in D_j} \frac{1}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} x_{ji} = 0 \quad , j=1, \dots, n \quad (3)$$

and that the existence of other solutions depends on the rank of the matrix $(\psi_{jk}(\beta))$. In particular if it is assumed that the ψ_j are not functionally related but that each is simply a parameter of the j -th distribution, (ψ_{jk}) is an identity matrix and (3) gives the unique solutions. When $p < n$ the rank of (ψ_{jk}) is at most p and there are solutions determined by its null-space as well as those from (3). Further (Watson and Leadbetter, 1964) hazard rates in general satisfy the following form of (3)

$$E \left[\frac{1}{h(x)} \right] = E[x] \quad (4)$$

thus if $\psi_j + h_0(t)$ is a hazard rate it should satisfy (4) whose empirical version is just (3). This means that solutions determined by the null-space of (ψ_{jk}) that do not also satisfy (3) do not yield hazard rates. Conversely solutions of (3) which are not in the null-space of (ψ_{jk}) do not allow the values of β to be estimated. Furthermore, for fixed β_k (3) must be satisfied by θ . In short there is an identifiability problem in which the values of ψ can be estimated, but not its parametric form.

7. Competing Risks

The competing risks model (David and Moeschberger, 1978) has two interpretations, the first, as its name suggests, it describes the lifetime of a system subject to statistically independent risks of failure, the second corresponds describes the lifetime

of a series system of components which fails as soon as one of the components fails. The occurrences of the potential failures can be regarded as a vector of independent random variables $(T_1, T_2, T_3 \dots T_n)$ so that the failure time is the minimum of the T_i . If the survival function for T_i is $R_i(x)$, the system survival function is easily found to be

$$R_{sys}(x) = \prod_{i=1}^{i=n} R_i(x)$$

with hazard rate

$$h_{sys}(x) = \sum_{i=1}^{i=n} h_i(x)$$

Competing risks arise naturally in reliability problems, in particular in the analysis of series systems and from "weakest link" arguments for system reliability. The β -factor method for dealing with dependency is also a version of a competing risks model since it in effect splits the system to be analysed into one part composed of independent components and a common-cause component in series (Lewis, 1987). The (non-)identifiability results reported in Crowder (1991) can be used constructively to construct competing risks models which correctly reflect the behaviour of the system.

8. Frailty or Mixtures

The idea of frailty or mixture models can be used in two ways, the first simply as a way of introducing an idea of heterogeneity into the construction of a model, and secondly as an object of interest in itself. In demography and econometrics the identification of a mixing distribution is of some importance, in reliability the use of mixing distributions is more commonly a step in model building, the most common reasons for using mixing distributions are that systems are built from components whose characteristics are random variables, and that data are frequently observed simply as a list of failure times which may be event times from a number of different but unknown processes.

Frailty (Vaupel, *et al.*, 1979) is defined rather like proportional hazards, but differs in that the relative risk factor is in this case a random variable. Vaupel *et al* define frailty λ in terms of the hazard rates of individuals in a population. They write

$$h(x;\lambda) = \lambda h_0(x)$$

for the hazard rate of an individual with frailty λ , h_0 is again a standardised, or baseline, hazard rate. If the frailty at time t has a density $w_t(\lambda)$, then the average hazard rate at time t is

$$\bar{h}(t) = \int_0^\infty h(t;\lambda)w_t(\lambda)d\lambda = h_0(t)\int_0^\infty \lambda w_t(\lambda)d\lambda = \bar{\lambda}_t h_0(t)$$

Now if the frailty λ_t decreases with time, the weakest die young, so will $\bar{\lambda}_t$ and we see the apparent effect of the average hazard rate for the population declining more rapidly than the hazard rate for individuals.

Mixture models also arise naturally in reliability (Lancaster, 1990; Littlewood and Verrall, 1973) when we write the hazard rate as a conditional hazard rate $h(t;z,\lambda)$ where as usual z is a covariate and λ is a random variable with density w , the unconditional density and survivor function for t are

$$f(t;z) = \int f(t;z,\lambda)w(\lambda)d\lambda = \int h(t;z,\lambda)\exp[-H(t;z,\lambda)]w(\lambda)d\lambda$$

$$R(t;z) = \int R(t;z,\lambda)w(\lambda)d\lambda = \int \exp[-H(t;z,\lambda)]w(\lambda)d\lambda$$

But note that the hazard rate defined in terms of frailty is not the hazard rate of the unconditional distribution, that is

$$h(t;z) = \frac{f(t;z)}{R(t;z)} \neq \int h(t;z,\lambda)w(\lambda)d\lambda = \bar{h}(t;z)$$

This observation also applies when we interpret the mixing model as a Bayesian model with prior $w(\lambda)$ for a parameter λ , for again care has to be taken over the choice of meaning for a hazard function, particularly when we wish to estimate a hazard function from data.

For example, the Littlewood–Verrall model (1973) mixes the density functions of the inter-failure times between the appearance of faults in a piece of software. However, the mixing process also seems to carry with it identification problems (Lancaster, 1990; Ridder, 1990).

9. Repairable Systems

The choice of an appropriate model depends on what is expected of the data. In a repairable system, a renewal process typifies the behaviour of system in which a failed component is replaced by an identical new component. A lamp with the replacement of failed bulbs is a perfect example of a renewal process. The two points about the renewal process are that the system returns to an "as new" state and ages in exactly the same way as before. If the ideas of a renewal process are extended, then the system can be returned to the working state, but in a condition between the new state and the failed state, and after a repair the system may age more rapidly than before. All of the above models can incorporate each of these features, although not always in one model. The questions that are asked about such a system are can the development of the system through time be predicted, can the effects of different operating conditions be incorporated in the model, for example, can the effects of modification or a different maintenance regime be modelled. The easiest assumption is that the interval lengths remain statistically independent, the use of covariates models changes in the distribution of lengths. If the process is described by an intensity function $\lambda(t)$ which gives the probability that the current interval ends in $[t, t+dt)$ as $\lambda(t)dt$ then when λ is a continuous function for all values of t we have a non-homogeneous Poisson process, and if the interval after event t_i has hazard rate $\lambda_i(t-t_i)$ the process can be called a *modulated renewal process* (Thompson, 1981; Cox, 1973; Prentice *et al.*, 1981). In this second case λ is a discontinuous function with a discontinuity at each failure time. It is also possible to have an intermediate situation in which the rate λ has discontinuities only at certain events and not at others (Brown and Proschan, 1981).

The techniques outlined for a single sample can be extended following Cox (1973) to a modulated renewal process by assuming that in place of the ordinary renewal process where the assumption of independent and identically distributed intervals is dropped and replaced that of interval length depending only on the state at the beginning of the interval. The assumption is that the j -th interval X_j is distributed as $F_{X_j}(x)$. The aim is use changes in the F_{X_j} to model the changes in the system as its history unfolds. Through such models it is hoped to discover whether the system improves or deteriorates through time, and for example, to determine optimal maintenance or replacement policies.

The simplifying assumptions that the j -th interval has distribution $F_{X_j}(x)$ means that the durations of the j -th interval can be treated simply as a sample from $F_{X_j}(x)$ and used to construct a log-likelihood $L^{(j)}$ for that interval alone, the overall log-likelihood is the sum

$$\mathcal{L} = \sum_{j=1}^{j=n} L^{(j)} .$$

This log-likelihood also provides log-likelihoods for the parameters of $F_{X_j}(x)$ as functions of explanatory variables. The estimators are, in principle, obtained as solutions of the likelihood equations obtained by setting the appropriate derivatives of \mathcal{L} equal to zero. Systematic use of the chain rule frequently produces simple relationships between the ordinary log-likelihoods and those where there is assumed to be an underlying model.

Techniques for the analysis of renewal processes can also be carried over, in principle, to the modulated renewal process, although explicit closed formulas for measures of interest are mostly not available. If the explanatory variables are deterministic, interval number for example, we can proceed as for a renewal process: the time of event n , t_n is simply the sum of the n independently distributed interval lengths x_i ,

$$t_n = \sum_{i=1}^{i=n} x_i$$

so that the Laplace transform of $g_n(t)$, the density of t_n , can be written as

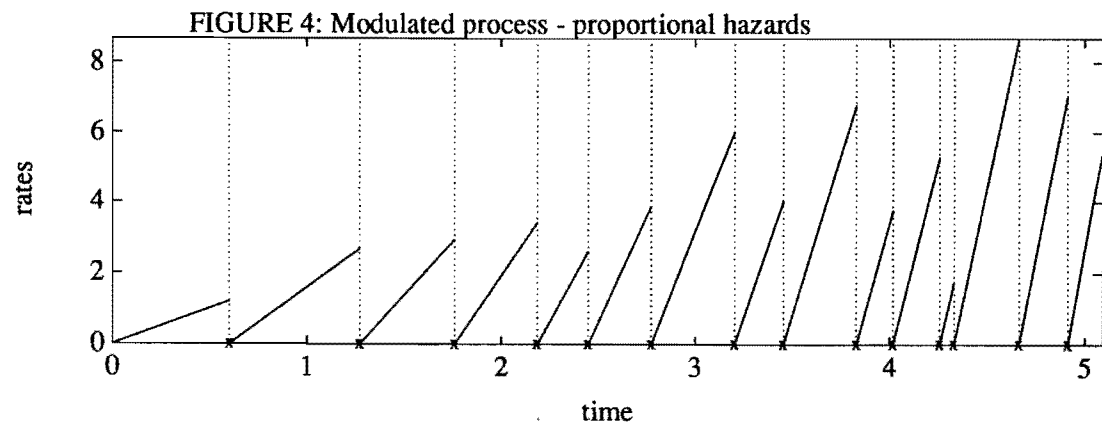
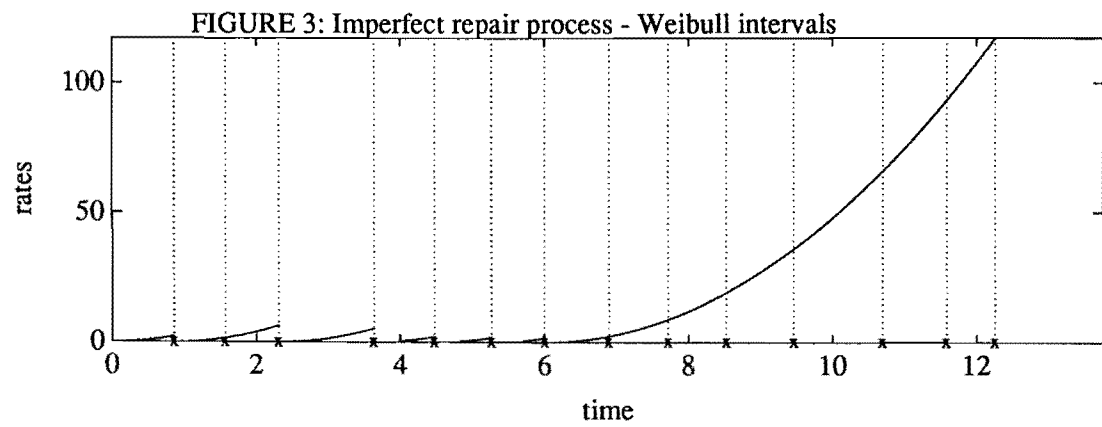
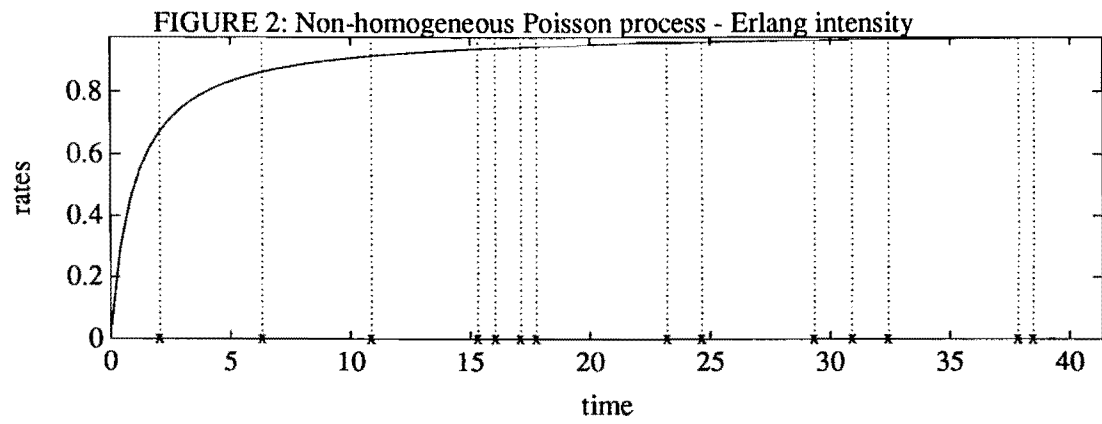
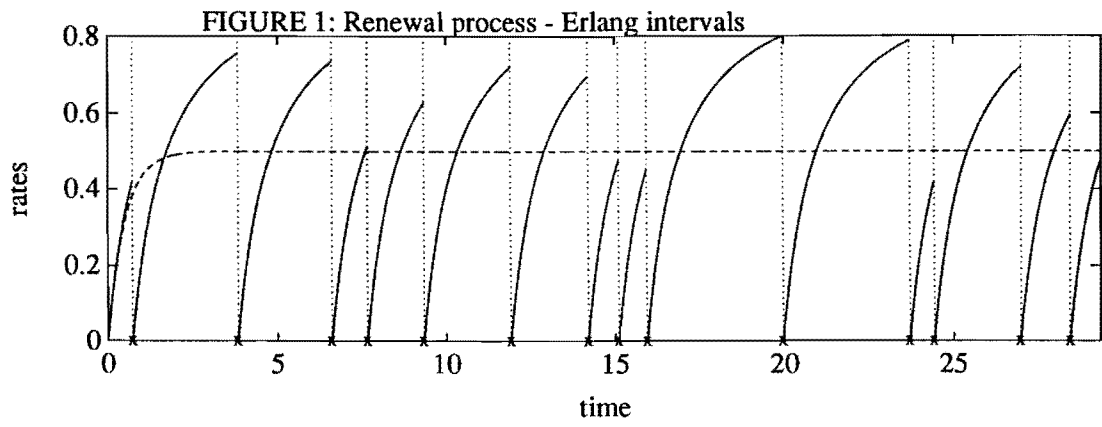
$$\overline{g}_n(s) = \prod_{i=1}^{i=n} \overline{f}_{X_i}(s)$$

and the renewal function $V(t)$ has Laplace transform

$$\overline{V}(s) = \frac{1}{s} \sum_{i=1}^{i=\infty} \overline{g}_n(s)$$

When it comes to using these building bricks a number of points are clear. Proportional hazards and accelerated failure time approaches can only model the rate of aging between events. However, if these models are based on a baseline hazard rate $h_0(t)$ with the property $h_0(0)=0$, every interval hazard rate h_j also satisfies $h_0(0)=0$. If the hazard rate is zero at time zero, the system is instantaneously as good as new after a repair, even though thereafter it may age faster. A means of generating interval hazard rates which have $h_0(0)>0$ is required: one is the additive hazards approach; the second is a version of the imperfect repair idea obtained through the use of a location parameter as a *virtual age*. It is worth remarking that hardly any of the commonly used distributions fulfil the requirements of a hazard rate (and thus density function) that is non-zero at time zero and has an increasing hazard rate. Apart from the exponential all the commonly used distributions have a zero or an infinite hazard rate at time zero.

A number of the possibilities are illustrated in Figures 1–8, in each figure an 'x' on the abscissa with a stippled line above it denotes an event time, the solid curves the interval hazard rates,. Figure 1 shows a standard renewal process and the hazard rate clearly repeats itself after each event. Figure 2 by contrast show a non-homogeneous Poisson process with a continuous intensity function, the hazard rates can be regarded as being patched together at the event times to make up the continuous intensity function. In



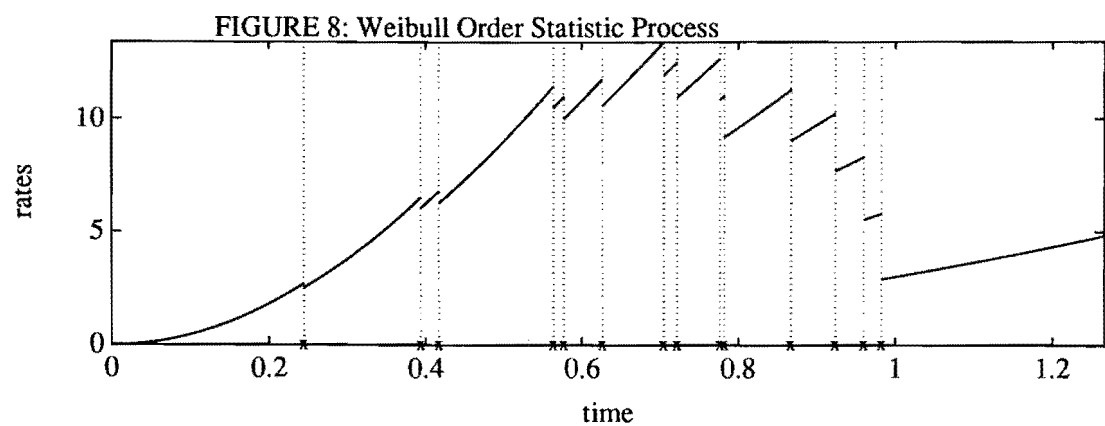
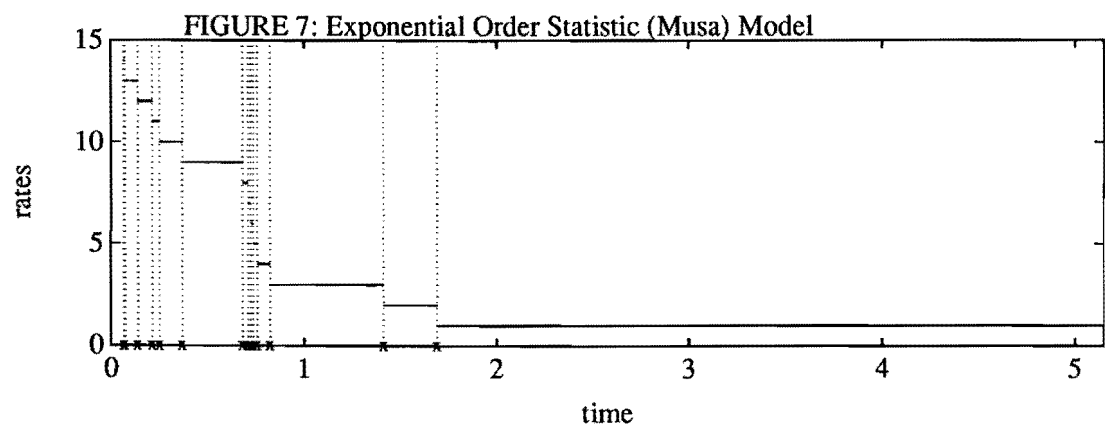
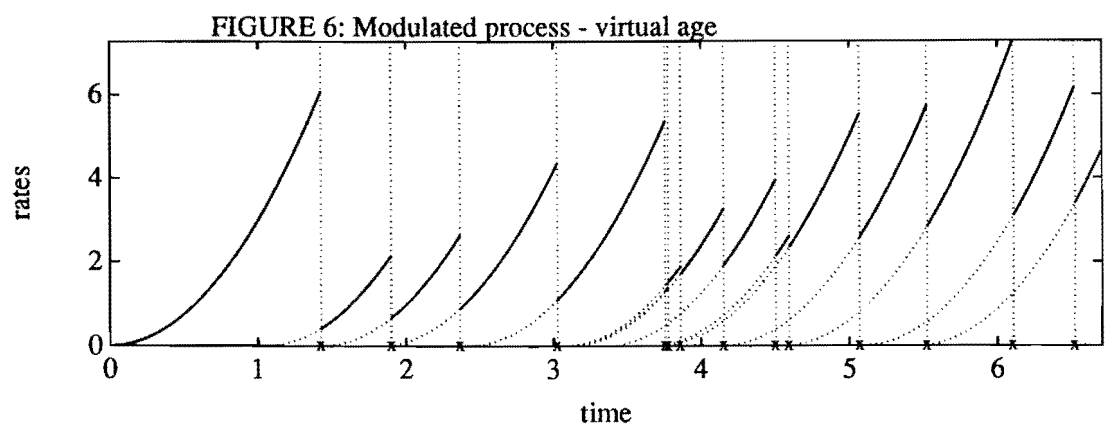
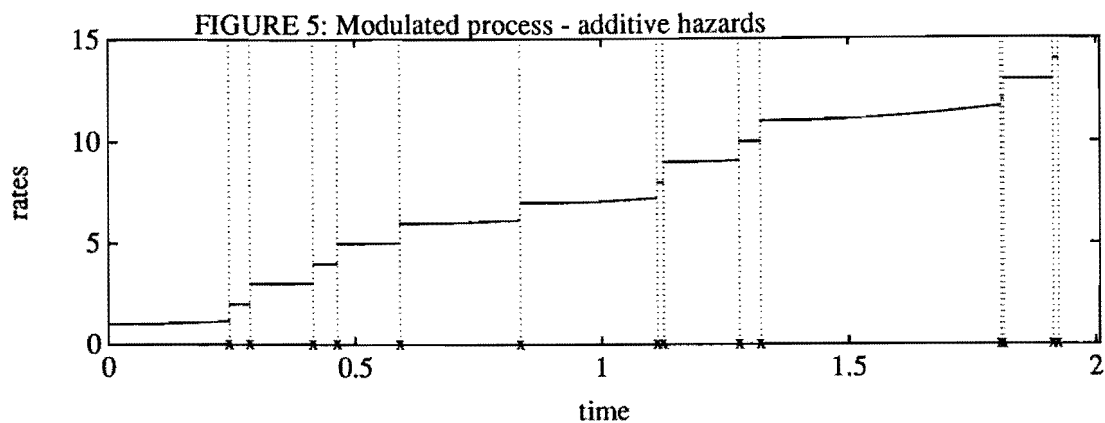


Figure 3 can be seen an imperfect repair process where sometimes the repair is perfect, and at other times it is just enough to restore the system to the state instantaneously before the failure; the hazard rate or intensity function repeats itself between the perfect repairs when it is restored to zero. Figure 4 gives an impression of a proportional hazards modulated renewal process, after each repair the hazard rate is restored to zero, but thereafter rise more steeply in each succeeding interval; thus the system is instantaneously as good as new after a repair, but has an increasing tendency to age between events. Additive hazards are used to show imperfect repair in Figure 5, there the rate of aging between events repeat itself, the aging is seen in the rising initial value for the interval hazard rate. In Figure 6 imperfect repair is shown, this time using a virtual age model, the aging pattern between events has a fixed functional form, but aging shows in the rising trend in the initial values of the interval hazard rate as well as in the increasing steepness of the curve. For completeness, two order statistic processes are given in Figures 7 and 8, one process for an exponentially distributed failure time, the Musa model (Littlewood, 1991) and an order statistic process for a Weibull distributed failure time with an increasing hazard rate. Clearly, the Weibull order statistic process shows two opposed trends, the initial value of the hazard rate after an event begins to decline, but the hazard rate between events becomes steeper.

There are a few simple point-process models which allow the kinds of analyses available for non-repairable systems to be extended to deal with repairable systems. The two simple models are a *non-homogeneous Poisson process* and a *renewal process* which have as the unique common model the ordinary Poisson process. These two models also enable some more complicated models, *imperfect repair* and *alternating renewal process*, to be analysed with the same techniques.

The history of the simplest processes is described by a sequence of times $t_1, t_2, t_3, \dots, t_n, \dots$ at which events, usually called failures, occur, in effect repair

times are ignored. The process can also be described in terms of the inter-failure times. Define $t_0=0$ for consistency, then the inter-failure times are

$$x_j = t_j - t_{j-1} \quad .$$

The type of repair considered here fall into three classes, a *perfect repair* which returns a system to its new state, a *minimal repair* which returns the system to working order but in exactly the same state as just before the failure, and an *imperfect repair* which produces a working state that is better than immediately before the failure but not as good as the original state.

10. The Non-Homogeneous Poisson Process

This process has two evocative names, *bad-as-old* and *minimal-repair* (Ascher and Feingold, 1984, their warnings about the ROCOF and hazard rate apply particularly here). The assumptions are that after a failure the system is instantaneously restored to the state in which it was immediately before the failure. That is only sufficient work is done (minimal repair) to restore the system to working order, and after repair the system is as exactly as it was before the breakdown. The meaning of this assumption is that if we look at a system immediately after a breakdown at time s we cannot distinguish it from a system of the same type which has had no breakdown up to time s ; all breakdowns can be regarded as the first breakdown after a period s with no breakdown. The survival function for the time to the first breakdown is $R(t)$. Thus the conditional survival function for the waiting time w to the next breakdown can immediately be written as

$$R_s(w) = \frac{R(s+w)}{R(s)}$$

The hazard rate for the waiting time w is

$$h_s(w) = \frac{d}{dw} \{ -\ln[R_s(w)] \} = \frac{d}{dw} \{ -\ln[R(s+w)] - \ln[R(s)] \} = h(s+w) \quad ,$$

in other words, the intensity for the process is just the current value of the hazard rate for the time to first failure regarded as a function and evaluated at the current time. A

standard argument (Cox and Lewis, 1978) shows that the process so defined is a non-homogeneous Poisson process (NHPP). If N_t denotes the number of events up to time t , the intensity $\lambda(t)$ and mean value $\Lambda(t)$ given by

$$\lambda(t) = h(t) , \quad \Lambda(t) = E[N_t] = H(t) = \int_0^t h(u)du$$

and the probability mass function is

$$\mathbb{P}[N_t=n] = \frac{\Lambda^n(t) \exp[-\Lambda(t)]}{n!}$$

The survivor function for the time to the first failure is

$$R(t) = \mathbb{P}[N_t=0] = \exp[-\Lambda(t)] ,$$

revealing again the connection between the mean value function and the time to first failure.

The NHPP is widely used in reliability growth modelling (the Duane model, Ascher and Feingold, 1984) and in software reliability growth modelling (Miller and Keiller, 1991). The strength of the model is that it is simple and relatively flexible, and the weakness is that it tends to be used as a black-box approach to data from repairable systems. It can provide a useful way to detect trends and dependency in data. The case of λ constant is a standard Poisson process, and many results from the standard Poisson process can be used by noting that in the time scale $\tau = \Lambda(t)$ the process becomes a standard Poisson process with unit rate. Time dependency is revealed through a non-constant λ . Conversely, an NHPP with intensity λ defines a *bad-as-old* or *minimal-repair* model. The important distinguishing feature is that the intensity λ is a continuous function of time (Thompson, 1979).

The scope for the use of covariates is somewhat limited, the problem is that because λ is a continuous function, for non-time varying covariates the effect applies once and for all

at the start of the process. When restricted to time independent covariates it is impossible to model the effect of changes in a system (for example maintenance policy, improved components) during its lifetime. Continuously time varying covariates can, however, appear simply in the rate function. A once and for all effect can be determined by comparing the ROCOF under different policies. What the ROCOF can tell us is whether the system performance changes over time, and this aspect is widely used in modelling reliability growth and software reliability. Moreover for this system the expected number of failures in a fixed interval is easy to calculate.

Both accelerated failure time and proportional hazards versions are easy to construct using a baseline intensity λ_0 , the models are

$$\lambda(x;z) = \frac{1}{\psi(z)}\lambda_0\left[\frac{x}{\psi(z)}\right] \quad \text{and} \quad \lambda(x;z) = \psi(z)\lambda_0(x) .$$

respectively. In the light of the above remarks it is clear that estimation of $\psi(z)$ requires data from a number of systems operating with different values of z , and a single process history will not allow ψ to be estimated unless λ_0 is known *a priori*.

11. Imperfect Repair

The *imperfect repair model* is easy to describe. It is assumed that two sorts of repair can take place after a failure: the first kind is a perfect repair that returns the system to its original state; the second type restores the system to the working state, but only to the state just before the failure (an imperfect repair). We shall not examine this model in detail, but remark that the sequences of time of a perfect repair are regeneration points for the process, and the process of intervals between imperfect repair is a renewal process (Brown and Proschan, 1983). The techniques applied to renewal and modulated renewal processes can be borrowed to determine many of the properties of this model. The treatment is analogous to the treatment of the alternating renewal process (Whittaker and Samaniego, 1989).

12. Additive Hazards Revisited

This approach arises naturally from the desire to model a system that after a repair is better than it was just before the repair, but not as good as new. Thus if a new system has interval hazard rate $h_0(x)$ after the $(j-1)$ -st failure the interval hazard rate is $\psi_j + h_0(x)$. The attraction of this model is that it appears as a simple additive analogue of the proportional hazards approach, and that each failure contributes something to the age of the system. Moreover, this model offers hazard rates that are not zero at time zero. However, in view of the problems of estimation described above it can only be used in a phenomenological way to measure the magnitude of the jumps ψ_j in the hazard rate, models for the ψ_j which make use of explanatory variables are unlikely to produce satisfactory estimators of the parameters (Pijnenburg, 1990; Sander, 1990). The statistical problems are unfortunate since in this case the Laplace transforms required for a renewal process approach are likely to be more readily obtained. The Laplace transform of $g_n(t)$, the density of t_n the time to the n -th event, can be written as

$$\bar{g}_n(s) = \prod_{i=1}^{i=n} \bar{f}_{X_i}(s) = \prod_{i=1}^{i=n} \bar{f}_0(s+\psi_i) \quad .$$

13. Virtual Age Model

Here after a repair the system returns to working order and the interval hazard rate is that of a system which is not new, but is better than a system which has undergone a minimal repair. This can be modelled by assuming that after j -th repair the system begins to operate as a system with an age τ_j . The parameter τ_j is the *virtual age* after a repair. This is also an approximation to Downton's model for the analysis of data from a non-homogeneous Poisson process (Downton, 1969) in which several copies of the process are observed, but the failure times and the identities of the processes are not recorded, only the inter failure times are known. The reliability function for the interval is

$$R_j(x) = \frac{R_0(x+\tau_j)}{R_0(\tau_j)}$$

If the model is written as

$$R_j(x) = \frac{R_0(x + R^{-1}[p_j])}{p_j} \quad ,$$

p_j represents the relative survival chance immediately after repair. The hazard rate is

$$h_j(x) = h_0(x + \tau_j)$$

Thus it can be seen that if $\tau_j = 0$ for all j the system is a renewal process, and if $\tau_j = t_j$ for all j the system becomes an NHPP. This makes the model attractive in that it lies naturally between the NHPP and the renewal process. The model can be extended in two ways using proportional hazards ideas or accelerated failure time ideas. The two versions are

$$h_j(x) = \psi_j h_0(x + \tau_j) \quad \text{and} \quad h_j(x) = \frac{1}{\psi_j} h_0\left(\frac{x + \tau_j}{\psi_j}\right)$$

respectively. With this proportional hazards model the presence of the τ_j means that a parametric likelihood function must be used to estimate the parameters of ψ and τ .

Later the accelerated failure time version will be illustrated. The parameter τ is a measure of accumulated aging and the parameter ψ is a measure of the rate at which aging occurs. In its most general form the ψ and τ are assumed to be functions of explanatory variables, for instance a simple trend model would have ψ constant and $\tau = \tau(\alpha + \beta j)$ or *vice versa*. However, just as with the additive hazards model there are, as yet unresolved, identification problems.

This model also solves a trivial seeming difficulty with hazard rates. For the most commonly occurring distributions, apart from the exponential, the hazard rate and density at time zero are either zero or infinite. On examining data from repairable systems the interval density almost always appears to be non-zero at time zero. Noting that the j -th interval density is

$$f_j(x) = \frac{f_0(x+\tau_j)}{R_0(\tau_j)}$$

shows that

$$f_j(0) = \frac{f_0(\tau_j)}{R_0(\tau_j)} = h_j(0) = h_0(\tau_j) \quad .$$

The mean and variance of the j -th interval are clearly the mean residual life for the baseline distribution and the variance of the mean residual life of the baseline distribution calculated at τ_j . Explicit expressions for the mean residual life of a number of standard distributions are given by the author (Newby, 1988).

14. Examples

Software reliability

The two most common forms of model used in software reliability are versions of the NHPP (Goel and Okumoto, 1979) and mixture models based on order statistics (Littlewood, 1991). In this example a family of NHPP models based on a mean value function is used to analyse a data set from Musa (1980). The distinguishing feature of many analyses of software failure is that it is almost always possible to obtain a good fit to the data, but the forecasting power of the models is poor. The class of models used is described by Al Ayoubi *et al.* (1990) and Miller *et al.* (1991), the mean value for a software failure process is bounded because there is a finite initial number of faults, indeed the process generated is indistinguishable from an order statistics process. The observation about the boundedness of the failure rate suggest that the mean value can be written as

$$E[N_t] = \alpha F(t)$$

where $F(t)$ is a distribution function and α is the unknown initial number of faults. In this case the accelerated failure time approach is natural and the scale parameter of the distribution F is then a measure of the relative performance of the development of a particular piece of software.

Using the scale parameter model the mean value function is

$$\mu(t) = \alpha F(t/b; \beta)$$

and the intensity function is

$$\lambda(t) = \frac{d\mu}{dt} = \frac{\alpha}{b} f(t/b; \beta)$$

Since the intensity function is a density function it must eventually decline to zero, suggesting that these models represent a situation in which the intervals between faults are increasing but that the expected time to the discovery of the last fault is infinite. The following familiar models are recovered by particular choices of distribution:

$$\mu(t) = \alpha \left[1 - \frac{1}{(1+t/b)^\beta} \right] ; \text{ Pareto distribution, resembles the Littlewood model}$$

$$\mu(t) = \alpha \left[1 - e^{-(t/b)^\beta} \right] ; \text{ Weibull, Goel and Goel \& Okumoto}$$

$$\mu(t) = \alpha e^{-(t/b)^\beta} ; \text{ extreme value distribution}$$

The likelihood, in this case for grouped data , $\{(f_i, t_i)\}_{i=1}^{i=s}$ with f_i failures in the interval (t_{i-1}, t_i) with $n = \sum f_i$, is

$$L(\alpha, \beta, b) \propto \alpha^n \exp[-\alpha F(u^*)] \prod_{i=1}^{i=s} \left\{ F(u_i) - F(u_{i-1}) \right\}^{f_i}$$

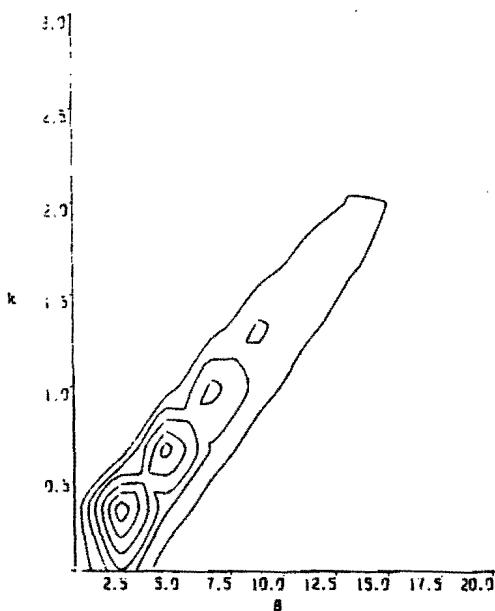
with $u = t/b$ and u^* the end of the observation period. In all cases the estimator for α is

$$\hat{\alpha} = n/F(u^*)$$

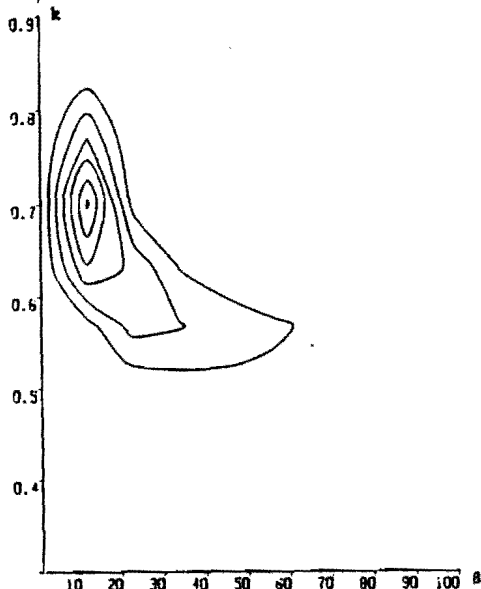
which can be used to give a profile likelihood

$$L^*(\beta, b) \propto \prod_{i=1}^{i=s} \left\{ \frac{F(u_i) - F(u_{i-1})}{F(u^*)} \right\}^{f_i}$$

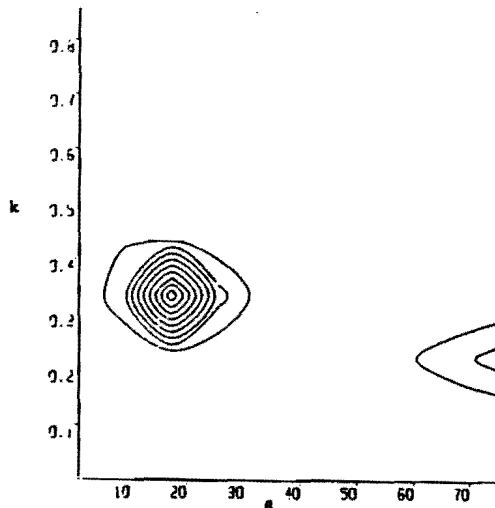
Figure 9: Contour Maps for the Musa Data



Contours of profile likelihood for Pareto model.



Contours of profile likelihood for Weibull model.



Contours of profile likelihood for extreme value model.

The profile likelihood was maximised using NAG routine E04KBF and the results were rather unstable. The model is illustrated using a data set from Musa. What is striking about the likelihood is that there are multiple maxima and that the estimators are highly correlated. Although the model can yield a good fit to the data the interpretation requires some care. The results are summarised in Table 1 and the profile likelihoods can be seen in the contour maps in Figure 9.

TABLE 1: Summary of results

	Pareto	Weibull	ext. value
log-likelihood	-143.7	-143.9	-143.8
$\hat{\alpha}$	377.5	160.1	502.7
\hat{b}	1.9	10.7	72.5
$\hat{\beta}$	0.2	0.7	0.2
s.d. $\hat{\alpha}$	346.6	21.0	197.6
s.d. \hat{b}	1.2	3.5	100.7
s.d. $\hat{\beta}$	0.2	0.1	0.1
corr ($\hat{\alpha}, \hat{b}$)	-0.8	0.7	0.97
corr ($\hat{\alpha}, \hat{\beta}$)	-0.9	-0.6	-0.9
corr ($\hat{b}, \hat{\beta}$)	0.9	-0.7	-0.96
Kolmogorov-Smirnov statistic	0.1889	0.2004	0.1999

What is remarkable in the maps is that the likelihood for the Pareto based model exhibits multiple maxima along a ridge at about 45°. This ridge show also the high correlation between the shape and scale parameter in agreement with the estimated correlation in Table 1. The extreme value also gives a likelihood which indicates a second maximum for the log-likelihood function. The predictions from the three models also vary widely, but all the models fit the data satisfactorily as measured by the Kolmogorov-Smirnov statistic which does not reject the hypothesis that the models fit at the 5% level. Miller (Miller *et al.* 1991) included these models in a "super model" and showed that after different lengths of time different simple models where chosen on the basis of fit. What can be

seen from Table 1 is that there is again an identification problem in that the likelihood values are all more or less the same and that in this particular case, the fits are also more or less the same. The "super model" fails to distinguish between the simple models on the basis of likelihood.

McCollin *et al.* (1989) discuss more fully the use of explanatory variables to analyse the times between failure for a number of pieces of software. They indicate that the most widely used approach is the proportional hazards in combination with one of the other basic models described in this paper. Their conclusions were somewhat tentative but indicated that only two rather simple covariates could be used in modelling, the age of the software, and the sequence number of the fault. Although they fitted proportional hazards models in some cases, the fits were always marginal. With these two covariates the model reduces directly to one of the kind discussed above.

The investigations of Pul (1990, 1991) into the likelihoods for the Musa and Littlewood models demonstrates that the number of observations needed to attain asymptotic normality in the maximum likelihood estimators is extremely large, of the order of thousands of observations. Moreover, not all samples yielded acceptable estimators of the parameters of interest. The problem appears to be two fold: firstly there is an identification problem in that the models cannot distinguish between a system with a large number of faults with a small rate of occurrence per fault and a system with few faults with a high rate of occurrence per fault (Wright & Hazelhurst, 1988); and secondly the likelihoods themselves are ill-conditioned.

A mechanical system

Downton (1969) studied the failure pattern of a fleet of buses, from data originally published by Davis (1952), see Table 2. Indeed, until now this data seems to have resisted statistical analysis. Pijnenburg (1990) showed that on the basis of a graphical

TABLE 2: Frequency table of distances between repair
(Davis 1952; Downton 1969)

distance (1000 miles)		X_1	X_2	X_3	X_4	X_5
0	10	3	8	12	15	19
10	20	3	11	15	19	15
20	30	6	9	13	9	17
30	40	5	4	3	11	15
40	50	7	6	9	2	8
50	60	9	8	9	13	6
60	70	9	6	6	12	4
70	80	16	9	7	3	4
80	90	14	7	5	3	2
90	100	20	8	6	5	2
100	110	25	2	9	1	1
110	120	21	6	2	2	1
120	130	23	1	1	1	
130	140	10	4	3		
140	150	11	2	0		
150	160	5	0	0		
160	170	0	1	0		
170	180	2	0	0		
180	190	1	1	1		
190	200	0	1			
200	210	0	1			
210	220	1				
total		191	105	101	96	94
mean distance		96.62	70.05	53.61	41.35	32.98
variance		1403.14	2084.85	1558.06	951.83	657.16
mean and variance of fitted distributions ($\alpha=4.7$)						
		96.34	70.10	53.65	41.37	32.87
		1393.17	2093.74	1556.24	946.31	677.31
χ^2 ($\alpha=4.7$)						
sig level		1.00	0.15	0.07	0.07	0.80
relative survival chance						
$\alpha=4.70$		1.00	0.69	0.43	0.40	0.23

Figure 10: Histograms and Fitted Density Functions

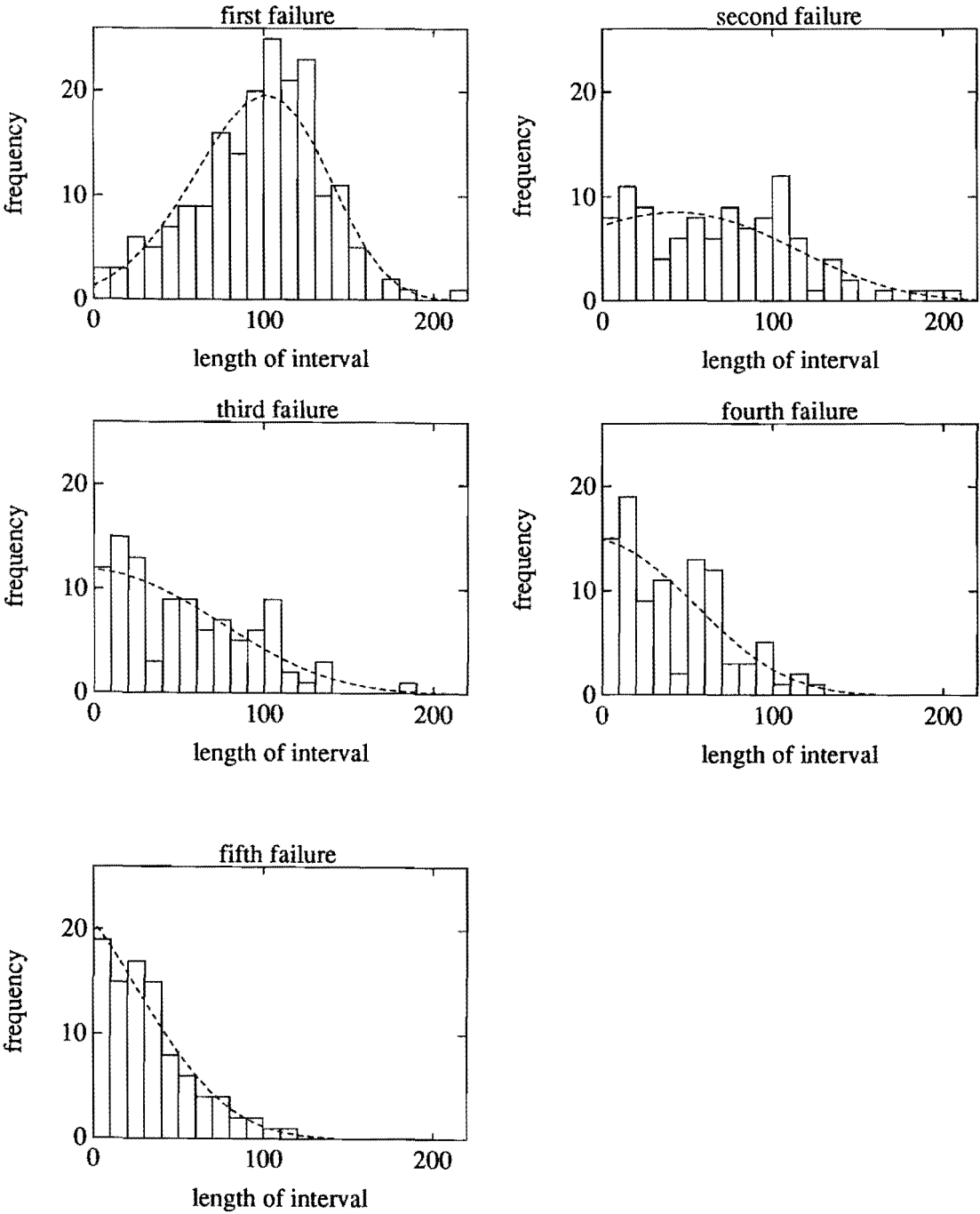


Figure 11: Empirical and Fitted Survival Functions

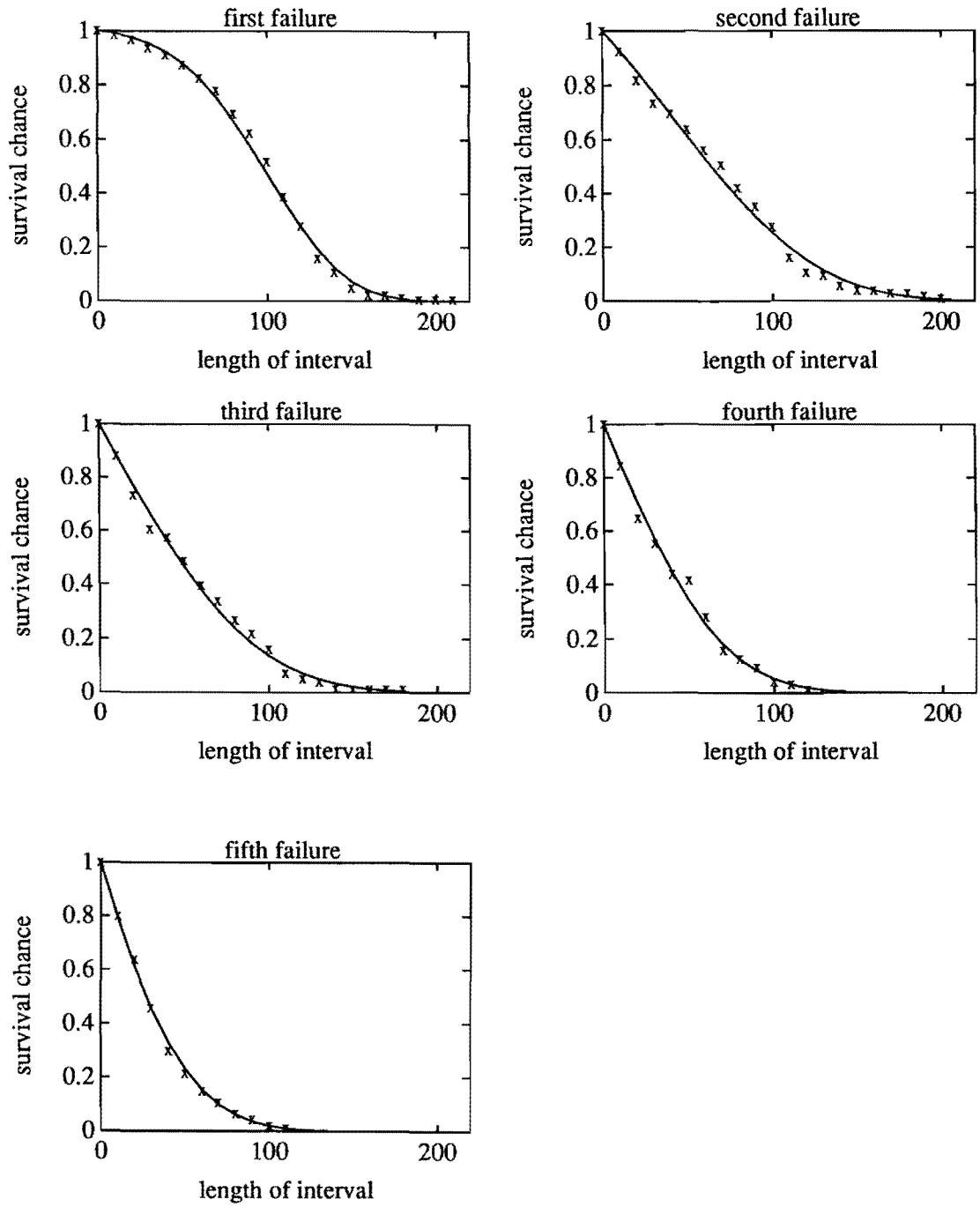


Figure 12: Empirical and Fitted Hazard Rates

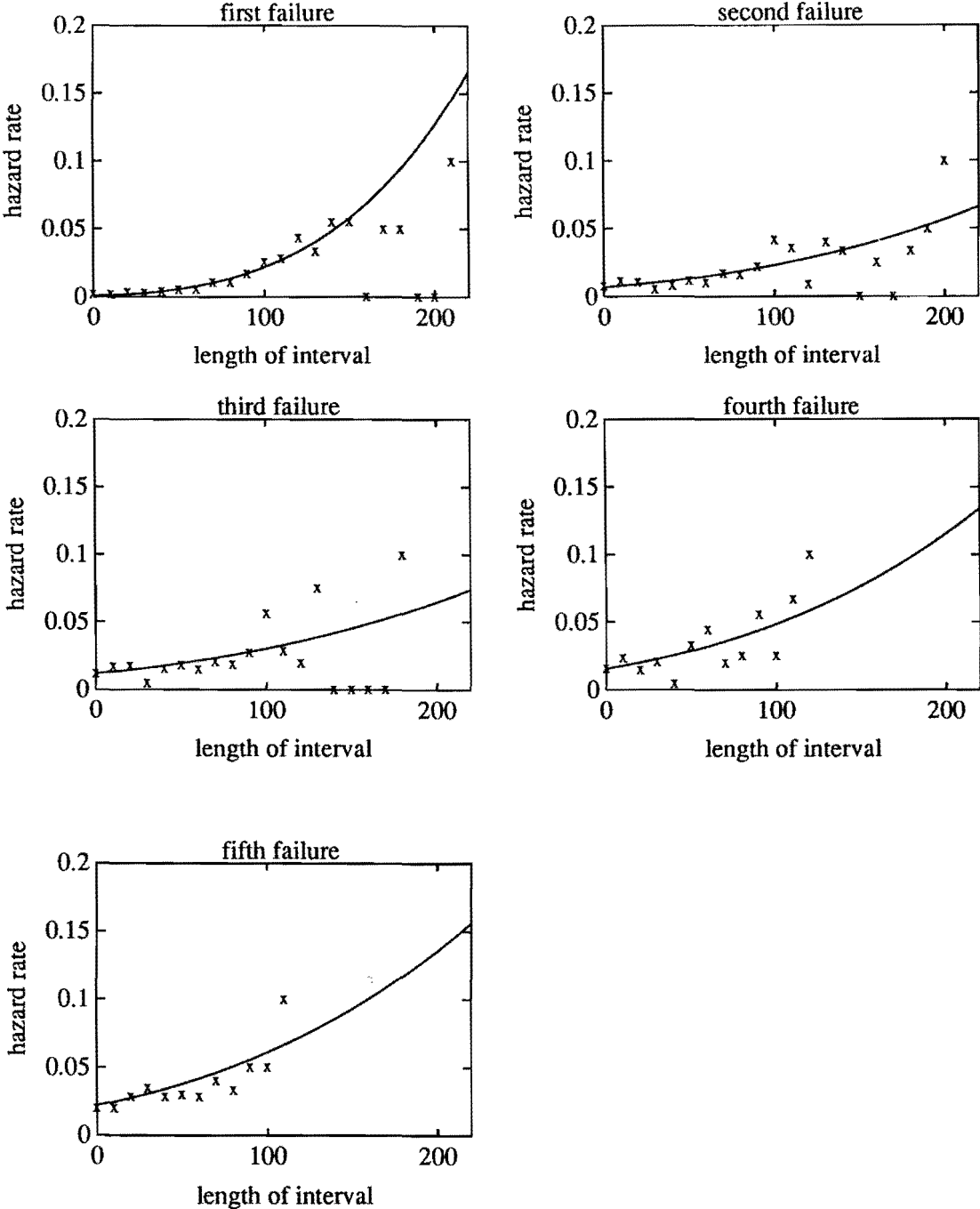


TABLE 3: Estimated parameters and likelihoods

α	4.70		5.04		5.50	
j	ψ_j	τ_j	ψ_j	τ_j	ψ_j	τ_j
1	173.65	63.47	179.32	68.10	202.46	91.92
2	319.85	260.52	258.59	198.58	388.47	331.89
3	352.57	340.47	370.03	355.61	432.33	424.81
4	283.19	278.20	401.38	435.58	346.99	345.62
5	289.69	314.45	308.52	331.57	361.12	392.12
likelihood	-1506.54		-1513.73		-1506.33	

analysis an additive hazards model seemed plausible with the sequence number of the failure as an explanatory variable. Pijnenburg failed to find estimators for the additive hazards model

$$h_j(x) = (\beta_1 + \beta_2 j) + h_0(x)$$

The reasons for the lack of estimators are now clear from the above results. In the light of the difficulties with the additive hazards model, a virtual age model seems a reasonable alternative. Indeed, the interval lengths from a non-homogeneous Poisson process show a similar pattern. The model is formulated precisely as in the accelerated failure time version above with a Weibull as the underlying distribution. The model is

$$F_0(x;\alpha) = 1 - e^{-x^\alpha} \qquad f_0(x;\alpha) = \alpha x^{\alpha-1} e^{-x^\alpha}$$

As with many other data sets no covariates were reported and so the simplest version of the model was taken with interval hazard rate

$$h_j(x) = \frac{1}{\psi_j} h_0\left(\frac{x + \tau_j}{\psi_j} ; \alpha\right)$$

A grouped a grouped likelihood was used. The likelihood function appears to have multiple maxima and the two parameters τ_j and ψ_j are also highly correlated. The likelihood was maximised using the *PC-MATLAB* package. Two different sets of parameters produced equal

values of the likelihood, but only one of these sets was not rejected on the basis of chi-square tests. Thus there are once more indications of an identifiability problem. The results are reported in Table 3 and illustrated in the Figures 10–12.

15. Summary

The discussion and examples given here show that although the parametric models can be fitted to failure data from repairable systems there are frequently identification problems, something not usually discussed in reliability analysis, and the unsatisfactory behaviour of the likelihood function and estimators. Remarkably, the conclusions seem to be that data analysis based on non-homogeneous Poisson processes or proportional hazards is likely to yield most of the information available in the data, even though they do not necessarily truly represent the underlying process and may even seem unlikely in certain situations. In particular proportional hazards appears very robust and requires few assumptions. Thus while the proportional hazards may not give an adequate representation of the process, it can frequently give a useful of the relative importance and influence of explanatory variables and allows the detection of dependencies.

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