

## The lost sales inventory model

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## **The lost sales inventory model**

W.G.M.M. Rutten, K. van Donselaar, A.G. de Kok and G.J.K. Regterschot

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Graduate School of Industrial Engineering and Management Science  
Eindhoven University of Technology  
P.O. Box 513, Paviljoen F11  
NL-5600 MB Eindhoven  
The Netherlands  
Phone: +31.40.473828

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# THE LOST SALES INVENTORY MODEL

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## 1. INTRODUCTION

There are many articles which describe an inventory policy in which backordering is allowed. The number of articles on the situation with lost sales, however, is far less. Most authors neglect the difference between lost sales and backordering stating that the difference is negligible if the service level is high [Silver, 1981; Tijms & Groenevelt, 1984]. But what to do when the service level is not high?

In practice a low service level may be beneficial. To illustrate this, the following example is given. Two products (X1 and X2) are sold. Product X1 is made out of raw material A, but may also be made out of raw material B. Raw material B is slightly better and therefore slightly more expensive compared to raw material A. Product X2 is made out of raw material B (see Figure 1). Both raw materials A and B are stored. The demand for X1 and X2 is very uncertain. In this situation it may well be worthwhile to limit the amount of inventory for raw material A and to use raw material B instead of raw material A if raw material A runs out of stock. At that moment demand for raw material A is lost. So in order to control this type of situation, methods to determine the order quantity are needed for the lost sales model with low service levels.

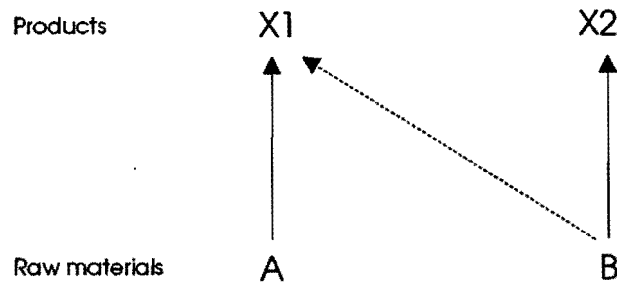


Figure 1. The product structures of products X1 and X2.  
 The straight lines indicate the standard recipes,  
 the dotted line represents the alternative recipe.

Nahmias and Cohen remarked that the periodic review leadtime lost sales inventory problem has not received much attention in the literature [Nahmias, 1979; Cohen et al., 1988]. This lack of attention in the literature is due to the complex nature of the problem. In contrast to the case with backordering, where the sum of inventory on hand and on order gives enough information to calculate the reorder quantity, in case of lost sales the total vector of inventory on hand and on order needs to be considered in order to calculate the reorder quantity.

In case backordering is allowed and an “order-up-to” policy is used, it is easy to find the order-up-to level [Silver & Peterson, 1985; Tijms & Groenevelt, 1984; Schneider, 1978]. If we wish to realize a service level  $\alpha$ , which we define as the probability of no stockout in a period, the order-up-to level  $S$  can be calculated. This  $S$  value will result in a service level of  $\alpha$  for the policy *with* backordering. However, if we don’t allow for backordering (in other words: if we are dealing with the lost sales model), using the “backorder” order-up-to level will result in a higher  $\alpha$  service level than planned, for inventory on hand during the leadtime is higher in case of lost sales compared to the case of backordering (see Figure 2).

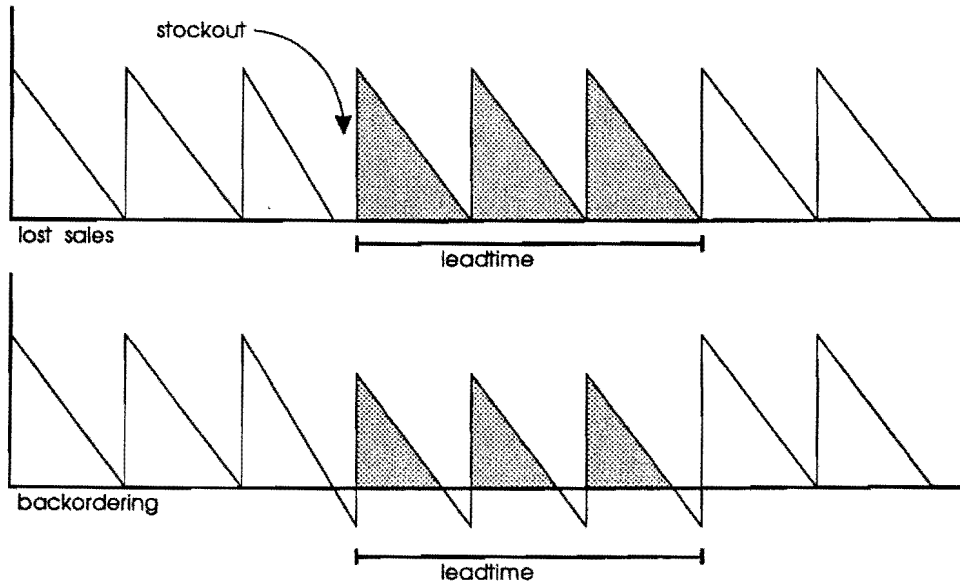


Figure 2. Mean stock during the leadtime in case of lost sales and in case of backordering

Simulation results show that the underestimation of the  $\alpha$  service level depends on the leadtime and on the target service level (see Figure 3). The larger the leadtime and the smaller the target service level, the higher the underestimation of the service level will be. In conclusion, using the backorder policy in case of lost sales as proposed in the literature gives a significant deviation for smaller service levels.

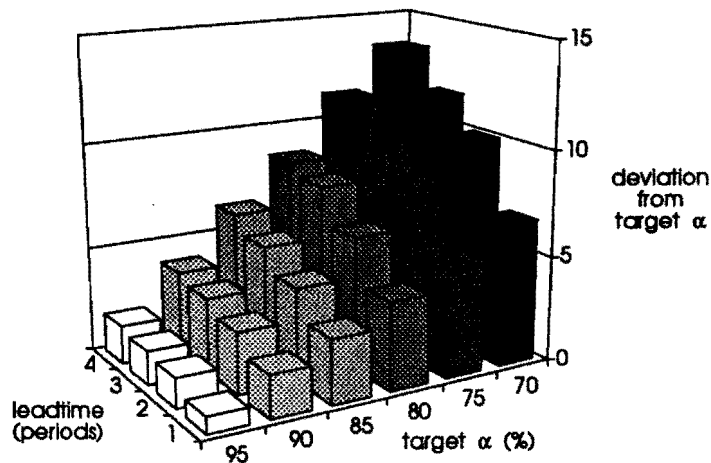


Figure 3. Deviation from the target  $\alpha$  when using the backorder  $S$ -level in case of lost sales [Rutten, 1991]

For the lost sales case, Karlin and Scarf succeeded in finding an explicit expression for the distribution function of the inventory on hand for two specific situations: (1) situations with a leadtime equal to one period and (2) situations with negative exponentially distributed demand and a leadtime larger than one period [Karlin & Scarf, 1958]. The distribution function of the inventory on hand enabled them to determine an appropriate reorder quantity in these specific situations.

This paper aims at an exact inventory policy for the lost sales case with leadtimes larger than one period. Therefore, we will develop a general formula for the probability of a stockout in case of lost sales. In section three, we will show how to calculate the reorder quantity for a Erlang distributed demand based on this probability (the Erlang distribution is equal to the Gamma distribution with a scaling parameter  $\lambda$  and an *integer* parameter  $\eta$ ). Section four will present an approximate formula for the case of lost sales and finally we will give some conclusions and suggestions for further research.

## 2. THE PROBABILITY OF STOCKOUT IN CASE OF LOST SALES

We will use the following notations and reorder cycle:

Notations:

- $I_t$  inventory on hand at the start of period  $t$ , before an order arrives
- $k$  leadtime in periods
- $Q_{t-k}$  quantity ordered at start of period  $t-k$  (which, by definition, will arrive at start of period  $t$ )
- $\xi_t$  demand during period  $t$
- $\alpha$  desired  $\alpha$  service level; probability that in a period demand is smaller than or equal to available inventory at the start of the period.

The reorder cycle (see Figure 4) in the periodic review inventory model is as follows:

1. starting inventory on hand (equals the resulting inventory on hand in  $t-1$ ) is  $I_t$
2. the reorder quantity  $Q_t$  is determined and the reordered quantity  $Q_{t-k}$  is received
3. the inventory on hand for satisfying demand during period  $t$  equals  $I_t + Q_{t-k}$
4. the demand during the period is met as long as inventory is available.

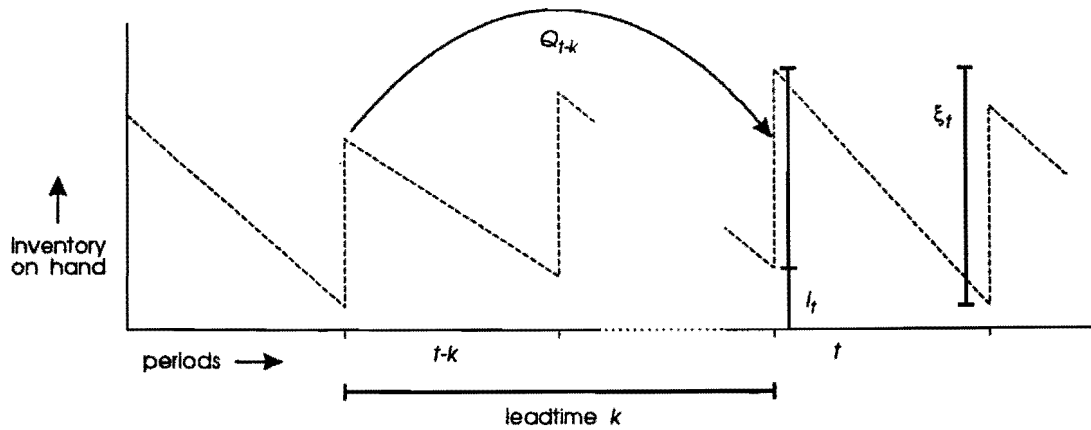


Figure 4. Inventory in time

In our inventory policy, backordering is not allowed. Moreover, the order-up-to level is dynamic; every period  $t$  the order-up-to level is chosen such that after reordering the expected service level in period  $t+k$  equals  $\alpha$ . In other words, the reorder quantity  $Q_t$  is taken as large as is required to assure that the probability of a stockout occurrence in period  $t+k$  is  $1-\alpha$ . For leadtime  $k$  this gives the following probability:

$$Pr\{ \xi_{t+k} > I_{t+k} + Q_t \} = 1 - \alpha \quad (1)$$

It should be noted that there is no proof available that the form of this reorder policy is optimal. The policy is also more complex compared to e.g. the policy of the backorder model or the policy suggested by Morton [1969] or Nahmias [1979]. The advantage of this policy however is the fact that the service level is known beforehand.

Essential with the above probability (1) is the determination of the value of  $I_{t+k}$ . First we will show how these probabilities can be calculated in case the leadtime equals zero periods, one period and two periods. The structure of the calculation scheme for these cases will help to determine a general formula for calculating the probability for leadtime  $k$ ,  $k > 2$ .



### Case $k=0$

If the leadtime  $k=0$ , then

$$Pr\{ \xi_t > I_t + Q_t \} = 1 - \alpha \quad (2)$$

It is easy to see that this equation is equivalent to the equation for a backorder-case. In other words, if the leadtime equals zero there is no difference between the order-up-to level in case of backordering ( $S$ ) and the level in case of lost sales ( $I_t + Q_t = S$ ); only the reorder quantity can be different.

### Case $k=1$

If the leadtime  $k=1$ , we know the values of  $I_t$  and  $Q_{t-1}$ . The inventory at the start of period  $t+1$  is as follows:

$$I_{t+1} = \begin{cases} 0 & \text{if } \xi_t \geq I_t + Q_{t-1} \\ I_t + Q_{t-1} - \xi_t & \text{if } \xi_t < I_t + Q_{t-1} \end{cases} \quad (3)$$

The probability of a stockout is (according to equation (1)):

$$Pr\{ \xi_{t+1} > I_{t+1} + Q_t \} = 1 - \alpha \quad (4)$$

From (3) and (4) follows:

$$Pr\{ [\xi_{t+1} > Q_t \cap \xi_t \geq I_t + Q_{t-1}] \cup [\xi_{t+1} > I_t + Q_{t-1} + Q_t - \xi_t \cap \xi_t < I_t + Q_{t-1}] \} = 1 - \alpha$$

This can be written as (see Appendix A):

$$Pr\{ \xi_{t+1} > Q_t \cap \xi_{t+1} + \xi_t > I_t + Q_{t-1} + Q_t \} = 1 - \alpha \quad (5)$$

### Case $k=2$

If the leadtime  $k=2$ , then  $I_t$  and  $Q_{t-2}$  and  $Q_{t-1}$  are known. The inventory for period  $t+1$  is calculated as follows:

$$I_{t+1} = \begin{cases} 0 & \text{if } \xi_t \geq I_t + Q_{t-2} \\ I_t + Q_{t-2} - \xi_t & \text{if } \xi_t < I_t + Q_{t-2} \end{cases}$$

and the inventory for period  $t+2$  is then:

$$I_{t+2} = \begin{cases} 0 & \text{if } \xi_t \geq I_t + Q_{t-2} \text{ and } \xi_{t+1} \geq Q_{t-1} \\ 0 & \text{if } \xi_t < I_t + Q_{t-2} \text{ and } \xi_{t+1} \geq I_t + Q_{t-2} + Q_{t-1} - \xi_t \\ Q_{t-1} - \xi_{t+1} & \text{if } \xi_t \geq I_t + Q_{t-2} \text{ and } \xi_{t+1} < Q_{t-1} \\ I_t + Q_{t-2} + Q_{t-1} - \xi_t - \xi_{t+1} & \text{if } \xi_t < I_t + Q_{t-2} \text{ and } \xi_{t+1} < I_t + Q_{t-2} + Q_{t-1} - \xi_t \end{cases}$$

The probability of a stockout in case  $k=2$  is (according to equation (1)):

$$\Pr\{\xi_{t+2} > I_{t+2} + Q_t\} = 1 - \alpha$$

This leads to:

$\Pr\{$

$$[\xi_{t+2} > Q_t \cap \xi_t \geq I_t + Q_{t-2} \cap \xi_{t+1} \geq Q_{t-1}] \cup$$

$$[\xi_{t+2} > Q_t \cap \xi_t < I_t + Q_{t-2} \cap \xi_{t+1} \geq I_t + Q_{t-2} + Q_{t-1} - \xi_t] \cup$$

$$[\xi_{t+2} > Q_{t-1} + Q_t - \xi_{t+1} \cap \xi_t \geq I_t + Q_{t-2} \cap \xi_{t+1} < Q_{t-1}] \cup$$

$$[\xi_{t+2} > I_t + Q_{t-2} + Q_{t-1} + Q_t - \xi_t - \xi_{t+1} \cap \xi_t < I_t + Q_{t-2} \cap \xi_{t+1} < I_t + Q_{t-2} + Q_{t-1} - \xi_t]$$

$$\} = 1 - \alpha$$

Analogous to the case with  $k=1$  (equation (5)) this probability can be rewritten as:

$$\Pr\{\xi_{t+2} > Q_t \cap \xi_{t+1} + \xi_{t+2} > Q_{t-1} + Q_t \cap \xi_t + \xi_{t+1} + \xi_{t+2} > I_t + Q_{t-2} + Q_{t-1} + Q_t\} = 1 - \alpha \quad (6)$$

If we study the equations (2), (5) and (6) we can observe a general structure which results in postulating the general probability (7) for leadtime  $k, k \geq 0$ :

$$Pr\left( \xi_{t+k} > Q_t \cap \xi_{t+k} + \xi_{t+k-1} > Q_t + Q_{t-1} \cap \dots \right. \\ \left. \cap \sum_{i=0}^{k-1} \xi_{t+k-i} > \sum_{i=0}^{k-1} Q_{t-i} \cap \sum_{i=0}^k \xi_{t+k-i} > I_t + \sum_{i=0}^k Q_{t-i} \right) = 1 - \alpha \quad (7)$$

The formal proof of the correctness of (7) is given in [Regterschot et al., 1993]. Here only an intuitive explanation of (7) is given. Consider the probability that there will be no stockout in period  $t+k$ . No stockout occurs if either the demand in the last period ( $\xi_{t+k}$ ) is smaller than or equal to the last order which arrived at the stock point ( $Q_t$ ), or if the demand in the last two periods ( $\xi_{t+k} + \xi_{t+k-1}$ ) is smaller than or equal to the last two orders which arrived at the stock point ( $Q_t + Q_{t-1}$ ), or ... or if the demand in all periods ( $\sum_{i=0}^k \xi_{t+k-i}$ ) is smaller than or equal to the content of the entire system ( $I_t + \sum_{i=0}^k Q_{t-i}$ ). In formula this yields:

$$Pr\left( \xi_{t+k} \leq Q_t \cup \xi_{t+k} + \xi_{t+k-1} \leq Q_t + Q_{t-1} \cup \dots \cup \sum_{i=0}^k \xi_{t+k-i} \leq I_t + \sum_{i=0}^k Q_{t-i} \right) = \alpha$$

The transformation from this equation to equation (7) is trivial.

### 3. CALCULATING REORDER QUANTITIES

In this section we will work out the above mentioned probabilities into equations for the reorder quantity. We assume that the demand function follows an Erlang distribution with parameters  $\lambda$  and  $\eta$ . The Erlang distribution is very appropriate in inventory control [Burgin, 1975; Tijms & Groenevelt, 1984]. The parameter  $\eta$  is kept integer. If  $\eta$  is not integer, a simple interpolation technique can be used (see [Cox, 1962], page 20) to apply the results given in this paper. The leadtime is taken  $k$  periods. Demand per period is independent. Equation (8) gives the probability density function for the demand during the leadtime plus review period.

$$\varphi(\xi) = \frac{\lambda^{(k+1)\eta}}{((k+1)\eta-1)!} \xi^{(k+1)\eta-1} e^{-\lambda\xi} \quad (8)$$

In case backordering is allowed and an order-up-to policy is used, it is easy to find the order-up-to level. If we wish to realize a service level  $\alpha$ , which is defined as the probability of no stockout, the order-up-to level  $S$  can be calculated with equation (9) in case demand has a Erlang distribution function [Mood et al., 1974].

$$\int_0^S \varphi(\xi) d\xi = \alpha \quad \Rightarrow \quad \sum_{i=0}^{(k+1)\eta-1} \frac{(\lambda S)^i}{i!} e^{-\lambda S} = 1 - \alpha \quad (9)$$

As we have seen in section 1, using this formula in case of lost sales will result in an underestimation of the service level  $\alpha$  (remember Figure 3). Therefore, we will develop a specific equation for the case of lost sales.

If the leadtime is less than or equal to two periods it is relatively easy to proof the expressions, which can be used to determine the reorder quantity for the lost sales case given a target service level  $\alpha$ .

If  $k=0$ :

$$\sum_{i=0}^{\eta-1} \frac{\lambda^i (I_t + Q_t)^i}{i!} e^{-\lambda(I_t + Q_t)} = 1 - \alpha \quad (10)$$

If  $k=1$ :

$$\sum_{i=0}^{\eta-1} \frac{(\lambda Q_t)^i}{i!} \sum_{j=0}^{2\eta-1-i} \frac{\lambda^j (I_t + Q_{t-1})^j}{j!} e^{-\lambda(I_t + Q_{t-1} + Q_t)} = 1 - \alpha \quad (11)$$

If  $k=2$ :

$$\sum_{i=0}^{\eta-1} \frac{(\lambda Q_t)^i}{i!} \sum_{j=0}^{2\eta-1-i} \frac{(\lambda Q_{t-1})^j}{j!} \sum_{r=0}^{3\eta-1-i-j} \frac{\lambda^r (I_t + Q_{t-2})^r}{r!} e^{-\lambda(I_t + Q_{t-2} + Q_{t-1} + Q_t)} = 1 - \alpha \quad (12)$$

Equation (10) follows from the observation that the lost sales and backorder cases are identical for  $k=0$ .

Equations (10) to (12) show a clear pattern. This pattern suggests that the general formula for any leadtime  $k$  is as follows:

$$1 - \alpha = \sum_{i_1=0}^{\eta-1} \frac{(\lambda Q_t)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda Q_{t-1})^{i_2}}{i_2!} \dots \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda Q_{t-k+1})^{i_k}}{i_k!} \sum_{i_{k+1}=0}^{(k+1)\eta-1-\sum_{j=1}^k i_j} \frac{\lambda^{i_{k+1}} (I_t + Q_{t-k})^{i_{k+1}}}{i_{k+1}!} e^{-\lambda \left( I_t + \sum_{i=0}^k Q_{t-i} \right)} \quad (13)$$

The exact proof of equation (13) is given in Appendix B.

## 4. APPROXIMATE FORMULA FOR THE CASE OF LOST SALES

Because solving the exact lost sales formula (13) requires a lot of CPU-time for large leadtimes, we derived a simpler formula. The easiest approximation is the backorder formula, but it gives a significant underestimation of  $\alpha$  for larger leadtimes. If we compare the probability for the backorder case to the probability for the lost sales case (equation (7)), we see that the last term of (7) is equal to the complete backorder formula.

$$\Pr\left\{ \xi_{t+k} > Q_t \cap \xi_{t+k} + \xi_{t+k-1} > Q_t + Q_{t-1} \cap \dots \right. \\ \left. \cap \sum_{i=0}^{k-1} \xi_{t+k-i} > \sum_{i=0}^{k-1} Q_{t-i} \cap \sum_{i=0}^k \xi_{t+k-i} > I_t + \sum_{i=0}^k Q_{t-i} \right\} = 1 - \alpha \quad (7)$$

Since the backorder formula gives an underestimation of  $\alpha$ , using more terms from the exact lost sales probability should give a better approximation. In accordance with the approximation of Morton [1971] the first term is chosen as the extra term. So, by using the first and last term of (7), we expect to achieve a better approximation compared to the backorder formula. Equation (14) gives the simpler formula.

$$\Pr\left\{ \xi_{t+k} > Q_t \cap \sum_{i=0}^k \xi_{t+k-i} > I_t + \sum_{i=0}^k Q_{t-i} \right\} = 1 - \alpha \quad (14)$$

In order to compare formula (14) with the approximation of both Morton [1971] and Nahmias [1979] we introduce the sets  $Z_i(Q_t), i = 0, 1, \dots, k$ :

$$Z_i(Q_t) = \left\{ \sum_{j=0}^i \xi_{t+k-j} > \sum_{j=0}^i Q_{t-j} \right\} \text{ for } i < k \text{ and}$$

$$Z_k(Q_t) = \left\{ \sum_{j=0}^k \xi_{t+k-j} > I_t + \sum_{j=0}^k Q_{t-j} \right\}.$$

The order quantity can be determined by one of the following equations:

$$1 - \alpha = \Pr\{ Z_0(Q_t) \cap Z_1(Q_t) \cap \dots \cap Z_k(Q_t) \} \quad (\text{leading to } \alpha \text{ (exact)})$$

$$1 - \alpha = \Pr\{ Z_k(Q_t) \} \quad (\text{leading to } \alpha \text{ (backorder)})$$

$$1 - \alpha = \Pr\{ Z_0(Q_t) \cap Z_k(Q_t) \} \quad (\text{leading to } \alpha \text{ (formula14)})$$

Morton and Nahmias both use an approximation which consists of two steps: First they determine an order quantity  $Q_{t,1}$  by solving the equation  $\Pr\{ Z_0(Q_t) \} = 1 - \alpha$ . Likewise they derive  $Q_{t,2}$  from the equation  $\Pr\{ Z_k(Q_t) \} = 1 - \alpha$ . Their ultimate order quantity then is equal to  $Q_t$  with  $Q_t = \min(Q_{t,1}, Q_{t,2})^+$ .

Now it is easy to see that

$$\alpha(\text{backorder}) \geq \alpha(\text{Morton}) \geq \alpha(\text{formula14}) \geq \alpha(\text{exact})$$

Calculating the reorder quantity formula for an Erlang distributed demand according to equation (14) results in equation (15) (see Proposition 2 in Appendix B):

$$1 - \alpha = \sum_{i=0}^{\eta-1} \frac{(\lambda Q_t)^i}{i!} \sum_{j=0}^{(k+1)\eta-1-i} \frac{(\lambda E_t)^j}{j!} e^{-\lambda(E_t+Q_t)} \quad (15)$$

where  $E_t = I_t + \sum_{i=1}^k Q_{t-i}$ .

Note that (15) equals (13) in case the leadtime is equal to zero or one period. In Figure 5 simulation results are given for a small value (70%) of the target service level. It can be seen that the difference between the target and the resulting service level increases as the leadtime increases.

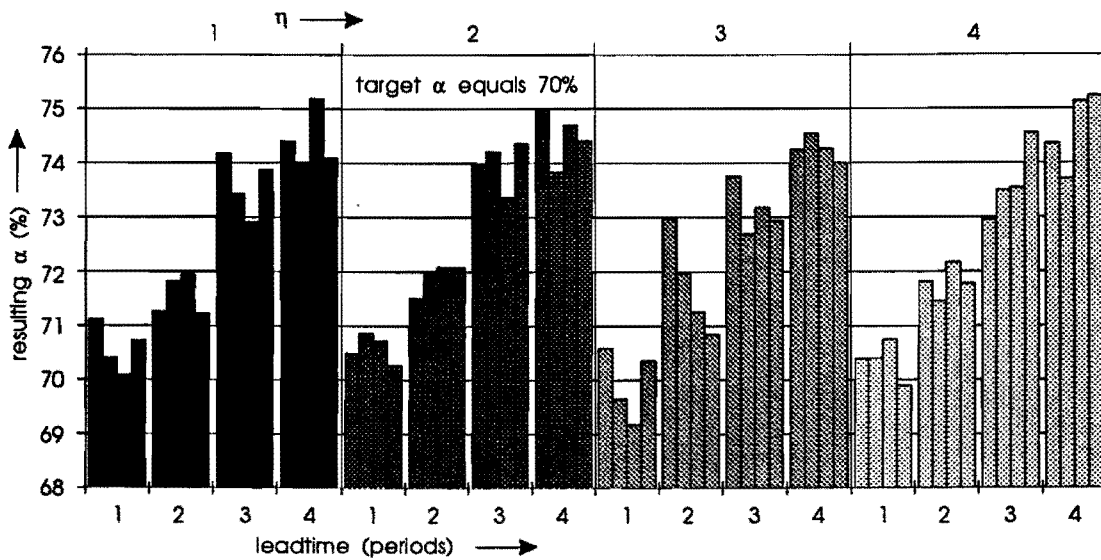


Figure 5. Resulting  $\alpha$  when using formula (15)  
Four simulation results are given per  $\eta$ /leadtime combination [Rutten, 1991]

The underestimation of the service level, however, is far less than in case the backorder order-up-to level would be used as can be seen in Figure 6.

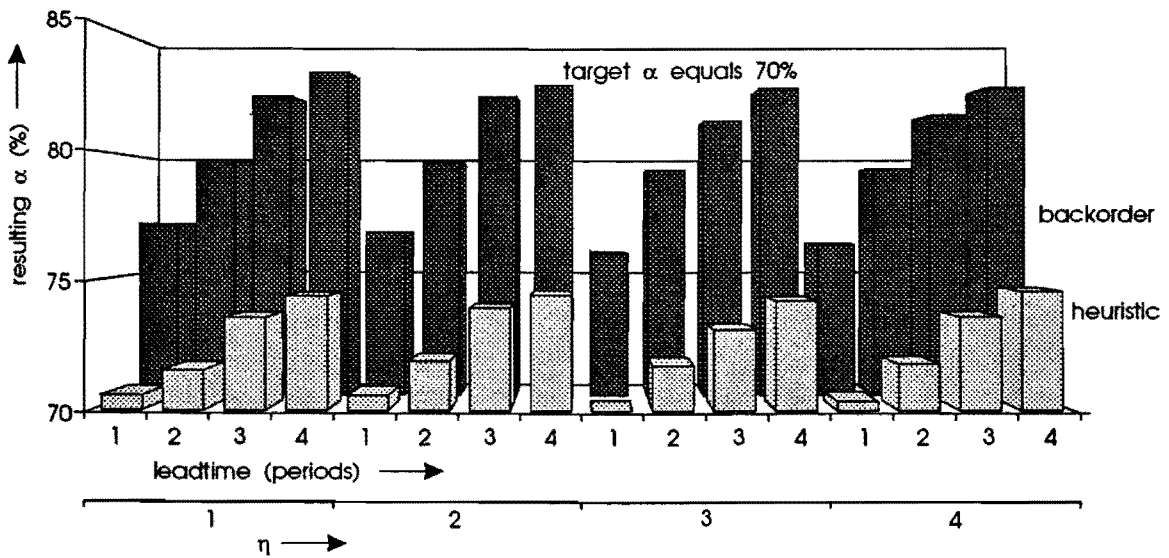


Figure 6. Resulting  $\alpha$  when using formula (15) and the backorder order-up-to level (9) in case of lost sales [Rutten, 1991].



## 5. CONCLUSIONS AND FURTHER RESEARCH

For small values of the service level  $\alpha$  and long leadtimes, the backorder policy is not a good model for a situation with lost sales. Therefore, we defined an order-up-to policy with a dynamic order-up-to level and we found a general formula, which expresses the service level in terms of the inventory on hand and on order. Furthermore we found an expression which can be used to determine the reorder quantity in a lost sales model if demand is assumed to be Erlang distributed with integer value for the parameter  $\eta$ .

Since solving the exact lost sales formula is very time-consuming, we suggested an approximation. Simulation results show that the suggested approximation performs significantly better than the traditional approximation (which is: using the results of the backorder policy in a lost sales environment). However, as may be expected, the approximation does have a significant underestimation of the service level for systems with both a small service level  $\alpha$  and a large leadtime.

In order to extend our results to general Gamma distributions, a simple interpolation technique as indicated by Cox [1962] might be helpful. Further research could be devoted to evaluating the resulting approximation for general Gamma distributions.

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## APPENDIX A.

We have two equations:

$$Pr\{ [\xi_{t+1} > Q_t \cap \xi_t \geq I_t + Q_{t-1}] \cup [\xi_{t+1} + \xi_t > I_t + Q_{t-1} + Q_t \cap \xi_t < I_t + Q_{t-1}] \} = 1 - \alpha \quad (\text{App. 1})$$

$$Pr\{ \xi_{t+1} > Q_t \cap \xi_{t+1} + \xi_t > I_t + Q_{t-1} + Q_t \} = 1 - \alpha \quad (\text{App. 2})$$

Is equation (App. 1) equal to (App. 2)?

Denote:

$$A = \{ \xi_{t+1} > Q_t \}$$

$$B = \{ \xi_t \geq I_t + Q_{t-1} \}$$

$$C = \{ \xi_t + \xi_{t+1} > I_t + Q_{t-1} + Q_t \}$$

Then equation (App. 1) can be written as:  $Pr\{ [A \cap B] \cup [C \cap \neg B] \}$

and equation (App. 2) can be written as:  $Pr\{ A \cap C \}$

These two probabilities are graphically shown in Figure 7.

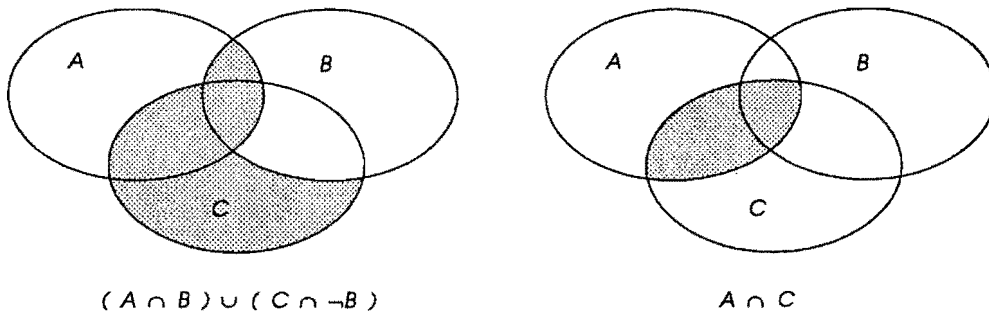


Figure 7. Graphically exhibited probabilities

It can be seen that equation (App. 1) has two "surfaces" more than equation (App. 2):

$$A \cap B \cap \neg C = \{ \xi_{t+1} > Q_t \cap \xi_t \geq I_t + Q_{t-1} \cap \xi_t + \xi_{t+1} \leq I_t + Q_{t-1} + Q_t \} = \emptyset$$

$$C \cap \neg A \cap \neg B = \{ \xi_t + \xi_{t+1} > I_t + Q_{t-1} + Q_t \cap \xi_{t+1} \leq Q_t \cap \xi_t < I_t + Q_{t-1} \} = \emptyset$$

q.e.d.

## APPENDIX B.

### Lemma 1

For  $\eta \geq 1$ ,  $\eta$  integer:

$$\int_A^B x^{\eta-1} (a-x)^i dx = \sum_{j=0}^{\eta-1} \frac{-x^j (a-x)^{\eta+i-j}}{j! (\eta+i-j)!} i! (\eta-1)! \Big|_A^B$$

### Proof

(by means of complete induction)

For  $\eta=1$  the result is trivial. Suppose Lemma 1 holds for a certain  $\eta$ . We want to prove

Lemma 1 for  $\eta+1$ . By means of partial integration we get:

$$\begin{aligned} \int_A^B x^{\eta+1} (a-x)^i dx &= \frac{-x^{\eta+1} (a-x)^{i+1}}{i+1} \Big|_A^B + \frac{\eta+1}{i+1} \int_A^B x^{\eta} (a-x)^{i+1} dx \\ &= \frac{-x^{\eta+1} (a-x)^{i+1}}{i+1} + \frac{\eta+1}{i+1} \sum_{j=0}^{\eta} \frac{-x^j (a-x)^{\eta+i+2-j}}{j! (\eta+i+2-j)!} (i+1)! \eta! \Big|_A^B \\ &= \sum_{j=0}^{\eta+1} \frac{-x^j (a-x)^{\eta+i+2-j}}{j! (\eta+i+2-j)!} i! (\eta+1)! \Big|_A^B \end{aligned}$$

q.e.d.

### Proposition 1

For a lost sales system with leadtime  $k$  and Erlang distributed demand (with parameters  $\eta$  and  $\lambda$ ), the stock-out probability is equal to:

$$\begin{aligned} 1-\alpha &= \sum_{i_1=0}^{\eta-1} \frac{(\lambda Q_t)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda Q_{t-1})^{i_2}}{i_2!} \dots \\ &\dots \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda Q_{t-k+1})^{i_k}}{i_k!} \sum_{i_{k+1}=0}^{(k+1)\eta-1-\sum_{j=1}^k i_j} \frac{\lambda^{i_{k+1}} (I_t + Q_{t-k})^{i_{k+1}}}{i_{k+1}!} e^{-\lambda \left( I_t + \sum_{i=0}^k Q_{t-i} \right)} \end{aligned}$$

After substitution this can be written as:

$$Q1-\alpha = \sum_{i_1=0}^{\eta-1} \frac{(\lambda y_1)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda y_2)^{i_2}}{i_2!} \dots \sum_{i_{k+1}=0}^{(k+1)\eta-1-\sum_{j=1}^k i_j} \frac{(\lambda y_{k+1})^{i_{k+1}}}{i_{k+1}!} e^{-\lambda \sum_{i=1}^{k+1} y_i} \quad (\text{App. 3})$$

where

$$y_{k+1} = I_t + Q_{t-k} ; y_j = Q_{t+1-j} , j = 1, \dots, k$$

$$x_j = \xi_{t+1+k-j} \quad j = 1, \dots, k+1$$

### Proof

Using the same substitution as above, formula (7) can be written as:

$$1-\alpha = 1 - \Pr \left\{ x_1 \leq y_1 \cup x_1+x_2 \leq y_1+y_2 \cup \dots \cup \sum_{i=1}^{k+1} x_i \leq \sum_{i=1}^{k+1} y_i \right\}$$

This can be written as:

$$1-\alpha = 1 - \Pr \left\{ (((\dots((y_{k+1}-x_{k+1})^+ + y_k - x_k)^+ \dots)^+ + y_2 - x_2)^+ + y_1 - x_1)^+ > 0 \right\} \quad (\text{App.4})$$

with  $(a-b)^+ := \max(0, a-b)$

Proposition 1 is proved by means of complete induction: For  $k = 0$  the lost sales model and the backorder model are the same. So the service level is equal to

$$\alpha = 1 - \sum_{i=0}^{\eta-1} \frac{\lambda^i (I_t + Q_t)^i}{i!} e^{-\lambda(I_t + Q_t)} = 1 - \sum_{i_1=0}^{\eta-1} \frac{(\lambda y_1)^{i_1}}{i_1!} e^{-\lambda y_1}$$

This is equal to (App. 3) since  $\sum_{j=1}^0 i_j = 0$  by definition.

In order to proof Proposition 1 for leadtime  $k$ , we assume Proposition 1 holds for leadtime  $k-1$ . For a system with leadtime  $k$  formula (App. 4) applies. We may transform formula (App. 4) into:

$$1-\alpha = 1 - \int_0^{y_{k+1}} \Pr\left\{ (((\dots(y_{k+1}-x_{k+1}+y_k-x_k)^+\dots)^++y_2-x_2)^++y_1-x_1)^+ > 0 \right\} dF_{x_{k+1}}(x_{k+1}) \\ - \int_{y_{k+1}}^{\infty} \Pr\left\{ (((\dots(y_k-x_k)^+\dots)^++y_2-x_2)^++y_1-x_1)^+ > 0 \right\} dF_{x_{k+1}}(x_{k+1}) \quad (\text{App. 5})$$

The first probability in equation (App. 5) is equal to the service level of a lost sales system with leadtime  $k-1$  and with inventory on hand equal to  $y_{k+1} + y_k - x_{k+1}$ . The second probability in equation (App. 5) is equal to the service level of a lost sales system with leadtime  $k-1$  and with inventory on hand equal to  $y_k$ . Since we assume that Proposition 1 holds for leadtime  $k-1$ , equation (App. 3) may be used to rewrite equation (App. 5) as follows:

$$1-\alpha = \int_0^{y_{k+1}} \sum_{i_1=0}^{\eta-1} \frac{(\lambda y_1)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda y_2)^{i_2}}{i_2!} \dots \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{\lambda^{i_k}}{i_k!} \times \\ \times (y_{k+1}-x_{k+1}+y_k)^{i_k} e^{-\lambda \left( \sum_{i=1}^{k+1} y_i - x_{k+1} \right)} dF_{x_{k+1}}(x_{k+1}) \\ + \int_{y_{k+1}}^{\infty} \sum_{i_1=0}^{\eta-1} \frac{(\lambda y_1)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda y_2)^{i_2}}{i_2!} \dots \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda y_k)^{i_k}}{i_k!} e^{-\lambda \sum_{i=1}^k y_i} dF_{x_{k+1}}(x_{k+1}) \\ = \sum_{i_1=0}^{\eta-1} \dots \sum_{i_{k-1}=0}^{(k-1)\eta-1-\sum_{j=1}^{k-2} i_j} (I_1+I_2) \quad (\text{App. 6})$$

with

$$I_1 = \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \int_0^{y_{k+1}} \frac{\lambda^{i_k} (y_{k+1}-x_{k+1}+y_k)^{i_k}}{i_k!} e^{-\lambda \left( \sum_{i=1}^{k+1} y_i - x_{k+1} \right)} \frac{\lambda^\eta x_{k+1}^{\eta-1} e^{-\lambda x_{k+1}}}{(\eta-1)!} dx_{k+1}$$



and

$$\begin{aligned}
 I_2 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda y_k)^{i_k}}{i_k!} e^{-\lambda \sum_{i=1}^k y_i} \int_{y_{k+1}}^{\infty} dF_{x_{k+1}}(x_{k+1}) \\
 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda y_k)^{i_k}}{i_k!} e^{-\lambda \sum_{i=1}^k y_i} \sum_{i_{k+1}=0}^{\eta-1} \frac{e^{-\lambda y_{k+1}} (\lambda y_{k+1})^{i_{k+1}}}{i_{k+1}!}
 \end{aligned} \tag{App. 7}$$

Using Lemma 1,  $I_1$  can be rewritten as follows:

$$\begin{aligned}
 I_1 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{\lambda^{i_k+\eta}}{i_k! (\eta-1)!} e^{-\lambda \sum_{i=1}^{k+1} y_i} \int_0^{y_{k+1}} (y_{k+1}-x_{k+1}+y_k)^{i_k} x_{k+1}^{\eta-1} dx_{k+1} \\
 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \sum_{j=0}^{\eta-1} \frac{-x_{k+1}^j (y_{k+1}-x_{k+1}+y_k)^{\eta+i_k-j}}{j! (\eta+i_k-j)!} \Bigg|_0^{y_{k+1}} \times \lambda^{i_k+\eta} e^{-\lambda \sum_{i=1}^{k+1} y_i} \\
 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \left[ \frac{(\lambda (y_{k+1}+y_k))^{\eta+i_k}}{(\eta+i_k)!} - \sum_{j=0}^{\eta-1} \frac{(\lambda y_{k+1})^j (\lambda y_k)^{\eta+i_k-j}}{j! (\eta+i_k-j)!} \right] e^{-\lambda \sum_{i=1}^{k+1} y_i} \\
 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \left[ \sum_{j=0}^{\eta+i_k} \frac{(\lambda y_{k+1})^j (\lambda y_k)^{\eta+i_k-j}}{j! (\eta+i_k-j)!} - \sum_{j=0}^{\eta-1} \frac{(\lambda y_{k+1})^j (\lambda y_k)^{\eta+i_k-j}}{j! (\eta+i_k-j)!} \right] e^{-\lambda \sum_{i=1}^{k+1} y_i}
 \end{aligned}$$

After subtracting the second sum from the first sum and after using the transformation

$l = \eta + i_k - j$  this can be written as:

$$\begin{aligned}
 I_1 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \sum_{l=0}^{i_k} \frac{(\lambda y_{k+1})^{i_k+\eta-l} (\lambda y_k)^l}{(i_k+\eta-l)! l!} e^{-\lambda \sum_{i=1}^{k+1} y_i} \\
 &= \sum_{l=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \sum_{i_k=l}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \frac{(\lambda y_{k+1})^{i_k+\eta-l}}{(i_k+\eta-l)!} \times \frac{(\lambda y_k)^l}{l!} e^{-\lambda \sum_{i=1}^{k+1} y_i}
 \end{aligned}$$

(using the transformation  $i_{k+1} = i_k + \eta - l$ .)

$$\begin{aligned}
 &= \sum_{l=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \sum_{i_{k+1}=\eta}^{k\eta-1-\sum_{j=1}^{k-1} i_j+\eta-l} \frac{(\lambda y_{k+1})^{i_{k+1}}}{(i_{k+1})!} \times \frac{(\lambda y_k)^l}{l!} e^{-\lambda \sum_{i=1}^{k+1} y_i} \\
 &= \sum_{i_k=0}^{k\eta-1-\sum_{j=1}^{k-1} i_j} \sum_{i_{k+1}=\eta}^{(k+1)\eta-1-\sum_{j=1}^k i_j} \frac{(\lambda y_k)^{i_k}}{i_k!} \times \frac{(\lambda y_{k+1})^{i_{k+1}}}{i_{k+1}!} e^{-\lambda \sum_{i=1}^{k+1} y_i}
 \end{aligned}$$

This result together with formulas (App. 6) and (App. 7) yields:

$$1 - \alpha = \sum_{i_1=0}^{\eta-1} \frac{(\lambda y_1)^{i_1}}{i_1!} \sum_{i_2=0}^{2\eta-1-i_1} \frac{(\lambda y_2)^{i_2}}{i_2!} \dots \sum_{i_{k+1}=0}^{(k+1)\eta-1-\sum_{j=1}^k i_j} \frac{(\lambda y_{k+1})^{i_{k+1}}}{i_{k+1}!} e^{-\lambda \sum_{i=1}^{k+1} y_i}$$

q.e.d.

## Proposition 2

In case  $\xi_{t+k-i}$  with  $i = 0, 1, \dots, k$  is Erlang distributed, then:

$$\begin{aligned}
 \Pr\{ \xi_{t+k} > Q_t \cap \sum_{i=0}^k \xi_{t+k-i} > E_t + Q_t \} = \\
 \sum_{j=0}^{\eta-1} \frac{(\lambda Q_t)^j}{j!} \sum_{i=0}^{(k+1)\eta-1-j} \frac{(\lambda E_t)^i}{i!} e^{-\lambda(E_t+Q_t)}
 \end{aligned}$$

(App. 8)

## Proof

The probability in equation (App. 8) can be written as follows:

$$1-\alpha = \int_{Q_t}^{E_t+Q_t} \int_{E_t+Q_t-x}^{\infty} \frac{\lambda^{(k+1)\eta} x^{\eta-1} y^{k\eta-1}}{(\eta-1)!(k\eta-1)!} e^{-\lambda(x+y)} \partial y \partial x +$$

$$+ \int_{E_t+Q_t}^{\infty} \int_0^{\infty} \frac{\lambda^{(k+1)\eta} x^{\eta-1} y^{k\eta-1}}{(\eta-1)!(k\eta-1)!} e^{-\lambda(x+y)} \partial y \partial x$$

Since  $\int_B^{\infty} \frac{\lambda^{k\eta} y^{k\eta-1}}{(k\eta-1)!} e^{-\lambda y} \partial y = \sum_{i=0}^{k\eta-1} \frac{(\lambda B)^i}{i!} e^{-\lambda B}$ , the last equation can be written as:

$$1-\alpha = \int_{Q_t}^{E_t+Q_t} \frac{\lambda^{\eta} x^{\eta-1}}{(\eta-1)!} \sum_{i=0}^{k\eta-1} \frac{\lambda^i (E_t+Q_t-x)^i}{i!} e^{-\lambda(E_t+Q_t)} \partial x +$$

$$+ \sum_{i=0}^{\eta-1} \frac{\lambda^i (E_t+Q_t)^i}{i!} e^{-\lambda(E_t+Q_t)}$$
(App. 9)

Using Lemma 1, the integral in equation (App. 9) can be written as:

$$\lambda^{\eta+i} e^{-\lambda(E_t+Q_t)} \sum_{i=0}^{k\eta-1} \sum_{j=0}^{\eta-1} \frac{Q_t^j E_t^{\eta+i-j}}{j!(\eta+i-j)!}$$

By substitution of  $r=\eta+i-j$ , this becomes:

$$e^{-\lambda(E_t+Q_t)} \sum_{j=0}^{\eta-1} \sum_{r=\eta-j}^{k\eta-1-j} \frac{(\lambda Q_t)^j}{j!} \frac{(\lambda E_t)^r}{r!}$$
(App. 10)

Replacing the integral in equation (App. 9) by (App. 10) and using the fact that

$$\sum_{j=0}^{\eta-1} \sum_{i=0}^{\eta-1-j} \frac{x^i}{i!} \frac{y^j}{j!} = \sum_{i=0}^{\eta-1} \frac{(x+y)^i}{i!},$$

it follows that:

$$1-\alpha = e^{-\lambda(E_t+Q_t)} \sum_{j=0}^{\eta-1} \frac{(\lambda Q_t)^j}{j!} \sum_{i=0}^{(k+1)\eta-1-j} \frac{(\lambda E_t)^i}{i!}$$

q.e.d.