The RUBIN CMS procedure for general state-space models

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ABSTRACT

The dynamic analysis of structural systems should frequently be based on systems with a very limited number of degrees of freedom. This is for example the case when there are non-linearities involved or when the mathematical model has to play a central role in a control strategy.
To reduce the order of the system (the number of degrees of freedom) methodologies have been developed based on fysical understanding, the so-called Component-Mode-Synthesis (CMS) techniques.
An important step in the development of these procedures is the assumption that the (sub)systems should be weakly damped or at least be based on so-called proportional or Rayleigh damping.
In many situations this will not be so such as in the case of models for mechatronic systems, motorcar suspension models etc. In that case a formalism based on **real** vibration modes will give very bad results.
In this report a well-known CMS-procedure (the RUBIN-CMS technique) will be generalized for the case of **general** damping leading to a transformation based on (residual) flexibility modes and free-free dynamic modes, (Complex RITZ-vectors). The fysical meaning of these modes will be illustrated and an example will be presented and discussed showing the potential of this extension of the CMS-procedure.

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Chapter 1

INTRODUCTION

In the analysis of nonlinear dynamical systems we frequently have to deal with systems which in general will be described sufficiëntly accurate by linear models and where the nonlinearities will have a local character.
This is for example the case in complex vehicle models where frame, cabin, etc. can be modelled by a linear model but where the (local) nonlinear behaviour of the engine-suspension has to be taken into account.
The number of degrees of freedom (dof's) of the assembled model should be very limited in order to make a numerical evaluation not excessive time-consuming. Therefore reduction techniques such as the CMS (Component Mode Synthesis) technique have been developed (see for example Fey [1] and v.d. Vorst [2]);
In such a reduction technique a structure will be subdivided into several components (or substructures) which will be reduced individually. After the reduction of each (linear) component, the components are coupled, nonlinearities are added and the final, nonlinear equations are solved.

In general, for a linear component the equations of motions can be written as :

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t) \tag{1.1}$$

where

- n=number of degrees of freedom (dof's)
- q(t)=column (n*1), containing the dof's
- M = (n*n) symmetric Massmatrix
- B = (n*n) symmetric Dampingmatrix
- K = (n*n) symmetric Stiffnessmatrix
- f(t) = column of external loads

In general, reduction of the number of degrees of freedom (n) is done by writing the dof's "q" as a linear combination of a (much) smaller number (n_p) of generalized coordinates p by

means of a transformation matrix T ($n * n_p$):

$$q(t) = Tp(t) \tag{1.2}$$

Introduction of this transformation in the expression for kinetic energy, potential energy and virtual work of the external loads leads to a reduced set of differential equations:

$$M^{red}\ddot{p}(t) + B^{red}\dot{p}(t) + K^{red}p(t) = f^{red}(t) \tag{1.3}$$

where

$$\begin{aligned} M^{red} &= T^T MT \\ B^{red} &= T^T BT \\ K^{red} &= T^T KT \\ f^{red}(t) &= T^T f(t) \end{aligned} \tag{1.4}$$

This reduction proces can be seen as a special variant of the Ritz-method. The columns of the transformation matrix T are often called the Ritz-vectors.
In the literature a number of methods are presented to select these Ritz vectors, but only two of them seem to be really important, namely the **Graig-Bampton method** (see [5]) and the **Rubin method** (see[2]). In chapter 2 the standard Rubin method will be presented. In chapter 3 a modification, mainly consisting of a formulation in first order differential equations (so-called state-space form) together with an extrapolation of the strategy will be given. It will also give an illustration of the state flexibility modes and the special attention which is needed for the state rigid body modes.
In chapter 4 an example will be discussed to demonstrate that the method is able to generate reduced state-space models, accurate in a preselected frequency band.
In chapter 7 finally some conclusions and recommendations are given.

Chapter 2

THE RUBIN METHOD

The RUBIN reduction method (or CMS method) in its standard form is based on the **undamped** n-dof system described by the equations:

$$M\ddot{q}(t) + Kq(t) = f(t) \tag{2.1}$$

If we suppose for free vibrations the solution:

$$q(t) = ue^{\lambda t} \tag{2.2}$$

we get the eigenvalue problem:

$$[\lambda^2 M + K]u = 0 \tag{2.3}$$

For a component with **Rigid Body Modes** some of the eigenvalues λ_k (k=1,2, ...n) will be zero. Suppose we have r of these rigid body modes u_r^i, (i=1,2,...r) for each of which:

$$Ku_r^i = 0 \tag{2.4}$$

we can make the matrix U_r :

$$U_r = [u_r^1, u_r^2, ...u_r^r] \tag{2.5}$$

The rigid-body modes are assumed to be "mass-normalized" which means that:

$$U_r^T M U_r = I_{rr} \tag{2.6}$$

with I_{rr} a (r*r) identity matrix.

The total set of dof's q(t) is devided in socalled boundary dof's $q_b(t)$ (number is b) and internal dof's $q_i(t)$ (number=i, i=n-b):

$$q = \begin{bmatrix} q_b \\ q_i \end{bmatrix} \tag{2.7}$$

Then eq. (2.1) can be written:

$$\begin{bmatrix} M_{bb} & M_{bi} \\ M_{ib} & M_{ii} \end{bmatrix} \begin{bmatrix} \ddot{q}_b \\ \ddot{q}_i \end{bmatrix} + \begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{bmatrix} q_b \\ q_i \end{bmatrix} = \begin{bmatrix} f_b \\ f_i \end{bmatrix} \tag{2.8}$$

We assume that forces are only present at the interface dof's, so $f_i = 0$.
For the definition of socalled **Inertia Relief Modes** we assume a set of interface forces f_b leading to rigid-body accelerations of the substructure. Taking:

$$\ddot{q}_r = U_r \ddot{\xi} \tag{2.9}$$

we get:

$$\ddot{q}_r = U_r U_r^T f_b \tag{2.10}$$

The Inertia Relief Modes are assumed to be the quasi-static deformation modes corresponding to a force-system f_b and $-M\ddot{q}_r$ (which is in equilibrium):

$$K u_{inert} = f_b - M\ddot{q}_r = (I - MU_rU_r^T) f_b \tag{2.11}$$

If we put a unity force at each of the interface dof's whereas the other interface dof's are unloaded we can write:

$$F^B = \begin{bmatrix} I_{bb} \\ O_{ib} \end{bmatrix} \tag{2.12}$$

where O_{pq} means a (p*q) nulmatrix.
So, we get,

$$KU_{inert} = (I - MUU^T)F^B = PF^B \tag{2.13}$$

Due to the singularity of the stiffnessmatrix K, system (2.13) cannot be solved directly. For this an alternative procedure will be chosen.
If we split the internal dof-set q_i in a set q_w (with w elements), and a set q_r (with r elements, i=w+r, r=n-b-w):

$$q = \begin{bmatrix} q_b \\ q_i \end{bmatrix} = \begin{bmatrix} q_b \\ q_w \\ q_r \end{bmatrix} \tag{2.14}$$

we can write (2.13) as:

$$\begin{bmatrix} K_{bb} & K_{bw} & K_{br} \\ K_{wb} & K_{ww} & K_{wr} \\ K_{rb} & K_{rw} & K_{rr} \end{bmatrix} \begin{bmatrix} U_{bb} \\ U_{wb} \\ O_{rb} \end{bmatrix} = \begin{bmatrix} P_{bb} & P_{bw} & P_{br} \\ P_{wb} & P_{ww} & P_{wr} \\ P_{rb} & P_{rw} & P_{rr} \end{bmatrix} \begin{bmatrix} I_{bb} \\ O_{wb} \\ O_{rb} \end{bmatrix} + \begin{bmatrix} O_{bb} \\ O_{wb} \\ R_{rb} \end{bmatrix} \tag{2.15}$$

The **r-set q_r has been suppressed** . If these dof's have been chosen well, the stiffness matrix K will become positive-definite and due to the equilibrium of the force-system (f_b and $-M\ddot{q}_r$, the reaction forces R_{rb} will be zero.
The **Flexibility matrix G** is now defined as:

$$G = \begin{bmatrix} K_{ee}^{-1} & O_{er} \\ O_{re} & O_{rr} \end{bmatrix} \text{ with } K_{ee} = \begin{bmatrix} K_{bb} & K_{bw} \\ K_{wb} & K_{ww} \end{bmatrix} \tag{2.16}$$

It can be shown that:

$$G_e = P^T G P = U_e \Lambda_{ee}^{-1} U_e^T \tag{2.17}$$

where U_e contains the vibration modes of the eigenvalueproblem (2.3) corresponding to the **non-zero** eigenvalues (stored in the diagonalmatrix Λ_{ee}).

If we divide these nonzero-eigenvalues in a number k which is kept in the analysis and a number d which will be deleted (based on some frequency criterion), so:

$$U_e = \begin{bmatrix} U_k & U_d \end{bmatrix}; \; \Lambda_{ee} = \begin{bmatrix} \Lambda_{kk} & O_{kd} \\ O_{dk} & \Lambda_{dd} \end{bmatrix} \tag{2.18}$$

we can write :

$$G_e = U_k \Lambda_{kk}^{-1} U_k^T + U_d \Lambda_{dd}^{-1} U_d^T \tag{2.19}$$

The **Residual Flexibility Modes** are now defined as:

$$\Phi = \left[U_d \Lambda_{dd}^{-1} U_d^T \right] F^B = \left[G_e - U_k \Lambda_{kk}^{-1} U_k^T \right] F^B \tag{2.20}$$

For these modes not the complete eigenvalueproblem has to be solved. They can be extracted from one matrix-inverse and the modes U_k which have to be preserved in the analysis. The transformation matrix T $(n * (r + b + k))$ will be:

$$T = [\Phi, U_r, U_k] \tag{2.21}$$

The influence of U_k in Φ can also be removed, so instead of using residual flexibility modes, we can also use flexibility modes Ψ:

$$\Psi := G_e F^B \tag{2.22}$$

This has some advantages:

- Using the coupling procedure (see next) of Martinez the numerical proces will be more robust numerically.
- When taking more and more modes into account the flexibility matrix will not change, so updating can be avoided.

SUBSTRUCTURE COUPLING

For the coordinate-transformation we have found:

$$\begin{bmatrix} q_b \\ q_i \end{bmatrix} = \begin{bmatrix} \Phi & U_r & U_k \end{bmatrix} P = \begin{bmatrix} \Phi_{bb} & U_{bm} \\ \Phi_{ib} & U_{im} \end{bmatrix} \begin{bmatrix} p_b \\ p_m \end{bmatrix} = Tp \tag{2.23}$$

where m=r+k.
Coupling of substructures using the generalized coordinates p is rather difficult. Therefore we use a procedure for re-introduction of the fysical boundary degrees of freedom q_b.
From (2.23) we can see:

$$q_b = \Phi_{bb} p_b + U_{bm} p_m \rightarrow p_b = \Phi_{bb}^{-1} q_b - \Phi_{bb}^{-1} U_{bm} p_m \tag{2.24}$$

We can define a second coordinate transformation:

$$\begin{bmatrix} p_b \\ p_m \end{bmatrix} = \begin{bmatrix} \Phi_{bb}^{-1} & -\Phi_{bb}^{-1} U_{bm} \\ O_{mb} & I_{mm} \end{bmatrix} \begin{bmatrix} q_b \\ p_m \end{bmatrix} = T_2 p_2 \tag{2.25}$$

The total coordinate-transformation will be:

$$q = \begin{bmatrix} q_b \\ q_i \end{bmatrix} = T T_2 p_2 = T_t \begin{bmatrix} q_b \\ p_m \end{bmatrix} \tag{2.26}$$

In the final dof-set, the interface dof's q_b are still present which makes the substructure-coupling straightforward.
It can be shown that Φ_{bb} is regular if n-r-k > b. For flexibility modes instead of residual flexibility modes we use the inverse Ψ_{bb}^{-1} which is independent of the number of modes U_k which will be taken into account in the CMS procedure.
It can also be shown that in case of flexibility modes the reduced matrices will become more "sparse", so less computer memory is needed (see [2]).

Chapter 3

GENERALIZED RUBIN REDUCTION

For a dynamic system with n degrees of freedom under the assumption of viscous damping a system of n coupled, 2^{nd} order differential equations can be given:

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t) \tag{3.1}$$

where:

- M: Massmatrix, (n*n), positive definite
- B: Viscous dampingmatrix, (n*n)
- K: Stiffness matrix, (n*n), positive definite
- q(t): (n*1), column of generalized coordinates
- f(t): (n*1), column of generalized forces

The n, 2^{nd} order differential equations (3.1) can be transformed into 2n, 1^{st} order differential equations by adding the n equations: $M\dot{q}(t) = M\dot{q}(t)$ and introducing the state-vector $y(t)$:

$$y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{3.2}$$

We than get :

$$C\dot{y}(t) + Dy(t) = g(t) \tag{3.3}$$

where:

$$C = \begin{bmatrix} B & M \\ M & O \end{bmatrix}; D = \begin{bmatrix} K & O \\ O & -M \end{bmatrix}; g(t) = \begin{bmatrix} f(t) \\ o \end{bmatrix} \tag{3.4}$$

If the stiffnessmatrix K is singular, the system will have a number (r) of **Rigid Body Modes** u_r^i (i=1,2,..r), which are collected in the matrix :$U_r = [u_r^1, u_r^2, ...u_r^r]$. For these modes:

$$KU_r = O_{nr} \tag{3.5}$$

For the homogeneous part of (3.3), we can write the eigenvalue problem:

$$[\lambda_k C + D]v_k = o \tag{3.6}$$

It can be shown that the existance of r rigid body modes leads to some eigenvalues λ_k equal to zero. The number of zero-eigenvalues (R) will be $\leq 2r$. For the eigenvalue problem (3.6) we can write :

$$CV\Lambda + DV = O \tag{3.7}$$

Using the rigid body modes u_r defined before in chapter 2, it can be shown (see appendix A) that two situations can exist:

Situation 1: A double root $\lambda_k = 0$ and $Bu_r = o$

In this case we have an eigenvalue $\lambda_k = 0$ with multiplicity 2 leading to one regular eigenvector and one generalized eigenvector. This is obvious the case for undamped systems but can also be valid for damped systems. In this case the matrix Λ will not be a diagonal matrix but will have Jordan blocks on the diagonal.

Situation 2: $Bu_r \neq o$

The system is nondefective, there is no generalized eigenvector corresponding to $\lambda_k = 0$. In fact $\lambda_k = 0$ is not a repeated root but a single root of the eigenvalue problem. In this case the matrix Λ will be a diagonalmatrix with the eigenvalue λ_k (which is zero) on the diagonal.

We assume that all the state-rigid body modes (regular or generalized) can be determined and stored in the matrix V_R, (2n*R) matrix ,$R \leq 2r$.

For a certain substructure the "n" dof's q can be split up into the "b" boundary dof's q_b and the "i" internal dof's q_i with n=i+b:

$$q = \begin{bmatrix} q_b \\ q_i \end{bmatrix} \tag{3.8}$$

as indicated in Fig. 3.1.

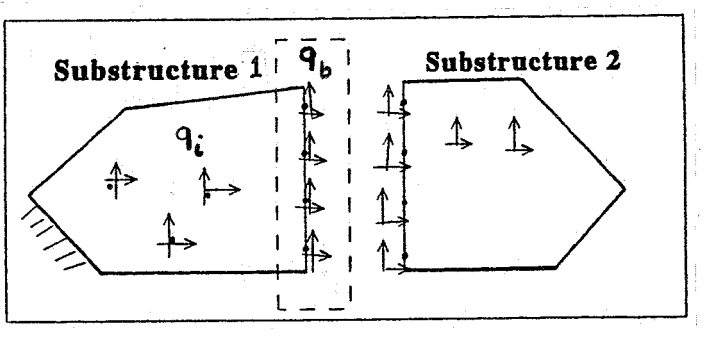

Fig 3.1. Substructures with internal and boundary dof's.

In general, internal dof's where external forces are acting or with prescribed displacements will also be selected in the set q_b.

After renumbering the dof's according to (3.8) a (N*1) state-vector $z(t)$ can be formed:

$$z(t) = \begin{bmatrix} q_b(t) \\ \dot{q}_b(t) \\ q_i(t) \\ \dot{q}_i(t) \end{bmatrix} = \begin{bmatrix} z_B(t) \\ z_I(t) \end{bmatrix} \tag{3.9}$$

where:

$$N = 2n; B = 2b; I = 2i; n = b + i; N = B + I \tag{3.10}$$

The differential equations for the dynamic behaviour of the substructure can, according to (3.3) now be written as:

$$C^*\dot{z}(t) + D^*z(t) = g^*(t) \tag{3.11}$$

where the matrices C^* and D^* follow from the corresponding matrices C and D by simply shifting rows and columns.
In the partitioned form we than get:

$$\begin{bmatrix} C^*_{BB} & C^*_{BI} \\ C^*_{IB} & C^*_{II} \end{bmatrix} \begin{bmatrix} \dot{z}_B \\ \dot{z}_I \end{bmatrix} + \begin{bmatrix} D^*_{BB} & D^*_{BI} \\ D^*_{IB} & D^*_{II} \end{bmatrix} \begin{bmatrix} z_B \\ z_I \end{bmatrix} = \begin{bmatrix} g^*_B \\ g^*_I \end{bmatrix} \tag{3.12}$$

where we assume that $g^*_I = o$.
For simplicity we further omit the $*$ symbol. For the corresponding eigenvalue problem R eigenvalues $\lambda_r = 0$ will be found with corresponding eigenvectors (regular or generalized) $v^i_r, (i = 1, 2, ..R)$. These vectors are put in the matrix V_R:

$$V_R = [v^1_r, v^2_r, ...v^R_r] \tag{3.13}$$

We define a matric C_{RR} (not necessarily diagonal) as:

$$C_{RR} = V^T_R C V_R \tag{3.14}$$

INERTIA RELIEF MODES

We assume a set of interface forces g_b leading to "rigid body accelerations" in a state-space sence:

$$C\dot{z}_r = g_b \tag{3.15}$$

If we assume that we can write these rigid body accelerations as a linear combination of the colums of the rigid body matrix V_R :

$$\dot{z}_r = V_R\dot{\xi} \tag{3.16}$$

we get

$$CV_R\dot{\xi} = g_b \tag{3.17}$$

and

$$\dot{z}_r = V_R C^{-1}_{RR} V^T_R g_b \tag{3.18}$$

The inertia relief modes are assumed to be the **quasi-static** modes according to a force system of g_b and $-C\dot{z}_r$ (which is in equilibrium):

$$Dz_{inert} = g_b - C\dot{z}_r = [I - CV_R C_{RR}^{-1} V_R^T] g_b := P g_b \tag{3.19}$$

All the inertia relief modes can be found by successively putting a force on each interface dof:

$$DZ_{inert} = PF_B \tag{3.20}$$

where

$$F_B = \begin{bmatrix} I_{BB} \\ O_{IB} \end{bmatrix} \tag{3.21}$$

Due to the singularity of K, the equations (3.20) cannot be solved directly.
The internal dof-set z_I will therefore be split-up in a set z_W with W components and a set z_R with R components, (with $N = B + I = B + W + R$):

$$z = \begin{bmatrix} z_B \\ z_I \end{bmatrix} = \begin{bmatrix} z_B \\ z_W \\ z_R \end{bmatrix} \tag{3.22}$$

We can than write:

$$\begin{bmatrix} D_{BB} & D_{BW} & D_{BR} \\ D_{WB} & D_{WW} & D_{WR} \\ D_{RB} & D_{RW} & D_{RR} \end{bmatrix} \begin{bmatrix} Z_{BB} \\ Z_{WB} \\ Z_{RB} \end{bmatrix} = \begin{bmatrix} P_{BB} & P_{BW} & P_{BR} \\ P_{WB} & P_{WW} & P_{WR} \\ P_{RB} & P_{RW} & P_{RR} \end{bmatrix} \begin{bmatrix} I_{BB} \\ O_{WB} \\ O_{RB} \end{bmatrix} \tag{3.23}$$

If the dof's z_R are chosen well and suppressed, the system can be solved.
The **Flexibility matrix G** is defined as

$$G = \begin{bmatrix} D_{EE}^{-1} & O_{ER} \\ O_{RE} & O_{RR} \end{bmatrix} \text{ with } D_{EE} = \begin{bmatrix} D_{BB} & D_{BW} \\ D_{WB} & D_{WW} \end{bmatrix} \tag{3.24}$$

where $E = B + W = N - R$.
The socalled elastic flexibility matrix G_E can be defined as:

$$G_E = V_E \Lambda_{EE}^{-1} C_{EE}^{-1} V_E^T \tag{3.25}$$

where Λ_{EE} contains the **nonzero** eigenvalues, V_E contains the corresponding eigenmodes and C_{EE} is defined as:

$$C_{EE} = V_E^T C V_E \tag{3.26}$$

It can be shown (see appendix B) that G_E can also be written as:

$$G_E = P^T G P \tag{3.27}$$

RESIDUAL FLEXIBILITY MODES

For the flexibility matrix G_E according to (3.25) we can write:

$$G_E = V_K \Lambda_{KK}^{-1} C_{KK}^{-1} V_K^T + V_D \Lambda_{DD}^{-1} C_{DD}^{-1} V_D^T \tag{3.28}$$

where V_K contains the eigenvectors from V_E which are kept in the analysis and V_D the eigenvectors of V_E which will be deleted, base on some frequency considerations.
The diagonal matrices Λ_{KK} respectively Λ_{DD} contain the corresponding (nonzero) eigenvalues.
The residual flexibility matrix will now be defined as:

$$\begin{aligned} \Phi_B = & \quad [\quad V_D \Lambda_{DD}^{-1} C_{DD}^{-1} V_D^T] F_B \\ \text{or} & \quad [\quad G_E - V_K \Lambda_{KK}^{-1} C_{KK}^{-1} V_K^T] F_B \end{aligned} \tag{3.29}$$

The final transformationmatrix T_{N_T} is based on rigid body modes V_R, residual flexibility modes Φ_B and free-free "kept" eigenmodes V_K:

$$T = [\Phi_B, V_R, V_K] \tag{3.30}$$

As already mentioned in chapter 2 the effect of the modes V_K in the matrix Φ_B can also be neglected. In that case we use **flexibility modes** Ψ_B:

$$\Psi_B = G_E F_B \tag{3.31}$$

and the transformation matrix reads:

$$T = [\Psi_B, V_R, V_K] \tag{3.32}$$

3.1 Illustration of the flexibility modes

In the preceeding chapter socalled (residual) flexibility modes have been determined as components of the transformation matrix T for the reduction of a substructure.
In this paragraph these components will be studied in detail in order to get some physical understanding as in the case of the classical, undamped reduction strategy.
As an example we look at a straight bar in longitudinal vibration modelled with 3 equal (standard) bar-elements: (see also appendix A).

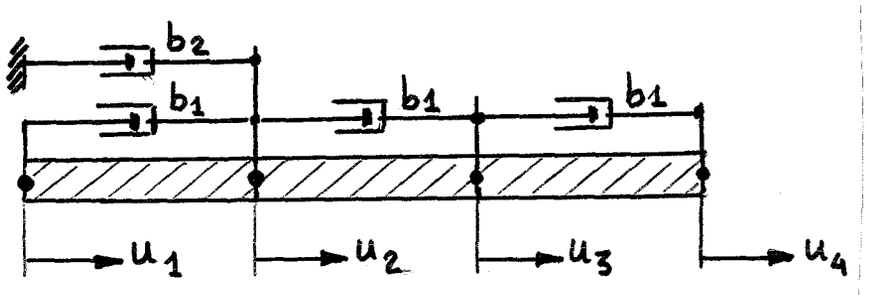

Fig. 3.2. Simple 4-dof bar problem.

As boundary dof's we select the dof's 1 and 4, so the internal dof's are the nr's 2 and 3, which leads to the state-space column:

$$z^T = [u_1, \dot{u}_1, u_4, \dot{u}_4, u_2, \dot{u}_2, u_3, \dot{u}_3] = [z_B^T, z_I^T]$$

CASE 1 : $b1 = 0.1 mboxandb2 = 0$
We can see that: $B = \alpha * K \Rightarrow R = 2$

$$V_R^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}; C_{RR} = \begin{bmatrix} 0 & 18 \\ 18 & 0 \end{bmatrix}$$

So we have one regular and also one generalized eigenvector for the state-space eigenvalue problem. We take: $z_R^T = [u_3, \dot{u}_3]$, so $[z_B^T, z_W^T] = [u_1, \dot{u}_1, u_4, \dot{u}_4, u_2, \dot{u}_2]$
See Fig 3.3.

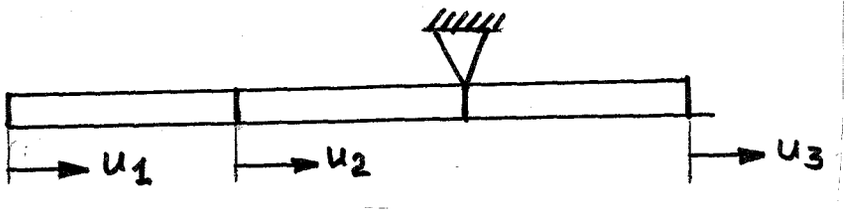

Fig 3.3. Constraint problem for flexibility evaluation.

For the flexibility matrix we can find:

$$G = \frac{1}{14}\begin{bmatrix} 28 & 0 & 0 & 0 & 14 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 14 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 \\ 14 & 0 & 0 & 0 & 14 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and for the flexibility modes: $\Psi_B = [P^T G P] F_B$ we get the results shown in Fig 3.4

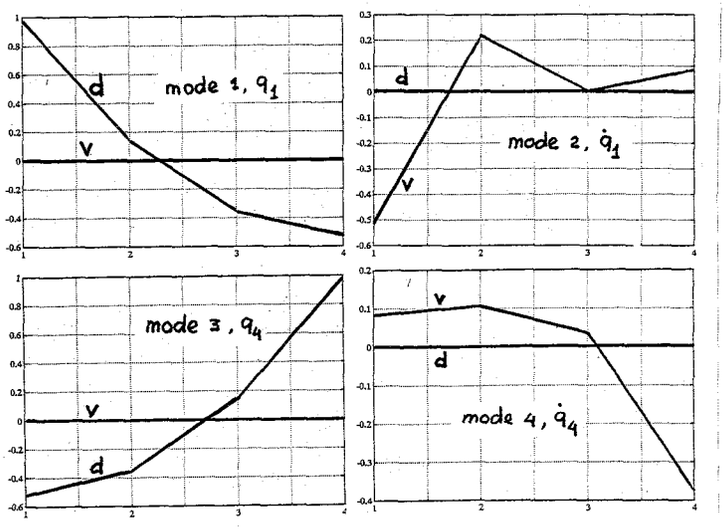

Fig. 3.4. Plot of the 4 flexibility modes. (d) indicates displacement dof's and (v) the corresponding velocity dof's

The modes 1 and 3 (corresponding to the end displacements) show only a displacement contribution and can be seen as a combination of a pure flexibility and some rigid body displacement. The modes 2 and 4 (corresponding to the velocity dof's) only have a velocity contribution and they look like a flexibility mode but not based on the stiffness matrix, but on the massmatrix (due to the inversion of matrix D).

CASE 2 : $b1 = 0.1$ and $b2 = 0.2 \Rightarrow R = 1$,
It is shown in Appendix A that we have only one (regular) eigenvector with zero-eigenvalue for the state space eigenvalue problem, namely the obvious, fysical rigid body mode. In this

case we take only $z_R = [u_3]$, the resulting flexibility modes are shown in Fig 3.5.

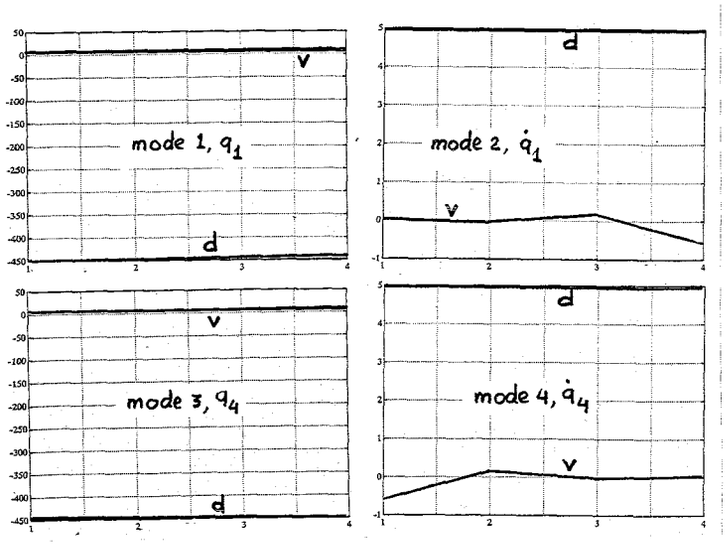

Fig. 3.5.Plot of the 4 flexibility modes for b=0.1. (d) indicates displacement dof's and (v) the corresponding velocity dof's

One can see that in this case each flexibility mode has a "displacement part" as well as a "velocity part", although in two cases they have a different order of magnitude (modes 1 and 3). Inspite of suppressing the fysical rigid body mode by displacement dof nr. 3 the res.flex.displacement part seems to show a large contribution from this single rigid body mode. However it is not the rigid body mode but the eigenvector corresponding to the nonzero eigenvalue, degenerated from the previous-generalized-eigenvector which is responsible for this. As given in appendix A, this eigenvector looks like: $v^T = [1, 1, 1, 1, 0.01, 0.01, 0.01, 0.01]$. The fysical interpretation of the flexibility modes of case 2 however is becoming difficult already.

3.2 Coupling procedure of Martinez

The starting point are the equations of motion of the substructure (3.11):

$$C\dot{z}(t) + Dz(t) = g(t) \tag{3.33}$$

We have defined a transformation matrix T for going from the (N) fysical coordinates z to a (much) smaller set of (p) generalized coordinates p, where P=R+K+B:

$$z = Tp \tag{3.34}$$

$$\begin{bmatrix} z_B \\ z_I \end{bmatrix} = \begin{bmatrix} \Psi_B & V_R & V_K \end{bmatrix} z_r = \begin{bmatrix} \Psi_{BB} & V_{BM} \\ \Psi_{IB} & V_{IM} \end{bmatrix} \begin{bmatrix} p_B \\ p_M \end{bmatrix} \tag{3.35}$$

We used $V_M = [V_R, V_K]$ with $M = R + K < N - B$. From (3.35) we can write:

$$p_B = \Psi_{BB}^{-1} z_B - \Psi_{BB}^{-1} V_{BM} p_M \tag{3.36}$$

The second coordinate transformation now becomes:

$$\begin{bmatrix} p_B \\ p_M \end{bmatrix} = \begin{bmatrix} \Psi_{BB}^{-1} & -\Psi_{BB}^{-1} V_{BM} \\ O_{MB} & I_{MM} \end{bmatrix} \begin{bmatrix} z_B \\ p_M \end{bmatrix} = T_2 p_2 \tag{3.37}$$

The total transformation reads:

$$z = \begin{bmatrix} z_B \\ z_I \end{bmatrix} = T T_2 p_2 = T_t \begin{bmatrix} z_B \\ p_M \end{bmatrix} \tag{3.38}$$

where T_t is a $N * (B + M)$matrix:

$$T_t = \begin{bmatrix} I_{BB} & O_{BM} \\ \Psi_{IB}\Psi_{BB}^{-1} & V_{IM} - \Psi_{IB}\Psi_{BB}^{-1} V_{BM} \end{bmatrix} \tag{3.39}$$

In the new coordinate set the fysical coordinates z_B (displacement and velocities of the boundary dof's) are present which makes the substructure coupling straightforward.

3.3 The reduced system

We start from the basic expression for the dynamical behaviour of the substructure in state-space formulation:

$$C\dot{z}(t) + Dz(t) = g(t) \tag{3.40}$$

In chapter 3 we have developed a coordinate transformation for the state vector $z(t)$:

$$z(t) = T_t p_2(t) \tag{3.41}$$

The transformationmatrix T_t (size (N*r) with r=B+M) was based on rigid body modes, (residual) flexibility modes and free-free normal modes developed for symmetric system matrices. Substitution of this coordinate transformation (4.1) into expression (3.10) and premultiplication with the transposed of the transformation matrix gives:

$$T_t^T C T_t \dot{p}_2(t) + T_t^T D T_t p_2(t) = T_t^T g(t) \tag{3.42}$$

or

$$C_p \dot{p}_2(t) + D_p p_2(t) = g_p(t) \tag{3.43}$$

with

$$\begin{aligned} C_p &= T_t^T C T_t, \ (\text{r*r}) \\ D_p &= T_t^T D T_t, \ (\text{r*r}) \\ g_p(t) &= T_t^T g(t), \ (\text{r*1}) \end{aligned} \tag{3.44}$$

Now a state-space form of the reduced system has been got, based on a generalized coordinate column in which the substructure boundary dof's have been preserved. The coupling of the substructures will now be a straightforward procedure.

Chapter 4

TESTPROBLEMS

For a numerical evaluation of the proposed substructure-reduction procedure we look at the bending vibration of a beam, clamped at one side and devided into two substructures (I and II) as illustrated in Fig 4.1.

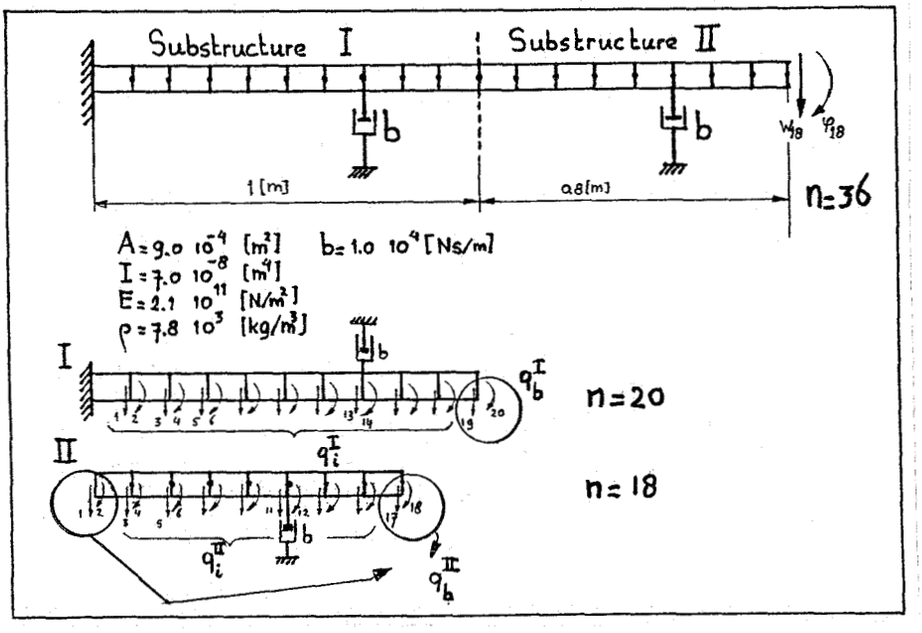

Fig. 4.1 TESTPROBLEM, Beam problem using 2 substructures

The interface dof's for substructure I are w_{10} and ϕ_{10} and for substructure II the displacements w_1 and w_9 and the rotations ϕ_1 and ϕ_9. Substructure II has 2 fysical rigid body modes (one translation and one rotation).
Fig. 4.2 gives the 72 eigenvalues for the coupled structure in the complex plane and Fig 4.3 the absolute values of the real and imaginairy parts (when an imaginary part is missing it means that it has been zero).

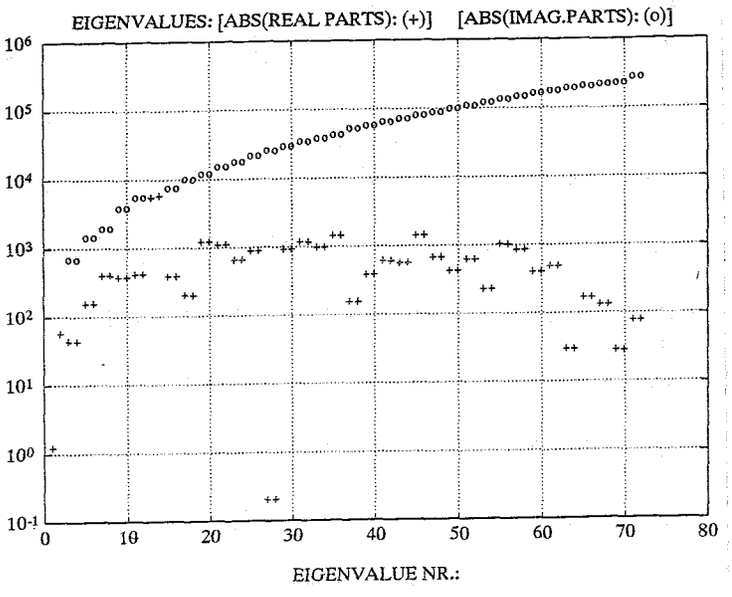

Fig. 4.2 Eigenvalues for the coupled system

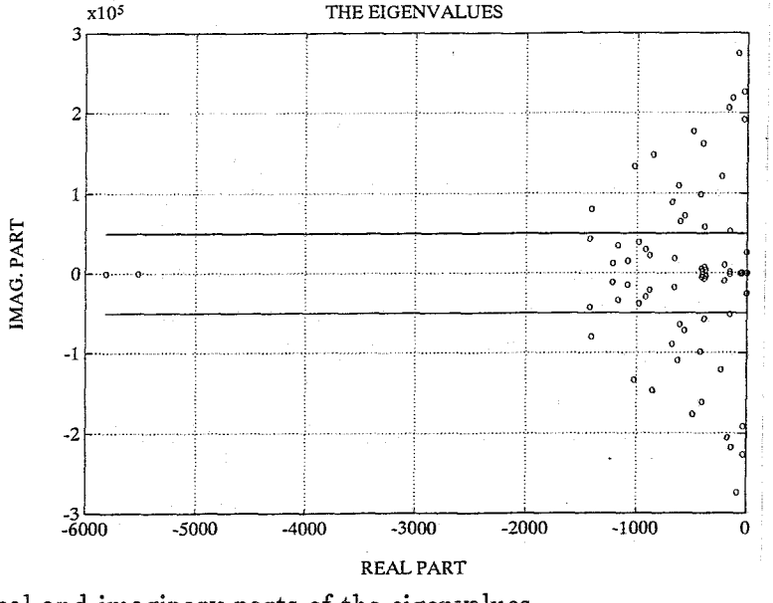

Fig. 4.3 Real and imaginary parts of the eigenvalues

It can be seen that we have to deal with a lot of complex conjugate eigenvalues pairs, some almost real eigenvalues and 4 eigenvalues without any imaginary part. For both the substructures we used a selection criterion in which an eigenvalue and corresponding eigenvector are selcted when $|imag(\lambda_k)| \leq 50.000[rad/s]$, indicated by the two straight lines in Fig. 4.3

Substructure I has 4 flexibility modes corresponding to the chosen boundary dof's (end translation and -rotation and their time-derivatives). They are shown in Fig. 4.4.

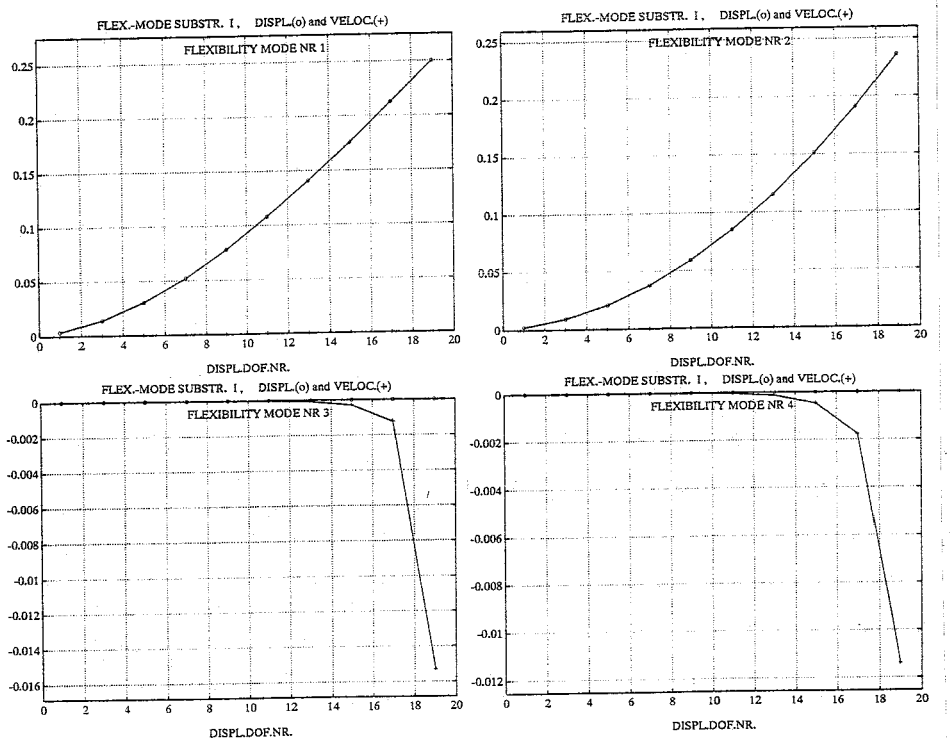

Fig. 4.4 Flexibility modes of substructure I

The flexibility modes have been split up in a displacement part and a velocity part. In can be noticed that two of the modes consist of a pure translation and two of them are pure velocity modes.
The free-free eigenvalues of substructure I are given in Fig. 4.5 whereas the "kept" eigenvalues are given in Fig. 4.6.

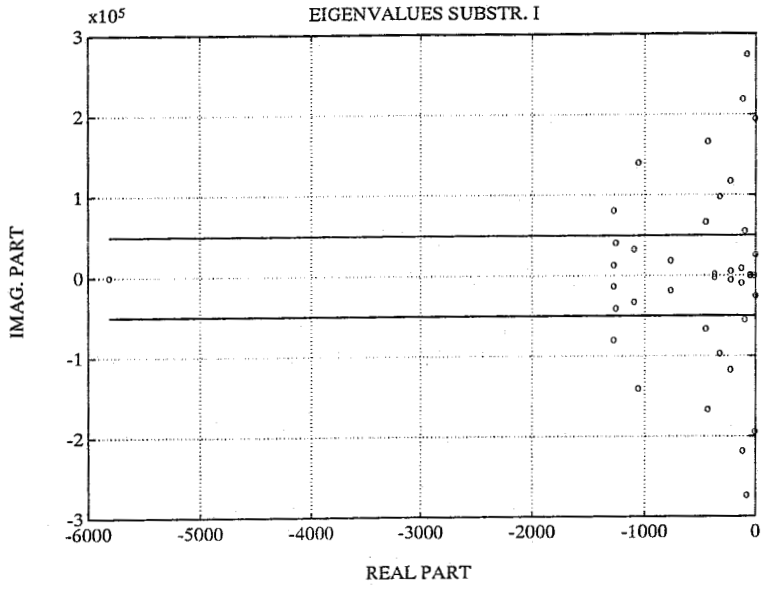

Fig. 4.5 Free-free eigenvalues of substructure I

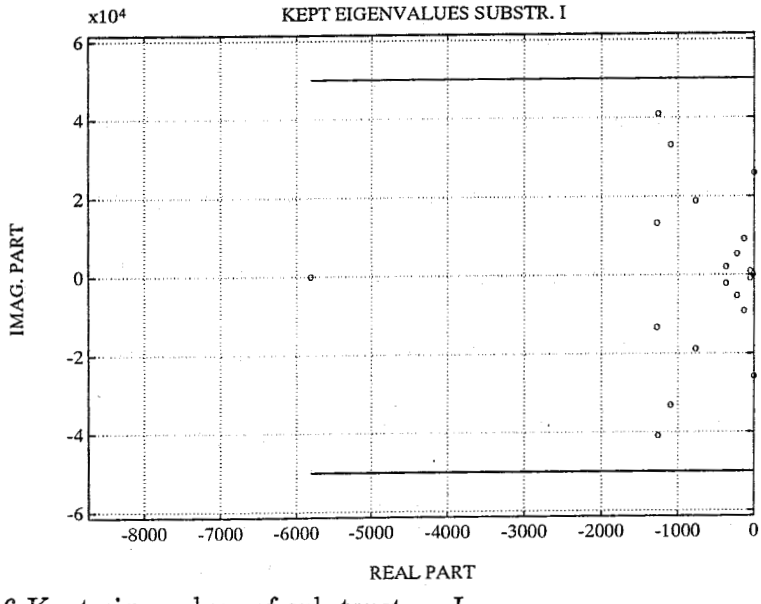

Fig. 4.6 Kept eigenvalues of substructure I

The number of kept eigenvalues is 20 so the total number of dof's of substr. I after reduction will be 24.

Substructure II has 2 fysical rigid body modes. In state-space form this resulted for this system in one single zero-eigenvalue (the translation mode) and one double zero-eigenvalue (the rotation mode). The translation r.b.m is shown in fig 4.7a whereas the regular- and generalized rotational rigid body mode are shown in the Fig's 4.7b and 4.7c.

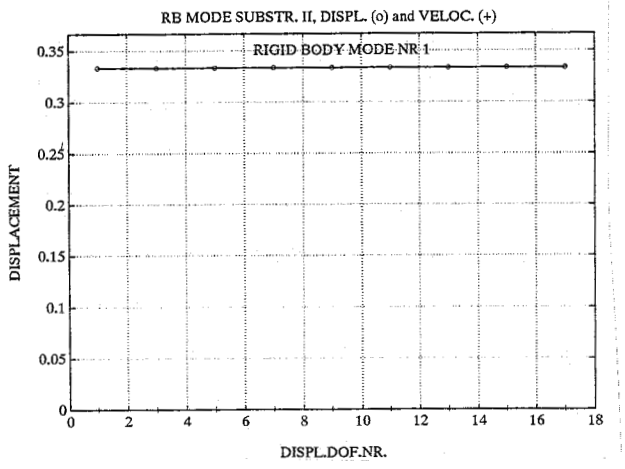

Fig. 4.7a Regular translational rbm.

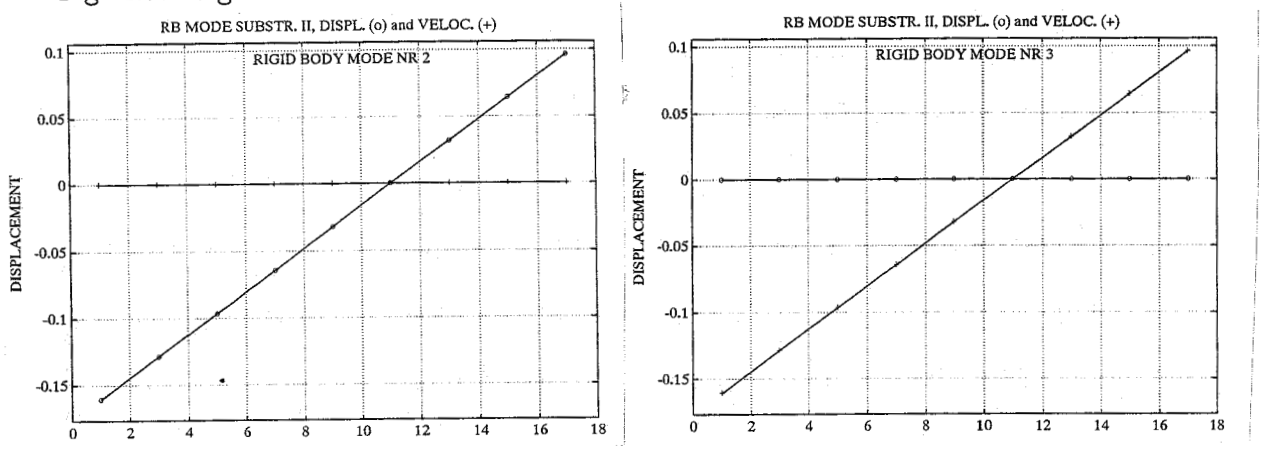

Fig. 4.7 Regular- (b) and generalized- (c) rotational rbm.

Substructure II has 8 flexibility modes which are shown in Fig 4.8.

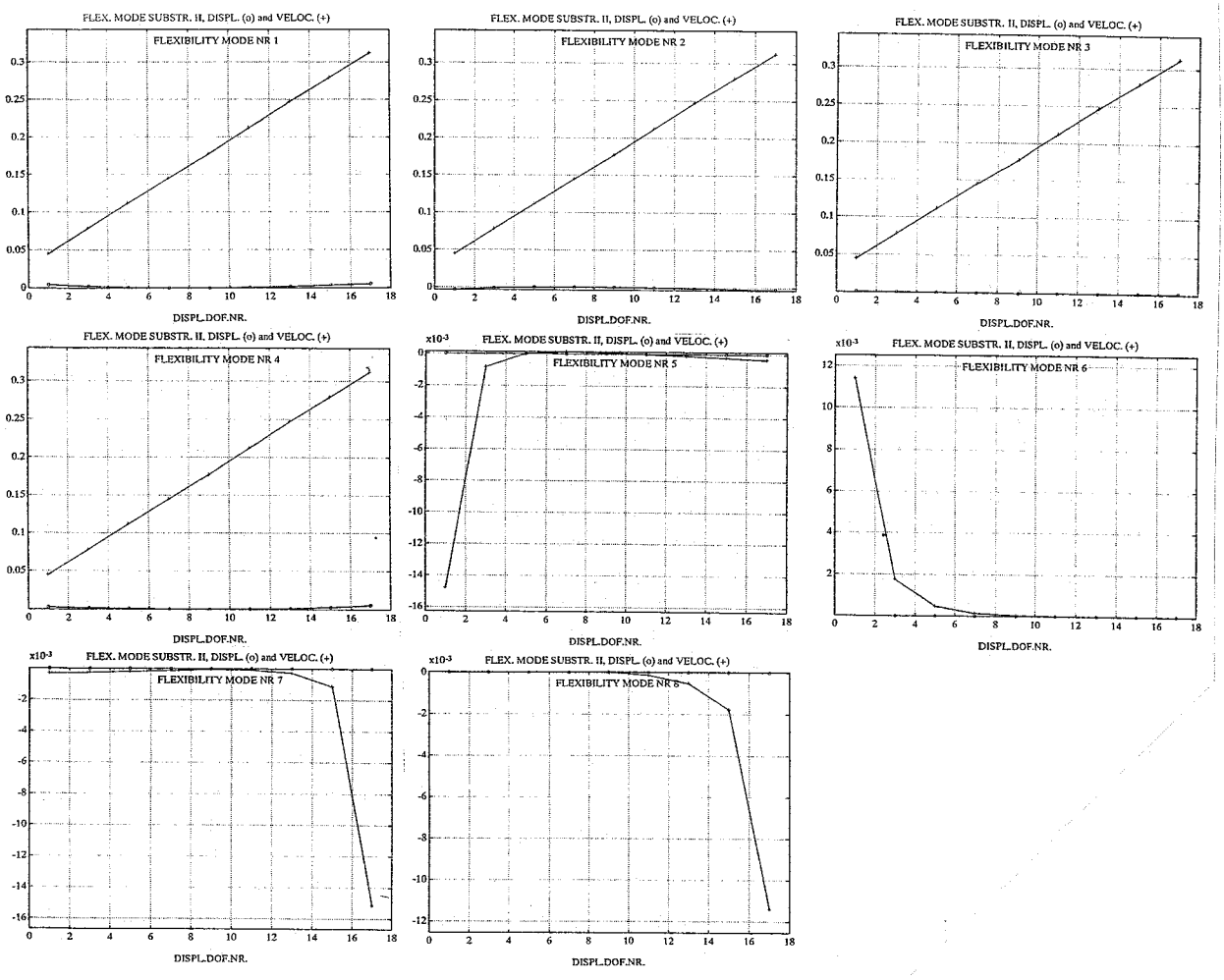

Fig. 4.8 Flexibility modes of substructure II.

It can be seen that for substructure II the flexibility modes not only have a displacement part but also a velocity part. In the flex.modes 1 t/m 4 the influence of the velocity-rigid-body-mode is clearly visible.
The free-free (nonzero !) eigenvalues of substr. II are shown in Fig. 4.9, the kept eigenvalues

are shown in Fig. 4.10.

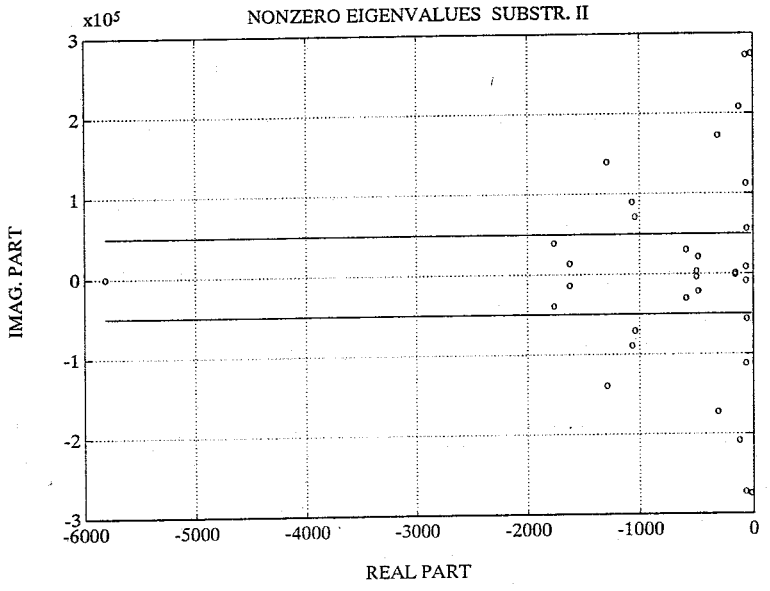

Fig. 4.9 Free-Free eigenvalues substr. II

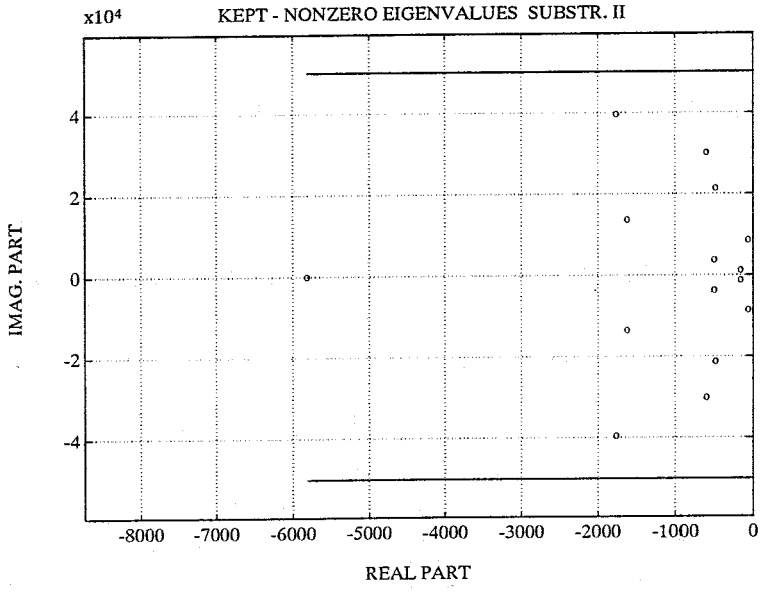

Fig. 4.10 Kept eigenvalues substr. II

The number of kept eigenvalues for substr. II is 15, so the total nr. of dof's after reduction will be 26.
After coupling the two substructures (leading to a system with 46 dof's) the eigenvalues have been calculated. The result is shown in Fig. 4.11 which gives the eigenvalues of the unreduced (o) and the reduced (+) system.

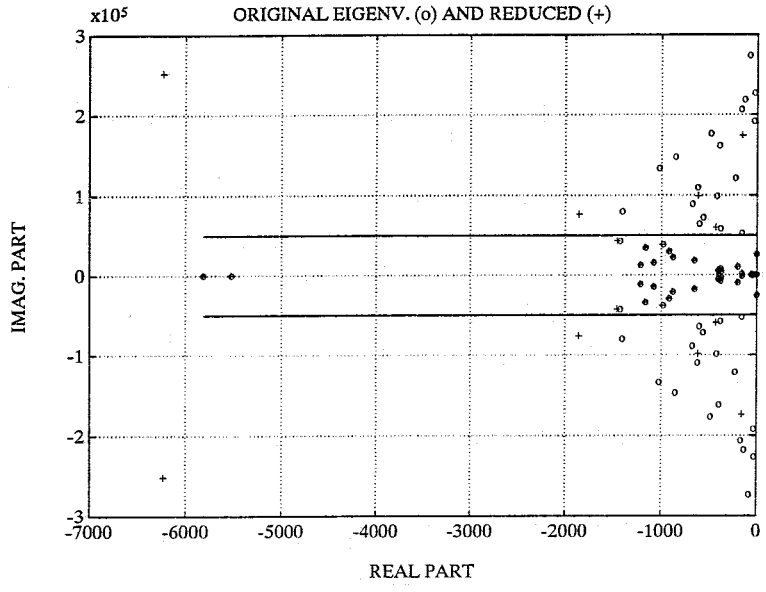

Fig. 4.11 Eigenvalues of the reduced and unreduced system

Figuur 4.12 gives a zoomplot of the eigenvalues of the reduced and unreduced system in the frequencyband of interest.

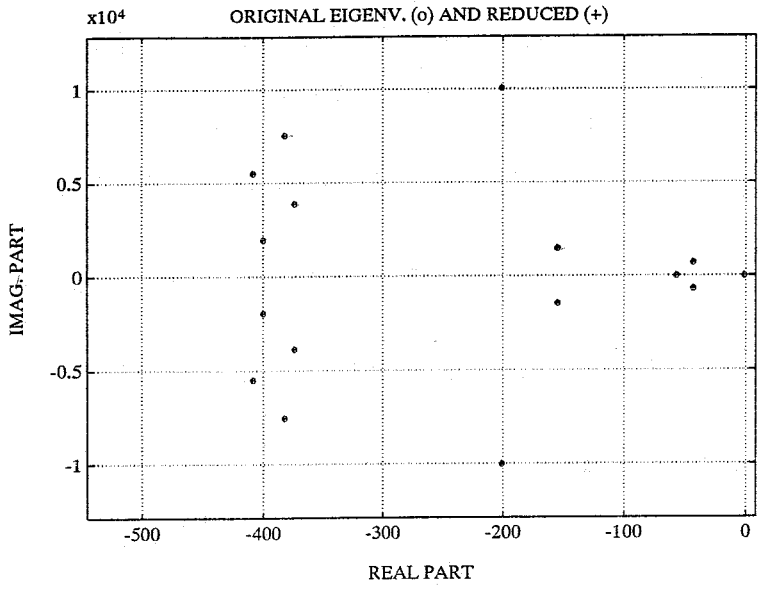

Fig. 4.12 Eigenvalues of the reduced and unreduced system, zoomplot

It can be seen that within the frequency band the correspondence of the reduced and unreduced eigenvalues is very good.
Finally Fig. 4.13 gives the Frequency Respons Function for excitation and respons in the displacement dof. of the free end of the coupled beam. Here the same quality of approximation

can be noticed below the cutt-off frequency of 50.000 [rad/s].

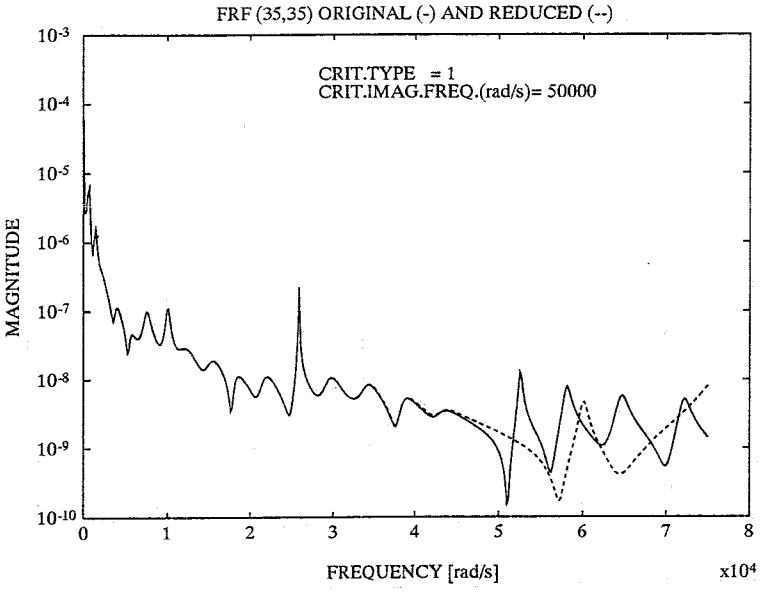

Fig. 4.13 Frequency respons function with and without reduction.

The same system has also been analysed using an extension of the classical dynamic reduction, thus by only using eigenmodes (including rigid body modes) but without residual flexibility modes. The original and reduced eigenvalues in this case are shown in the Fig's 4.14 and 4.15.

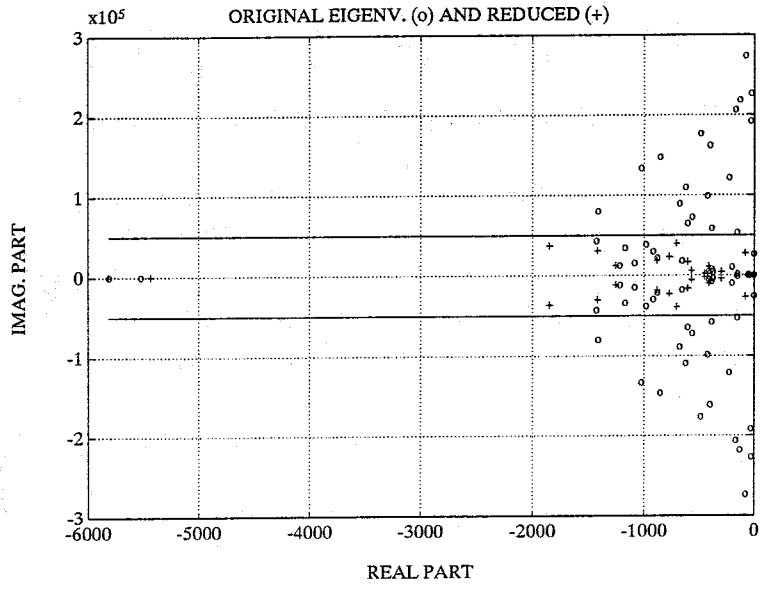

Fig. 4.14 Eigenvalues of the reduced and unreduced system without residual flexibility modes.

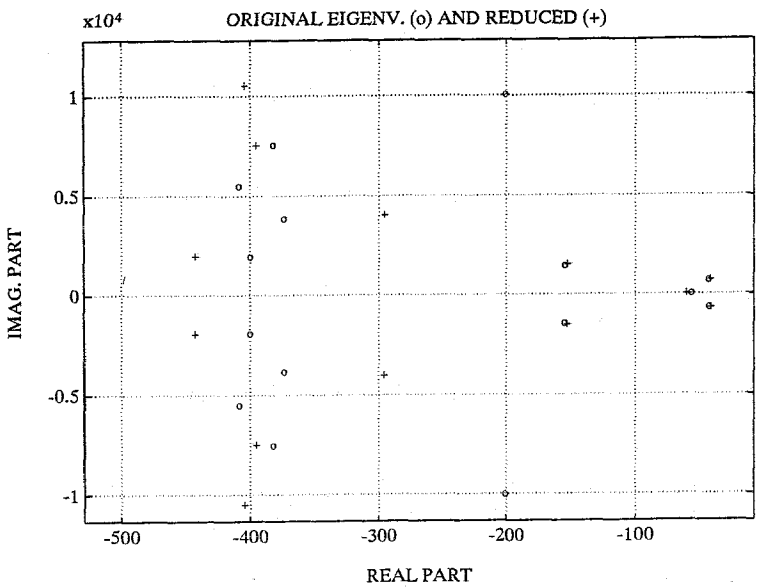

Fig. 4.15 Eigenvalues of the reduced and unreduced system without residual flexibility modes, zoomplot

The Frequency Response Function again is shown in Fig. 4.16.

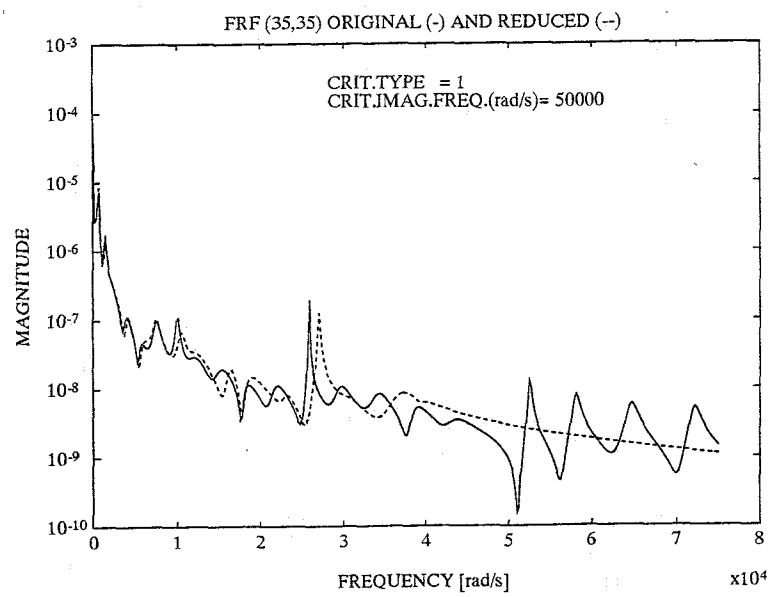

Fig. 4.16 Frequency respons function with and without reduction.

From these figures it is clear that the residual flexibility modes (also in this generalised form) play a very important role in getting an accurate reduced model.

As a final test the frequency criterion was changed. Now we choose a limit for the absolute value of the imaginairy parts 20.000 [rad/s] instead of 50.000 [rad/s]. The number of

dof's (state space variables) of the reduced system now was 32.
The reduced and unreduced eigenvalues are given in Fig. 4.17 and 4.18.

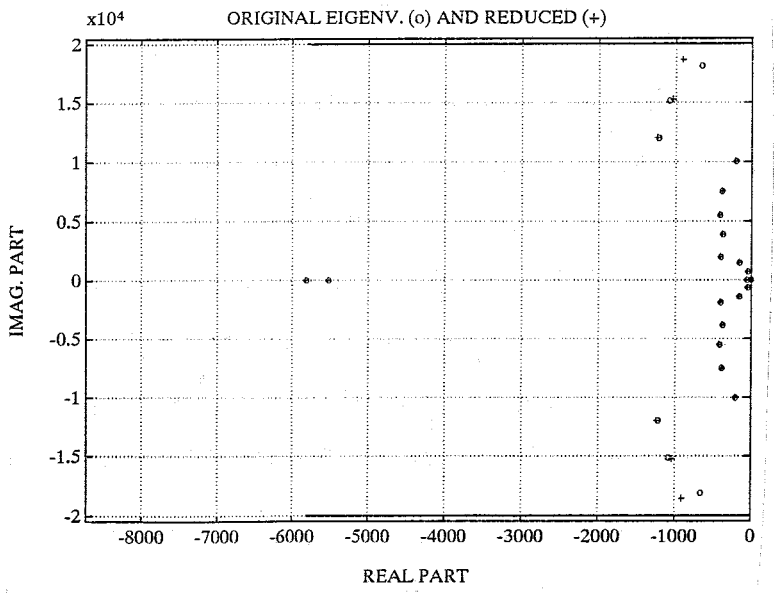

Fig. 4.17 Eigenvalues of the reduced and unreduced system with cutt-off frequency of 20.000 [rad/s].

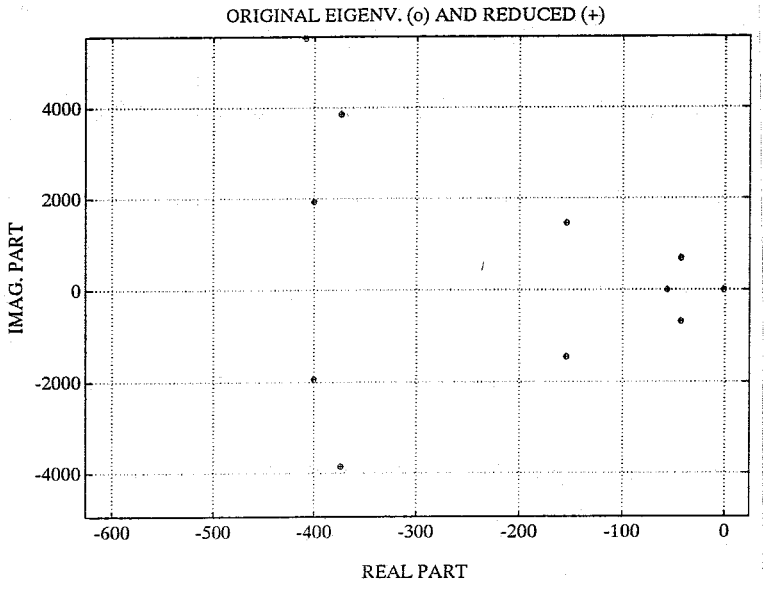

Fig. 4.18 Eigenvalues of the reduced and unreduced system with cutt-off frequency of 20.000 [rad/s], zoomplot

The Frequency Response Function for this situation is given in Fig. 4.19

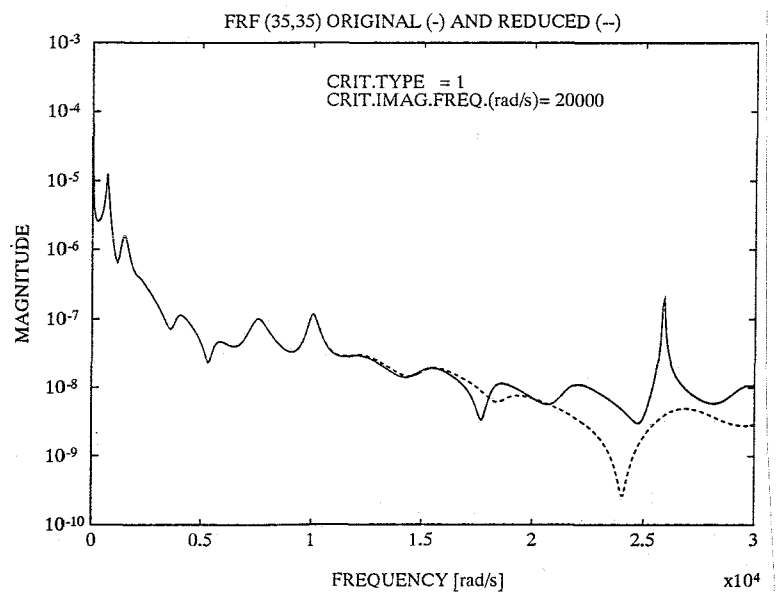

Fig. 4.19 Frequency respons function with and without reduction.

It can be seen that within the smaller frequency band the results relatively have the same accuracy, which means that the chosen frequency criterion can be used very well in this reduction scheme.

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

In this report the standard RUBIN CMS procedure is extended to the case of systems with general viscous damping, but still with symmetric system matrices. In its standard form the transformation matrix used for this reduction is build from real static modes (rigid body modes and residual flexibility modes) and a frequency dependent selection of real dynamic modes.
In the extended form (based on a state-space formulation in first order differential equations) the rigid body mode concept had to be modified with the possibility of generalized eigenvectors corresponding to multiple zero eigenvalues and the use of complex, free-free state-space eigenmodes.
The application of the procedure to a 2-substructure beamproblem showed:

- The use of (residual) flexibility modes additional to complex eigenmodes is essential for getting an accurate, reduced model,
- A simple frequency selection criterion can be used very well for the generation of a reduced model in an preselected frequency band.
- The presence of fysical rigid body modes needs special attention when using a state-space description for a structural system.

Finally it should be recommended that the possibility of the CMS extension for nonsymmetric system matrices (for example in the case of rotor-bearing systems) should be investigated. In that case however not only right eigenvectors but also left eigenvectors should be distinguished as well as left- and right (residual) flexibility modes.

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Appendix A

State rigid body modes

For a system with n dof's we can write:

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t) \tag{A.1}$$

We assume that the Massmatrix is positive definite and that the stiffnessmatrix is singular with rank n-r. Therefore the system wil have r rigid body modes (rbm's) u_r for which:

$$Ku_r = o \tag{A.2}$$

The rbm's u_r are linearly independent. The State-space formulation of the problem is:

$$C\dot{y}(t) + Dy(t) = g(t) \tag{A.3}$$

with:

$$C = \begin{bmatrix} B & M \\ M & O \end{bmatrix}; D = \begin{bmatrix} K & O \\ O & -M \end{bmatrix}; y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}; g(t) = \begin{bmatrix} f(t) \\ o \end{bmatrix} \tag{A.4}$$

The eigenvalue problem corresponding to the homogeneous part is:

$$[\lambda_k C + D]v_k = o \tag{A.5}$$

For each fysical rbm there will be **one or two** eigenvalues $\lambda_k = 0$ if we solve the eigenvalue problem (A.5).
For $\lambda_k = 0$ we get:

$$Dv_k = o \Rightarrow \begin{bmatrix} K & O \\ O & -M \end{bmatrix} \begin{bmatrix} v_k^d \\ v_k^v \end{bmatrix} = o \tag{A.6}$$

For a positive definite massmatrix M this will lead to r **regular state rigid body modes (srbm's)** v_r:

$$v_r = \begin{bmatrix} u_r \\ o \end{bmatrix} \tag{A.7}$$

If the eigenvalue $\lambda_k = 0$ has multiplicity 2 there will also be a generalized srbm w_r. In this case it will not be possible to diagonalize C and/or D so there will **not** exist an eigenvectormatrix U such that:

$$CU\Lambda + DU = O \tag{A.8}$$

where Λ is a diagonalmatrix.
But it will be possible to find a linearly independent set of **generalized eigenvectors** which transform C or D into the almost-diagonal **JORDAN-form** :

$$CVJ + DV = O \tag{A.9}$$

where:

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & \lambda_2 & 1 & 0 & 0 & 0 & . \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & \lambda_4 & 0 & . \\ . & . & . & . & . & . & . \end{bmatrix} \tag{A.10}$$

for example for one eigenvalue λ_2 with multiplicity 2.

The Jordan-matrix J will have Jordan-blocks on the diagonal having the multiple zero-eigenvalue on its diagonal and ones on the first co-diagonal.
In this case, a repeated root $\lambda_k = 0$ will then lead to:

$$C \begin{bmatrix} v_r & w_r \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + D \begin{bmatrix} v_r & w_r \end{bmatrix} = \begin{bmatrix} o & o \end{bmatrix} \tag{A.11}$$

or:

$$Dv_r = o \tag{A.12}$$

which we already had, and:

$$Cv_r + Dw_r = o \tag{A.13}$$

with:

$$w_r = \begin{bmatrix} w_r^d \\ w_r^v \end{bmatrix} \tag{A.14}$$

this gives:

$$Bv_r^d + Mv_r^v + Kw_r^d = o \tag{A.15}$$

and:

$$Mv_r^d - Mw_r^v = o \tag{A.16}$$

For a regular M (A.16) gives:

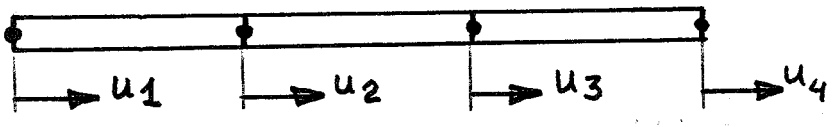
$$w_r^v = v_r^d = u_r, \text{ and } Kw_r^d = -Bu_r \tag{A.17}$$

- **IF:** $Bu_r = o$, we can take $w_r^d = o$ or $w_r^d = u^r$. This gives the generalized state-rigid-body mode:

$$w_r = \begin{bmatrix} o \\ u_r \end{bmatrix}, \text{ or } w_r = \begin{bmatrix} u_r \\ u_r \end{bmatrix}, \tag{A.18}$$

- **IF:** $Bu_r \neq o$, there will not be a solution for corresponding columns of Kw_r^d and hence the assumption that $\lambda_k = 0$ is a double root leading to both regular and generalized rigid body modes is not valid in this case.

EXAMPLE



$$q^T = [u1, u2, u3, u4];$$

$$M = m\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}; \text{ and } K = k\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix};$$

We take $m = 1$ and $k = 1$.

- Case I: $B = O$ or$B = \alpha * K \Rightarrow B * u_r = o$

$$[\lambda_k C + D]v_k = o \Rightarrow \text{ repeated root } \lambda_k = 0 \text{ with multiplicity } = 2.$$

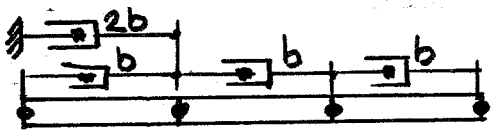
$$V_R = \begin{bmatrix} u_r & o \\ o & u_r \end{bmatrix} \text{ and } CV_R\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + DV_R = O$$

For $C_{RR} = V_R^T C V_R$ and $D_{RR} = V_R^T D V_R$ we find:

$$C_{RR} = \begin{bmatrix} O_{rr} & u_r^T M u_r \\ u_r^T M u_r & O_{rr} \end{bmatrix} = \begin{bmatrix} 0 & 18 \\ 18 & 0 \end{bmatrix}; D_{RR} = \begin{bmatrix} O_{rr} & O_{rr} \\ O_{rr} & -u_r^T M u_r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -18 \end{bmatrix}$$

The **Non-diagonal** C_{RR} will be regular for positive definite massmatrix M.

- Case II: For b=0.1 we get:



$$B = 0.1\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$Bu_r = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \neq o \Rightarrow \lambda_k = 0 \text{ with multiplicity } 1$$

$$\lambda_1 = 0; v_1^T = [1, 1, 1, 1, 0, 0, 0, 0];$$

$$\lambda_2 = -0.011; v_2^T = [1, 1.0004, 0.9993, 0.9989, -0.011, -0.011, -0.011, -0.011];$$

$$V_R = \begin{bmatrix} u_r \\ o \end{bmatrix}, \text{ r=1 !}$$

$$C_{RR} = V_R^T C V_R = u_r^T B u^r = 2$$

$$D_{RR} = V_R^T D V_R = u_r^T K u^r = 0$$

Appendix B

Proof of statement (3.27)

Starting point:

$$C\dot{z} + Dz = g \tag{B.1}$$

Suppose $z = ve^{\lambda t}$:

$$[\lambda C + D]v = o \tag{B.2}$$

The solution can be written as:

$$\Lambda = \begin{bmatrix} \Lambda_{EE} & O_{ER} \\ O_{RE} & \Lambda_{RR} \end{bmatrix}; V = [V_E; V_R] \tag{B.3}$$

where Λ_{RR} contains the **zero-eigenvalues** and V_R the corresponding regular- or generalized state rigid body modes (number = R). Λ_{EE} contains the **nonzero** eigenvalues.
One can write:

$$V^T C V = C^* \rightarrow C^{*-1} = V^{-1} C^{-1} V^{-T} \tag{B.4}$$

so:

$$CVC^{*-1}V^T = CV(V^{-1}C^{-1}V^{-T}V^T = I \tag{B.5}$$

With:

$$V = [V_R, V_E] \text{ and } C^* = \begin{bmatrix} C^*_{RR} & O_{RE} \\ O_{ER} & C^*_{EE} \end{bmatrix} \tag{B.6}$$

we can write:

$$C[V_R, V_E] \begin{bmatrix} {C^*_{RR}}^{-1} & O_{RE} \\ O_{ER} & {C^*_{EE}}^{-1} \end{bmatrix} \begin{bmatrix} V_R^T \\ V_E^T \end{bmatrix} = I \tag{B.7}$$

or:

$$C[V_R {C^*_{RR}}^{-1} V_R^T + V_E {C^*_{EE}}^{-1} V_E^T] = I \tag{B.8}$$

This gives:

$$P = [I - CV_R {C^*_{RR}}^{-1} V_R^T] = CV_E {C^*_{EE}}^{-1} V_E^T \tag{B.9}$$

From

$$CV_E\Lambda_{EE} + DV_E = O \tag{B.10}$$

we get:

$$P = -DV_E \Lambda_{EE}^{-1} {C^*_{EE}}^{-1} V_E^T \tag{B.11}$$

For the elastic flexibility matrix $G_E = P^T G P$ we than can write:

$$G_E = (-V_E {C^*_{EE}}^{-1} \Lambda^{-1}_{EE} V_E^T D^T) G (-D V_E \Lambda^{-1}_{EE} {C^*_{EE}}^{-1} V_E^T) \tag{B.12}$$

With:

$$D = \begin{bmatrix} D_{EE} & D_{ER} \\ D_{RE} & D_{RR} \end{bmatrix} \text{ and } D^T G D = \begin{bmatrix} D_{EE} & D_{ER} \\ D_{RE} & D_{RE} D^{-1}_{EE} D_{ER} \end{bmatrix} \tag{B.13}$$

The matrix D was singular, so we can write:

$$\begin{bmatrix} D_{ER} \\ D_{RR} \end{bmatrix} = \begin{bmatrix} D_{EE} \\ D_{RE} \end{bmatrix} \Theta \tag{B.14}$$

where Θ is an (N*R) matrix.
This gives: $D_{RE} D^{-1}_{EE} D_{ER} = D_{RE} \Theta = D_{RR}$.
Finally we get:

$$D^T G D = D \tag{B.15}$$

Equation (B.14) is only true when the number of suppressed dof's (R) is equal to the number of zero-eigenvalues.
Using (B.15) we can write for (B.12):

$$G_E = V_E {C^*_{EE}}^{-1} \Lambda^{-1}_{EE} V_E^T D V_E \Lambda^{-1}_{EE} {C^*_{EE}}^{-1} V_E^T \tag{B.16}$$

with:

$$V_E^T D V_E = V_E^T C V_E \Lambda_{EE} = C^*_{EE} \Lambda_{EE} \tag{B.17}$$

we can write:

$$G_E = V_E {C^*_{EE}}^{-1} \Lambda^{-1}_{EE} V_E^T = V_E \Lambda^{-1}_{EE} {C^*_{EE}}^{-1} V_E^T \tag{B.18}$$