Dirac bases

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A measure theoretical concept of basis

based on Carleman operators

by

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DIRAC BASIS: A MEASURE THEORETICAL CONCEPT OF BASIS

BASED ON CARLEMAN OPERATORS

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Abstract

The notion of Dirac basis is introduced in the setting of a Sobolev-like triple $R(X) \subset X \subset R^{-1}(X)$. It is a continuum substitute of the notion of orthonormal basis for Hilbert spaces. As a side result a generalization of the Sobolev embedding theorem is given in terms of Carleman operators and geometric measure theory. Finally, the concept of canonical Dirac basis is introduced. It is the basic concept for solving the generalized eigenvalue problem and corresponding generalized expansion problems.

0. A conceptual introduction

Our papers [EG 1] contain a mathematical interpretation of several aspects of Dirac's formalism. Herein we needed a continuum substitute of the ordinary notion of orthonormal basis. Therefore, we introduced the notion of Dirac basis in the setting of the Gelfand triple

$$S_{X,A} \subset X \subset T_{X,A} \ .$$

Here X denotes a separable Hilbert space, and A a positive self-adjoint unbounded operator in X. The space $S_{X,A}$ is the inductive limit

$$\bigcup_{t>0} e^{-tA}(X) \ .$$

The space $T_{X,A}$ is a projective limit; it consists of mappings $F : (0,\infty) \to X$ with the property $F(t + \tau) = e^{-\tau A} F(t), \quad t,\tau > 0.$

In the present paper we introduce the notion of Dirac basis in the bare setting of a triple of Hilbert spaces

$$R(X) \subseteq X \subseteq R^{-1}(X)$$

where R denotes a bounded positive operator. It turns out that the concepts of Dirac basis and of Carleman operators are indissolubly connected.

This introduction contains the preliminary concepts.

The first one is the concept of Sobolev triple. Let R denote a positive bounded operator on the separable Hilbert space X with possibly unbounded inverse R^{-1}. The dense subspace $R(X)$ of X is the maximal domain of R^{-1}. In $R(X)$ we introduce the non-degenerate sequilinear form

$$(w,v)_1 = (R^{-1}w, R^{-1}v) \ , \qquad w,v \in R(X)$$

with $(\cdot,\cdot)$ the inner product of X. Then $R(X)$ is a Hilbert space with $(\cdot,\cdot)_1$ as its inner product.

Let $R^{-1}(X)$ denote the completion of X with respect to the norm $f \mapsto \|Rf\|$, $f \in X$. Then R extends to an isometry from $R^{-1}(X)$ into X. The non-degenerate form $(\cdot,\cdot)_{-1}$ on $R^{-1}(X)$ is defined by

$$(F,G)_{-1} = (R F, RG) \ , \quad F,G \in R^{-1}(X) \ .$$

Thus $R^{-1}(X)$ becomes a Hilbert space.

It yields the triple of Hilbert spaces

$$R(X) \subseteq X \subseteq R^{-1}(X)$$

The spaces $R(X)$ and $R^{-1}(X)$ establish a dual pair. Their pairing $<\cdot,\cdot>$ is given by

$$<w,G> = (R^{-1}w, RG), \quad w \in R(X), \quad G \in R^{-1}(X) \ .$$

We note that $R(X) = X = R^{-1}(X)$ if and only if R^{-1} is bounded.

The second concept is the notion of Carleman operator.

(0.1) <u>Definition</u>

Let X denote a separable Hilbert space, and M a measure space with σ-finite measure μ. An operator $T : X \to L_2(M,\mu)$ is called a Carleman operator, if there exists a measurable function $k : M \to X$ with the following property:

For each $f \in D(T)$, the function

$$\alpha \rightarrow (f, k(\alpha))$$

is a representant of the class $Tf \in L_2(M,\mu)$. (Cf. [Wei] .)

A Carleman operator $T : X \rightarrow L_2(M,\mu)$ is Hilbert-Schmidt iff the function $\alpha \mapsto \|k(\alpha)\|^2$ is μ-integrable. Then the Hilbert-Schmidt norm equals $(\int_M \|k(\alpha)\|^2 d\mu_\alpha)^{\frac{1}{2}}$.

The following result is straightforward.

(0.2) <u>Lemma</u>

Let $T : X \rightarrow L_2(M,\mu)$ denote a Carleman operator induced by the measurable function $k : M \rightarrow X$. Let $(v_n)_{n \in \mathbb{N}}$ be an orthonormal basis in $D(T)$. Fix representants $(Tv_n)^\wedge$ for each $n \in \mathbb{N}$. Then there exists a null set $N \subset M$ such that for all $\alpha \in M \setminus N$

$$\sum_{n=1}^\infty |(Tv_n)^\wedge(\alpha)|^2 = \|k(\alpha)\|^2 .$$

Put differently $k(\alpha) = \sum_{n=1}^\infty \overline{(Tv_n)^\wedge(\alpha)} v_n , \quad \alpha \in M \setminus N$

The third fundamental concept is a generalization by Federer of Vitali's differentiation theorem for the Lebesque measure on $\mathbb{R}^n$. Federer considers a topological measure space M metrized by the metric d, and a regular Borel measure μ on M, such that bounded subsets of M have finite μ-measure. In the monograph [Fe], conditions are introduced on the metric space (M,d) which lead to the following result.

(0.3) <u>Theorem</u>

Let the function $f : M \to \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set N_f such that for all $r > 0$ and all $\alpha \in M \setminus N_f$ the closed ball $B(\alpha, r)$ with radius r and centre α has positive measure. Further, the limit

$$\widetilde{f}(\alpha) = \lim_{r \downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} f \; d\mu$$

exists for all $\alpha \in M \setminus N_f$. The function $\alpha \mapsto \widetilde{f}(\alpha)$ defines a μ-measurable function with $f = \widetilde{f}$ almost everywhere.

<u>Proof</u>. Cf. [Fe], Theorem 2.8.18. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Examples of such metric spaces are the following

- Finite dimensional vector spaces M with $d(x,y) = v(x - y)$ where v is any norm on M.
- A Riemannian manifold (of class ≥ 2) with its usual metric.
- The disjoint union of metric spaces (M_j, d_j), $j \in \mathbb{N}$, which satisfy Federer's conditions. Here d is defined by

$$\begin{cases} d(x_\ell, y_j) = 1 & , \; \ell \neq j \; . \\ d(x_\ell, y_\ell) = d_\ell(x_\ell, y_\ell) \end{cases}$$

1. The concept of Dirac basis

Let $R > 0$ be a bounded operator on X. We consider the triple

$R(X) \subseteq X \subseteq R^{-1}(X)$. Further, let M denote a measure space with σ-finite

measure μ. A function $\Phi : M \to R^{-1}(X)$ is called (weakly) μ-measurable

if for each $w \in R(X)$ the function $\alpha \to <w, \Phi(\alpha)>$ is μ-measurable.

In the space of measurable functions from M to $R^{-1}(X)$ we introduce the

natural equivalence relation

$$\Phi \sim \Psi : \Leftrightarrow \Phi(\alpha) = \Psi(\alpha)$$

almost everywhere.

In order to arrive at a proper introduction of the concept of Dirac basis,

the following auxilliary result is needed.

1.1. Lemma

There exists an orthonormal basis $(u_n)_{n \in \mathbb{N}}$ in X which is a Schauder basis

in $R(X)$, i.e. for all $w \in R(X)$ the series $\sum_{n=1}^{\infty} (w, u_n) u_n$ converges in $R(X)$.

Proof.

We may as well assume that $0 < R \leq I$. If R has pure point spectrum than

an orthonormal basis of eigenvectors satisfies the requirements.

Now suppose that R has no pure point spectrum. We construct an operator

$\tilde{R}$ as follows. Let $(E_\lambda)_{\lambda \in \mathbb{R}}$ denote the spectral resolution of the identity

associated to R. Put $\tilde{R} = \sum_{n=1}^{\infty} \frac{1}{n+1} P_n$ with

$$P_n = \int_{\mathbb{R}} \chi_{(\frac{1}{n+1}, \frac{1}{n}]} dE_\lambda .$$

We have $\|R^{-1}\tilde{R}\| = 1$ and $\|\tilde{R}^{-1}R\| = 2$.

Hence $\tilde{R}(X) = R(X)$ as Hilbert spaces. Finally, observe that $\tilde{R}$ has a pure

point spectrum. □

1.2. Definition

Let (G,M,μ,R,X) denote an equivalence class of μ-measurable functions

$\hat{G} : M \rightarrow R^{-1}(X)$. Then (G,M,μ,R,X) is called a Dirac basis if for a certain

orthonormal basis $(u_k)_{k\in\mathbb{N}}$ in X, which is a Schauder basis in $R(X)$, the

following relations are valid.

$$(1.2') \qquad \int_M <u_\ell,\hat{G}_\alpha>\overline{<u_k,\hat{G}_\alpha>}d\mu_\alpha = \delta_{k\ell} , \qquad k,\ell \in \mathbb{N} .$$

Now let (G,M,μ,R,X) be a Dirac basis. Let $\hat{G} \in (G,M,\mu,R,X)$ and $(u_k)_{k\in\mathbb{N}}$

be an orthonormal basis, which is a Schauder basis in $R(X)$ such that

$(1.2')$ is satisfied. Then for each $k \in \mathbb{N}$, the function $\hat{\phi}_k : \alpha \mapsto <u_k,\hat{G}_\alpha>$

is square integrable. Let $\phi_k \in L_2(M,\mu)$ denote the equivalence class of

$\hat{\phi}_k$, $k \in \mathbb{N}$. We introduce the operator V on the linear span $<\{u_k|k \in \mathbb{N}\}>$

by

$$Vu_k = \phi_k , \qquad k \in \mathbb{N} .$$

From $(1.1')$ it follows that $(\phi_k,\phi_\ell) = \delta_{k\ell}$. So V can be extended to an

isometry from X into $L_2(M,\mu)$. Let $w \in R(X)$, and put $(Vw)^\wedge = \sum_{k=1}^{\infty} (w,u_k)\hat{\phi}_k$.

It is clear that $(Vw)^\wedge \in Vw$. Moreover, $<w,\hat{G}_\alpha> = \sum_{k=1}^{\infty} (w,u_k)\hat{\phi}_k(\alpha)$.

Hence $(Vw)^\wedge = \sum_{k=1}^{\infty} (w,u_k)\hat{\phi}_k$ where the convergence is pointwise. For

$w_1,w_2 \in R(X)$,

$$(w_1, w_2) = \int_M (Vw_1)^\wedge \overline{(Vw_2)^\wedge} d\mu = \int_M \langle w_1, \hat{G}_\alpha \rangle \overline{\langle w_2, \hat{G}_\alpha \rangle} d\mu_\alpha \quad .$$

(1.3.) Theorem

Let (G, M, μ, R, X) be a Dirac basis.

(i)   For each representant $\hat{G} \in (G, M, \mu, R, X)$, and all $w_1, w_2 \in R(X)$

$$(w_1, w_2) = \int_M \langle w_1, \hat{G}_\alpha \rangle \overline{\langle w_2, \hat{G}_\alpha \rangle} d\mu_\alpha \quad . \qquad \text{(Plancherel)}$$

There exists an isometry $V : X \mapsto L_2(M, \mu)$ with the properties:

(ii)   For each representant $\widetilde{G} \in (G, M, \mu, R, X)$ and each $w \in R(X)$ the function

$(\alpha \mapsto \langle w, \widetilde{G}_\alpha \rangle) \in Vw$.

(So the definition of V does not depend on the choice of $\hat{G}$).

(iii) The operator VR is a bounded Carleman operator.

Proof

Part (i) follows from the observations above this theorem. To prove part

(ii), let $\hat{G} \in (G, M, \mu, R, X)$. As we have seen, there exists an isometry

$V : X \to L_2(M, \mu)$ such that for all $w \in R(X)$ the function $\alpha \mapsto \langle w, \hat{G} \rangle$ is

a representant of Vw. For each $\widetilde{G} \in (G, M, \mu, R, X)$, there exists a null

set $\widetilde{N}$ such that $\widetilde{G}_\alpha = \hat{G}_\alpha$ for all $\alpha \in M \setminus N$. It follows that

$(\alpha \mapsto \langle w, \widetilde{G}_\alpha \rangle) \in Vw, \quad w \in R(X)$.

Let $\hat{G} \in (G, M, \mu, R, X)$, and let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis in X.

Then for all $\alpha \in M$

$$\|VR\hat{G}_\alpha\|^2_{L_2} = \|R\hat{G}_\alpha\|^2 = \sum_{k=1}^{\infty} |<Re_k,\hat{G}_\alpha>|^2 = \sum_{k=1}^{\infty} |(VRe_k)^\wedge(\alpha)|^2 \quad .$$

Put $k : \alpha \mapsto \sum_{k=1}^{\infty} (VRe_k)^\wedge(\alpha)e_k$. Then k induces VR as a Carleman operator. $\square$

The reverse of the previous theorem is also valid. So there exists a one-to-one correspondence between Dirac bases (G,M,μ,R,X) and isometries $V : X \to L_2(M,\mu)$ such that VR is a Carleman operator.

(1.4) __Theorem__

Let V denote an isometry from X into $L_2(M,\mu)$. Suppose the operator VR is Carleman. Then there exists a Dirac basis (G,M,μ,R,X) such that for each $\hat{G} \in (G,M,\mu,R,X)$ and all $w \in R(X)$

$$(\alpha \mapsto <w,\hat{G}_\alpha>) \in Vw \quad .$$

__Proof__

Let $(e_k)_{k\in \mathbb{N}}$ denote an orthonormal basis in X, and let $(VR\ e_k)^\wedge$ denote a representant of the class $VR\ e_k \in L_2(M,\mu)$, $k \in \mathbb{N}$. By Lemma (0.2) there exists a null set $\hat{N}$ such that for all $\alpha \in M \setminus \hat{N}$

$$\sum_{k=1}^{\infty} |(VR\ e_k)^\wedge(\alpha)|^2 < \infty \quad .$$

Now define $\hat{G} : M \to R^{-1}(X)$ by

$$
\left[
\begin{array}{l}
G_\alpha = 0 \qquad\quad \text{if} \quad \alpha \in \hat{N} \\[3mm]
G_\alpha = \sum_{k=1}^{\infty} \overline{(VR\, e_k)^{\wedge}(\alpha)} \quad R^{-1}\, e_k
\end{array}
\right.
$$

Let $w \in R(X)$. Then

$$
<w,\hat{G}_\alpha> \;=\; \sum_{k=1}^{\infty} (R^{-1}w,\, e_k)(VR\, e_k)^{\wedge}(\alpha) \;\;.
$$

Hence $\alpha \mapsto <w,\hat{G}_\alpha>$ is measurable and $(\alpha \mapsto <w,\hat{G}_\alpha>) \in Vw$. It follows that $\hat{G}$ is weakly measurable. Let $w_1, w_2 \in R(X)$. Then we have

$$
(w_1, w_2) = (Vw_1, Vw_2)_{L_2} = \int <w_1, \hat{G}_\alpha> \overline{<w_2, \hat{G}_\alpha>} d\mu_\alpha \;\;.
$$

If (G,M,μ,R,X) denotes the equivalence class of $\hat{G}$, then (G,M,μ,R,X) is the wanted Dirac basis.

Remark:

If the support of the measure μ consists of atoms only, then any Dirac basis (G,M,μ,R,X) is an orthogonal basis.

Let (G,M,μ,R,X) be a Dirac basis, and let $\hat{G} \in (G,M,\mu,R,X)$. From the Placherel-type result stated in Theorem (1.3) we obtain the weak expansion

$$w = \int_M <w,G_\alpha>G_\alpha \, d\mu_\alpha \, , \qquad w \in \mathcal{R}(X)$$

in the sense that for all $v \in \mathcal{R}(X)$

$$(w,v) = \int_M <w,G_\alpha>\overline{<v,G_\alpha>}d\mu_\alpha \, .$$

A sharper result is valid, if $\mathcal{R}$ is a Hilbert-Schmidt operator on X.

(1.5) <u>Theorem</u>

Let $\mathcal{R} > 0$ be a Hilbert-Schmidt operator. Let $\hat{G}$ be a representant of the Dirac basis $(G,M,\mu,\mathcal{R},X)$. Then for each $w \in \mathcal{R}(X)$ the function

$$\alpha \mapsto \| <w,\hat{G}_\alpha> \hat{G}_\alpha \|_{-1}$$

is μ-integrable. So we get the strong expansion

$$w = \int_M <w,\hat{G}_\alpha>\hat{G}_\alpha \, d\mu_\alpha$$

i.e. in strong X-sense

$$\mathcal{R}w = \int_M <w,\hat{G}_\alpha>\mathcal{R}\hat{G}_\alpha \, d\mu_\alpha \, .$$

<u>Proof</u>

Let V denote the corresponding isometry from X into $L_2(M,\mu)$, and let

$(e_k)_{k \in \mathbb{N}}$ denote an orthonormal basis in X. Then

$$\|R\hat{G}_\alpha\|^2 = \sum_{k=1}^{\infty} |(VR\ e_k)^\wedge(\alpha)|^2$$

with $(VR\ e_k)^\wedge(\alpha) = (e_k, R\hat{G}_\alpha)$, $\alpha \in M$, $k \in \mathbb{N}$.

Thus we obtain

$$\int_M \|R\hat{G}_\alpha\|^2 d\mu_\alpha = \sum_{k=1}^{\infty} \int_M |(VR\ e_k)^\wedge(\alpha)|^2 d\mu_\alpha = \sum_{k=1}^{\infty} \|Re_k\|^2 < \infty.$$

Hence

$$\int_M \|<w,\hat{G}_\alpha>R\hat{G}_\alpha\| d\mu_\alpha \leq \left(\int_M |<w,G_\alpha>|^2 d\mu_\alpha \right)^{\frac{1}{2}} \cdot \left(\int_M \|R\hat{G}_\alpha\|^2 d\mu_\alpha \right)^{\frac{1}{2}}. \qquad \square$$

Another problem concerns the existence of Dirac bases for each (M,μ). This problem is solved in the following lemma.

(1.6) <u>Lemma</u>

Let R be a positive bounded operator in X, and let M be a measure space with σ-finite measure μ.

Let the densely defined operator R^{-1} be unbounded. Then there exists an isometry V from X into $L_2(M,\mu)$ such that VR is a Carleman operator. Let also R^{-1} be bounded. Then there exists an isometry V from X into $L_2(M,\mu)$ such that VR is Carleman iff the support of μ consists of atoms only.

Proof

The proof can be obtained from [Wei], Theorem 7.1 and 7.2.　　□

Consequently, we obtain

(1.7) Corollary

Let R be a positive bounded operator in X, and M a measure space with σ-finite measure μ.

(i)　　Let R^{-1} be unbounded. Then there exists a Dirac basis (G,M,μ,R,X) with respect to the triple $R(X) \subseteq X \subseteq R^{-1}(X)$.

(ii)　　Let R^{-1} be bounded. Then the only Dirac bases are the orthogonal bases (Note that $R(X) = X = R^{-1}(X)$).

(iii) Let R be Hilbert-Schmidt. Then any isometry $V : X \to L_2(M,\mu)$ gives rise to a Dirac basis.

If we put restrictions on the measure space (M,μ), a so called canonical choice can be made in each equivalence class (G,M,μ,R,X). In the next section, we clarify this statement. We prove a measure theoretical Sobolev lemma based on Carleman operators.

2. A measure theoretical Sobolev lemma

Let $R > 0$ be a bounded operator, and let M be a metrizable topological measure space with regular Borel measure μ. We assume that the pair (M,μ)

satisfies Federer's conditions, i.e. Theorem (0.3) is valid. Let

$\mathcal{D}$: X → L_2(M,μ) be a densely defined operator with $\mathcal{R}$(X) contained in

its domain.

On the pair $\mathcal{D}$,$\mathcal{R}$ we impose the following conditions

(2.1.i) The operator $\mathcal{D}$·$\mathcal{R}$: X → L_2(M,μ) is a bounded Carleman operator.

Let the function k : M → X induce $\mathcal{D}\mathcal{R}$. (Cf. Definition (0.1).)

(2.1.ii) The function $\alpha \mapsto \|k(\alpha)\|^2$ is integrable on bounded Borel sets.

Remarks

- Condition (2.1.ii) is not redundant. To show this, consider the following

  example: Define k : IR → L_2(IR,dx) by

$$
k(t) = \begin{cases} \chi_{(0,\,|t|^{-1}]} & t > 0 \\[2em] 0 & t = 0 \ . \end{cases}
$$

Then for t ≠ 0, $\|k(t)\|^2 = |t|^{-1}$ and hence condition (2.1.ii) is not

satisfied.

- In our paper [EG 2] a measure theoretical Sobolev lemma has been

  proved based on Hilbert-Schmidt operators. So we started with a positive

  Hilbert-Schmidt operator $\mathcal{R}$, and on $\mathcal{D}$ we imposed the condition that $\mathcal{D}$·$\mathcal{R}$

  is Hilbert-Schmidt. In that case the condition (2.1.ii) is always ful-

  filled because $(\alpha \mapsto \|k(\alpha)\|^2) \in L_2$(M,μ) .

Now let $(v_k)_{k \in IN}$ denote an orthonormal basis in X. Since $\mathcal{D}\mathcal{R}v_k \in L_2$(M,μ)

there exists a null set N_1 such that for all k ∈ IN

(2.2.i) $\phi_k(\alpha) = \lim\limits_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} \mathcal{D}Rv_k \, d\mu$, $\alpha \in M \setminus N$

exists. Each function $\alpha \mapsto \phi_k(\alpha)$ extends to a representant of the class $\mathcal{D}Rv_k$, $k \in \mathbb{N}$.

Since $|\mathcal{D}Rv_k|^2 \in L_1(M,\mu)$ for all $k \in \mathbb{N}$, Theorem (0,3) yields a null set N_2 such that for all $k \in \mathbb{N}$ and $\alpha \in M \setminus N_2$

(2.2.ii) $|\phi_k(\alpha)|^2 = \lim\limits_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} |\mathcal{D}Rv_k|^2 d\mu$.

Finally, since the measurable function $\alpha \mapsto \|k(\alpha)\|^2$ is integrable on bounded Borel sets, Lemma (0.2) and Theorem (0.3) yield a null set N_3 such that for all $\alpha \in M \setminus N_3$

(2.2.iii) $\sum\limits_{k=1}^{\infty} |\phi_k(\alpha)|^2 = \lim\limits_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} (\sum\limits_{k=1}^{\infty} |\mathcal{D}Rv_k|^2) d\mu$.

Throughout this section N denotes the null set $N_1 \cup N_2 \cup N_3$.
We put $\phi_k(\alpha) = 0$ for all $\alpha \in N$, $k \in \mathbb{N}$.

(2.3) <u>Lemma</u>

(i) Let $\alpha \in M$. Define $e_\alpha = \sum\limits_{k=1}^{\infty} \overline{\phi_k(\alpha)} Rv_k$. Then $e_\alpha \in R(X)$.

(ii) Let $\alpha \in M \setminus N$. Then for all $r > 0$

$$e_\alpha(r) = \mu(B(\alpha,r))^{-1} \sum_{k=1}^{\infty} (\int_{B(\alpha,r)} (\overline{\mathcal{D}\mathcal{R}v_k})d\mu)\mathcal{R}v_k$$

belongs to $\mathcal{R}(X)$. Furthermore

$$\lim_{r\downarrow 0} \|e_\alpha - e_\alpha(r)\|_1 = 0 .$$

Proof

(i) Let $\alpha \in \mathsf{N}$. Then $e_\alpha = 0$.

Let $\alpha \in \mathsf{M} \setminus \mathsf{N}$. Then $\sum_{k=1}^{\infty} |\phi_k(\alpha)|^2 < \infty$, whence $e_\alpha \in \mathcal{R}(X)$.

(ii) Let $\alpha \in \mathsf{M} \setminus \mathsf{N}$. The Hölder inequality yields

$$\sum_{k=1}^{\infty} |\mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\overline{\mathcal{D}\mathcal{R}v_k})d\mu|^2 \leqq$$

$$\leqq \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\sum_{k=1}^{\infty} |\mathcal{D}\mathcal{R}v_k|^2)d\mu .$$

Because of condition (2.1.ii) the latter expression is finite, whence

$e_\alpha(r) \in \mathcal{R}(X)$.

Let $\varepsilon > 0$. Take a fixed $n_0 \in \mathsf{IN}$ so large that

$$(*) \qquad \sum_{k=n_0+1}^{\infty} |\phi_k(\alpha)|^2 < \frac{\varepsilon^2}{4} .$$

Next take $r_0 > 0$ so small that for all r, $0 < r < r_0$

(**) $\quad |\phi_k(\alpha) - \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} (\mathcal{D}Rv_k)d\mu| < \varepsilon/\sqrt{n_0}$

for all $n = 1,2,\ldots,n_0$, and also

(***) $\quad \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} (\sum\limits_{k=n_0+1}^{\infty} |\mathcal{D}Rv_k|^2)d\mu < \varepsilon^2.$

(Cf. 2.2.(i)-(iii) and (*).)

We estimate as follows

$$\|e_\alpha - e_\alpha(r)\|_1^2 = (\sum_{k=1}^{n_0} + \sum_{k=n_0+1}^{\infty}) |\overline{\phi_k(\alpha)} - \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} \overline{\mathcal{D}Rv_k}d\mu|^2 .$$

By (**)

$$\sum_{k=1}^{n_0} |\phi_k(\alpha) - \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} (\mathcal{D}Rv_k)|^2 < \varepsilon^2$$

and by (*) and (***)

$$\sum_{k=n_0+1}^{\infty} |\phi_k(\alpha) - \mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} \mathcal{D}Rv_k \, d\mu|^2 \leq$$

$$\leq 2 \sum_{k=n_0+1}^{\infty} |\phi_k(\alpha)|^2 + 2\mu(B(\alpha,r))^{-1} \int\limits_{B(\alpha,r)} (\sum_{k=n_0+1}^{\infty} |\mathcal{D}Rv_k|^2)d\mu < 3\varepsilon^2$$

Thus we have proved that for all r, $0 < r < r_0$

$$\|e_\alpha - e_\alpha(r)\|_1 < 2\varepsilon \ .$$

(2.4) Theorem (Measure theoretical Sobolev lemma)

For each $w \in R(X)$ a representant $\widetilde{\mathcal{D}w}$ can be chosen with the following properties

(i) $\widetilde{\mathcal{D}w} = \sum\limits_{k=1}^{\infty} (R^{-1}w, v_k)\phi_k$ with pointwise convergence.

(ii) Let $\alpha \in M$. Then the linear functional $w \to \widetilde{\mathcal{D}w}(\alpha)$ is continuous on $R(X)$; its Riesz representant in $R(X)$ equals e_α.

(iii) Let $\alpha \in M \setminus N$. Then

$$\widetilde{\mathcal{D}w}(\alpha) = \lim_{r \downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\mathcal{D}w)\,d\mu \ .$$

(iv) Suppose $\alpha \mapsto \|k(\alpha)\|^2$ is essentially bounded on M. Then there exists a null set N_0 such that the convergence in (i) is uniform on $M \setminus N_0$. Further,

$$\exists_{K>0} \ \forall_{\alpha \in M \setminus N_0} \ : \ |\widetilde{\mathcal{D}w}(\alpha)| \leq K\|w\|_1 \ .$$

Proof

Let $w \in R(X)$, and put $\widetilde{\mathcal{D}w} = \sum\limits_{k=1}^{\infty} (R^{-1}w, v_k)\phi_k$. Then $\widetilde{\mathcal{D}w} \in \mathcal{D}w$, because

$$\mathcal{D}w = \sum\limits_{k=1}^{\infty} (R^{-1}w, v_k)\mathcal{D}Rv_k \ .$$

(i)   Since $(w,e_\alpha)_1 = \sum\limits_{k=1}^{\infty} (R^{-1}w,v_k)\phi_k$, and since $e_\alpha \in R(X)$ the series

converges pointwise on M.

(ii) Trivial, because $\mathcal{D}w(\alpha) = (w,e_\alpha)_1$.

(iii) Let $\alpha \in M \setminus N$. Then by Lemma (2.3)

$$\widetilde{\mathcal{D}}w(\alpha) = \lim_{r\downarrow 0} (w,e_\alpha(r))_1 =$$

$$= \lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \sum_{k=1}^{\infty} (R^{-1}w,v_k)\left(\int_{B(\alpha,r)} \mathcal{D}Rv_k \, d\mu \right) .$$

We show that summation and integration can be interchanged.

$$\left(\sum_{k=1}^{\infty} \int_{B(\alpha,r)} \right) \mu(B(\alpha,r))^{-1}((R^{-1}w,v_k))|\mathcal{D}Rv_k|d\mu \leq$$

$$\leq \left(\sum_{k=1}^{\infty} |(R^{-1}w,v_k)|^2 \right)^{\frac{1}{2}} \left(\mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} \left(\sum_{k=1}^{\infty} |\mathcal{D}Rv_k|^2 \right) d\mu \right)^{\frac{1}{2}} .$$

This yields

$$\widetilde{\mathcal{D}}w(\alpha) = \lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} \left(\sum_{k=1}^{\infty} (R^{-1}w,v_k)\mathcal{D}Rv_k \right) d\mu ,$$

which proves statement (2.4.iii) .

(iv) There exists $K > 0$ and a null set $N_0 \subset M$ such that

$$\sum_{k=1}^{\infty} |\phi_k(\alpha)|^2 \leq K^2 , \qquad \alpha \in M \setminus N_0 .$$

So for all $L_1 < L_2$, $L_1, L_2 \in \mathbb{N}$.

$$\left| \sum_{k=L_1}^{L_2} (R^{-1}w, v_k) \phi_k(\alpha) \right| \leq K \left(\sum_{k=L_1}^{L_2} ((R^{-1}w, v_k))^2 \right)^{\frac{1}{2}} . \qquad \square$$

Illustration: "The classical Sobolev embedding theorem on $\mathbb{R}^n$"

Let $dx = dx_1 \ldots dx_n$ denote the Lebesque measure on $\mathbb{R}^n$.

Let Δ denote the positive self-adjoint operator

$$\Delta = 1 - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}$$

in $L_2(\mathbb{R}^n, dx)$. For each $m > 0$, put $R_m = \Delta^{-m/2}$. Then R_m is positive and bounded. It is well-known that the Sobolev space $H^m(\mathbb{R}^n))$ of order m equals $R_m(L_2(\mathbb{R}^n))$. We have the classical Sobolev triple $H^m(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \subset H^{-m}(\mathbb{R}^n)$

(2.5) Corollary (Cf. [Yo], p.174.)

Let $m > n/2$ and let $0 \leq \ell < m - n/2$, $\ell \in \mathbb{N} \cup \{0\}$. Then there is a null set $N_n^{(\ell)}$ such that for each $u \in H^m(\mathbb{R})$ there exists a representant $\tilde{u}$ with the following property: For all $s \in (\mathbb{N} \cup \{0\})^m$, $|s| \leq \ell$, there

exists $\gamma_s > 0$ such that

$$\forall_{x \in \mathbb{R}^n \setminus N_m^{(\ell)}} \; : \; |(D^s u)\tilde{\,}(x)| \leq \gamma_s \|u\|_m \; .$$

Here $D^s = \left(\dfrac{\partial}{\partial x_1}\right)^{s_1} \dots \left(\dfrac{\partial}{\partial x_n}\right)^{s_n}$, $|s| = s_1 + \dots + s_n$ and $\|\cdot\|_m$ denotes the

norm of $H^m(\mathbb{R}^n)$.

Proof

Let $\mathbb{F}_n$ denote the Fourier transform on $L_2(\mathbb{R}^n)$. The operator $D^s R_m$ is a

bounded Carleman operator induced by the function $k_{s,m} : \mathbb{R}^n \to L_2(\mathbb{R}^n)$,

$$k_{s,m}(x;y) = (\mathbb{F}_n g_{s,m})(x - y) \; , \qquad x,y \in \mathbb{R}^n$$

with

$$g_{s,m}(y) = (i)^{|s|} y_1^{s_1} \cdot \dots \cdot y_n^{s_n} / (1 + y_1^2 + \dots + y_n^2)^{-m/2} \; .$$

So for all $x \in \mathbb{R}^n$,

$$\|k_{s,m}(x;\cdot)\|_{L_2(\mathbb{R}^n)} = \|g_{s,m}\|_{L_2(\mathbb{R}^n)} \; . \qquad\qquad \square$$

3. Canonical Dirac basis with applications to the generalized eigenvalue

problem

In Section 1 we have defined a Dirac basis (G,M,μ,R,X) as an equivalence

class of μ-measurable functions from M to $R^{-1}(X)$. No restrictions have

been put on the measure space M and on the σ-finite measure μ. However,

if we restrict to topological measure spaces M and regular Borel measures μ, which satisfy Federer's conditions, then for certain $R > 0$, a canonical choice can be made in the equivalence class (G,M,μ,R,X). Such a choice is called a canonical Dirac basis.

(3.1) <u>Definition</u>

Let (G,M,μ,R,X) be a Dirac basis. A representant $\widetilde{G} \in (G,M,\mu,R,X)$ is called a *canonical* Dirac basis if there exists a null set $\widetilde{N}$ such that for all $w \in R(X)$ and all $\alpha \in M \setminus \widetilde{N}$

$$\lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} <w,\widetilde{G}_\beta>d\mu_\beta = <w,\widetilde{G}_\alpha> \quad .$$

We use the notation $(\widetilde{G}_\alpha)_{\alpha\in M}$.

Throughout this section we assume that M is a metrizable topological measure space which satisfies Federer's conditions, and μ a regular Borel measure on M such that bounded sets have finite μ-measure. So Theorem (0.3) is valid.

(3.2) <u>Lemma</u>

Let $V : X \to L_2(M,\mu)$ be an isometry with the property that VR is a Carleman operator. Assume that the function $k : M \to \mathbb{C}$ which induces VR,

satisfies (2.1.ii). Then in the Dirac basis (G,M,μ,R,X) associated to

V (cf. Theorem (1.4)) there exists a canonical representant $(\widetilde{G}_\alpha)_{\alpha\in M}$.

<u>Proof</u>

Following Lemma (2.3) and Theorem (2.4) there are $\widetilde{g}_\alpha \in R(X)$, $\alpha \in M$,

and there is a null set $\widetilde{N}$ such that for all $w \in R(X)$

$$(\alpha \mapsto (w,\widetilde{g}_\alpha)_1) \in Vw$$

and

$$(w,\widetilde{g}_\alpha)_1 = \lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} <w,\widetilde{g}_\beta>d\mu_\beta .$$

For each $\alpha \in M$, we define $\widetilde{G}_\alpha \in R^{-1}(X)$ by

$$(w,\widetilde{g}_\alpha)_1 = <w,\widetilde{G}_\alpha>, \qquad w \in R(X) .$$

Then $(\widetilde{G}_\alpha)_{\alpha\in M} \in (G,M,\mu,R,X)$ (cf. Theorem (1.4)), and $(\widetilde{G}_\alpha)_{\alpha\in M}$ is canonical

Dirac basis. $\qquad\square$

(3.3) <u>Remarks</u>

Let $(u_k)_{k\in\mathbb{N}}$ be an orthonormal basis in X. For each $\alpha \in M$, we put

$$g_\alpha(r) = \mu(B,\alpha,r))^{-1} \sum_{k=1}^{\infty} (\int_{B(\alpha,r)} \overline{VRu_k} \, d\mu)u_k .$$

Then by Lemma (2.3) there exists a null set $\widetilde{N}$ such that

$$\lim_{r \downarrow 0} \|\widetilde{g}_\alpha - \widetilde{g}_\alpha(r)\|_1 = 0 , \quad \alpha \in M \setminus \widetilde{N} .$$

We observe that $\widetilde{g}_\alpha(r) = \mu(B(\alpha,r))^{-1} RV^* \chi_{B(\alpha,r)}$ where $\chi_{B(\alpha,r)}$ denotes the characteristic function of $B(\alpha,r)$.

In the next theorem we present a sufficient condition which guarantees that a Dirac basis (G,M,μ,R,X) contains a canonical representant.

(3.4) <u>Theorem</u>

Let (G,M,μ,R,X) be a Dirac basis, and $\hat{G}$ denote a representant. Assume that the measurable function $\alpha \mapsto \|R\hat{G}_\alpha\|^2$ is integrable on bounded Borel sets. Then (G,M,μ,R,X) contains a canonical Dirac basis $(\hat{G}_\alpha)_{\alpha \in M}$.

<u>Proof</u>

Theorem (1.3) yields an isometry V from X into $L_2(M,\mu)$ such that VR is Carleman, and

$$(\alpha \mapsto < VRf, \hat{G}_\alpha >) \in VRf, \quad f \in X.$$

It is clear that the Carleman operator VR is induced by the function $\hat{k} : \alpha \mapsto R\hat{G}_\alpha$. By assumption $\hat{k}$ satisfies (2.1.ii). Hence from Lemma (3.2) the assertion follows. □

In a natural way the notion of canonical Dirac basis is associated to

the generalized eigenvalue problem. We describe this connection here.

Let ϕ be a complex valued measurable function on M, which is bounded

on Borel sets. In $L_2(M,\mu)$ we define the multiplication operator M_ϕ by

$$D(M_\phi) = \{h \in L_2(M,\mu) \mid \int_M |\phi h|^2 d\mu < \infty\}$$

and

$$M_\phi h = \phi h , \qquad h \in D(M_\phi) .$$

Because of the conditions on ϕ, $\chi_{B(\alpha,r)} \in D(M_\phi)$ for all $r > 0$ and all

$\alpha \in M$. We note that M_ϕ is a normal operator.

(3.5) <u>Lemma</u>

Let (G,M,μ,R,X) denote a Dirac basis. Let $\hat{G}$ be a representant such that

the function $\alpha \mapsto \|R\hat{G}_\alpha\|^2$ is integrable on bounded Borel sets. Further let

$V : X \to L_2(M,\mu)$ denote the isometry associated to (G,M,μ,R,X). Then

there exists a canonical representant $(\widetilde{G}_\alpha)_{\alpha \in M}$ and a null set $\widetilde{N}$ such that

for all $\alpha \in M \setminus \widetilde{N}$

$$\lim_{r \downarrow 0} \|\widetilde{G}_\alpha - \mu(B(\alpha,r))^{-1}\{V^*\chi_{B(\alpha,r)}\}\|_{-1} = 0 .$$

Let $\phi : M \to \mathbb{C}$ be a measurable function which is bounded on bounded Borel

sets. Then there exists a null set N_ϕ such that for all $\alpha \in M \setminus N_\phi$

$$\lim_{r \downarrow 0} \|\phi(\alpha)\widetilde{G}_\alpha - \mu(B(\alpha,r))^{-1}\{V^*M_\phi \chi_{B(\alpha,r)}\}\|_{-1} = 0 .$$

Proof

The first statement follows immediately from the previous theorem and remark (3.3.i).

With respect to the second assertion, we are ready if we can prove that there exists a null set N_2 such that for all $\alpha \in M \setminus N_2$

(*) $\lim\limits_{r \downarrow 0} \mu(B(\alpha,r))^{-1} \left\| V^*(\phi(\alpha)I - M_\phi) \chi_{B(\alpha,r)} \right\|_{-1} = 0$.

Consider the estimation for all $\alpha \in M$ with $\mu(B(\alpha,r)) \neq 0$, $r > 0$.

$$\mu(B(\alpha,r))^{-2} \left\| V^*(\phi(\alpha)I - M_\phi) \chi_{B(\alpha,r)} \right\|_{-1}^2 =$$

$$= \sum_{k=1}^{\infty} \left| \mu(B(\alpha,r))^{-1} \left(\int_{B(\alpha,r)} (\phi(\alpha) - \phi(\lambda)) VR u_k \, d\mu_\lambda \right) \right|^2 \leq$$

$$\leq (\mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} |\phi(\alpha) - \phi(\lambda)|^2 d\mu_\lambda)(\mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\sum_{k=1}^{\infty} |VR u_k|^2) d\mu)$$

Since ϕ is bounded on bounded Borel sets, both ϕ and $|\phi|^2$ are μ-integrable on bounded Borel sets. Hence there exists a null set N_{21} such that for all $\alpha \in M \setminus N_{21}$

$$\lim_{r \downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} |\phi(\alpha) - \phi(\lambda)|^2 d\mu_\lambda = 0 \ .$$

Further, for all $r > 0$ and all α

$$\int_{B(\alpha,r)} (\sum_{k=1}^{\infty} |VRu_k|^2)d\mu = \int_{B(\alpha,r)} \| R\widetilde{G}_\lambda \|^2 d\mu_\lambda \ .$$

So there exists a null set N_{22} such that for all $\alpha \in M \setminus N_{22}$

$$\lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\sum_{k=1}^{\infty} |VRu_k|^2)d\mu = \| R\widetilde{G}_\alpha \|^2 \ .$$

Now put $N_2 = N_{21} \cup N_{22}$, then (*) follows. $\qquad\qquad\qquad \square$

(3.6) <u>Corollary</u>

Suppose V is unitary and $V^*M_\phi V$ extends to a closable operator in $R^{-1}(X)$, i.e. $RV^*M_\phi V^{-1}$ is closable in X. Then we have

$$\overline{V^*M_\phi V}\ \widetilde{G}_\alpha = \phi(\alpha)G_\alpha$$

where $\overline{V^*M_\phi V}$ denotes the closure in $R^{-1}(X)$ of $V^*M_\phi V$.

<u>Proof</u>

We have $\qquad \lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1}RV^*\chi_{B(\alpha,r)} = R\widetilde{G}_\alpha$

and

$$\lim_{r\downarrow 0} \mu(B(\alpha,r))^{-1}RV^*M_\phi\chi_{B(\alpha,r)} = \phi(\alpha)R\widetilde{G}_\alpha \ .$$

Hence

$$\overline{V^*M_\phi V}\,\widetilde{G}_\alpha \;=\; \phi(\alpha)\widetilde{G}_\alpha \;.$$

□

An application of the previous results is the following.

Let T be a self-adjoint operator in X with a simple spectrum. Then there

exists a finite Borel measure μ on $\mathbb{R}$ and a unitary operator $U : X \to L_2(\mathbb{R},\mu)$

such that UTU^* equals the self-adjoint operator of multiplication by the

identity function. It is clear that $(\mathbb{R},\mu)$ satisfies Federer's conditions.

Now let R be a bounded positive operator with the property that

$UR : X \to L_2(\mathbb{R},\mu)$ is a Carleman operator satisfying Condition (2.1.ii).

Then following Lemma (3.2) and (3.5), there exists a canonical Dirac

basis $(\widetilde{E}_\alpha)_{\alpha \in \mathbb{R}}$ and a null set N_T such that for all $\alpha \in \mathbb{R} \setminus N_T$

$$\lim_{r\downarrow 0} \left\| \widetilde{E}_\alpha - \mu(B(\alpha,r))^{-1}\{U^*\chi_{B(\alpha,r)}\} \right\|_{-1} = 0$$

and

$$\lim_{r\downarrow 0} \left\| \alpha\widetilde{E}_\alpha - \mu(B(\alpha,r))^{-1}\{TU^*\chi_{B(\alpha,r)}\} \right\|_{-1} = 0 \;.$$

(Here $B(\alpha,r) = [\alpha - r,\ \alpha + r]$.) So for each $\alpha \in \mathbb{R} \setminus N_T$, E_α is a candidate

(generalized) eigenvector.

If RTR^{-1} is closable in X, then the closure $\overline{T}$ of T in $R^{-1}(X)$ exists, and

for all $\alpha \in \mathbb{R} \setminus N_T$

$$\overline{T}\,\widetilde{E}_\alpha \;=\; \alpha\widetilde{E}_\alpha$$

So the rather mild condition that RTR^{-1} is closable in X yields 'genuine'

eigenvectors.

(3.7) <u>Remarks</u>

(i) If R is a positive Hilbert-Schmidt operator, then for any unitary

operator V, the operator VR is Hilbert-Schmidt. It follows that

for each self-adjoint operator in X with simple spectrum. there

exists a canonical Dirac basis of candidate (generalized) eigen-

vectors. In our paper [EG 5] we have proved the same result for

any self.adjoint operator.

(ii) Along similar lines as in [EG 5] the following can be proved:

Let T be any self-adjoint operator in X. Let U be its diagonalizing

unitary operator in the sense of the multiplicity theorem (Cf.

Theorem 1.2 in [EG 5]), and let $R > 0$ such that UR is a Carleman

operator which satisfies (1.2.ii). Then the unitary operator U

gives rise to a canonical Dirac basis $(\widetilde{G}_\alpha)_{\alpha \in M}$. Here M denotes the

disjoint union

$$M = \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} \mathbb{R}_{m,j} \cup \left(\bigcup_{j=1}^{\infty} \mathbb{R}_{\infty,j} \right)$$

where each $\mathbb{R}_{m,j}$, $m = \infty, 1, 2, \ldots$, $1 \le j < m+1$, is a copy of $\mathbb{R}$.

To almost each point λ in the spectrum $\sigma(T)$ of T with multiplicity

m_λ there belong m_λ candidate (generalized) eigenvectors which are

elements of $\{G_\alpha \mid \alpha \in M\}$.

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