On the correct form of rate-type constitutive equations for elastic behavior

ON THE CORRECT FORM OF RATE-TYPE CONSTITUTIVE EQUATIONS FOR ELASTIC BEHAVIOR.

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SUMMARY.

Deformations are elastic if the stresses depend on the total deformation and not on the way in which this deformation has been reached. From this concept a class of rate–type constitutive models is developed for elastic deformations. It is proved that earlier proposed models, like those using the Jaumann and the Dienes rate, in general do not result in a correct description of elastic material behavior.

1. INTRODUCTION.

In this paper attention is focused mainly on the description of elastic behavior, starting from a rate–type constitutive equation of the type

$$\overset{\nabla}{\sigma} = {}^4\mathbf{C}:\mathbf{D}, \tag{1.1}$$

where σ is the Cauchy stress tensor and ∇ denotes an objective rate, like the Jaumann [1], the Truesdell [2], the Cotter–Rivlin [3] or the Dienes rate. Furthermore, $\mathbf{D}$ is the deformation rate tensor and ${}^4\mathbf{C}$ is a fourth order elasticity tensor, which does not depend on $\mathbf{D}$ nor on the rate of σ but can be a function of σ. In the geometrical linear theory the deformations and rotations are very small and the objective rate of σ may be approximated by the material rate of σ. Then, for a given state at time t_0 and a given deformation path in the time interval $[t_0,t]$ the current stress tensor $\sigma(t)$ can be determined by integration of ${}^4\mathbf{C}:\mathbf{D}$ over that interval. In the geometrical non–linear theory, however, it is not allowed to approximate the objective rate by the material rate.

The reason to choose (1.1) for the description of elastic behavior is that this form is often used as the starting point for the derivation of constitutive equations for elastic–plastic behavior. In section 4 it turns out that this kind of behavior can also be described by (1.1) if ${}^4\mathbf{C}$ is replaced by the elastic–plastic material tensor ${}^4\mathbf{L}$.

Nagtegaal and de Jong [5] showed that (1.1) yields unacceptable results in the simple shear test if the Jaumann rate is used. Other authors [6,7,8] tried to get acceptable results by using other objective rates. Lee, Mallet and Wertheimer [6] used a modification of the Jaumann rate in their analysis of the shear test of kinematic hardening materials. However. a generalization of their procedure to arbitrary deformation patterns is not trivial. Atluri [8] used symmetrical and non–symmetrical objective rates that can be written as the Jaumann rate plus an objective function of the tensor $\sigma\cdot\mathbf{D} + \mathbf{D}\cdot\sigma$. His analysis of the influence of this function on the resulting stresses in the simple shear test leads, among others, to the conclusion that the Truesdell and the Cotter–Rivlin rate yield acceptable results. His conclusion that other objective rates are superfluous is not really proved.

In this paper a class of objective rates is studied and it is examined which rates can result in a correct description of elastic behavior. The earlier mentioned rates belong to this class. It will be shown that for each of these rates there exists a tensor **A**, such that

$$\sigma = \mathbf{A}^{-1}\cdot\mathbf{S}\cdot\mathbf{A}^{-c}; \quad \overset{\nabla}{\sigma} = \mathbf{A}^{-1}\cdot\dot{\mathbf{S}}\cdot\mathbf{A}^{-c} \tag{1.2}$$

Here, the index ^{c} denotes conjugation, i.e.

$$\vec{a}\cdot(\mathbf{A}\cdot\vec{b}) = \vec{b}\cdot(\mathbf{A}^c\cdot\vec{a}) \quad \text{for all } \vec{a}, \vec{b} \text{ and } \mathbf{A}. \tag{1.3}$$

The tensor S is invariant under rigid body rotations. Combination of (1.1) and (1.2) yields a relation for the material rate of S:

$$\dot{\mathbf{S}} = \mathbf{A}\cdot({}^4\mathbf{C}:\mathbf{D})\cdot\mathbf{A}^c \tag{1.4}$$

Integration of this relation yields S, whereupon σ can be determined. This procedure is used in section 5 to investigate which objective rates can result in a correct description of elastic material behavior. For isotropic elasticity tensors ${}^4\mathbf{C}$ it will be shown that this is not the case for the Jaumann rate and the Dienes rate. The discussion in section 5 leads to the introduction of a special class of objective rates, each of which can result in a correct description of some kind of elastic behavior. The Truesdell and the Cotter–Rivlin rate belong to this class. In section 6 the objective rates of this class are used in the analysis of the torsion of elastic bars.

2. SOME KINEMATIC NOTIONS.

Let **F** be the deformation tensor of the current configuration of the body with respect to a reference configuration. The determinant J of **F** equals the current volume per unit reference volume:

$$J = \det(\mathbf{F}); \quad J>0 \tag{2.1}$$

Right polar decomposition of **F** leads to the right Cauchy strain tensor **C**, the right stretch tensor **U** and the rotation tensor **R**:

$$\mathbf{F} = \mathbf{R}\cdot\mathbf{U}; \quad \mathbf{C} = \mathbf{F}^c\cdot\mathbf{F} = \mathbf{U}^2; \quad \mathbf{R}^c\cdot\mathbf{R} = \mathbf{I} \tag{2.2}$$

Similarly, left polar decomposition of **F** leads to the left Cauchy strain tensor **B** and the left stretch tensor **V**:

$$\mathbf{F} = \mathbf{V}\cdot\mathbf{R}; \quad \mathbf{B} = \mathbf{F}\cdot\mathbf{F}^c = \mathbf{V}^2 \tag{2.3}$$

Since **V** is symmetric and positive definite the eigenvalues λ_1, λ_2 and λ_3 are real and positive. Hence **V** and **B** can be written as

$$\mathbf{V} = \sum_{i=1}^{3}(\lambda_i\vec{n}_i\vec{n}_i); \quad \mathbf{B} = \sum_{i=1}^{3}(\lambda_i^2\vec{n}_i\vec{n}_i), \tag{2.4}$$

where $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are mutually orthogonal unit eigenvectors. The principal logarithmic strains ϵ_1, ϵ_2 and ϵ_3 are defined by

$$\epsilon_i = \ln(\lambda_i) \quad \text{for } i = 1,2,3 \tag{2.5}$$

The deformation rate tensor **D** and the spin tensor **Ω** follow from

$$\mathbf{D} + \mathbf{\Omega} = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1}; \quad \mathbf{D} = \mathbf{D}^c; \quad \mathbf{\Omega} = -\mathbf{\Omega}^c \tag{2.6}$$

and from these definitions it is readily seen that

$$\dot{\mathbf{B}} = (\mathbf{D} + \mathbf{\Omega})\cdot\mathbf{B} + \dot{\mathbf{B}}\cdot(\mathbf{D} + \mathbf{\Omega})^c \tag{2.7}$$

A second relation for $\dot{\mathbf{B}}$ can be derived from (2.4) if the eigenvectors of $\mathbf{B}$ are differentiable functions of t. Then there is a tensor $\mathbf{N}$, such that

$$\mathbf{N} = -\mathbf{N}^c; \quad \dot{\vec{n}}_i = \mathbf{N}\cdot\vec{n}_i \quad \text{for } i = 1,2,3 \tag{2.8}$$

and $\dot{\mathbf{B}}$ can also be written as

$$\dot{\mathbf{B}} = [\sum_{i=1}^{3}(\dot{\epsilon}_i\vec{n}_i\vec{n}_i) + \mathbf{N}]\cdot\mathbf{B} + \mathbf{B}\cdot[\sum_{i=1}^{3}(\dot{\epsilon}_i\vec{n}_i\vec{n}_i) + \mathbf{N}]^c \tag{2.9}$$

Comparison of this result with (2.7) yields that $\mathbf{N}$ has to satisfy

$$\mathbf{N} = \mathbf{D} - \sum_{i=1}^{3}(\dot{\epsilon}_i\vec{n}_i\vec{n}_i) + \mathbf{\Omega} + \mathbf{B}\cdot\mathbf{W} \tag{2.10}$$

where $\mathbf{W}$ is a skew-symmetrical tensor. Because $\mathbf{D}$ is symmetrical while $\mathbf{N}$ and $\mathbf{\Omega}$ are skew-symmetrical it is seen that

$$\mathbf{D} = \sum_{i=1}^{3}(\dot{\epsilon}_i\vec{n}_i\vec{n}_i) + \tfrac{1}{2}(\mathbf{W}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{W}^c) \tag{2.11}$$

$$\mathbf{N} = \mathbf{\Omega} + \tfrac{1}{2}(\mathbf{W}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{W}^c) \tag{2.12}$$

Furthermore it is seen that the trace of $\mathbf{D}$, tr($\mathbf{D}$), is equal to

$$\mathrm{tr}(\mathbf{D}) = \mathbf{I}:\mathbf{D} = \frac{\dot{J}}{J} = \dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3 \tag{2.13}$$

A motion of the body is called an objective equivalent motion if it is generated from the real motion by a rigid body translation and/or a rigid body rotation $\mathbf{Q}$, where $\mathbf{Q}$ is a rotation tensor:

$$\mathbf{Q}^c\cdot\mathbf{Q} = \mathbf{I}; \quad \det(\mathbf{Q}) = 1 \tag{2.14}$$

The transformation of the real motion to a objective equivalent motion is called an objective transformation. Rigid body translations are irrelevant in the sequel and are left out of consideration. Hence, objective transformations are characterized by the tensor $\mathbf{Q}$ and can be denoted by $\mathcal{J}(\mathbf{Q})$. Quantities, related to an objective equivalent motion, are labeled with *. A quantity that remains unchanged for every $\mathcal{J}(\mathbf{Q})$ is called invariant. A scalar quantity ϕ, a vectorial quantity $\vec{\phi}$ and a second order tensorial quantity $\boldsymbol{\phi}$ are objective if for every $\mathcal{J}(\mathbf{Q})$ holds

$$\phi^* = \phi; \quad \vec{\phi}^* = \mathbf{Q}\cdot\vec{\phi}; \quad \boldsymbol{\phi}^* = \mathbf{Q}\cdot\boldsymbol{\phi}\cdot\mathbf{Q}^c; \tag{2.15}$$

A fourth order tensorial quantity ${}^4\boldsymbol{\phi}$ is objective if for every $\mathcal{J}(\mathbf{Q})$ and every second order tensor $\mathbf{M}$ holds

$${}^4\boldsymbol{\phi}^*{:}\mathbf{M} = \mathbf{Q}\cdot[{}^4\boldsymbol{\phi}{:}(\mathbf{Q}^c\cdot\mathbf{M}\cdot\mathbf{Q})]\cdot\mathbf{Q}^c \tag{2.16}$$

Every invariant scalar quantity is also objective. The unit tensor $\mathbf{I}$ is the only second order tensor that is both invariant and objective. There are three fourth order tensors that are both invariant and objective. These are denoted by ${}^4\mathbf{I}$, ${}^4\mathbf{I}^c$ and $\mathbf{II}$ and are defined by the requirement that for every $\mathbf{M}$ holds

$${}^4\mathbf{I}{:}\mathbf{M} = \mathbf{M}; \quad {}^4\mathbf{I}^c{:}\mathbf{M} = \mathbf{M}^c; \quad \mathbf{II}{:}\mathbf{M} = \mathrm{tr}(\mathbf{M})\mathbf{I} \tag{2.17}$$

From their definitions it follows that J and $\mathbf{U}$ are invariant, that $\mathbf{D}$ and $\mathbf{V}$ are objective and that $\mathbf{F}$, $\mathbf{R}$ and $\boldsymbol{\Omega}$ are neither invariant nor objective:

$$\mathbf{F}^* = \mathbf{Q}\cdot\mathbf{F}; \quad \mathbf{R}^* = \mathbf{Q}\cdot\mathbf{R} \tag{2.18}$$

$$J^* = J; \quad \mathbf{U}^* = \mathbf{U} \tag{2.19}$$

$$\mathbf{D}^* = \mathbf{Q}\cdot\mathbf{D}\cdot\mathbf{Q}^c; \quad \boldsymbol{\Omega}^* = \dot{\mathbf{Q}}\cdot\mathbf{Q}^c + \mathbf{Q}\cdot\boldsymbol{\Omega}\cdot\mathbf{Q}^c \tag{2.20}$$

It is tacitly assumed here that $\mathbf{Q} = \mathbf{I}$ in the reference state.

3. OBJECTIVE AND INVARIANT STRESS QUANTITIES.

The mechanical power, currently supplied to an infinitesimal small material element with current volume dV, is equal to $\boldsymbol{\sigma}{:}\mathbf{D}\mathrm{d}V$. The mechanical power per unit of reference volume π is given by

$$\pi = J\boldsymbol{\sigma}{:}\mathbf{D} \tag{3.1}$$

The principle of objectivity states that π is invariant, i.e. that $\pi^* = \pi$ for every $\mathcal{J}(\mathbf{Q})$. Because $\mathbf{D}$ and J are objective $\boldsymbol{\sigma}$ must be objective too:

$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^c \quad \text{for every } \mathcal{J}(\mathbf{Q}) \tag{3.2}$$

However, the material rate of $\boldsymbol{\sigma}$ is not objective since

$$(\dot{\boldsymbol{\sigma}}^*) = \dot{\mathbf{Q}}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^c + \mathbf{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\mathbf{Q}^c + \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\dot{\mathbf{Q}}^c \neq \mathbf{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\mathbf{Q}^c \tag{3.3}$$

if $\dot{\mathbf{Q}} \neq \mathbf{O}$. With (2.20) $\dot{\mathbf{Q}}$ can be eliminated, yielding

$$(\dot{\boldsymbol{\sigma}}^*) - \boldsymbol{\Omega}^*\cdot\boldsymbol{\sigma}^* - \boldsymbol{\sigma}^*\cdot(\mathbf{Q}^*)^c = \mathbf{Q}\cdot(\dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{\Omega}^c)\cdot\mathbf{Q}^c \tag{3.4}$$

and this shows that the Jaumann or Zaremba rate $\overset{\nabla}{\boldsymbol{\sigma}}_J$ of $\boldsymbol{\sigma}$, given by

$$\overset{\nabla}{\boldsymbol{\sigma}}_J = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{\Omega}^c, \tag{3.5}$$

is objective. Furthermore, it follows that every rate $\overset{\nabla}{\boldsymbol{\sigma}}$ of the type

$$\overset{\nabla}{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}}_J - \mathbf{M} \tag{3.6}$$

is objective if $\mathbf{M}$ is objective. It is not necessary for $\mathbf{M}$ to be symmetric. Atluri [8] uses non–symmetrical tensors but it is not clear whether the resulting non–symmetrical objective rates offer any advantage in formulating constitutive equations. Here, only symmetrical tensors $\mathbf{M}$ of the type

$$\mathbf{M} = \tilde{\mathbf{H}}\cdot\sigma + \sigma\cdot\tilde{\mathbf{H}}^c \tag{3.7}$$

are considered with tensors $\tilde{\mathbf{H}}$ of the form

$$\tilde{\mathbf{H}} = \mathbf{H} + \sigma\cdot\mathbf{P} \tag{3.8}$$

Here **H** must be objective and **P** must be skew–symmetrical to guarantee objectivity of **M** :

$$\mathbf{H}^* = \mathbf{Q}\cdot\mathbf{H}\cdot\mathbf{Q}^c \quad \text{for every } \mathcal{I}(\mathbf{Q}); \quad \mathbf{P} = -\,\mathbf{P}^c \tag{3.9}$$

Combination of (3.5), (3.6), (3.7) and (3.8) finally yields

$$\overset{\nabla}{\sigma} = \dot{\sigma} - (\mathbf{\Omega} + \mathbf{H})\cdot\sigma - \sigma\cdot(\mathbf{\Omega} + \mathbf{H})^c \tag{3.10}$$

Hence, each objective tensor **H** results in an objective rate. It will be shown that each of these rates is associated with the material rate of an invariant stress tensor. Let **A** be the solution of

$$\dot{\mathbf{A}} = -\,\mathbf{A}\cdot(\mathbf{\Omega} + \mathbf{H}) \quad \text{for } t>t_0; \quad \mathbf{A} = \mathbf{I} \quad \text{for } t=t_0 \tag{3.11}$$

Without any essential restriction it may be assumed that **A** is regular for all $t \geq t_0$. From (2.20) and (3.9) it is seen that

$$\dot{(\mathbf{A}^*\cdot\mathbf{Q})} = -\,(\mathbf{A}^*\cdot\mathbf{Q})\cdot(\mathbf{\Omega} + \mathbf{H}) \quad \text{for every } \mathcal{I}(\mathbf{Q}) \tag{3.12}$$

and this means that

$$\mathbf{A}^* = \mathbf{A}\cdot\mathbf{Q}^c \quad \text{for every } \mathcal{I}(\mathbf{Q}) \tag{3.13}$$

Because σ is objective the tensor **S**, defined by

$$\mathbf{S} = \mathbf{A}\cdot\sigma\cdot\mathbf{A}^{c}, \tag{3.14}$$

is invariant and from (3.10) and (3.11) it follows that

$$\mathring{\mathbf{S}} = \mathbf{A}\cdot\overset{\nabla}{\sigma}\cdot\mathbf{A}^{c} \tag{3.15}$$

It will be clear that there are two methods to arrive at objective rates of the considered type: the direct method, based on the choice of an objective tensor **H**, and the indirect method, based on the choice of a tensor **A** which satisfies (3.13). To illustrate the direct method **H** is chosen as

$$\mathbf{H} = -\gamma\mathrm{tr}(\mathbf{D})\mathbf{I} \tag{3.16}$$

where γ is a constant. Together with (3.10) this results in

$$\overset{\nabla}{\sigma} = \dot{\sigma} - \boldsymbol{\Omega}\cdot\sigma - \sigma\cdot\boldsymbol{\Omega}^{c} + 2\gamma\mathrm{tr}(\mathbf{D})\sigma = J^{-2\gamma}(J^{-2\gamma}\overset{\nabla}{\sigma})_{J} \tag{3.17}$$

If $\gamma = 0$ then $\overset{\nabla}{\sigma} = \overset{\nabla}{\sigma}_{J}$, i.e. the Jaumann rate is obtained if $\mathbf{H} = \mathbf{O}$. For $\gamma \neq 0$ it is seen from (3.11) that **A** is given by

$$\mathbf{A} = J^{\gamma}\cdot\mathbf{P}^{c} \tag{3.18}$$

where the rotation tensor **P** is the solution of

$$\dot{\mathbf{P}} = \boldsymbol{\Omega}\cdot\mathbf{P} \quad \text{for } t>t_0; \quad \mathbf{P} = \mathbf{I} \quad \text{for } t=t_0 \tag{3.19}$$

Other choices for **H** have been given by e.g. Lee c.s. [6] and Atluri [8].

To illustrate the indirect method **A** is chosen as

$$\mathbf{A} = J^{\gamma}\cdot\mathbf{R}^{c} \tag{3.20}$$

where γ is a constant and **R** is the rotation tensor from the decomposition (2.2). The associated objective rate is given by

$$\overset{\nabla}{\sigma} = \overset{\nabla}{\sigma}_D + 2\gamma \mathrm{tr}(\mathbf{D})\sigma; \quad \overset{\nabla}{\sigma}_D = \dot{\sigma} - \dot{\mathbf{R}}\cdot\mathbf{R}^c\cdot\sigma - \sigma\cdot(\dot{\mathbf{R}}\cdot\mathbf{R}^c)^c \tag{3.21}$$

The tensor $\overset{\nabla}{\sigma}_D$ is the Dienes rate of σ. It is easily seen that the choice

$$\mathbf{A} = J^{\gamma}\mathbf{F}^c \tag{3.22}$$

leads to the Cotter–Rivlin rate $\overset{\nabla}{\sigma}_C$:

$$\overset{\nabla}{\sigma} = \overset{\nabla}{\sigma}_C + 2\gamma \mathrm{tr}(\mathbf{D})\sigma; \quad \overset{\nabla}{\sigma}_C = \dot{\sigma} - (\mathbf{\Omega}-\mathbf{D})\cdot\sigma - \sigma\cdot(\mathbf{\Omega}-\mathbf{D})^c \tag{3.23}$$

Finally, if **A** is chosen as

$$\mathbf{A} = J^{\gamma}\mathbf{F}^{-1} \tag{3.24}$$

the Truesdell or Green rate $\overset{\nabla}{\sigma}_G$ is found

$$\overset{\nabla}{\sigma} = \overset{\nabla}{\sigma}_G + 2\gamma \mathrm{tr}(\mathbf{D})\sigma; \quad \overset{\nabla}{\sigma}_G = \dot{\sigma} - (\mathbf{\Omega}+\mathbf{D})\cdot\sigma - \sigma\cdot(\mathbf{\Omega}+\mathbf{D})^c \tag{3.25}$$

From (3.14) it follows that the invariant stress tensor **S**, associated with the last choice of **A**, is the second Piola–Kirchhoff stress tensor if $\gamma = 0.5$.

The given examples show that the most widely used objective rates belong to the class of rates, specified by (3.10). In the last examples, resulting in the Dienes, the Cotter–Rivlin and the Truesdell rate, the tensor **A** at time t only depends on the deformation tensor **F** of the current configuration with respect to the reference configuration. This is not true for the first example since the solution **P**(t) of (3.19) will depend on $\mathbf{\Omega} = \mathbf{\Omega}(\tau)$ for $t_0 \leq \tau \leq t$.

4. SOME COMMENTS ON RATE–TYPE CONSTITUTIVE EQUATIONS.

Usually the derivation of the constitutive equation for elastic–plastic material behavior is based on the decomposition

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{4.1}$$

where $\mathbf{D}^e$ and $\mathbf{D}^p$ represent the elastic and the plastic part of the deformation rate tensor $\mathbf{D}$. The elastic part is defined by

$$\mathbf{D}^e = {}^4\mathbf{M}:\overset{\nabla}{\boldsymbol{\sigma}} \tag{4.2}$$

with an objective stress rate $\overset{\nabla}{\boldsymbol{\sigma}}$. The objective tensor ${}^4\mathbf{M}$ may depend on the current stress and deformation but not on the stress rate or on the deformation rate. Furthermore, ${}^4\mathbf{M}$ is left– and right–symmetrical, i.e.

$${}^4\mathbf{M}:\mathbf{Z} = {}^4\mathbf{M}:\mathbf{Z}^c; \quad \mathbf{Z}:{}^4\mathbf{M} = \mathbf{Z}^c:{}^4\mathbf{M} \quad \text{for every } \mathbf{Z} \tag{4.3}$$

It is assumed that there exists an objective, left– and right–symmetrical fourth order tensor ${}^4\mathbf{C}$, the elasticity tensor, such that

$${}^4\mathbf{C}:({}^4\mathbf{M}:\mathbf{Z}) = \mathbf{Z} \quad \text{for every } \mathbf{Z} \tag{4.4}$$

From (4.1) and (4.2) it then follows that

$$\overset{\nabla}{\boldsymbol{\sigma}} = {}^4\mathbf{C}:\mathbf{D} - {}^4\mathbf{C}:\mathbf{D}^p \tag{4.5}$$

It remains to relate $\mathbf{D}^p$ to $\mathbf{D}$ or $\boldsymbol{\sigma}$. For time–independent plasticity this relation is derived by the introduction of a yield criterion, a flow rule and a hardening model (Lehmann, [9]). This finally results in

$$\overset{\nabla}{\boldsymbol{\sigma}} = {}^4\mathbf{L}:\mathbf{D} \tag{4.6}$$

where the elastic-plastic tensor ${}^4\mathbf{L}$ is objective, left- and right-symmetrical and independent of both the stress rate and the deformation rate.

Straightforward integration of (4.6) is not possible since the stress rate in (4.6) is not the material rate of $\boldsymbol{\sigma}$. However, each of the objective rates in this paper is associated with a regular tensor $\mathbf{A}$ and an invariant stress tensor $\mathbf{S}$, such that

$$\mathbf{S} = \mathbf{A}\cdot\boldsymbol{\sigma}\cdot\mathbf{A}^c; \quad \dot{\mathbf{S}} = \mathbf{A}\cdot\overset{\nabla}{\boldsymbol{\sigma}}\cdot\mathbf{A}^c \tag{4.7}$$

Substitution in (4.6), followed by integration yields

$$\mathbf{S}(t) = \mathbf{S}(t_0) + \int_{t_0}^{t}[\mathbf{A}\cdot({}^4\mathbf{L}:\mathbf{D})\cdot\mathbf{A}^c]d\tau \tag{4.8}$$

Here $\mathbf{S}(t_0) = \boldsymbol{\sigma}(t_0)$ since $\mathbf{A} = \mathbf{I}$ for $t=t_0$. Hence, $\boldsymbol{\sigma}(t)$ is given by

$$\boldsymbol{\sigma}(t) = \mathbf{K}(t) + \mathbf{A}^{-1}(t)\cdot\boldsymbol{\sigma}(t_0)\cdot\mathbf{A}^{-c}(t) \tag{4.9}$$

where the symmetrical tensor $\mathbf{K}$ is defined by

$$\mathbf{K}(t) = \mathbf{A}^{-1}(t)\cdot\{\int_{t_0}^{t}[\mathbf{A}\cdot({}^4\mathbf{L}:\mathbf{D})\cdot\mathbf{A}^c]d\tau\}\cdot\mathbf{A}^{-c} \tag{4.10}$$

Similar expressions for $\boldsymbol{\sigma}(t)$ were used by Nagtegaal and Veldpaus [10] for the numerical integration of the constitutive equations for isotropic hardening elastic-plastic behavior. However, they only considered the Jaumann rate, whereas (4.9) and (4.10) are applicable to any objective rate that belongs to the class defined by (3.10).

If the outlined procedure to determine $\boldsymbol{\sigma}$ is used in a general solution process for elastic-plastic problems it must be applicable for purely elastic problems too. It will be shown in the next section that this requirement restricts the allowable combinations of objective stress rates and fourth order elasticity tensors.

5. ELASTIC BEHAVIOR.

Before analyzing rate–type constitutive equations for the description of elastic behavior some definitions are given. It is assumed that the material is stress–free in the reference configuration.

A deformation from the reference configuration to the current configuration is called elastic if σ is determined completely by $\mathbf{F}(t)$. In this case the stress–strain relation must be of the form (Hunter, [11])

$$\sigma = \mathbf{F}\cdot\mathbf{G}(\mathbf{C})\cdot\mathbf{F}^c; \quad \mathbf{C} = \mathbf{F}^c\cdot\mathbf{F}; \quad \mathbf{G}(\mathbf{C}) = \mathbf{G}^c(\mathbf{C}) \tag{5.1}$$

An elastic material is isotropic if $\mathbf{G}=\mathbf{G}(\mathbf{C})$ is isotropic. Then the stress–strain relation becomes (Hunter, [11])

$$\sigma = \mathbf{F}\cdot\mathbf{G}(\mathbf{C})\cdot\mathbf{F}^c = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2 \tag{5.2}$$

where α_0, α_1 and α_3 are scalar functions of the invariants of $\mathbf{B}$. With the spectral representation (2.4) of $\mathbf{B}$ it follows that, for an isotropic material, σ is given by (i=1,2,3)

$$\sigma = \sum_{i=1}^{3}(\sigma_i\vec{n}_i\vec{n}_i); \quad \sigma_i = \alpha_1 + \alpha_2\lambda_i^2 + \alpha_3\lambda_i^4 \tag{5.3}$$

A material is Green–elastic if the stresses can be derived from an elastic potential π. For an isotropic Green–elastic material the principal Cauchy stresses σ_1, σ_2 and σ_3 in (5.3) follow from

$$\sigma_i = \frac{1}{J}\cdot\frac{\partial\pi}{\partial\epsilon_i}; \quad \pi = \pi(\epsilon_1,\epsilon_2,\epsilon_3) \tag{5.4}$$

where ϵ_1, ϵ_2 and ϵ_3 are the principal logarithmic strains.

Suppose that the procedure of the preceding section is used for the analysis of an elastic deformation from the stress–free reference configuration to the current configuration. Then the rate–type equation (4.5) reduces to

$$\overset{\nabla}{\sigma} = {}^4\mathbf{C}:\mathbf{D} \tag{5.5}$$

If both $\overset{\nabla}{\sigma}$ and ${}^4\mathbf{C}$ are specified $\sigma(t)$ follows from (4.9) and (4.10) if ${}^4\mathbf{L}$ is replaced by ${}^4\mathbf{C}$. Because $\sigma = \mathbf{O}$ for $t=t_0$ this yields

$$\sigma = \mathbf{K} = \mathbf{A}^{-1}\cdot\{ \int_{t_0}^{t}[\mathbf{A}\cdot({}^4\mathbf{C}:\mathbf{D})\cdot\mathbf{A}^c]d\tau \}\cdot\mathbf{A}^{-c} \tag{5.6}$$

This relation represents some kind of elastic behavior if $\mathbf{K}(t)$ is independent of the deformation path and is determined completely $\mathbf{F}(t)$. A few examples are considered here. First of all it is assumed that ${}^4\mathbf{C}$ is an isotropic tensor with the earlier required symmetry properties:

$${}^4\mathbf{C} = \tfrac{1}{2}.\mu_1({}^4\mathbf{I} + {}^4\mathbf{I}^c) + \mu_2.\mathbf{II} \tag{5.7}$$

The scalar quantities μ_1 and μ_2 may be functions of the invariants of $\mathbf{B}$. The constitutive equation (5.5) then becomes

$$\overset{\nabla}{\sigma} = \mu_1\mathbf{D} + \mu_2\mathrm{tr}(\mathbf{D})\mathbf{I} \tag{5.8}$$

and it is readily seen that

$$\mathbf{K} = \mathbf{A}^{-1}\cdot[\int_{t_0}^{t}(\mu_1\mathbf{A}\cdot\mathbf{D}\cdot\mathbf{A}^c + \mu_2\mathrm{tr}(\mathbf{D})\mathbf{A}\cdot\mathbf{A}^c)d\tau]\cdot\mathbf{A}^{-c} \tag{5.9}$$

If $\overset{\nabla}{\sigma}$ is the Dienes rate $\mathbf{A}$ is given by (3.20). With $\mathbf{D}=\frac{1}{2}\mathbf{R}^c\cdot(\dot{\mathbf{U}}\cdot\mathbf{U}^{-1}+ \mathbf{U}^{-1}\cdot\dot{\mathbf{U}})\cdot\mathbf{R}$ it follows from (5.9) that

$$\mathbf{K}=J^{-2\gamma}\mathbf{R}\cdot[\int_{t_0}^{t}J^{2\gamma}\{\tfrac{1}{2}\mu_2(\dot{\mathbf{U}}\cdot\mathbf{U}^{-1}+\mathbf{U}^{-1}\cdot\dot{\mathbf{U}})d\tau+\mu_2\mathrm{tr}(\mathbf{D})\mathbf{I}\}d\tau]\cdot\mathbf{R}^c \tag{5.10}$$

$\mathbf{K}$ is deformation path independent if the integral is path independent. A necessary condition is that there exists a function $f=f(J)$ such that

$$\mu_2 J^{2\gamma} = J\,\frac{df}{dJ} \tag{5.11}$$

This condition is not sufficient: by choosing different deformation paths it can be shown that the integral is path dependent if $\mu_1 \neq 0$. This implies that the Dienes rate, combined with the constitutive equation (5.8), in general cannot result in a correct description of elastic behavior. The same conclusion can be drawn if the stress rate in (5.8) is replaced by the Jaumann rate. For the Truesdell rate it turns out that **K** is path independent if μ_1 and μ_2 satisfy

$$(\mu_1 J^{2\gamma})^{\bullet} + 2\mathrm{tr}(\mathbf{D})\mu_2 J^{2\gamma} = 0; \quad \dot{J} = J\mathrm{tr}(\mathbf{D}) \tag{5.12}$$

Since μ_1 and μ_2 only depend on the invariants of **B** it can be shown that (5.12) can be satisfied for every deformation path only if there exists a function f=f(J) with f(1)=0, such that

$$\mu_2 J^{2\gamma} = J.\frac{df}{dJ}; \quad \mu_1 J^{2\gamma} = 2(G_0 - f) \tag{5.13}$$

where G_0 is a constant. In that case $\boldsymbol{\sigma}$ and **K** are given by

$$\boldsymbol{\sigma} = \mathbf{K} = J^{-2\gamma}[f\mathbf{I} + G_0(\mathbf{B} - \mathbf{I})] \tag{5.14}$$

These results show that the combination of the Truesdell rate with the elasticity tensor ${}^4\mathbf{C}$ as specified by (5.7) results in a correct description of some kind of elastic behavior if (5.13) is satisfied. The stress–strain relation associated with this combination is given by (5.14).

If the Cotter–Rivlin rate is used in (5.8) it turns out that **K** is path independent if there exists a constant G_0 and a function f=f(J) such that

$$f(1) = 0; \quad \mu_2 J^{2\gamma} = J\frac{df}{dJ}; \quad \mu_1 J^{2\gamma} = 2(G_0 + f) \tag{5.15}$$

The associated elastic stress–strain relation is then given by

$$\boldsymbol{\sigma} = \mathbf{K} = J^{-2\gamma}[f\mathbf{I} + G_0(\mathbf{I} - \mathbf{B}^{-1})] \tag{5.16}$$

The same conclusions about these rates can be derived in a different way. In the approach outlined above it is assumed that both the elasticity tensor and the stress rate are specified and it is questioned whether or not this combination yields a correct description of elastic behavior. In the alternative approach it is assumed that ${}^4\mathbf{C}$ and the elastic stress–strain relation (5.1) are specified and it is investigated which objective stress rates of the type (3.10) will result in that stress–strain relation. This means that a tensor **H** must be determined such that

$$\dot{\sigma} - (\Omega + \mathbf{H})\cdot\sigma - \sigma\cdot(\Omega + \mathbf{H})^c = {}^4\mathbf{C}:\mathbf{D} \qquad (5.17)$$

In any vector basis this symmetrical tensor equation yields a set of six equations for the nine components of the matrix representation $\underline{H}$ of H in that basis. Hence, these components are not determined completely by (5.17). This is in agreement with section 3, where it was shown that an objective stress rate of the type (3.10) does not change if **H** is replaced by $\mathbf{H} + \sigma\cdot\mathbf{P}$ with skew–symmetrical tensor **P**. In general it is possible to determine a class of tensors H such that (5.17) is satisfied for the given stress–strain relation and the given elasticity tensor ${}^4\mathbf{C}$. In practice this is not of importance, but (5.17) can be used as a starting point for a further investigation of the commonly used objective rates like the Jaumann, the Dienes, the Truesdell and the Cotter–Rivlin rate. For simplicity only isotropic elastic behavior is considered and, as usual in literature, it is assumed that ${}^4\mathbf{C}$ is given by (5.7) with as yet unspecified scalars μ_1 and μ_2. By taking the material rate of (5.3) and use of (2.8) for the rate of the eigenvectors and of (2.11) for **D** (5.17) can be transformed into

$$[\mathbf{H} - \tfrac{1}{2}(\mathbf{W}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{W}^c)]\cdot\sigma + \sigma\cdot[\mathbf{H} - \tfrac{1}{2}(\mathbf{W}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{W}^c)]^c =$$
$$= \sum_{i=1}^{3}[\dot{\sigma}_i - \mu_1\epsilon_i - \mu_2\mathrm{tr}(\mathbf{D})]\vec{n}_i\vec{n}_i - \tfrac{1}{2}\mu_1(\mathbf{W}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{W}^c) \qquad (5.18)$$

Let $H_{ij} = \vec{n}_i\cdot\mathbf{H}\cdot\vec{n}_j$ (i.j=1,2,3) be the components of the representation $\underline{H}$ of **H** in the vector basis, spanned by the eigenvectors of **B**. From (5.18) it then follows for the diagonal components H_{11} , H_{22} and H_{33}:

$$2H_{ii}\sigma_i = \dot{\sigma}_i - \mu_1\dot{\epsilon}_i - \mu_2\mathrm{tr}(\mathbf{D}) \qquad (5.19)$$

For an isotropic material σ_1, σ_2 and σ_3 are functions of ϵ_1, ϵ_2 and ϵ_3 only. Hence, with (2.13) for the tr(**D**) it follows from (5.19) that

$$2H_{ii}\sigma_i = \sum_{j=1}^{3}\left(\frac{\partial\sigma_i}{\partial\epsilon_j} - \mu_1\delta_{ij} - \mu_2\right)\dot{\epsilon}_j \quad \text{for } i=1,2,3 \tag{5.20}$$

This must hold for all strain rates, so H_{ii} must be a linear function of these rates. This is true for each of the commonly used objective stress rates. For these rates **H** is given by

$$\mathbf{H} = \beta\mathbf{D} - \gamma\mathrm{tr}(\mathbf{D})\mathbf{I} + \mathbf{X}; \quad \mathbf{X} = -\mathbf{X}^c \tag{5.21}$$

where β=0 and **X**=**O** for the Jaumann rate, β=+1 and **X**=**O** for the Truesdell rate and β=-1 and **X**=**O** for the Cotter–Rivlin rate. For the Dienes rate β=0 while **X** follows from (i,j=1,2,3)

$$X_{ij} = \tfrac{1}{2}(\lambda_i - \lambda_j)^2 \cdot W_{ij}; \quad W_{ij} = \vec{n}_i \cdot W \cdot \vec{n}_j \tag{5.22}$$

In the sequel only tensors **H** of the type (5.21) are considered. The associated objective stress rates, given by

$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - (\boldsymbol{\Omega}+\mathbf{H})\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\boldsymbol{\Omega}+\mathbf{H})^c - \beta(\mathbf{B}\cdot\mathbf{D}+\mathbf{D}\cdot\mathbf{B}) + 2\gamma\mathrm{tr}(\mathbf{D})\boldsymbol{\sigma} \tag{5.23}$$

are called (β,γ)–type objective rates. It is not trivial that the combination of a rate of this type and the rate–type constitutive equation (5.8) or the equivalent form (5.18) can describe any kind of isotropic elastic behavior at all: for every deformation path the components of $\underline{H}$ have to satisfy the six differential equations that can be derived from (5.18) and this is impossible unless some special requirements are fulfilled. An evaluation of these requirements is given in the remainder of this section. For simplicity only constant factors β and γ are considered.

From (5.21) it is seen that $H_{ii}=\beta\dot{\epsilon}_i-\gamma(\dot{\epsilon}_1+\dot{\epsilon}_2+\dot{\epsilon}_3)$. These components of $\underline{H}$ must satisfy (5.20) for every strain rate. Therefore

$$\frac{\partial \sigma_i}{\partial \epsilon_j} = (\mu_1 + 2\beta\sigma_i)\delta_{ij} + \mu_2 - 2\gamma\sigma_i \tag{5.24}$$

must hold for i,j=1,2,3. With $J=\det(\mathbf{F})=\lambda_1\lambda_2\lambda_3$ and $\epsilon_i=\ln(\lambda_i)$ for i=1,2,3 these equations can be written as (i,j=1,2,3)

$$\frac{\partial}{\partial \epsilon_j}(J^{2\gamma}\lambda_i^{-2\beta}\sigma_i) = J^{2\gamma}\lambda_i^{-2\beta}(\mu_2 + \mu_1\delta_{ij}) \tag{5.25}$$

This set has a solution if and only if a function f=f(J) exists such that

$$f(1)=0; \quad \mu_1 J^{2\gamma} = 2(G_0 - f); \quad \mu_2 J^{2\gamma} = J\,\frac{df}{dJ} \tag{5.26}$$

The differential equations (5.25) then become

$$\frac{\partial}{\partial \epsilon_j}(J^{2\gamma}\lambda_i^{-2\beta}\sigma_i) = \frac{\partial}{\partial \epsilon_j}(\lambda_i^{-2\beta}\cdot f) + 2G_0\cdot\lambda_i^{-2\beta}\cdot\delta_{ij} \tag{5.27}$$

and the solution for the stress tensor is given by

$$\boldsymbol{\sigma} = J^{-2\gamma}[f\mathbf{I} + \frac{1}{\beta}\cdot G_0(\mathbf{B}^{\beta} - \mathbf{I})] \tag{5.28}$$

If $\beta \to 0$, as for the Jaumann and the Dienes rate, the solution becomes

$$\boldsymbol{\sigma} = J^{-2\gamma}[f\mathbf{I} + G_0\ln(\mathbf{B})] \tag{5.29}$$

For $\beta=+1$ and $\beta=-1$ the stress–strain relations (5.14) and (5.16) for the Truesdell and the Cotter–Rivlin rate are found again. The trivial case $G_0=0$ is not considered anymore.

Based on these results, it is concluded that the constitutive equation (5.8) with a (β,γ)–type objective stress rate may result in a correct description of isotropic elastic behavior only if μ_1 and μ_2 satisfy (5.26). This condition, however, might not be sufficient because up to now only three of the six independent equations, that can be derived from (5.18), are taken into account: only the main diagonal components of $\underline{H}$ have been considered. With (2.11) for $\mathbf{D}$ and (5.28) or (5.29) for $\boldsymbol{\sigma}$

the other equations result in three independent requirements for the non-trivial components of the matrix representation of the skew-symmetrical tensor **X** in the chosen vector basis:

$$X_{ij} = \tfrac{1}{2}\,[\lambda_i^2 + \lambda_j^2 - \beta(\lambda_j^2 - \lambda_i^2)\cdot\frac{\lambda_j^{2\beta} + \lambda_i^{2\beta}}{\lambda_j^{2\beta} - \lambda_i^{2\beta}}]\,W_{ij} \tag{5.30}$$

for i,j=1,2,3 (j>i) and for every W_{ij}. However, $X_{ij}=0$ for the Jaumann rate while X_{ij} is given by (5.22) for the Dienes rate. Hence, these rates cannot result in a correct description of isotropic elastic behavior. Furthermore, if $\beta=+1$ or $\beta=-1$ it is seen from (5.30) that $X_{ij}=0$ and, according to (5.21), **H** is equal to **D**−γtr(**D**)**I**. This confirms the earlier derived conclusion that the combination of the Truesdell rate ($\beta=+1$) and the Cotter–Rivlin rate ($\beta=-1$) with the constitutive equation (5.8) and μ_1 and μ_2 according to (5.26) yields a correct description of isotropic elastic behavior. The corresponding stress–strain relation is given by (5.28). If $\beta.\beta\neq1$ and a stress–strain relation of the type (5.28) must be represented correctly by (5.8) then a (β,γ)–type objective rate (5.23) must be used with a tensor **X** as specified by (5.30).

Up to now no statements on the value of γ have been made. For Cauchy elasticity γ may be arbitrary. For Green elasticity, however, the principal Cauchy stresses σ_1, σ_2 and σ_3 must be derivable from an elastic potential π as specified by (5.4). For a stress–strain relation of the type (5.28) this is true only if $\gamma=0.5$. This conclusion can be derived by differentiation of (5.4) with respect to ϵ_j and substitution of the result in (5.24).

6. TENSION–TORSION OF CYLINDRICAL BARS.

To evaluate a given rate–type constitutive equation physical and numerical experiments have to be done. Quite popular nowadays is the so–called simple shear test ([4], [5], [6], [8] etc.). This is a theoretical test and no experimental data are available for really large deformations. An alternative is the torsion test of cylindrical bars or the combined tension–torsion test, which is much easier to realize than the simple shear test. As will be shown, phenomena like oscillations of the stresses for the Jaumann rate also occur in the tension–torsion test.

Let (r_0,φ_0,z_0) and (r,φ,z) be the cylindrical co–ordinates of a material point of the bar in the reference configuration and the current configuration, respectively. The position vectors of this point in these configurations are given by

$$\vec{x}_0 = r_0\vec{e}_r(\varphi_0) + z_0\vec{e}_z; \quad \vec{x} = r\vec{e}_r(\varphi) + z\vec{e}_z \tag{6.1}$$

where $\vec{e}_z$ is the unit axial vector and $\vec{e}_r(\varphi)$ is the unit radial vector for a point with circumferential co–ordinate φ. It is assumed that the strain field in the bar is axi–symmetric and independent of the axial co–ordinate z_0. For the current co–ordinates (r,φ,z) this results in

$$r = r(r_0,t); \quad \varphi = \varphi_0 + \alpha(t)z_0; \quad z = [1 + \epsilon(t)]z_0 \tag{6.2}$$

The deformation tensor $\mathbf{F}(t)$ with respect to the reference configuration and the determinant of $\mathbf{F}$ are given by:

$$\mathbf{F} = \frac{\partial r}{\partial r_0}\vec{e}_r\vec{e}_{r_0} + \frac{r}{r_0}\vec{e}_\varphi\vec{e}_{\varphi_0} + r\alpha\vec{e}_\varphi\vec{e}_z + [1+\epsilon]\vec{e}_z\vec{e}_z \tag{6.3}$$

$$J = \det(\mathbf{F}) = (1+\epsilon)\frac{r}{r_0}\cdot\frac{\partial r}{\partial r_0} \tag{6.4}$$

Here, $\vec{e}_\varphi=\vec{e}_\varphi(\varphi)$ is the unit vector in circumferential direction and $\vec{e}_r(\varphi_0)$ and $\vec{e}_\varphi(\varphi_0)$ are denoted, for brevity, by $\vec{e}_{r_0}$ and $\vec{e}_{\varphi_0}$. From (6.3) the eigenvectors and the corresponding eigenvalues of the left Cauchy strain tensor $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^c$ can be determined. This yields

$$\lambda_1 = \frac{\partial r}{\partial r_0} \tag{6.5a}$$

$$\lambda_2 = \tfrac{1}{2}\left[\sqrt{r^2\alpha^2 + (1+\epsilon+\frac{r}{r_0})^2} + \sqrt{r^2\alpha^2 + (1+\epsilon-\frac{r}{r_0})^2}\right] \tag{6.5b}$$

$$\lambda_3 = \frac{1+\epsilon}{\lambda_2}\frac{r}{r_0} \tag{6.5c}$$

$$\vec{n}_1 = \vec{e}_r \tag{6.6a}$$

$$\vec{n}_2 = \cos(\psi)\vec{e}_\varphi + \sin(\psi)\vec{e}_z \tag{6.6b}$$

$$\vec{n}_3 = -\sin(\psi)\vec{e}_\varphi + \cos(\psi)\vec{e}_z \tag{6.6c}$$

where ψ, the angle between $\vec{e}_\varphi$ and $\vec{n}_2$, follows from

$$\sin(2\psi)=2\frac{(1+\epsilon)r\,\alpha}{\lambda_2^2 - \lambda_3^2}; \quad \cos(2\psi)=\frac{r^2(1+r_0^2\alpha^2)-r_0^2(1+\epsilon)^2}{r_0^2(\lambda_2^2 - \lambda_3^2)} \tag{6.7}$$

For an isotropic elastic material σ_1, σ_2 and σ_3 are completely determined by λ_1, λ_2 and λ_3 while the eigenvectors of $\boldsymbol{\sigma}$ coincide with the eigenvectors of **B**. With (6.6) and the spectral representation (5.3) of $\boldsymbol{\sigma}$ the components σ_{ij} (i,j=r,φ,z) of the matrix representation of $\boldsymbol{\sigma}$ can be determined:

$$\sigma_{rr} = \sigma_1 \tag{6.8a}$$

$$\sigma_{\varphi\varphi} = \tfrac{1}{2}[(\sigma_2+\sigma_3) + (\sigma_2-\sigma_3)\cos(2\psi)] \tag{6.8b}$$

$$\sigma_{zz} = \tfrac{1}{2}[(\sigma_2+\sigma_3) - (\sigma_2-\sigma_3)\cos(2\psi)] \tag{6.8c}$$

$$\sigma_{\varphi z} = \tfrac{1}{2}(\sigma_2-\sigma_3)\sin(2\psi) \tag{6.8d}$$

$$\sigma_{rz} = \sigma_{r\varphi} = 0 \tag{6.8e}$$

These stresses do not depend on z_0 or φ_0, so the relevant equilibrium equation and boundary condition are given by

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0; \quad \sigma_{rr} = 0 \quad \text{for } r_0 = R_0 \tag{6.9}$$

where R_0 is the radius of the bar in the reference configuration. The axial force N and the torque T in the end planes $z_0 = -\ell_0$ and $z_0 = +\ell_0$ of the bar follow from

$$N = 2\pi \int_0^R \sigma_{zz} r dr; \quad T = 2\pi \int_0^R \sigma_{\varphi z} r^2 dr \tag{6.10}$$

where R is the current radius of the bar.

The given set of equations must be completed by the constitutive equations. Only rate–type equations of the type (5.8) with a (β,γ)–type objective stress rate and scalars μ_1 and μ_2 according to (5.26) are considered and it is assumed that the skew–symmetrical tensor **X** in (5.23) satisfies the conditions (5.31). This excludes the Jaumann and the Dienes rate. The stress–strain relation is then given by (5.28) and σ_i (i=1,2,3) is determined by

$$\sigma_i = J^{-2\gamma}[f + \tfrac{1}{\beta}G_0(\lambda_i^{2\beta} - 1)] \tag{6.11}$$

Here f=f(J) must be a known function of J with f(1)=0. With (6.5), (6.7), (6.8) and (6.11) it is possible to derive from (6.9) and (6.10) a set of three equations for $r=r(r_0,t)$, $\epsilon=\epsilon(t)$, $\alpha=\alpha(t)$, N=N(t) and T=T(t). If either ϵ or N and α or T are prescribed as functions of time the remaining unknowns can be solved. Since the equations are highly non–linear they are simplified by assuming incompressibility. With J=1 it is seen from (6.4) that the current radius r(t) is related to the radius r_0 in the reference configuration by

$$r = \frac{r_0}{\sqrt{1 + \epsilon}} \tag{6.12}$$

The relations (6.5),...(6.10) remain valid but the stress–strain relations (6.11) have to be adjusted. It can be shown (Hunter [11]) that they must be replaced by

$$\sigma_i = -p + \tfrac{1}{\beta}G_0(\lambda_i^{2\beta} - 1) \quad \text{for } i=1,2,3 \tag{6.13}$$

where the unknown $p=p(r_0,t)$ has to be determined from the equilibrium equation. If $\beta=0$ it can be shown that p equals the hydrostatic pressure.

Two special cases are considered in more detail. The first case concerns pure tension of the bar. Then $\alpha(t)=0$, all stresses except σ_{zz} are equal to zero and σ_{zz} is given by

$$\sigma_{zz} = \frac{1}{\beta} G_0 [(1+\epsilon)^{2\beta} - (1+\epsilon)^{-\beta}] \tag{6.14}$$

In Figure 1 this stress is plotted as a function of β and ϵ for $-2\leq\beta\leq+2$ and $-\frac{1}{2}\leq\epsilon\leq+1$. It is noted again that $\beta=1$ and $\beta=-1$ correspond to the Truesdell and the Cotter–Rivlin rate, respectively.

The second case concerns torsion without axial loading, i.e. $N(t)=0$. Some of the results are given in Figure 2. In Figure 2.a the relative axial elongation is represented as a function of β and α for $-2\leq\beta\leq1$ and $1\leq\alpha\leq10$. Figure 2.b gives a plot of the dimensionless torque $T/(2\pi R_0^3 G_0)$ as a function of the same arguments. If experimental results for these two cases are available a choice for β can be made from these figures.

In section 5 it is concluded that neither the Jaumann nor the Dienes rate can result in a correct description of elastic behavior. To illustrate this, also these objective rates are used to analyze the tension–torsion test. The derivations for the case of incompressibility are given in Appendix A. To show that the results depend on the deformation path two calculations are made. In the first calculation axial displacements of the end planes of the bar are suppressed ($\epsilon=0$) while the bar is twisted up to $\alpha=\alpha_1$. In the second calculation the bar is twisted up to $\alpha=\alpha_1$ while the axial force N is kept zero. For both the Jaumann and the Dienes rate this results in an elongation of the bar. After $\alpha=\alpha_1$ is reached the bar is pushed back to its original length. The final deformation is characterized by $\epsilon=0$ and $\alpha=\alpha_1$, as is the case in the first calculation. However, the deformation paths are different. In Figure 3.a and Figure 3.b the dimensionless torque is plotted as a function of α_1 for the Jaumann and the Dienes rate, respectively. The solid lines gives the results of the first calculation, while the results of the second calculation are represented by the dotted lines. The solid and dotted lines coincide only for small values of α_1 and this again shows that these rates cannot correctly describe elastic behavior for large deformations.

It is noted that pure torsion of the bar ($\epsilon=0$, the first calculation) is very similar to the simple shear test. The results in Appendix A for pure torsion show that the shear stress is given by $\sigma_{\varphi z}=G_0\sin(\alpha r_0)$ if the Jaumann rate is used. Hence, $\sigma_{\varphi z}$ and

T oscillate (Figure 3.a). For the Dienes rate also an analytical expression for $\sigma_{\varphi z}$ can be derived. This expression is rather complicated but very similar to the expression for the simple shear test (Dienes, [4]).

7. CONCLUSIONS.

In this paper attention was focused on rate–type constitutive equations $\overset{\nabla}{\sigma}={}^4\mathbf{C}:\mathbf{D}$ with emphasis on isotropic elastic behavior. It was shown that the commonly used stress rates can be written as $\overset{\nabla}{\sigma}=\dot{\sigma}-(\Omega+\mathbf{H})\cdot\sigma-\sigma\cdot(\Omega+\mathbf{H})^c$, where $(\mathbf{H}\cdot\sigma+\sigma\cdot\mathbf{H}^c)$ is an objective tensor. For a given objective rate **H** is not determined uniquely: if **H** results in the given rate then $\mathbf{H}+\sigma\cdot\mathbf{P}$ will result in the same rate if **P** is skew-symmetrical. For every objective tensor **H** a tensor **A** can be found which is related to an invariant stress tensor S, such that $\mathbf{S}=\mathbf{A}\cdot\sigma\cdot\mathbf{A}^c$ and $\dot{\mathbf{S}}=\mathbf{A}\cdot\overset{\nabla}{\sigma}\cdot\mathbf{A}^c$. As outlined in section 4, these relations can be used as a starting point for the numerical integration of rate–type constitutive equations.

For special purposes, for example in elastic–plastic problems, it can be advantageous to characterize elastic behavior by the rate–type equation $\overset{\nabla}{\sigma}={}^4\mathbf{C}:\mathbf{D}$ with elasticity tensor ${}^4\mathbf{C}$. As soon as ${}^4\mathbf{C}$ and the objective rate are specified σ can be determined if **D** is given as a function of time. The resulting stress–strain relation must be independent of the deformation path. As shown in section 5 this is not the case for the Jaumann and the Dienes rate.

To describe isotropic elastic behavior the rate–type constitutive equation $\overset{\nabla}{\sigma}=\mu_1\mathbf{D}+\mu_2\mathrm{tr}(\mathbf{D})\mathbf{I}$ is used, where μ_1 and μ_2 may be scalar functions of the invariants of the left Cauchy strain tensor $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^c$. It was seen in section 5 that the diagonal components of the matrix representation $\underline{\mathrm{H}}$ of **H** in the eigenvector space of **B** must be linear functions of the principal logarithmic strain rates. This resulted in the introduction of (β,γ)–type objective stress rates with tensors **H** of the type $\mathbf{H}=\beta\mathbf{D}-\gamma\mathrm{tr}(\mathbf{D})\mathbf{I}+\mathbf{X}$ with skew–symmetrical **X**. It was shown that each of these rates results in a correct description of some isotropic elastic behavior if **X** satisfies some special requirements, which turned out to be fulfilled for the Truesdell and the Cotter–Rivlin rate. Furthermore, it was concluded that $\gamma=0.5$ must hold for an isotropic Green–elastic material.

If a (β,γ)–type objective rate is used for the description of the behavior of a given isotropic elastic material the value of β must be determined from data of large deformation experiments. Tension–torsion tests on cylindrical bars seem to be suitable for this purpose.

8. FINAL REMARKS.

The applicability of a rate–type constitutive equation for the description of elastic behavior is investigated by requiring that the resulting relation between stresses and strains does not depend on the deformation path. As stated in section 4, the constitutive equation for elastic–plastic behavior is also of the rate–type. If kinematic hardening has to be taken into account a second rate–type equation appears, namely

$$\overset{\nabla}{\boldsymbol{\alpha}} = \mu_3 \mathbf{D}^{\mathrm{p}}, \tag{8.1}$$

where $\boldsymbol{\alpha}$ is the shift or back stress tensor and $\mathbf{D}^{\mathrm{p}}$ is the plastic part of $\mathbf{D}$. In order to decide which objective rate has to be used in (8.1) a similar physical statement as in the case of elasticity would be very helpful. As long as this statement is lacking one has to consider different rates to find out which rate results in the best fit to experimental data. However, the objective rate to be used in (8.1) will probably differ from the one in (1.1).

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APPENDIX A.

The Jaumann rate–type constitutive equation for an incompressible, isotropic elastic material is given by

$$\dot{\boldsymbol{\sigma}}^{d} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}^{d} - \boldsymbol{\sigma}^{d}\cdot\boldsymbol{\Omega}^{c} = 2G_{0}\mathbf{D} \tag{A.1}$$

where $\boldsymbol{\sigma}^{d}$ is the deviatoric part of the Cauchy stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}^{d}; \quad p = -\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) \tag{A.2}$$

The hydrostatic pressure p has to be determined from the equilibrium equations and the condition of incompressibility.

In the tension–torsion test the radius r in the current configuration and the radius r_0 in the reference configuration are related by (6.12) if the material is incompressible. Hence, the matrix representations $\underline{D}$ of $\mathbf{D}$ and $\underline{\Omega}$ of $\boldsymbol{\Omega}$ in the vector basis $\{\vec{e}_r,\vec{e}_\varphi,\vec{e}_z\}$ are given by

$$\underline{D} = \begin{bmatrix} -\frac{1}{2}\dot{\epsilon}_a & 0 & 0 \\ 0 & -\frac{1}{2}\dot{\epsilon}_a & -\dot{\psi}_j \\ 0 & -\dot{\psi}_j & \dot{\epsilon}_a \end{bmatrix}; \quad \underline{\Omega} = \begin{bmatrix} 0 & -\dot{\alpha}z_0 & 0 \\ \dot{\alpha}z_0 & 0 & -\dot{\psi}_j \\ 0 & \dot{\psi}_j & 0 \end{bmatrix} \tag{A.3}$$

where ϵ_a and ψ_j are defined by

$$\epsilon_a = \ln(1+\epsilon); \quad \psi_j = -\tfrac{1}{2}r_0 \int_{t_0}^{t} \frac{\dot{\alpha}}{(1+\epsilon)^{\frac{3}{2}}}\, d\tau \tag{A.4}$$

Substitution in (A.1) yields that the matrix representation $\underline{\sigma}^{d}$ of $\boldsymbol{\sigma}^{d}$ in this vector basis has to satisfy

$$\begin{bmatrix} \dot{\sigma}^d_{rr} & 0 & 0 \\ 0 & \dot{\sigma}^d_{\varphi\varphi} & \dot{\sigma}^d_{\varphi z} \\ 0 & \dot{\sigma}^d_{\varphi z} & \dot{\sigma}^d_{zz} \end{bmatrix} + \dot{\psi}_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\sigma^d_{\varphi z} & \sigma^d_{zz}-\sigma^d_{\varphi\varphi} \\ 0 & \sigma^d_{zz}-\sigma^d_{\varphi\varphi} & -2\sigma^d_{\varphi z} \end{bmatrix} = 2G_0\underline{D} \tag{A.5}$$

so for the initial condition is $\underline{\sigma}^d = \underline{0}$ for $t=t_0$ the solution is

$$\sigma^d_{rr} = -G_0\epsilon_a \tag{A.6^a}$$

$$\sigma^d_{\varphi\varphi} = G_0\epsilon_a - \sigma^d_z \tag{A.6^b}$$

$$\dot{\sigma}^d_{zz} = +2\dot{\psi}_j\sigma^d_{\varphi z} + 2G_0\dot{\epsilon}_a \tag{A.6^c}$$

$$\dot{\sigma}^d_{\varphi z} = -2\dot{\psi}_j\sigma^d_{zz} + G_0\dot{\psi}_j(\epsilon_a-2) \tag{A.6^d}$$

The Dienes rate–type constitutive equation for an incompressible, isotropic elastic material is given by

$$\dot{\sigma}^d - (\dot{\mathbf{R}}\cdot\mathbf{R}^c)\cdot\sigma^d - \sigma^d\cdot(\dot{\mathbf{R}}\cdot\mathbf{R}^c)^c = 2G_0\mathbf{D}, \tag{A.7}$$

where the rotation tensor **R** follows from $\mathbf{F}=\mathbf{V}\cdot\mathbf{R}$ and $\mathbf{V}^2=\mathbf{B}$. With (2.4) and the results of section 6 both **V** and **R** can be determined. For the matrix representation $\underline{R}$ of **R** this results in

$$\underline{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi_d) & -\sin(\psi_d) \\ 0 & \sin(\psi_d) & \cos(\psi_d) \end{bmatrix} \cdot \begin{bmatrix} \cos(\psi_a) & -\sin(\psi_a) & 0 \\ \sin(\psi_a) & \cos(\psi_a) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.8}$$

where ψ_d and ψ_a are given by

$$\tan(\psi_d) = -\frac{\alpha r_0}{1+(1+\epsilon)^{\frac{3}{2}}}; \quad \psi_a = \alpha z_0 \tag{A.9}$$

Further elaboration of (A.7) and (A.8) yields equations for the non–trivial deviatoric stresses. These are given by (A.6) if ψ_j is replaced by ψ_d. However, there is a significant difference. From (A.9) it is seen that ψ_d is bounded ($|\psi_d|\leq\pi/2$)

while ψ_j is not. As a consequence, oscillations can appear in the stresses if the Jaumann rate is used but not if the Dienes rate is used.

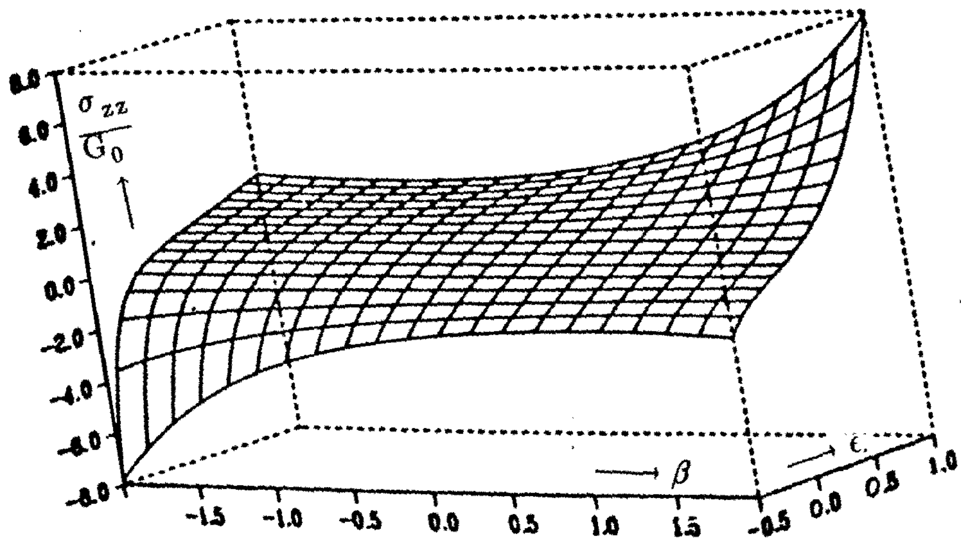

Figure 1.

The dimensionless axial stress σ_{zz}/G_0 as a function of β and ϵ in pure tension.

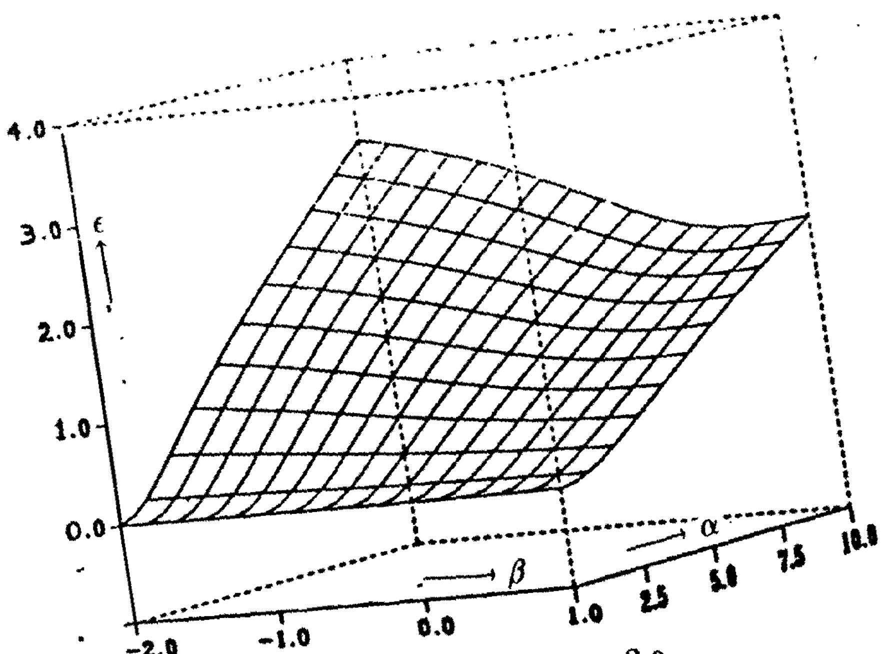

Figure 2.a

The relative elongation ϵ as a function of β and α in pure torsion.

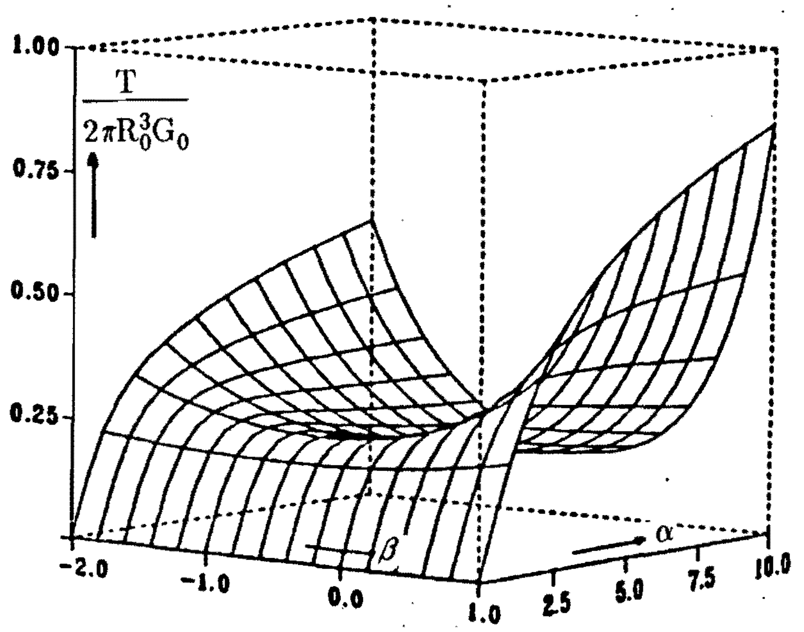

Figure 2.b

The dimensionless torque $T/(2\pi R_0^3 G_0)$ as a function of β and α in pure torsion.

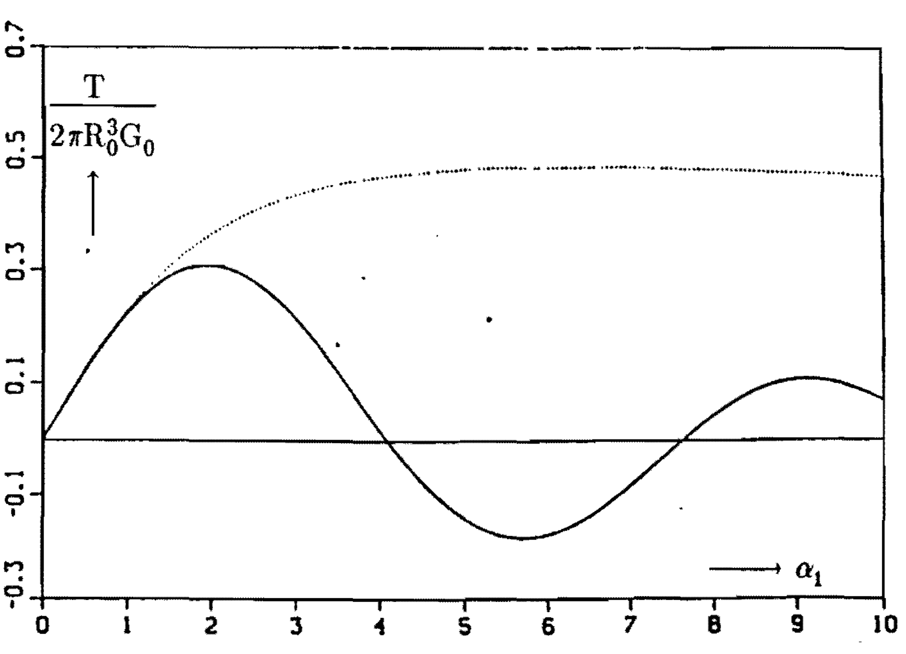

Figure 3.a

The dimensionless torque $T/(2\pi R_0^3 G_0)$ as a function of α_1 for the Jaumann rate–type constitutive equation.

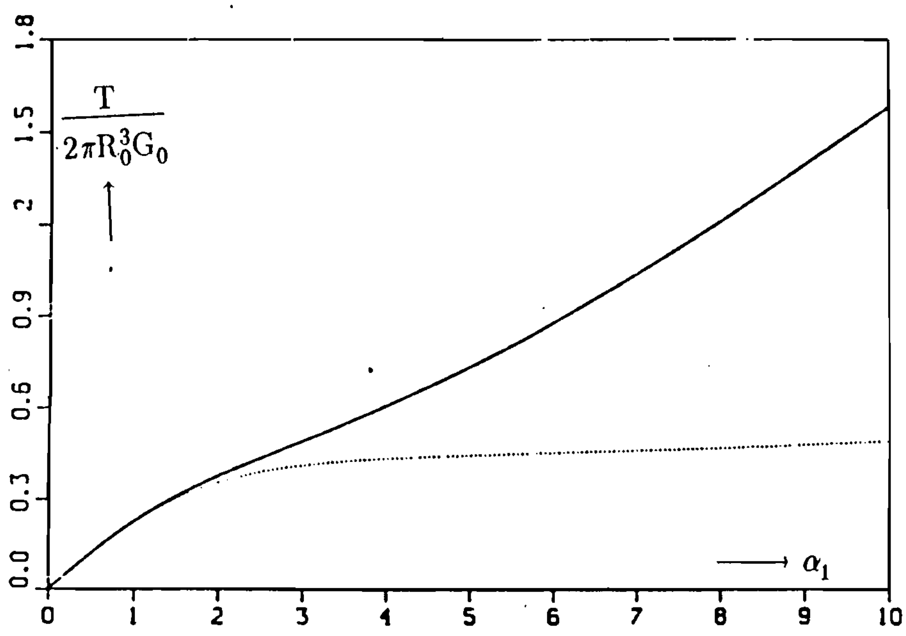

Figure 3.b

The dimensionless torque $T/(2\pi R_0^3 G_0)$ as a function of α_1 for the Dienes rate-type constitutive equation.