

# An electrolytic tank for instructional purposes representing the complex-frequency plane

**Citation for published version (APA):**

Eykhoff, P., Ophey, P. J. M., Severs, J., & Oome, J. O. M. (1968). *An electrolytic tank for instructional purposes representing the complex-frequency plane*. (EUT report. E, Fac. of Electrical Engineering; Vol. 68-E-04). Technische Hogeschool Eindhoven.

**Document status and date:**

Published: 01/01/1968

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

th

AN ELECTROLYTIC TANK FOR  
INSTRUCTIONAL PURPOSES REPRESENTING  
THE COMPLEX-FREQUENCY PLANE

BY

P. DUKHOFF, P. J. M. OPLEY, J. SEVENS, © M. COHEN

AN ELECTROLYTIC TANK FOR INSTRUCTIONAL PURPOSES  
REPRESENTING THE COMPLEX-FREQUENCY PLANE

by

P. Eykhoff, P.J.M. Ophay, J. Severs, J.O.M. Oome

TH-Report 68-E-04

Department of Electrical Engineering  
Technological University  
Eindhoven, the Netherlands

#### GROUP MEASUREMENT AND CONTROL

An electrolytic tank for instructional purposes representing the complex-frequency plane.

P.Eykhoff, P.J.M. Ophey, J.Severs,  
J.O.M. Oome.

#### Summary

After some general observations with respect to the complex-frequency plane a description is given of an electrolytic tank representing this plane. This tank was constructed with educational purposes in mind. If the complex function  $H(s)$  (transfer-, impedance-, or admittance function) is represented by current sources for its poles and by current sinks for its zeros then a potential proportional to  $\log |H(s)|$  can be measured. The  $j\omega$ -axis, representing the 'real' frequencies, can be scanned automatically. Using one of the Bode-relations a special instrumentation has been made which in addition provides  $\arg H(j\omega)$  for minimum phase systems.

Introduction.

There are a number of means available for the description of the dynamic behaviour of linear(ized) systems or processes. A schematic survey of these different types of descriptions is given in fig. 1 and in [ 1]. Among these means the pole-zero plot occupies a central position, because of the unique combination of theoretical insight and practical usefulness that is provided by these plots in the complex-frequency or s-plane [ 2]. Many different aspects -steady state behaviour, transient behaviour, the introduction of feedback, parameter sensitivity- can be easily studied using pole-zero plots.

Since in the electrical engineering and in the systems oriented curricula these concepts are of paramount importance, the use of analogons for the s-plane suggests itself.

The transfer-, impedance-, or admittance function  $H(s)$  is a function of the complex variable  $s = \sigma + j\omega$  (complex frequency). This can be written as

$$k \frac{\prod (s + d_m)}{\prod (s + e_n)} = H(s) = |H(s)| e^{j\varphi}$$

with

$$\varphi = \arg H(s)$$

The values  $s = -d_m$  and  $s = -e_n$  are the zeros and the poles of  $H(s)$  respectively. Fig. 2 shows some models made from plywood for simple pole/zero configurations. The cuts may represent  $|H(j\omega)|$  with a linear horizontal and a linear vertical scale. Instead of a linear representation it is worthwhile considering a logarithmic one:

$$\ln k + \sum \ln (s+d_m) - \sum \ln (s+e_n) = \ln H(s) = \ln |H(s)| + j\varphi.$$

This function is analytic in the whole s-plane, except in the poles and zeros. This can be shown by proving that the partial first derivatives are continuous and that they satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

where

$$u + jv = f(z) = f(x + jy)$$

which in our case corresponds with

$$\ln |H(s)| + j\varphi = \ln H(s) = \ln H(\sigma + j\omega)$$

One of the consequences is that  $\ln |H(s)|$  and  $\varphi$  are conjugate harmonic or logarithmic potential functions, satisfying Laplace's equations, i.e.

$$\frac{\partial^2}{\partial \sigma^2} \ln |H| + \frac{\partial^2}{\partial \omega^2} \ln |H| = \Delta \ln |H| = 0$$

This justifies the representation of  $\ln |H(s)|$  by a rubber sheet, c.f. fig. 3, or in an electrolytic tank. Such a tank may have linear  $\sigma$  and  $\omega$  scales [3]. The poles are represented by current sources, the zeros by current sinks. The equipotential lines then stand for the curves  $\ln |H(s)| = \text{constant}$ . Along the  $j\omega$ -axis a potential can be measured proportional to  $\ln |H(j\omega)|$ .

For engineering purposes it is an advantage to use a conformal mapping from the  $s$ -plane to the  $\ln s$ -plane [4]:

$$\ln s = \ln |s| + j\Psi$$

with

$$\Psi = \arg s$$

Fig. 4 illustrates this type of mapping. The  $j\omega$  axis is mapped on the line  $\Psi = \frac{\pi}{2}$ . The  $\omega$ -scale is divided logarithmically which enables the presentation of several decades of this scale. The mapping is periodic in  $\Psi$ ; only one period needs to be shown as there are no flow lines crossing the lines  $\Psi = n\pi$  with  $n = 0, \pm 1, \pm 2, \dots$ . This is due to the fact that the poles and zeros occur on the lines  $\Psi = 0, \pi, 2\pi$  or in complex conjugate pairs.

The following presentations have now been mentioned:

- $|H(s)|$  on a linear  $s$ -plane
- $\ln |H(s)|$  on a linear  $s$ -plane
- $\ln |H(s)|$  on a logarithmic  $s$ -plane

For a very simple example with one pole and one zero these functions are given schematically in fig. 5.

The determination of  $\arg H$  from the  $\ln |H|$  analog.

A very interesting approach to the determination of  $\arg H$  as a function of frequency can be found in ref. [5]. This implies the need for a special analog for the  $\arg H(s)$  function (which can also provide the root loci) A computer based on these principles is available commercially [5].

For demonstration purposes we have chosen another approach which is based on, and therefore is a direct illustration of, the Bode relations.

Consequently its use is limited to transfer-, impedance-, or admittance functions which have poles and zeros for  $\sigma \leq 0$  only (minimum phase type). According to Bode [6, pag. 312 ]:

$$\beta_0 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\alpha}{du} \ln \coth \frac{|u|}{2} du$$

with

$$\beta_0 = \arg H(j\omega_0)$$

$$\alpha = \ln |H(j\omega)|$$

$$u = \ln \frac{\omega}{\omega_0}$$

In the electrolytic tank, when measuring on the  $\ln \omega$  axis, a voltage  $V$  is available such that

$$V :: \ln |H(j\omega)|$$

The integral can be approximated using a finite number of measurements of  $V$  on the  $\ln \omega$  axis. To this end ten frequency intervals are chosen such that:

$$\Delta u_1 = \ln \frac{\omega_2}{\omega_1} = \ln \frac{\omega_{11}}{\omega_{10}} = \Delta u_{10}$$

$$\Delta u_2 = \ln \frac{\omega_3}{\omega_2} = \ln \frac{\omega_{10}}{\omega_9} = \Delta u_9$$

$$\Delta u_5 = \ln \frac{\omega_6}{\omega_5} = \ln \frac{\omega_7}{\omega_6} = \Delta u_6$$

For the derivative one may use the approximation

$$F_i = \frac{V_{i+1} - V_i}{\Delta u_i} \approx \frac{d \ln |H(j\omega)|}{d u} \quad \omega_i < \omega < \omega_{i+1}$$

The "weighting function"  $\ln \coth \frac{|u|}{2}$  is shown in fig. 6. This function is approximated as indicated in fig. 7 by weighting factors  $W_i$ . These factors are chosen such that

$$W_i = \frac{1}{\Delta u_i} \int_{u_i}^{u_{i+1}} \ln \coth \frac{|u|}{2} du$$

Consequently

$$\arg H(j\omega_0) = \beta_0 \approx \sum_{i=1}^{10} W_i F_i \Delta u_i = \sum_{i=1}^{11} W'_i V_i.$$

Of course several types of error have been introduced by these approximations:

- the weighting function is represented as changing stepwise instead of continuously. Furthermore it is truncated for  $u < \ln \frac{\omega_1}{\omega_0}$  and  $\ln \frac{\omega_{11}}{\omega_0} > u$
- the derivative is represented as a simple difference; in most cases this is not too serious as the functions  $\ln |H(j\omega)|$  are rather smooth provided there are no poles and zeros very close to the  $\ln \omega$  axis.

The equation approximating  $\arg H(j\omega_0)$  can be instrumented by measuring  $V_1, \dots, V_{11}$  and adding the results using appropriate coefficients, e.g. by means of an operational amplifier with a number of input resistors. This equation holds for  $\omega = \omega_0$ . By shifting the measuring probes along the  $\ln \omega$  axis the phase contribution or argument can be determined for any other value of  $\omega$ . Due to the logarithmic expressions the configuration of the probes does not need to be changed when travelling along the  $\ln \omega$  axis.

#### Constructional aspects.

The electrolytic tank is shown in fig. 8, both in operating condition and as a bottom-view. It is filled with distilled water to which a small amount of NaCl has been added.

The tank represents 17 decades of the  $\ln \omega$  axis. This high number is needed due to the fact that for  $|s| \rightarrow 0$  holds  $\ln |s| \rightarrow -\infty$ . Consequently the mapping of  $|s| = 0$  must be well-removed from the other part of the pole-zero pattern that is of interest. A similar reasoning holds for  $|s| \rightarrow \infty$ . Provided are eight platinum wire electrodes that can be positioned quite arbitrarily in the tank. Electrode polarization is avoided by using a 400 Hz voltage supply instead of D.C. Each electrode can either be disconnected, represent a pole or represent a zero by means of three-position switches. The choice between a pole and a zero is simply realized by phase reversal of the sinewave.

The same holds for the electrode at  $|s| = 0$ , which is a plate electrode. At the same time that any pole (zero) is switched on a zero (pole) is added to the plate electrode at  $|s| = \infty$ , making the number of poles equal to the number of zeros. This aspect is illustrated in fig. 9.

Also shown in fig. 9 is the simple motor-drive for the electrode configuration that scans the  $\ln \omega$  axis. Microswitches at both ends of the scanning interval reverse the direction of scanning.



The movable probe assembly is chain-driven by a 50 VA 50 Hz squirrel-cage induction motor at a rate of 10 cm per second via an appropriate reduction gear. Coupled to the probe drive is a ten turn linear potentiometer which supplies a DC voltage proportional to the distance of the probes from one end of the tank. This voltage is applied to the X-axis input terminals of an X-Y pen-recorder. The Y-input to the recorder is obtained from a full-wave phase sensitive rectifier connected to the balanced output of an impedance-transforming amplifier. A 3.5 - 0 - 3.5 volts moving-coil instrument across the Y-input serves as a useful indicator of the concentration of the electrolyte and as a warning measure against overloading the amplifier. The amplifier input can either be switched on to the central amplitude-probe or to the phase-sensing probe assembly. Having a real input impedance of over 5 megohms at 400 Hz, it represents a negligible load to the tank. The frequency response at 50 and at  $10^4$  Hz is down by a factor 30 with respect to the voltage gain of approx. 0.7 at 400 Hz. A phase-shifter, adjustable between + and -  $43^\circ$  limits, is incorporated in the amplifier. It enables the establishment, once and for all, of a proper phase relationship between the input- and gating-voltage to the phase-sensitive rectifier.

Two sets of five resistors are connected between the phase sensing probes and their appropriate summing amplifiers. Proper resistance values for an acceptable approximation of the arg. H function are shown in the circuit diagram fig. 9. The small size summing amplifiers  $-A_1$ ,  $-A_2$ , as well as the additional amplifier  $-A_3$ , are of the operational voltage-inverting type. As a precaution against oscillation low capacity condensers are shunted to the feedback resistors of  $-A_1$  and  $-A_2$ .

#### Additional details.

The liquid-container is made of plexiglass, enabling the  $\ln |s|$  and  $\Psi$  scales to be viewed through the bottom; the inward dimensions are 108 by 18 by 3 centimetres. It holds three litres of distilled water into which approx. one gramme of NaCl is dissolved; the solution should be well stirred to obtain isotropic conductivity. The mounting board has to be carefully levelled in order to ensure that the fluid layer is of constant height in every part of the tank. Although it is admittedly a drawback that the water will evaporate in the course of time, this objection is not felt to be serious. Loss of water will simply effectuate a higher conductivity of the electrolyte, which results in a lower output voltage, to be compensated for by increasing the gain of the recorder amplifiers.

When the evaporation process is judged to have gone too far the bath should be replenished with an appropriate quantity of distilled water.

It has been mentioned that the rotational direction of the motor field is reversed at each end of the track by means of microswitches. A second set of microswitches simultaneously energizes or de-energizes the pen-lifting mechanism of the recorder so that only rightward movements are recorded. In this way any effect of backlash in the moving system on the recordings is perfectly eliminated. Some specimen Bode diagrams produced by the instrument are shown in fig. 10.

#### Conclusions:

The electrolytic tank, described in this note, has been designed for demonstrational purposes.

It also can be used for system synthesis, although for such use its accuracy will rather soon become a limiting factor. Ways to improve the accuracy are easy to perceive. In view of the availability of special software for digital computers, however, the authors feel that for design purposes a rough approximation by presentation in the tank followed by accurate calculations on the digital computer is preferable. The tank has proved to be an excellent educational device for illustrating the complex frequency plane.

## List of Captions

- fig. 1. Different types of descriptions for linear(ized) processes.
- fig. 2. Models representing  $|H(s)|$  for some simple cases.
- fig. 3. Rubber sheet representing  $\ln |H(s)|$
- fig. 4. Mapping from the s-plane to the  $\ln s$ -plane.
- fig. 5. Some different representations of one simple pole/zero pattern.
- fig. 6. Weightingfunction  $\ln \coth \left| \frac{u}{2} \right|$
- fig. 7. Approximation of the weightingfunction.
- fig. 8. The electrolytic tank.
- fig. 9. Circuit diagram of the instrument.
- fig. 10. Some examples of recordings of amplitude and phase diagrams.

## References

1. Eykhoff, P. "Process Parameter Estimation"  
in: R.H. Macmillan et al. (ed.) Progress in Control Engineering, vol. 2.  
London, Heywood & Co., Ltd., 1964.
2. Blackman, P.F. "The Pole-Zero Approach to Systems Analysis".  
(based on a series of articles in Control)  
London, Rowse Muir Publ., 1961 ("Control Monograph" 2)
3. Boothroyd, A.R., E.C. Cherry, R. Makar, "An Electrolytic Tank for  
the Measurement of Steady-state Response ...",  
proc. IEEE, vol. 96, part I, p. 163, 1949.
4. Smith, O.J.M. "Feedback Control Systems"  
New York, McGraw Hill, 1958.
5. Morgan, M.L., J.C. Looney, "Design of the Esiac algebraic computer".  
IRE Trans. Electron.Comput., vol. EC-10, no. 3, 524-529, September 1961.
6. Bode, H.W., "Network Analysis and Feedback Amplifier Design".  
New York, 1945.

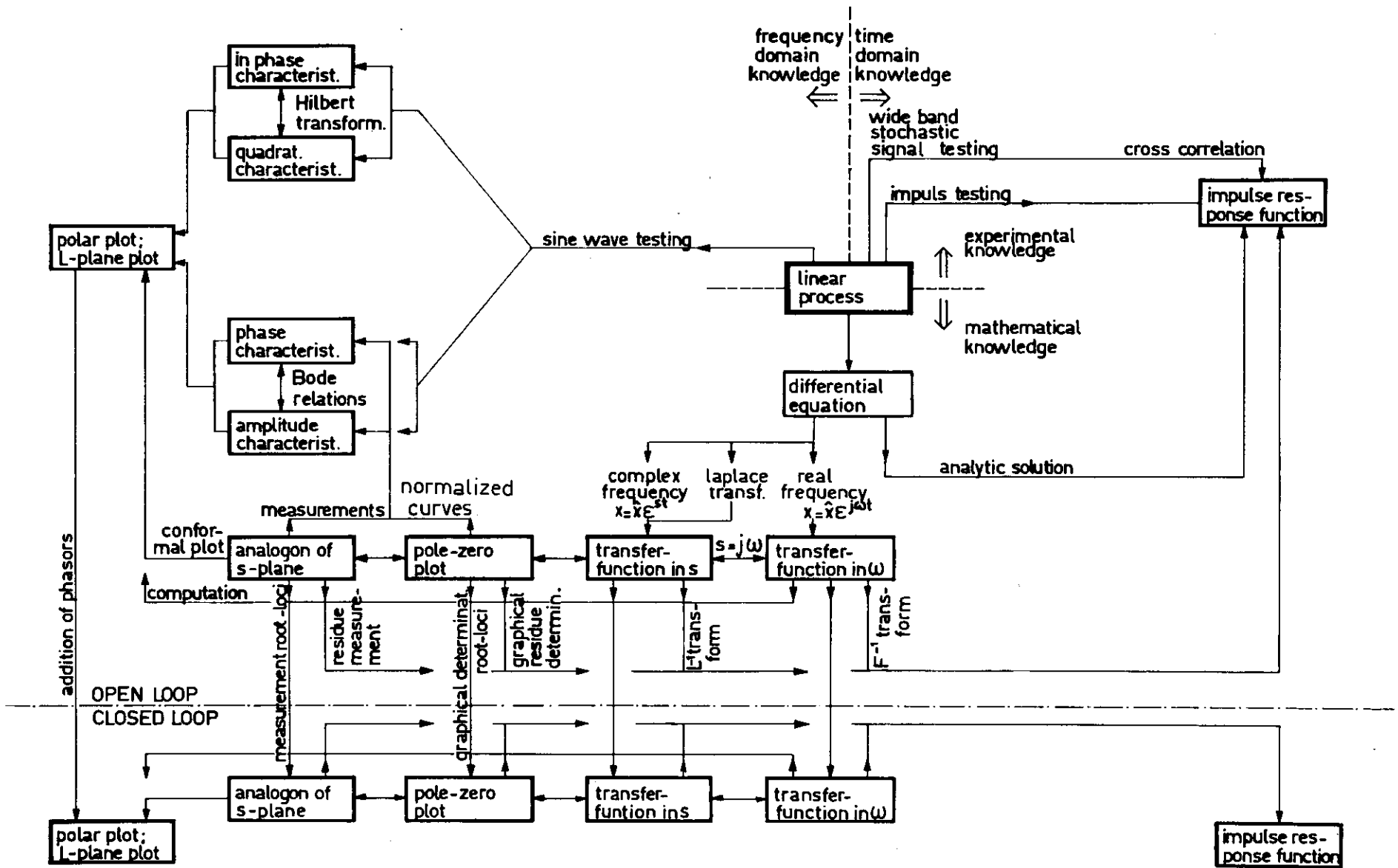


fig. 1

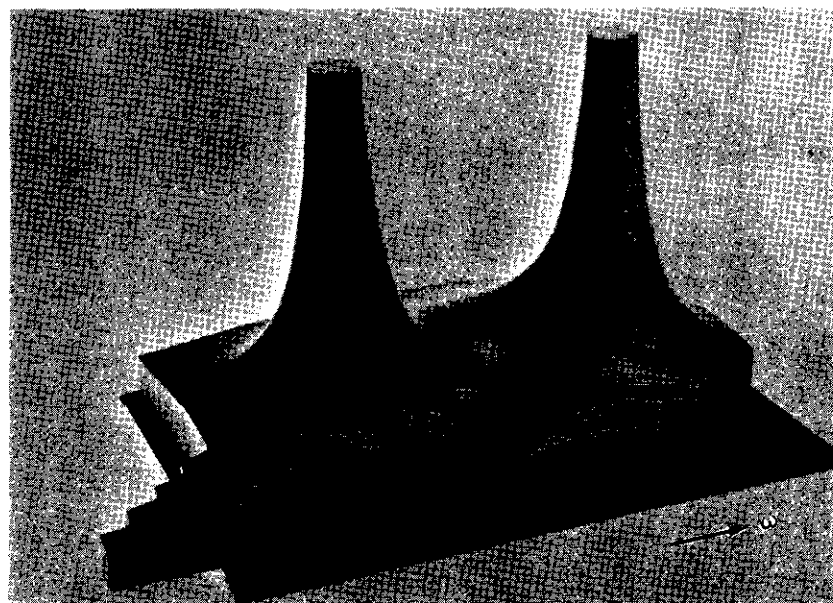
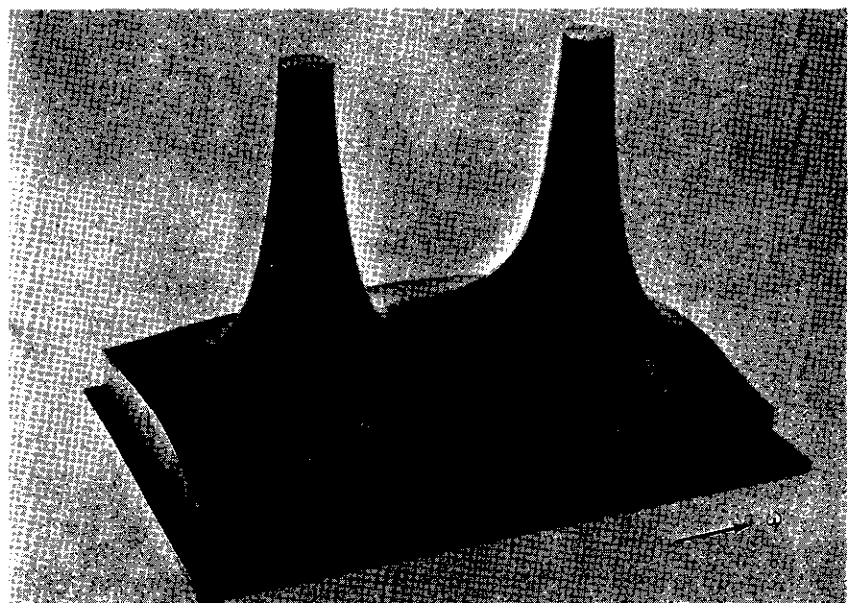
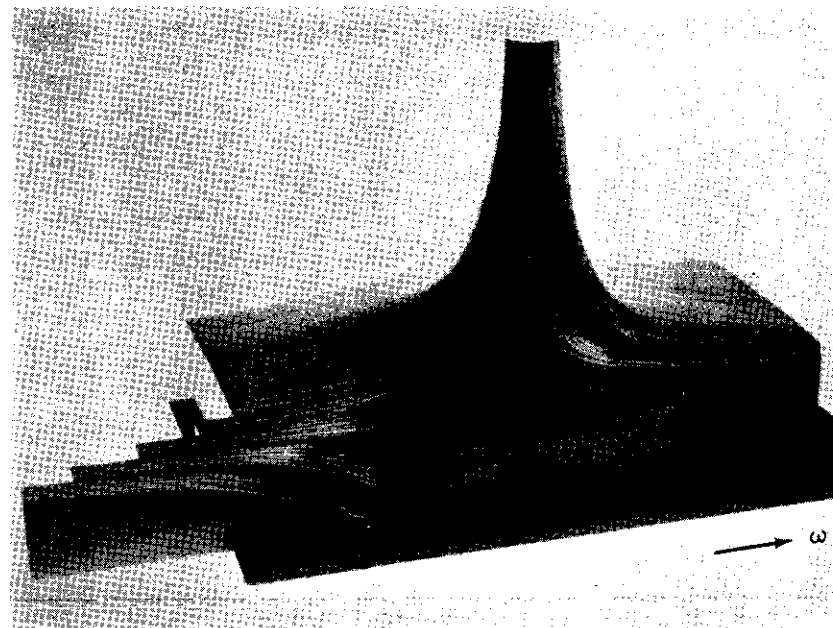
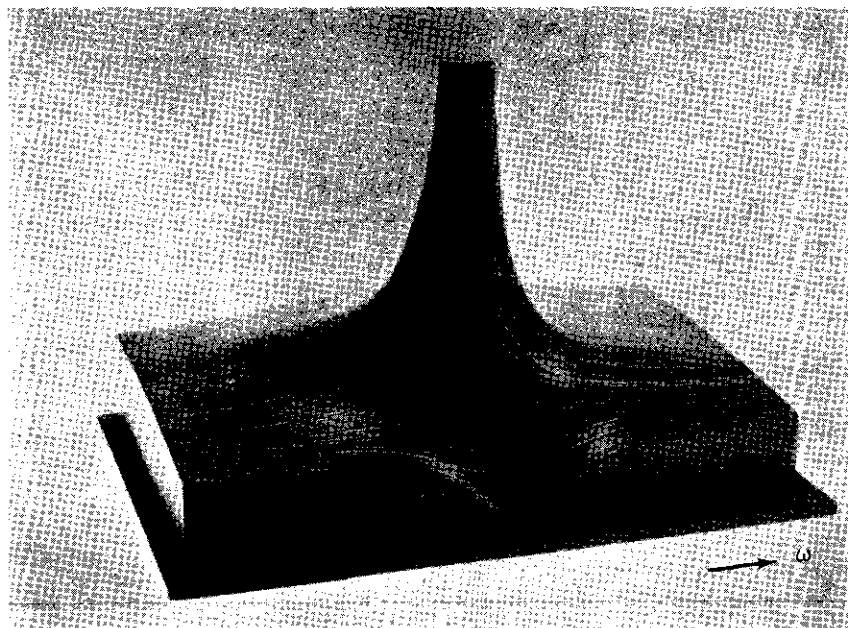


fig. 2

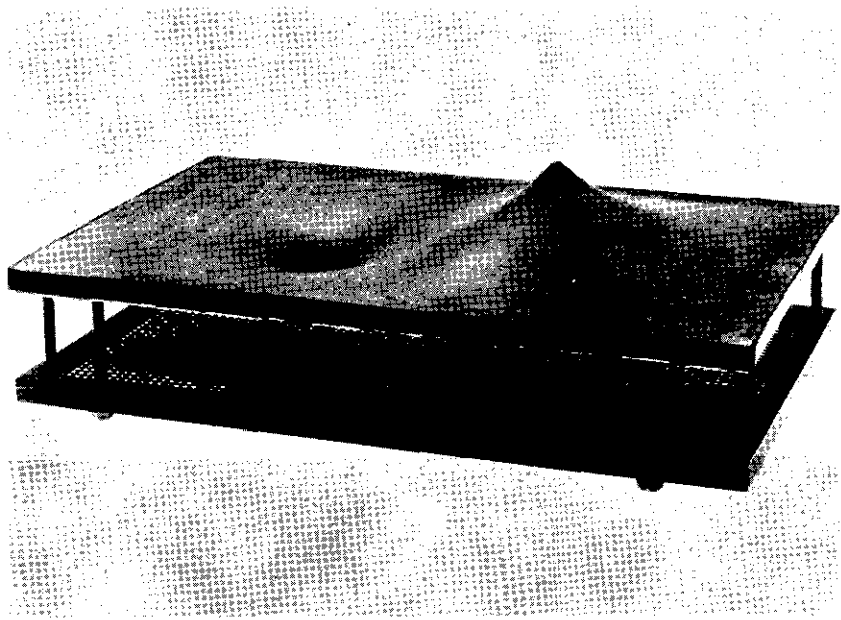


fig. 3

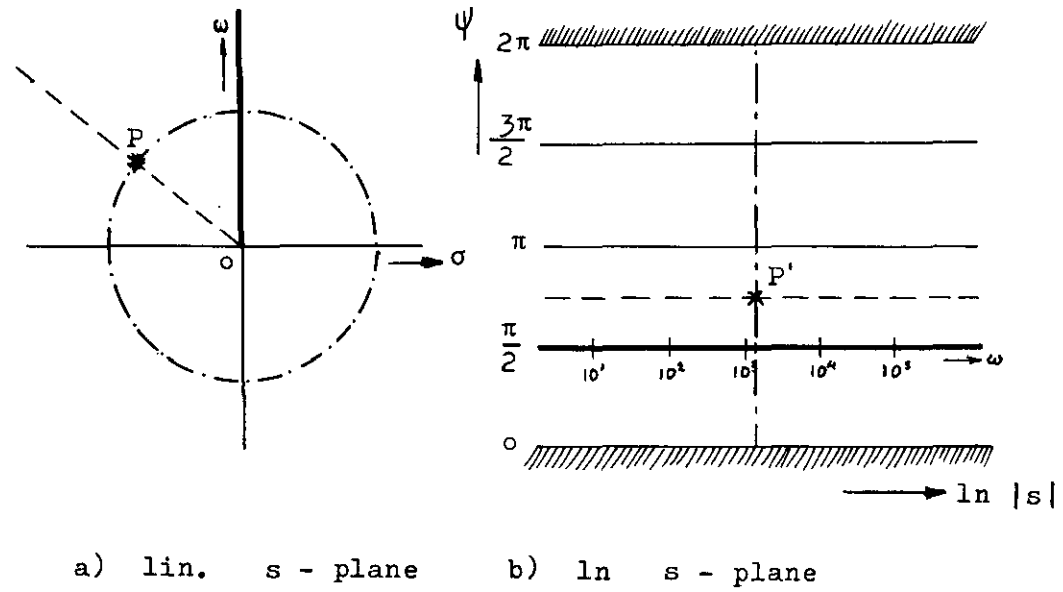
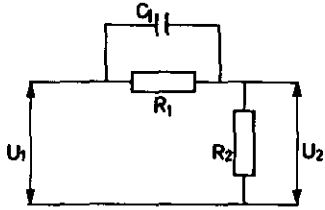


fig. 4

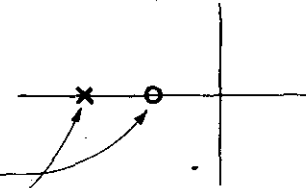
# POLES and ZERO'S



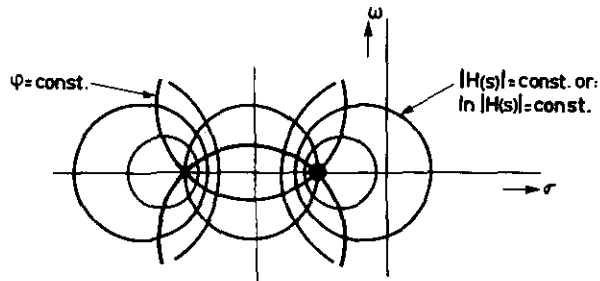
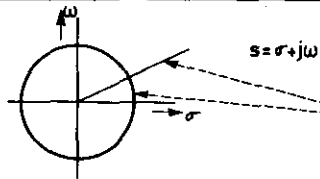
$$H(s) = \frac{U_2}{U_1} = \frac{R_2}{R_1 + \frac{1}{sC_1} + R_2} = \frac{R_2}{R_1 + R_2} \frac{sC_1 R_1 + 1}{sC_1 \frac{R_1 R_2}{R_1 + R_2} + 1}$$

numerator = 0 for  $s = -\frac{1}{C_1 R_1} \Rightarrow$  zero

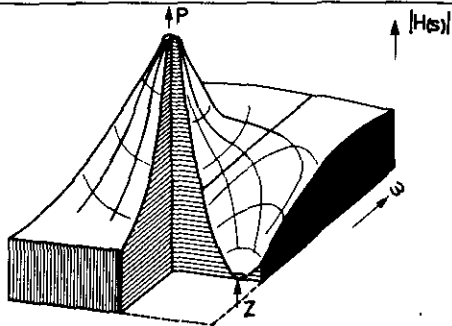
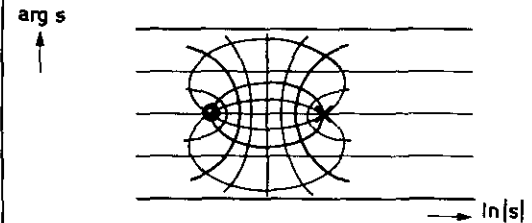
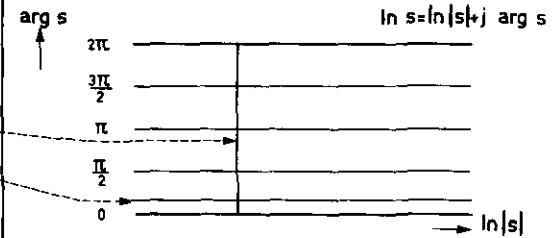
denominator = 0 for  $s = -\frac{R_1 + R_2}{C_1 R_1 R_2} \Rightarrow$  pole



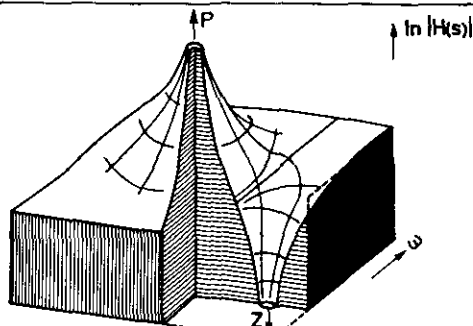
s-plane



ln s-plane



$$H(s) = |H(s)| e^{j\phi}$$



$$\ln H(s) = \ln |H(s)| + j\phi$$

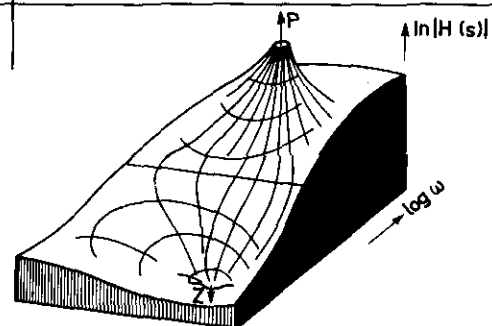


fig. 5



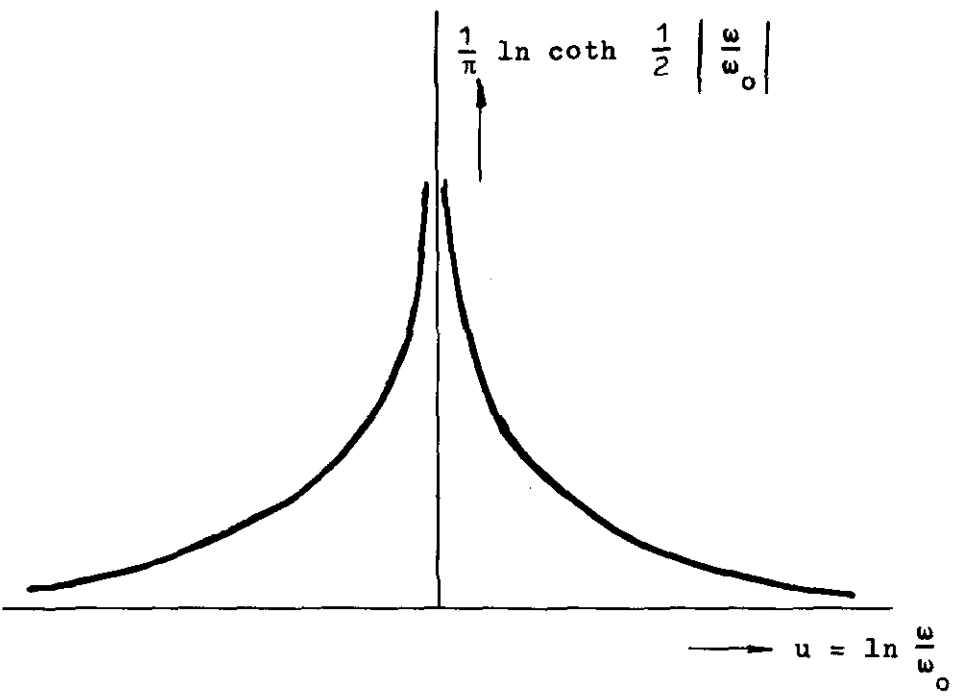


fig. 6

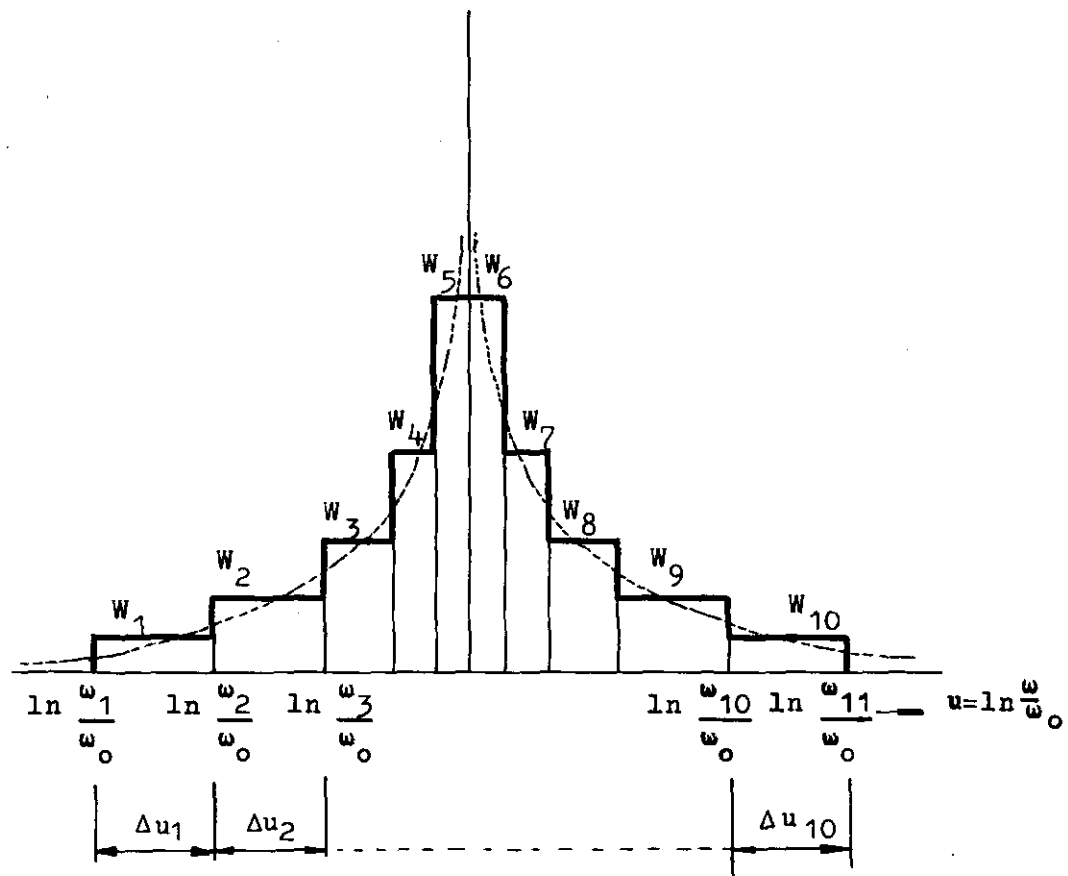


fig. 7

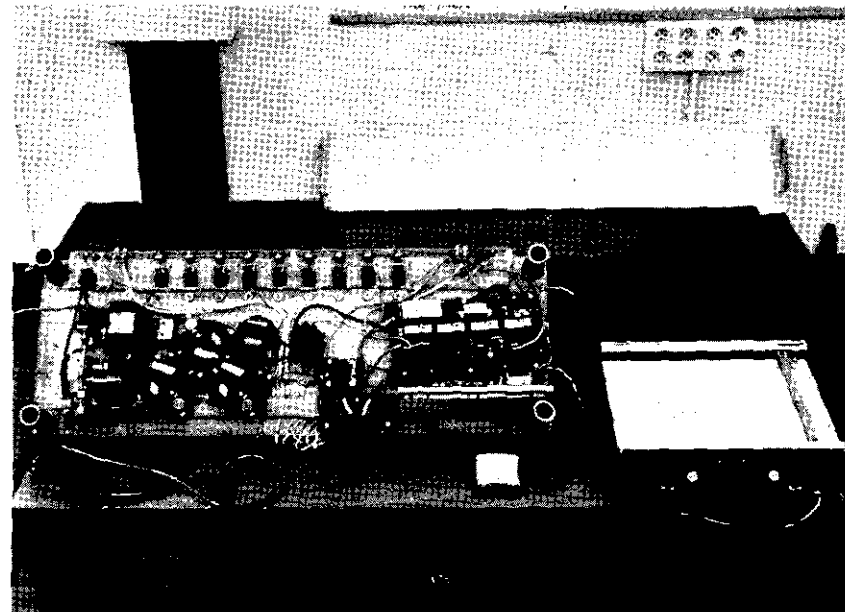
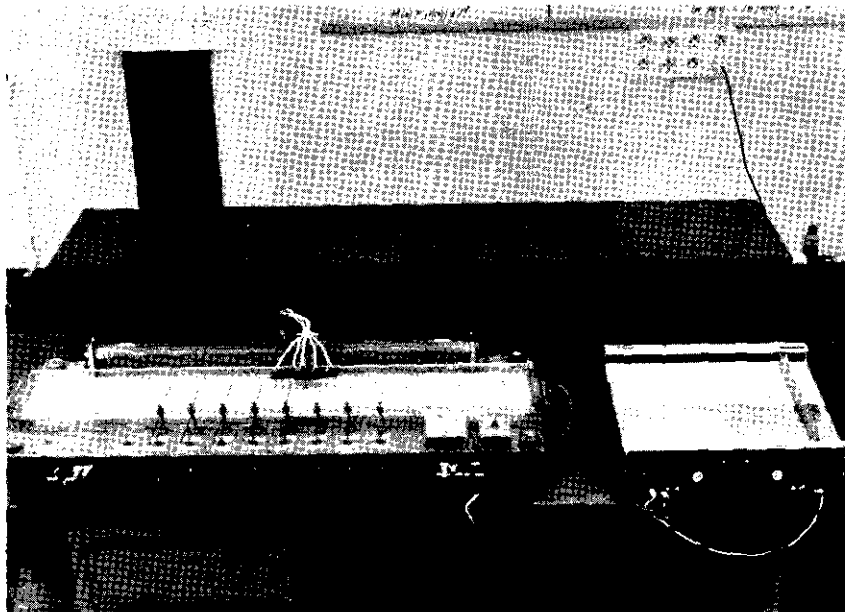


fig. 8

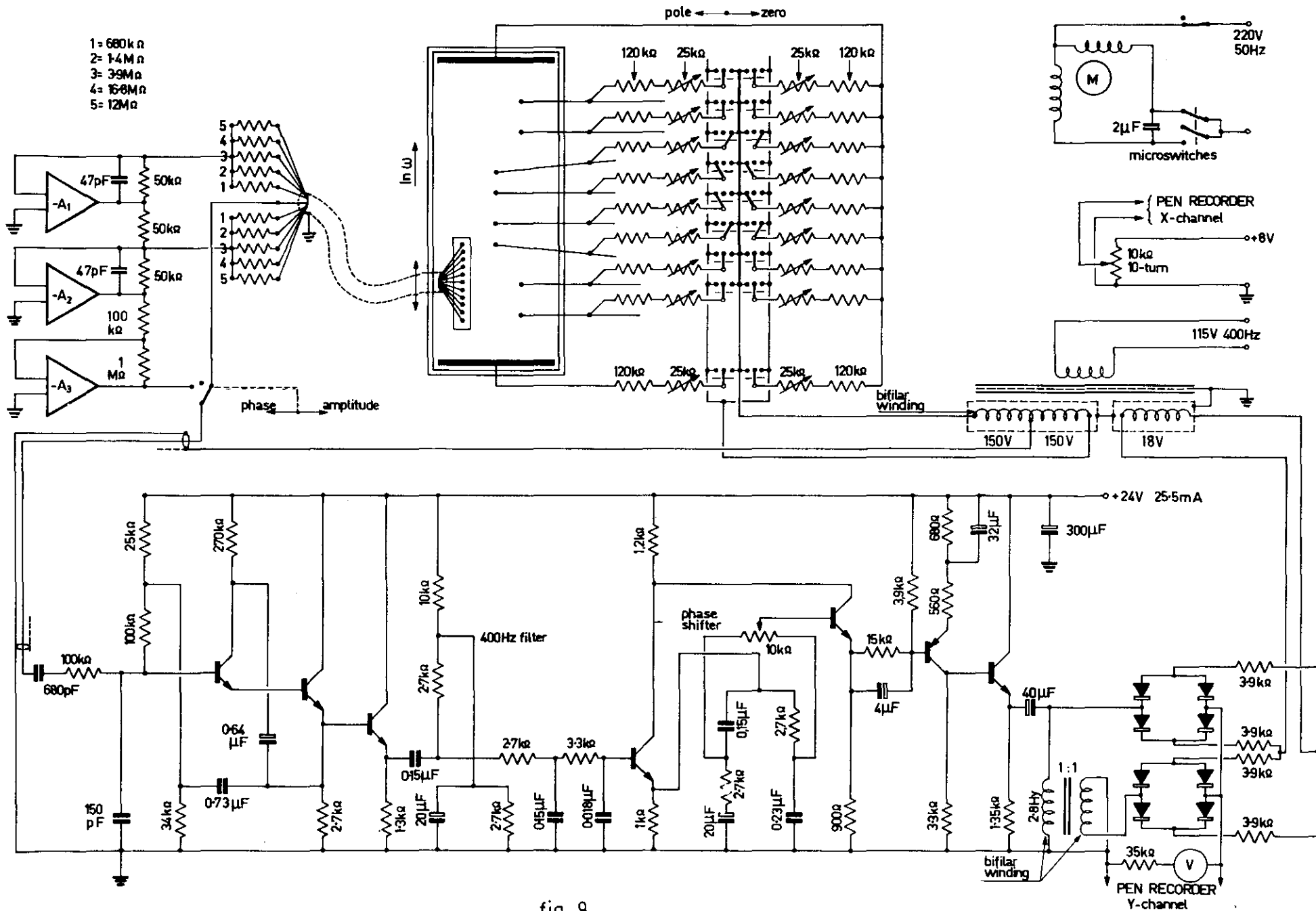


fig. 9

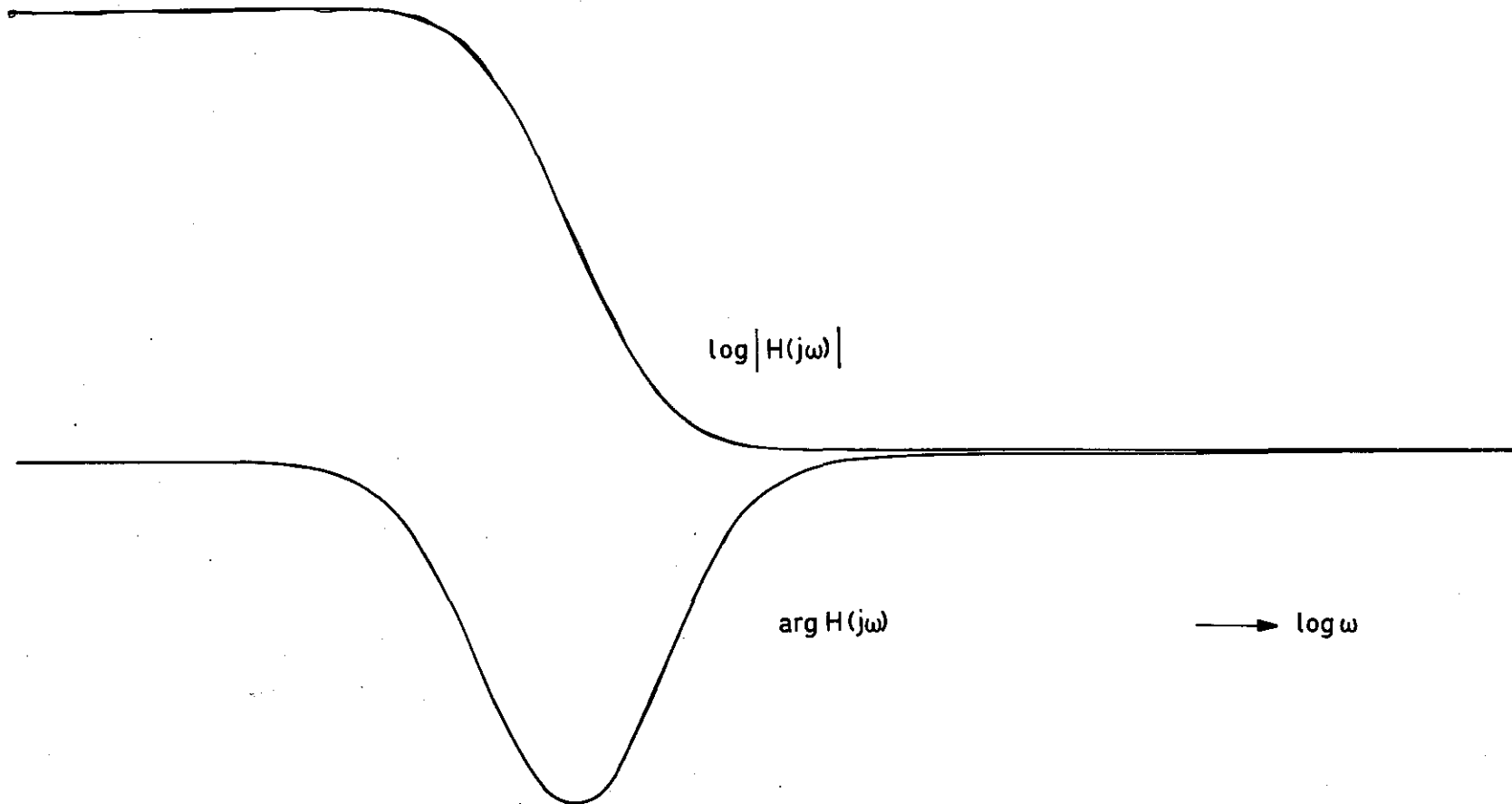


fig. 10a

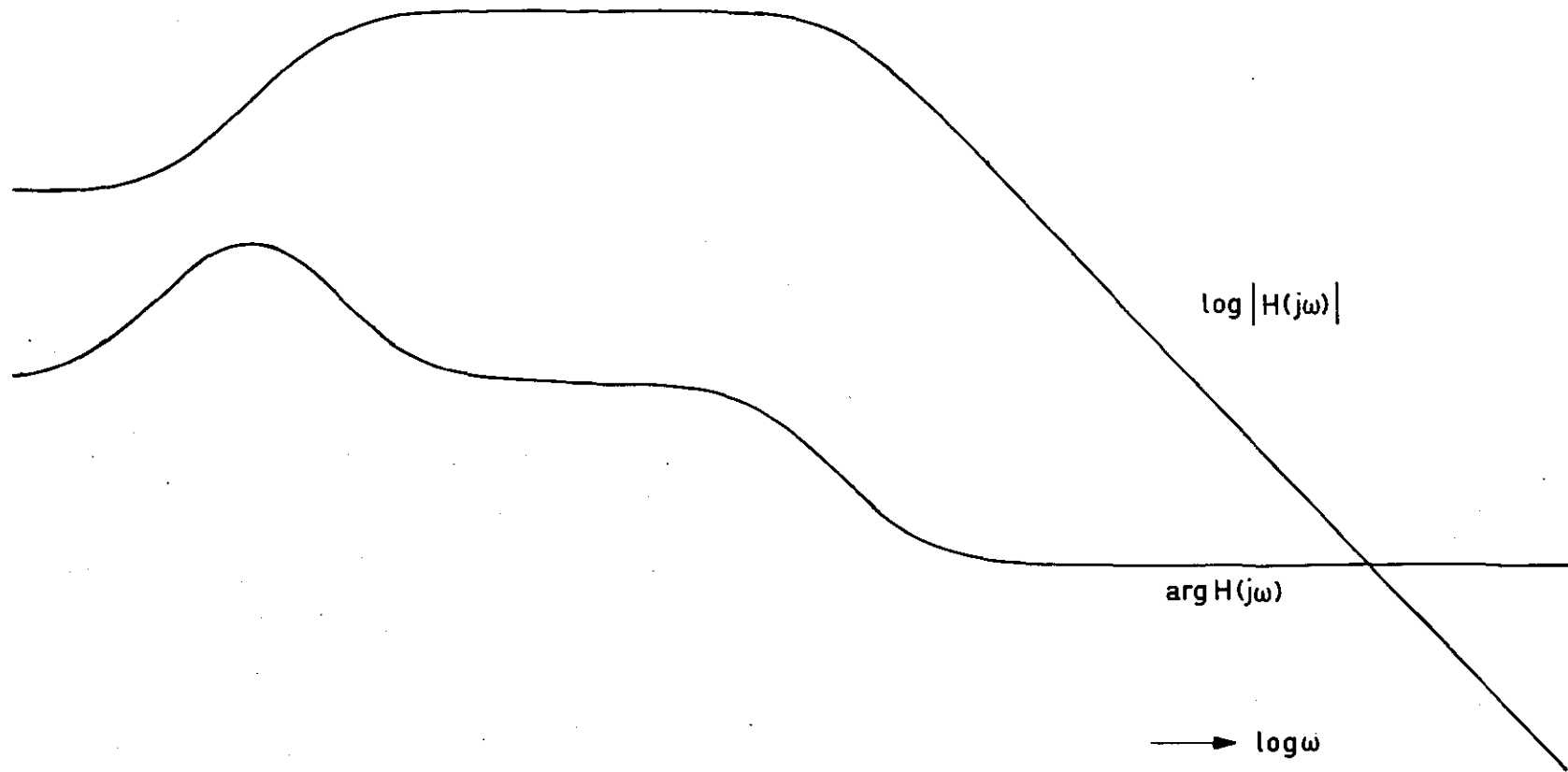


fig.10b

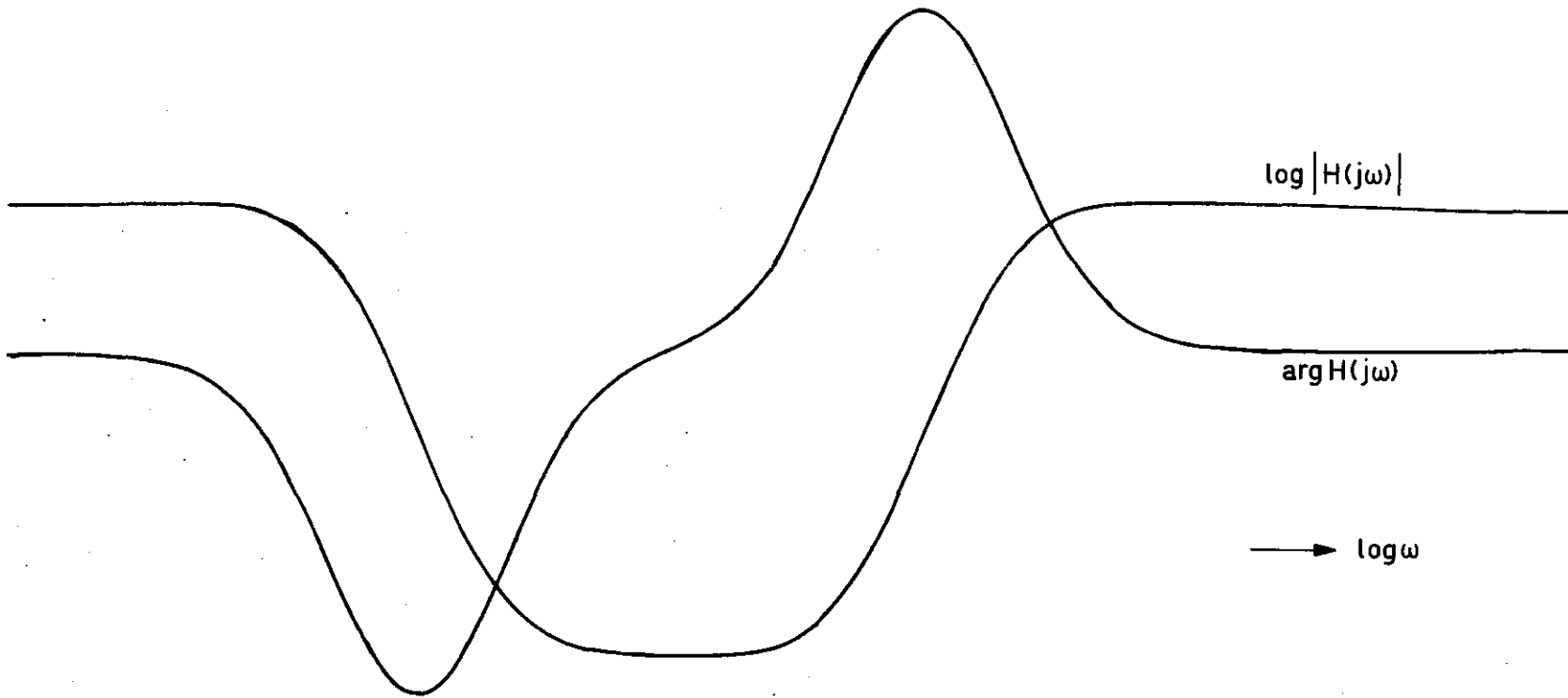


fig.10c