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Abstract

Given linear stochastic differential equations (sde's) with random coefficients (independent of place), the polynomial chaos decomposition can be used to solve these sde's numerically or analytically. In general this is possible if the distributions of those coefficients are known and can be inverted. In this report we will look at a concrete example and look how and under what choices of the coefficients the polynomial chaos decomposition can be applied.

1 Introduction

The work presented in this report takes place in a research program including several PhD research theses. A general title of the program could be: "Fast, efficient and robust optimalization of quiet and acoustically insensitive structures with uncertain parameters". TNO-TPD and the University of Eindhoven are participants in this project.
The main question to be answered in this report is given a sde, when is it allowed to use the polynomial chaos decomposition to solve this sde (either numerically or analytically). In this report we will focus on the following example as an illustration

$$\begin{cases} \frac{d^2}{dx^2}U(x,\omega) + \alpha\omega^2 U(x,\omega) = 0 & x \in]0,1[,\ \omega \in \mathbb{R}^+ \\ U(0,\omega) = 0 & \omega \in \mathbb{R}^+ \\ \frac{d}{dx}U(1,\omega) = \beta F & \omega \in \mathbb{R}^+ \end{cases} \tag{1}$$

with $\alpha = \frac{\rho}{E}$ and $\beta = \frac{1}{AE}$. This differential equation describes the longitudinal waves in a bar clamped on one side in a rigid wall excited by a harmonic force f. This results (after some time) in a harmonic displacement u (that's only true if there's damping present, but for simplicity we removed the damping). We define $F(\omega) = f(t)\exp(-I\omega t)$ and $U(x,\omega) = u(x,t)\exp(-I\omega t)$. Thus U is the displacement, A the cross-section, ρ the density and E the modulus of the bar. We will assume that α and β are random variables. It follows that the solution $U(x,\omega)$ will be a random variable for every $x \in [0,1]$ and $\omega \in \mathbb{R}^+$. In other words U will be a random field.

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It would be nice if we could determine the statistics (like the expectation and the variance) of the solution $U(x, \omega)$ given the probability densities of α and β. Note that α and β are not independent since it's in general not true that $\mathbb{E}[\alpha\beta] = \mathbb{E}[\alpha] \cdot \mathbb{E}[\beta]$ since α and β have a common factor. But it's not possible (at least we don't know how) to determine the statistics of the solution $U(x, \omega)$ using the polynomial chaos decomposition given arbitrary probability densities of α and β. However for some choices of α and β it's possible to determine the mean and variance of the solution $U(x, \omega)$ using the polynomial chaos decomposition. Before determining those solutions we will briefly discuss the polynomial chaos decomposition.

2 The polynomial chaos decomposition

The polynomial chaos decomposition uses the fact that the Hermite polynomials are orthogonal with respect to a weighting function (see for example (Janson, 1997)). This weighting function is just the Gaussian probability density. In other words

$$(h_n, h_m)_f = \int_{-\infty}^{\infty} h_n(x) h_m(x) f(x) dx = n!\ \delta_{n,m} \tag{2}$$

where the Hermite polynomials are given by

$$h_m(x) = \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \frac{m!}{2^r r!(m-2r)!} x^{m-2r}$$

(where $\lfloor m/2 \rfloor = m/2$ if m is even and $\lfloor m/2 \rfloor = (m-1)/2$ if m is odd) and the Gaussian probability density is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

To formulate it slightly different. Let ξ be a zero-mean Gaussian with variance one and define

$$h_m(\xi) = \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \frac{m!}{2^r r!(m-2r)!} \xi^{m-2r}$$

then it follows that the random variables $h_n(\xi)$ and $h_m(\xi)$ are orthogonal

$$(h_n(\xi), h_m(\xi)) = \mathbb{E}[h_n(\xi) h_m(\xi)] = n!\ \delta_{n,m}$$

The polynomial chaos decomposition states that a random variable X which satisfies the following two conditions

1. $\mathrm{Var}[X] < \infty$
2. X is defined on the same probability space as where ξ is defined

is a linear combination of the Hermite polynomials, that is

$$X = \sum_{j=0}^{\infty} g_j h_j(\xi), \quad g_j \in \mathbb{R} \tag{3}$$

(this sum holds in the second order sense meaning that the covariance and mean of both sides are equal, thus also in probability and in distribution, but not necessarily in sample-path). In other words the Hermite polynomials form a basis of the space of random variables which satisfy the above two conditions. Mathematically these two conditions mean that $X \in L^2(\Omega, \Sigma, \mathcal{P})$ where the probability space $(\Omega, \Sigma, \mathcal{P})$ is such that $\xi \in (\Omega, \Sigma, \mathcal{P})$. Returning to the sde in formula (1). As being said in the introduction, we want to use the polynomial chaos decomposition to solve the sde in formula (1). Thus assuming that the solution $U(x, \omega)$ admits a polynomial chaos decomposition we have

$$U(x, \omega) = \sum_{j=0}^{\infty} g_j(x, \omega) h_j(\xi)$$

and we have to determine the coefficients $\{g_j(x, \omega)\}_j$. This can be assumed if the solution $U(x, \omega)$ met the two conditions given above. We will see in the next section what kind of restrictions these conditions imply on the coefficients of the sde in formula (1).

3 Generation of random variables

In the previous section we looked at the polynomial chaos decomposition. In this section we will look what that means for the coefficients of the sde in formula (1) if we want to use the polynomial chaos decomposition to solve the sde in formula (1). Let's have a closer look at the second condition given in the previous section, what exactly does this condition mean? It turns out that if the random variable X is a continuous function of a random variable ξ then X is defined on the same probability space as where ξ is defined (see appendices A.3 and B.1). This result is illustrated/used in the following theorem and corollaries (see for example (Blum and Rosenblatt, 1972)). In the following theorem it is shown that a random variable with a continuous distribution function can be transformed into an arbitrary random variable if it has a continuous distribution function.

Theorem 1. *Generation of random variables*
Let $X : \Omega \to \mathbb{R}$ be a random variable with a continuous distribution function $F_X : \mathbb{R} \to [0,1]$ defined on a probability space $(\Omega, \Sigma, \mathcal{P})$ with $\mathcal{P} : \Sigma \to \mathbb{R}$ defined by $\mathcal{P}(X \leq x) = F_X(x)$. Let $F : \mathbb{R} \to [0,1]$ be a continuous distribution function (see appendix A.1) of a certain random variable. Then the following holds:
Define a random variable $Y : \Omega \to \mathbb{R}$ by $Y = F^{-1}(F_X(X))$ then Y has as distribution function F and Y is defined on the same probability space as X is.

Proof. It follows that

$$\begin{aligned} F_Y(y) &= \mathcal{P}(Y \leq y) = \mathcal{P}(F^{-1}(F_X(X)) \leq y) = \mathcal{P}(F_X(X) \leq F(y)) \\ &= \mathcal{P}(X \leq F_X^{-1}(F(y))) = F_X(F_X^{-1}(F(y))) = F(y) \end{aligned}$$

Hence Y has as distribution function F. Note that both inverses exist since the distributions are monotone increasing and continuous. See appendix A.3 for the last part of the theorem. □

As a special case of the theorem above, if this random variable X is Gaussian distributed we can express the uniformly distributed variable as an infinite polynomial in X.

Corollary 2. *Generation of a uniformly distributed variable*
Let X be a zero-mean Gaussian random variable with variance one and F the distribution function of an uniformly distributed random variable on $[0,1]$. It follows from theorem 1 if we define $U = F^{-1}(F_X(X))$ then U has as distribution function F and U is defined on the same probability space as X is. We will also show that $U = F^{-1}(F_X(X)) = F_X(X)$ and

$$F_X(X) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j X^{2j+1}}{(2j+1)2^j j!} \tag{4}$$

Proof. The probability density of U is given by

$$f_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \in \mathbb{R} \setminus [0,1] \end{cases}$$

The corresponding probability distribution is given by

$$F_U(x) = \int_{-\infty}^{x} f_U(y)dy = \begin{cases} 0 & x \leq 0 \\ x & x \in [0,1] \\ 1 & x \geq 1 \end{cases}$$

The inverse is given by $F_U^{-1}(x) = x$ for $x \in [0,1]$. It follows that $U = F^{-1}(F_X(X)) = F_X(X)$ since $F_X(X)$ takes values in $[0,1]$.
For the second part, we have

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy$$

When we make a Taylor expansion of this integral we get

$$F_X(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)2^j j!} \tag{5}$$

and finally

$$F_X(X) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j X^{2j+1}}{(2j+1)2^j j!}$$

□

As a special case of the theorem above, if this random variable X is Gaussian distributed we will express the exponentially distributed variable as an infinite polynomial in X.

Corollary 3. *Generation of an exponentially distributed variable*
Let X be Gaussian random variable and F the distribution function of a exponentially distributed random variable. It follows from theorem 1 if we define

$Y = F^{-1}(F_X(X))$ then Y has as distribution function F and Y is defined on the same probability space as X is. We will show that

$$Y = F_Y^{-1}(U) = \lambda^{-1} \sum_{j=1}^{\infty} \frac{1}{j} U^j \tag{6}$$

with according to corollary 4

$$U = F_X(X) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j X^{2j+1}}{(2j+1)2^j j!}$$

Proof. In order to show formula (6) we have to invert the distribution of the exponentially distributed random variable Y. The probability density of Y is given by $(\lambda > 0)$

$$f_Y(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The corresponding probability distribution is given by

$$F_Y(x) = \int_{-\infty}^{x} f_Y(y) dy = \begin{cases} 1 - \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The inverse is given by

$$F_Y^{-1}(x) = -\lambda^{-1} \log(1-x) = \lambda^{-1} \sum_{j=1}^{\infty} \frac{1}{j} x^j, \quad 0 \leq x \leq 1$$

It follows that

$$Y = F_Y^{-1}(U) = \lambda^{-1} \sum_{j=1}^{\infty} \frac{1}{j} U^j$$

□

To illustrate the corollaries above an example is shown below. First we generate 1500 samples of a zero-mean Gaussian variable with variance one (using Maple). We use formula (4) to generate 1500 samples of a uniformly distributed variable and finally we will invert the distribution of a exponentially variable and use formula (6) to generate 1500 samples of a exponentially distributed variable.

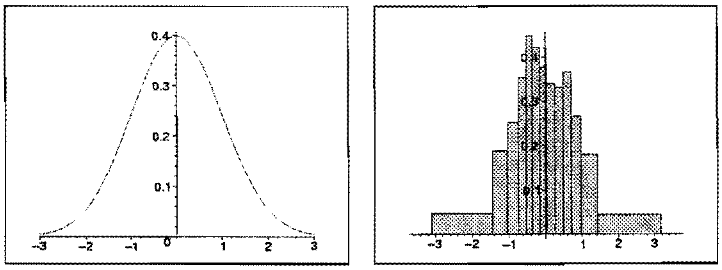
Continuing with our example we give the corresponding probability densities: The probability density of a Gaussian variable (zero mean and variance one) is given by

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \ x \in \mathbb{R}$$

The probability density is plotted in figure 1. We generated 1500 samples of a zero-mean Gaussian variable with variance one, plotted in the histogram in figure 1.

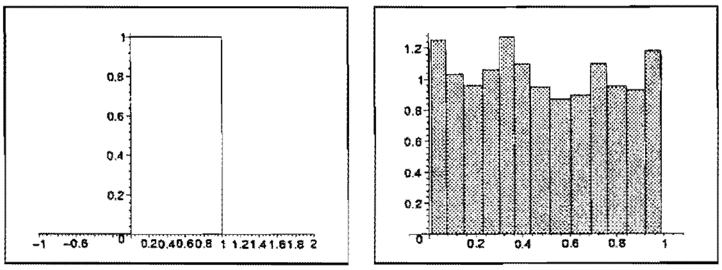
The probability density of a uniformly distributed variable on $[0,1]$ is given by

$$f_2(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \in \mathbb{R} \setminus [0,1] \end{cases}$$



(a) Gaussian density (b) histogram

Figure 1: Gaussian random variable



(a) Uniform density (b) histogram

Figure 2: Uniformly distributed random variable

The probability density is plotted in figure 2. We used formula (4) to generate 1500 samples of a uniformly distributed variable, presented in the histogram in figure 2.

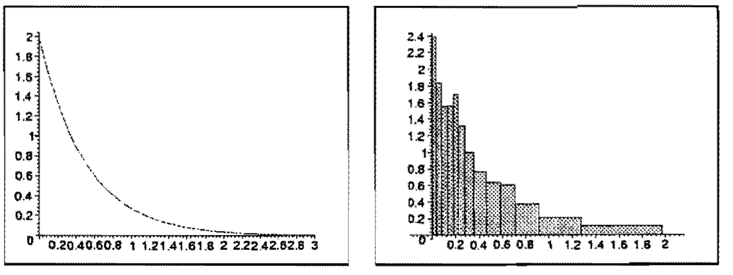
The probability density of a exponentially distributed process with parameter $\lambda = 2$ is given by

$$f_3(x) = \begin{cases} \lambda \exp(-\lambda x), \text{ with } \lambda = 2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The probability density is plotted in figure 3. Finally we used the expansion in formula (6) to generate 1500 samples of a exponentially distributed variable, presented in the histogram in figure 3.

If we want to use the sums in formula (4) and formula (6) we have to approximate them by their first couple of terms. By looking at these sums we see the convergence of the sum in formula (4) is quite good (since the denominator grows fast) while the convergence in formula (6) is quite bad (since the denominator grows slowly). In the first case we can approximate the sum in formula (4) by a few terms (say four) while in the second case we have to take more terms (say twenty).

Returning to the sde in formula (1). Suppose we have a random variable ξ which is Gaussian distributed with zero-mean and variance one. The solution of the sde has a polynomial chaos decomposition if the two conditions in section 2



(a) Exponential density (b) histogram

Figure 3: Exponentially distributed random variable

are met. In this section (theorem 1) we have seen that if one of the coefficients of the sde in formula (1) (say α) is a random variable with a continuous distribution F then it follows that $\alpha = F^{-1}(F_\xi(\xi))$ and α is defined on the same probability space as ξ is since ξ has a continuous distribution. In section 5 we will see that the solution $U(x,\omega)$ is a continuous function of ξ and thus of α. We will also see that the variance of $U(x,\omega)$ is finite implying that the solution $U(x,\omega)$ has a polynomial chaos decomposition. The main problem is determining the inverse of F, that is F^{-1}, this can only be done in a few cases. If for example the modulus E (see formula (1)) is Gaussian and the rest of the parameters (ρ and A) are constant then it's not possible to determine F^{-1}. Instead we will determine the necessary integrals (obtained in the solution process) in a direct way. We will compute them using asymptotic expansions which are briefly discussed in the following section.

4 Asymptotic expansions

In the previous section we looked under what conditions the polynomial chaos decomposition can be applied. In section this section we will see what happens if the modulus E is a Gaussian random variable and the rest of the parameters are deterministic. Practically it's not possible to express $1/E$ in a series of a Gaussian variable because we can't invert the corresponding probability distribution like we did in formula (6). However also in this case it's still possible to use the polynomial chaos decomposition to solve the sde in formula (1). This can be done using asymptotic expansions (asymptotic series). We say that a (finite) sum of functions f_j is an asymptotic expansion of a function g if this sum converge asymptotically to g, formally:

Definition 4. *asymptotic expansion*
Let $S_n(x) = \sum_{j=0}^{n} f_j(x)$ with $f_j(x) = a_j/x^j$. We say $S_n(x)$ that is an asymptotic expansion of g, denoted as $g(x) \sim S_n(x)$ if

$$\lim_{x\to\infty} x^n(g(x) - S_n(x)) = 0 \tag{7}$$

Before we look at an example we will state two lemmas:

Lemma 5. *If $g(x) \sim S_n(x)$ then it follows that*

$$\lim_{x\to\infty} x^j(g(x) - S_n(x)) = 0, \text{ for } j = 0\dots n$$

Proof. Suppose that $j \in \{0, \ldots, n-1\}$ Since $g(x) \sim S_n(x)$ it follows that $|x^n(g(x) - S_n(x))| \to 0$ and thus

$$|x^n(g(x) - S_n(x))| = |x^{n-j}||x^j(g(x) - S_n(x))| \to 0$$

Since $|x^{n-j}| \to \infty$ we have $|x^j(g(x) - S_n(x))| \to 0$. □

Lemma 6. *If $g(x) \sim S_n(x)$ then also $g(x) \sim S_j(x)$ for all $j < n$.*

Proof.

$$x^n(g(x) - S_n(x)) = x^n(g(x) - S_{n-1}(x)) - a_n$$

Since $x^n(g(x) - S_n(x)) \to 0$ it follows that $x^n(g(x) - S_{n-1}(x)) \to a_n$ and thus $x^{n-1}(g(x) - S_{n-1}(x)) \to 0$. If we repeat this $n-1$ times we end up with $g(x) - S_0(x) \to 0$. □

Typical behaviour of an asymptotic expansion is that the series $S_n(x)$ might diverge for $n \to \infty$ and x constant and finite. *The first lemma says (with $j = 0$) that the value $g(x)$ can be approximated by $S_n(x)$ with n small provided the mapping g depends on a large parameter x, or we can replace x by $1/a$ and the mapping must depend on a small parameter a. This approximation will become more accurate if we take n larger, however if we take n to large then the approximation will become less accurate and it will blow up to infinity.*

Let's look at an example as illustration. Suppose we want to determine the expectation of $1/E$ (with E a nonzero-mean Gaussian), the question is how can we use the asymptotic expansion to get an approximation of this expectation.

Example 7. *The expectation of $1/E$ is given by*

$$\mathbb{E}[1/E] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx \tag{8}$$

with $m = \mathbb{E}[E]$ and $\sigma^2 = \mathrm{Var}[E]$. Note that we must be a bit careful with this integral since this expectation doesn't exist in the usual sense because the integrand is singular at $x = 0$. But the integral in formula (8) *exists in the Cauchy principal value sense (see appendix B.2).*
In order to use the asymptotic expansion we need a small parameter a. As small parameter we take $a \equiv \sigma/m$ with m constant. It can be proven that (see appendix B.3)

$$\begin{aligned}\mathbb{E}[1/E] \sim \frac{1}{m}\sum_{j=0}^{n}(2j-1)!!\ a^{2j} &= \frac{1}{m}\sum_{j=0}^{n}\frac{(2j)!}{j!\ 2^j}a^{2j} \\ &= (1 + 1\cdot a^2 + 3\cdot 1\cdot a^4 + 5\cdot 3\cdot 1\cdot a^6 + \ldots)/m \end{aligned} \tag{9}$$

for $a \to 0+$ (a goes to zero from the positive side) and for any $n < \infty$. The double factorial $(.)!!$ is defined as $(2j-1)!! = (2j-1)\cdot(2j-3)\cdot\ldots\cdot 3\cdot 1$.

Proof. An heuristic proof (with ξ zero-mean Gaussian with variance one)

$$\mathbb{E}[\frac{1}{E}] = \mathbb{E}[\frac{1}{m+\sigma\xi}] = \frac{1}{m}\mathbb{E}[\frac{1}{1+a\xi}] = \frac{1}{m}\sum_{j=0}^{\infty} a^j \mathbb{E}[\xi^j] = \frac{1}{m}\sum_{j=0}^{\infty}(2j-1)!!\ a^{2j}$$

□

Suppose $a = 0.1$ and let's take $n = 0$ as approximation, how accurate is this approximation? For $n = 0$ we have

$$\mathbb{E}[1/E] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx \sim \frac{1}{m}$$

It is hardly a surprise that this will already be a good approximation since the Gaussian probability density has one sharp peak which is located around its mean m. This implies the following: The function given by $1/x$ is nearly constant around the mean m and thus we can replace $1/x$ by $1/m$ obtaining

$$\mathbb{E}[1/E] \approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{m} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx = \frac{1}{m} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx = \frac{1}{m}$$

The graphs of $\mathbb{E}[1/E]$ and of $1/m$ are plotted in figure 4 as function of m:

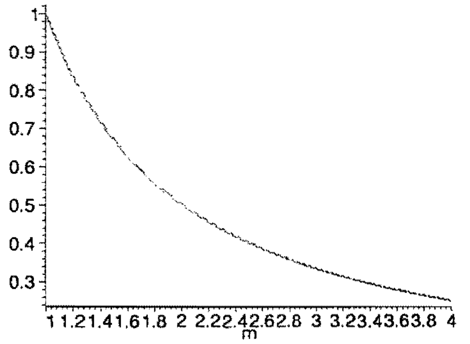


Figure 4: $\mathbb{E}[\frac{1}{E}]$ and $\frac{1}{m}$

As can be seen it turns out to be a very good approximation. This behaviour can be seen for all values of m not just the ones plotted. Figure 5 illustrates that the expansion will diverge for $n \to \infty$ when $a = 0.1$ (and $m = 1$). As can be seen

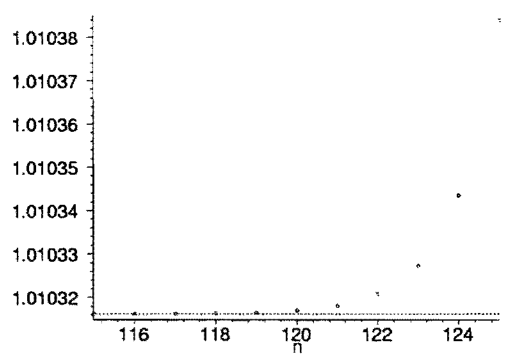


Figure 5: the expansion in formula (9)

the approximation is good for all $n \leq 118$, but blows up for larger n. We will *see in the following section how this can be used to solve the sde in formula* (1) *assuming that only the modulus E is random (and Gaussian).*

5 An example

The solution of the sde in formula (1) is given by

$$U(x,\omega) = \frac{\beta F \sin(x\omega\sqrt{\alpha})}{\omega\sqrt{\alpha}\cos(\omega\sqrt{\alpha})},\ \alpha = \frac{\rho}{E},\ \beta = \frac{1}{AE} \tag{10}$$

Suppose that one or more parameters ρ, E or A are random (Gaussian or uniformly) how can we use the polynomial chaos decomposition to solve the sde in formula (1). Some possibilities will be discussed below. The first one is well known and is treated in for example (Ghanem, 1999). I haven't seen any treatment about the other possibilities, so these results seem new to me.

1. Suppose that only ρ is a Gaussian random variable and the rest of the parameters are deterministic. In order to use the polynomial chaos decomposition we have to scale ρ to a Gaussian random variable with mean zero and variance one. Let

$$\xi = \frac{\rho - \mathbb{E}[\rho]}{\sqrt{\mathrm{Var}[\rho]}} \Rightarrow \rho = \xi\sqrt{\mathrm{Var}[\rho]} + \mathbb{E}[\rho]$$

   Then ξ is a zero-mean Gaussian random variable with variance one. As can be seen in formula (10) the displacement U is a continuous function of α, thus of ρ and thus of ξ, assuming that $\rho > 0$. This implies that U is defined on the same probability space as where ξ is defined (see appendix B.1). Since $\mathrm{Var}[U] < \infty$ (this is non-trivial, in fact is only finite in a certain sense, see appendix B.2) it follows that we can apply the polynomial chaos decomposition to U. Thus

$$U(x,\omega) = \sum_{j=0}^{\infty} g_j(x,\omega)h_j(\xi), \quad g_j(x,\omega) \in \mathbb{R} \tag{11}$$

   Substituting this into the sde in formula (1) gives

$$\begin{cases} \sum_{j=0}^{\infty} \frac{d^2}{dx^2} g_j(x,\omega)h_j(\xi) + \frac{\omega^2}{E}(\xi\sqrt{\mathrm{Var}[\rho]} + \mathbb{E}[\rho]) \sum_{j=0}^{\infty} g_j(x,\omega)h_j(\xi) = 0 \\ \sum_{j=0}^{\infty} g_j(0,\omega)h_j(\xi) = 0 \\ \sum_{j=0}^{\infty} \frac{d}{dx} g_j(1,\omega)h_j(\xi) = \beta F \end{cases} \tag{12}$$

   with $\frac{d}{dx}g_j(1,\omega) \equiv \frac{d}{dx}g_j(x,\omega)|_{x=1}$. From these equations we can determine $\{g_j(x,\omega)\}_j$ numerically or even analytically, but we won't expand on that here. (see for example in (DeBiesme, 2001) or (Ghanem, 1999))

2. Suppose that the density of the bar ρ is uniformly distributed instead of Gaussian distributed. Suppose we know for example that the density must be between two fixed values. In this case we can assume that it is

Gaussian distributed with an appropriate variance, but maybe it's better to assume that it's uniformly distributed with parameters y_{min}, y_{max}. It follows that the density is given by

$$f_\rho(x) = \begin{cases} \frac{1}{y_{max}-y_{min}} & x \in [y_{min}, y_{max}] \\ 0 & x \in \mathbb{R} \setminus [y_{min}, y_{max}] \end{cases}$$

Let

$$Z = \frac{\rho - y_{min}}{y_{max} - y_{min}}$$

One can check that Z is uniformly distributed on the interval $[0, 1]$. We can use formula (4) to express Z and thus ρ in an infinite polynomial in a zero-mean Gaussian variable ξ with variance one. We obtain

$$\rho = (y_{max} - y_{min})(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{\xi^{2j+1}}{(2j+1)2^j j!}) + y_{min} \qquad (13)$$

The density ρ is a continuous function of ξ (remember this sum is a Taylor series of ρ, this implies that ρ must certainly be differentiable and thus continuous), this implies that the solution is a continuous function of ξ. It follows that also in this case we can use the polynomial chaos decomposition to solve the sde in formula (1) when ρ is uniformly distributed. In practice we can take the first couple of terms (say three) of the sum in formula (13) as approximation (remember that the convergence of this sum is quite good as we noted at the end of section 3) and substitute this into the sde in formula (1).

3. Suppose that only the modulus E is a Gaussian random variable and the rest of the parameters are deterministic. In order to use the polynomial chaos decomposition we have to scale E to a Gaussian random variable with mean zero and variance one. Just like above we define

$$\xi = \frac{E - \mathbb{E}[E]}{\sqrt{\text{Var}[E]}} \Rightarrow E = \xi\sqrt{\text{Var}[E]} + \mathbb{E}[E] \equiv \xi\sigma + m$$

Then ξ is a zero-mean Gaussian random variable with variance one. Also in this case the displacement U is a continuous function of ξ and we can apply the polynomial chaos decomposition. The sde in formula (1) becomes

$$\begin{cases} \sum_{j=0}^{\infty} \frac{d^2}{dx^2} g_j(x,\omega) h_j(\xi) + \omega^2 \rho \frac{1}{\xi\sigma+m} \sum_{j=0}^{\infty} g_j(x,\omega) h_j(\xi) = 0 \\ \sum_{j=0}^{\infty} g_j(0,\omega) h_j(\xi) = 0 \\ \sum_{j=0}^{\infty} \frac{d}{dx} g_j(1,\omega) h_j(\xi) = \frac{1}{A} \frac{1}{\xi\sigma+m} F \end{cases} \qquad (14)$$

When want to apply the polynomial chaos decomposition we can do two things. The first thing is multiplying both sides with $\xi\sigma + m$ and then use the same procedure (as in (DeBiesme, 2001) or (Ghanem, 1999)) to determine the solution U, and the second thing is leaving this term on the right hand side. But in the latter case we need the following expectations (remember when solving this we have to multiply both sides with $h_k(\xi)$ and take the expectation):

$$\mathbb{E}[\frac{1}{E}h_j(\xi)h_k(\xi)] \text{ with } E = \xi\sigma + m$$

In section 4 we determined

$$\mathbb{E}[\frac{1}{E}h_0(\xi)h_0(\xi)] \approx 1/m \text{ with } h_0(\xi) = 1$$

We can use the same techniques to find the other expectations. Suppose we want to approximate

$$\mathbb{E}[\frac{1}{E}h_1(\xi)h_1(\xi)] \text{ with } h_1(\xi) = \xi$$

It follows that (with $E = \xi\sigma + m$ and thus $\xi = \frac{E-m}{\sigma}$)

$$\mathbb{E}[\frac{h_1(\xi)h_1(\xi)}{\sigma\xi + m}] = \mathbb{E}[\frac{1}{E}(\frac{E-m}{\sigma})^2] = \mathbb{E}[\frac{E}{\sigma^2} - 2\frac{m}{\sigma^2} + \frac{m^2}{E\sigma^2}] = -\frac{m}{\sigma^2} + \frac{m^2}{\sigma^2}\mathbb{E}[\frac{1}{E}] \tag{15}$$

In figure 6 three graphs are plotted, two of them are approximations (using formula (15) and formula (9)) of $\mathbb{E}[\frac{h_1(\xi)h_1(\xi)}{\sigma\xi+m}]$. The lowest one correspond to $n = 1$, the middle one to $n = 2$ and the third one to $\mathbb{E}[\frac{h_1(\xi)h_1(\xi)}{\sigma\xi+m}]$. The third one is hided behind the middle one. So the approximation with $n = 2$, that is

$$\mathbb{E}[\frac{1}{E}] \approx (1 + 1 \cdot a^2 + 3 \cdot 1 \cdot a^4)/m \text{ with } a = 0.1$$

is good enough.

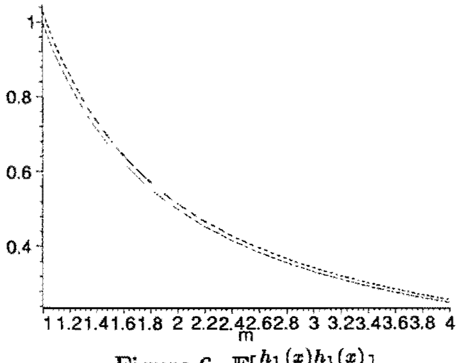


Figure 6: $\mathbb{E}[\frac{h_1(x)h_1(x)}{(mx+\sigma)}]$

If we use higher order Hermite polynomials we will need more terms in the asymptotic approximation. For example if we want to determine

$$\mathbb{E}[\frac{1}{E}h_4(\xi)h_4(\xi)] \text{ with } h_4(\xi) = \xi^4 - 6\xi^2 + 3 \tag{16}$$

we used seven terms to get the following approximation

$$\mathbb{E}[\frac{h_4(\xi)h_4(\xi)}{\sigma\xi + m}] = 24\frac{1}{m} + 216\frac{\sigma^2}{m^3} + 2952\frac{\sigma^4}{m^5} - 88695\frac{\sigma^6}{m^7} + 403515\frac{\sigma^8}{m^9} - 365715\frac{\sigma^{10}}{m^{11}} + 93555\frac{\sigma^{12}}{m^{13}}$$

In figure 7 are again three graphs plotted, two of them are approximations (using formula (16) and formula (9)) of $\mathbb{E}[\frac{h_1(\xi)h_1(\xi)}{\sigma\xi+m}]$. The lowest one correspond to $n = 6$, the middle one to $n = 7$ and the third one to $\mathbb{E}[\frac{h_1(\xi)h_1(\xi)}{\sigma\xi+m}]$. The third one is hided behind the middle one. So the approximation with $n = 7$, that is

$$\mathbb{E}[\frac{1}{E}] \approx \frac{1}{m}\sum_{j=0}^{7}(2j-1)!!\ a^{2j} \text{ with } a = 0.1$$

is good enough.

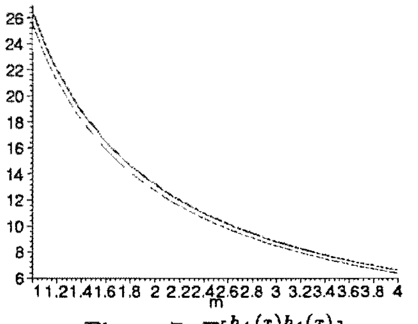


Figure 7: $\mathbb{E}[\frac{h_4(x)h_4(x)}{(mx+\sigma)}]$

4. Suppose that two parameters are (independent) Gaussian random variables, for example E and A. To use the polynomial chaos decomposition both random variables E and A have to be scaled to zero-mean Gaussians with variance one, say ξ_1 and ξ_2. In this case the displacement U is a continuous function of two random variables ξ_1 and ξ_2 and therefore we have to expand the displacement U using two-dimensional Hermite polynomials. Thus

$$U(x,\omega) = \sum_{j=0}^{\infty} g_j(x,\omega)h_j(\xi_1,\xi_2), \quad g_j \in \mathbb{R} \tag{17}$$

See appendix B.4 for determining these two-dimensional Hermite polynomials. Also in this case there are two ways of solving the sde in formula (1). The first thing is multiplying both sides with E (and the second boundary condition with E and A) and then use the same procedure (as in (DeBiesme, 2001) or (Ghanem, 1999)) to determine the solution U (this can also be done in case one (or both) of the parameters is uniformly distributed and the other one Gaussian), and the second thing leaving the term on the righthand side and determining the asymptotic expansions as in the previous case.

6 Conclusions

We obtained the following conclusions:

1. Returning to the sde in formula (1). It's not possible (at least we don't know how) to determine the solution $U(x, \omega)$ using the polynomial chaos decomposition given arbitrary probability densities of α and β. However for some choices of α and β it's possible to determine the mean and variance of the solution $U(x, \omega)$ using the polynomial chaos decomposition. We obtained solutions for the choices given in section 5. Basically if the parameters like E, A and ρ are Gaussian or Uniformly distributed we obtained the solution $U(x, \omega)$ using the polynomial chaos decomposition.

2. The polynomial chaos decomposition is very general. If the sde is linear and the coefficients have a continuous distribution which can explicitly be inverted, then the polynomial chaos decomposition can be applied (assuming that the variance of the solution will be finite, which is always the case when dealing with physical problems).

3. If there are n independent random parameters present in the sde then we need n dimensional Hermite polynomials to obtain the solution if we want to use the polynomial chaos decomposition. If one introduces more independent random parameters then it will take of course more computational time.

A Appendix

A.1 Properties of distribution function

Let Ω be a sample-space, Σ the σ-algebra of events on Ω, the function $\mathcal{P} : \Sigma \to [0, 1]$ is called a probability measure if the axioms of Kolmogorov are satisfied:

1. $\mathcal{P}(A) \geq 0$ for every $A \in \Sigma$
2. $\mathcal{P}(\Omega) = 1$
3. for every collection of disjunct events $A_1, A_2, \ldots$ we have $\mathcal{P}(\cup_n A_n) = \sum_n \mathcal{P}(A_n)$

Let $F : \mathbb{R} \to [0, 1]$ be a distribution function of a random variable X defined by $F(x) = \mathcal{P}(X \leq x)$. From the definition of probability measure given above the following properties follow:

1. F is right-continuous: $\lim_{h \to 0+} F(x + h) = F(x)$
   (h tends to zero from the right ...)
2. F is monotone increasing: $x \leq y \Rightarrow F(x) \leq F(y)$
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

Note that the distribution function of a random variable is always right-continuous, but not necessarily continuous (see formulation of theorem 3).

A.2 Borel-sets on $\mathbb{R}$

For the proof of theorem 3 we will need the notion of σ-algebra, Borel-set and a Σ-$\mathcal{B}$-measurable function:
A collection Σ of subsets of a set Ω is called a σ-algebra (in Ω) if:

1. $\Omega \in \Sigma$
2. $A \in \Sigma \Rightarrow A^c \in \Sigma$ with $A^c = \Omega \setminus A$
3. with a series $(A_j) \in \Sigma \Rightarrow \cup_{j=1}^{\infty} A_j \in \Sigma$

From these properties it follows that

1. $\emptyset \in \Sigma$
2. with a series $(A_j) \in \Sigma \Rightarrow \cap_{j=1}^{\infty} A_j \in \Sigma$

Four examples of σ-algebras:

Example 8. *The powerset*
Let $P(\Omega)$ be the powerset on Ω, that is $P(\Omega)$ consists of all the subsets of Ω. It's rather trivial that $P(\Omega)$ is a σ-algebra on Ω.

Example 9. *smallest σ-algebra*
There exists a smallest σ-algebra (denoted as $U(S)$) on a collection S of subsets of Ω with $S \subseteq U(S)$. That is for every σ-algebra U' on Ω with $S \subseteq U'$ we have $U(S) \subseteq U'$.

Proof. Let $U(S)$ be the intersection of all σ-algebras which contain S, thus $U(S) = \{\cap U' \mid S \subseteq U', U' \text{ a } \sigma\text{-algebra on } \Omega\}$. Using the definition of σ-algebra it can be shown that $U(S)$ is indeed a σ-algebra on Ω. This σ-algebra is by definition the smallest σ-algebra which contains S. $U(S)$ is called the σ-algebra generated by S and S the generator of $U(S)$. □

Example 10. *The Borel-σ-algebra $\mathcal{B}$ and Borel-set*
In the example above, let S be the set of all open subsets of $\Omega = \mathbb{R}$. Then as shown in the previous example there exists a smallest σ-algebra, denoted as $U(S)$, on Ω with $S \subseteq U(S)$. This σ-algebra $U(S)$ is called the Borel-σ-algebra on $\mathbb{R}$ and is denoted as $\mathcal{B}$. As a consequence of the properties of a σ-algebra it follows that $\mathcal{B}$ consists of all intervals on $\mathbb{R}$ (open, closed, half open, finite and infinite intervals) and countable unions, countable intersections and complements of those intervals. The members of $\mathcal{B}$ are called Borel-sets.

Definition 11. *Σ-$\mathcal{B}$-measurable mapping*
A mapping $X : \Omega \to \mathbb{R}$ is called a Σ-$\mathcal{B}$-measurable mapping if the following holds:

$$X^{-1}(B) \in \Sigma \text{ for all } B \in \mathcal{B}$$

Theorem 12. *Suppose we have a mapping $X : \Omega \to \mathbb{R}$ then we can always find a σ-algebra Σ such that X becomes a Σ-$\mathcal{B}$-measurable mapping. The σ-algebra $\sigma = X^{-1}(\mathcal{B})$ does the trick.*

Proof. First we will show that $X^{-1}(\mathcal{B})$ is actually a σ-algebra:

1. $\Omega \in X^{-1}(\mathcal{B})$ since $X(\Omega) \subseteq \mathbb{R}$ and $\mathbb{R} \in \mathcal{B}$
2. Suppose $A \in X^{-1}(\mathcal{B})$ then it follows that $A^c \in X^{-1}(\mathcal{B})$ since $(X^{-1}(B))^c = X^{-1}(B^c)$ for all $B \in \mathcal{B}$ with $B^c = \mathbb{R} \setminus B$
3. with a series $(A_j) \in X^{-1}(\mathcal{B}) \Rightarrow \cup_{j=1}^{\infty} A_j \in X^{-1}(\mathcal{B})$ since $\cup_j X^{-1}(B_j) = X^{-1}(\cup_j B_j)$ for all $\{B_j\}_j \in \mathcal{B}$

It follows that X becomes a $X^{-1}(\mathcal{B})$-$\mathcal{B}$-measurable mapping since $X^{-1}(B) \in X^{-1}(\mathcal{B})$ for all $B \in \mathcal{B}$. □

A.3 Generation of random variables

Recall that a random variable is defined as follows:

Definition 13. *Let $(\Omega, \Sigma, \mathcal{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ on this probability space is called a real-valued random variable if X is a Σ-$\mathcal{B}$-measurable function, which means that $\{X \in B\} \in \Sigma$ for every $B \subseteq \mathcal{B}$. Note that $\{X \in B\} \equiv \{s \in \Omega | X(s) \in B\} = X^{-1}(B)$. Where $\mathcal{B}$ is the Borel-σ-algebra on $\mathbb{R}$ consisting of the Borel-sets on $\mathbb{R}$ (see A.2).*

After stating this formal definition we can prove theorem 3: By assumption $X \in (\Omega, \Sigma, \mathcal{P})$ with Σ a σ-algebra on Ω. This implies that X is a Σ-$\mathcal{B}$-measurable function, which means that $\{X \in B\} \in \Sigma$ for every $B \in \mathcal{B}$ (see the definition above). First note that we defined a probability measure $\mathcal{P} : \Sigma \to [0, 1]$ (given a distribution function F_X) by

$$\mathcal{P}(X \in x) = F_X(x) \text{ with } \{X \in x\} \equiv \{s \in \Omega | X(s) \in]\infty, x]\} = X^{-1}(]\infty, x])$$

This is a well defined mapping since the sets $\{]-\infty, x]\}_x$ with $x \in \mathbb{R}$ generate the Borel-σ-algebra $\mathcal{B}$, (see (Rudin, 1987)). (The Borel-σ-algebra $\mathcal{B}$ are also generated by the open intervals, see example 10) Note that in general the σ-algebra $X^{-1}(\mathcal{B})$ isn't equal to Σ, but it's a sub-σ-algebra of Σ. Hence in theorem 1 we didn't prescribe the probability measure $\mathcal{P}$ fully, only it's restriction to the σ-algebra $X^{-1}(\mathcal{B})$.
We have to prove that $F^{-1} \circ F_X \circ X \in (\Omega, \Sigma, \mathcal{P})$. Note that $F^{-1} \circ F_X$ is the composition of two continuous functions hence continuous. It follows the composition $F^{-1} \circ F_X \circ X : \Omega \to [0, 1]$ is a Σ-$\mathcal{B}$-measurable function (see (Rudin, 1987), page 10), hence it is a random variable, which implies that $F^{-1} \circ F_X \circ X \in (\Omega, \Sigma, \mathcal{P})$.

A.4 Generation of a uniformly distributed variable

We derived that (see formula (5))

$$F_X(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)2^j j!} \tag{18}$$

Note this Taylor-series converges for all $x \in \mathbb{R}$: set $a_j = \frac{(-1)^j x^{2j+1}}{(2j+1)2^j j!}$ then

$$\lim_{j \to \infty} \frac{a_{j+1}}{a_j} = \lim_{j \to \infty} \frac{x^2 \cdot (2j+1)}{2(2j+3)(j+1)} = 0$$

Substituting $X(s)$ for x in the series in formula (18) gives

$$F_X(X(s)) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j X(s)^{2j+1}}{(2j+1)2^j j!}$$

since this holds for all $s \in \Omega$ it follows that

$$F_X(X) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j X^{2j+1}}{(2j+1)2^j j!}$$

B Appendix; Probability theory

B.1

See the first probability listed in section 5. The proof is similar as at the end of the proof of theorem A.3. U is a continuous function of ξ, say $U = f(\xi)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous. Suppose $\xi \in (\Omega, \Sigma, \mathcal{P})$ then ξ is a Σ-$\mathcal{B}$-measurable function, it follows that $f(\xi)$ and thus U is also a Σ-$\mathcal{B}$-measurable function which implies that $U \in (\Omega, \Sigma, \mathcal{P})$. Thus U is defined on the same probability space as ξ is.

B.2

Look at the integral in formula (8), that is the mean of $1/E$:

$$\mathbb{E}[1/E] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx$$

Note that $x = 0$ is a singularity of the integral, since the integrand is infinite for $x = 0$. It can be proven that this integral is ill-defined in the ordinary sense, meaning that

$$\mathbb{E}[1/E] =$$

$$\lim_{\epsilon_1,\epsilon_2 \to 0+} (\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{-\epsilon_1} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\epsilon_2}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx)$$

where ϵ_1 and ϵ_2 tend to zero from the right independently, doesn't exist. However it can be proven that the following integral is meaningful:

$$\lim_{\epsilon \to 0+} (\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{-\epsilon} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\epsilon}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx)$$

This is called the Cauchy principal value of the integral in formula (8). So we can define $\mathbb{E}[1/E]$ as this integral (assuming that all the usual properties of expectation hold for this generalization, see for example (Kolmogorov, 1956) for discussions about generalizations).

A bigger problem exists when looking at $\mathrm{Var}[U(x,\omega)]$, and also at $\mathbb{E}[U(x,\omega)]$. Look at the mean of U (the variance has exactly the same problem):

$$\mathbb{E}[U(x,\omega)] = \int_{-\infty}^{\infty} \frac{\beta F \sin(x\omega\sqrt{y})}{\omega\sqrt{y}\cos(\omega\sqrt{y})} f_\alpha(y) dy$$

with $\alpha = \frac{\rho}{E}$, $\beta = \frac{1}{AE}$ and f_α the probability density of α. Note that this integral is ill-defined in the ordinary sense since the integrand becomes infinite for $\omega\sqrt{y} = (k + 1/2)\pi$ with $k \in \mathbb{N}$ (and for $y = 0$). We think it's possible to make sense out of that integral (the main problem is that this integral has an infinite number of singularities (values where the integrand becomes infinite), but it decays to zero at infinity). This problem will disappear if one introduces damping. In this case the integrand has only imaginary (non-real) singular values.

B.3 Asymptotic expansion

We will prove formula (9). The expectation of $1/E$ is given by

$$\begin{aligned}
\mathbb{E}[1/E] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \exp(-\frac{(x-m)^2}{2\sigma^2}) dx \\
&= \exp(-\frac{m^2}{2\sigma^2}) \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \exp(-\frac{x^2}{2\sigma^2}) \exp(x\frac{m}{\sigma^2}) dx \\
&= \exp(-\frac{m^2}{2\sigma^2}) \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2\sigma^2}) x^{n-1} (\frac{m}{\sigma^2})^n \frac{1}{n!} dx \\
&= \exp(-\frac{m^2}{2\sigma^2}) \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2\sigma^2}) x^{n-1} (\frac{m}{\sigma^2})^n \frac{1}{n!} dx \\
&= \exp(-\frac{m^2}{2\sigma^2}) \sum_{n=1}^{\infty} (\frac{m}{\sigma^2})^n \frac{\sigma^{n-1}}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{(x/\sigma)^2}{2}) (x/\sigma)^{n-1} d(x/\sigma) \\
&= \exp(-\frac{m^2}{2\sigma^2}) \sum_{j=0}^{\infty} \frac{m^{2j+1}}{2^j \sigma^{2j+2} (2j+1) j!} = \sqrt{2} \exp(-\frac{m^2}{2\sigma^2}) \int_{t=0}^{\frac{m}{\sigma\sqrt{2}}} \exp(t^2) dt
\end{aligned}$$

Perform a change of variables $t = \frac{m}{\sigma\sqrt{2}} \cdot \sqrt{1-z}$ to obtain

$$\mathbb{E}[1/E] = \frac{\sqrt{2}}{2} \frac{m}{\sigma\sqrt{2}} \int_{t=0}^{1} \exp(-\frac{m}{\sigma\sqrt{2}} z) \sqrt{1-z} dz$$

Watson's lemma (a generalization of it) (see (Wong, 1989), page 22 or theorem 1 page 58) basically says change the top endpoint to infinity and integrate term by term to obtain the asymptotic expansion. It follows that

$$\mathbb{E}[1/E] \sim \frac{1}{m} \sum_{j=0}^{n} (2j-1)!!\, a^{2j} = \frac{1}{m} \sum_{j=0}^{n} \frac{(2j)!}{j!\, 2^j} a^{2j}$$

as $a = \frac{\sigma}{m} \to 0$.

B.4 Multi-dimensional Hermite polynomials

See the last probability in section 5. We will show how to derive the multi-dimensional Hermite polynomials. See for example (Ghanem, 1999).
Let $\{\xi_j\}_{j=1\dots n}$ be zero-mean uncorrelated Gaussians with variance one and let $N = \{1, 2, \dots, n\}$ (corresponding to the indices of the Gaussians). Let G be the set of all $\mathbb{N}$-valued functions on N (thus $G = \{p \mid p : N \to \mathbb{N}\}$) and let

$$|p| = \sum_{k \in N} p(k) = \sum_{k=1}^{n} p(k)$$

with $p \in G$. Let $p \in G$ with $|p| = m$ then the corresponding n-dimensional Hermite polynomials of order m are given by

$$h_p(\xi_1, \dots, \xi_n)|_{|p|=m} = \prod_{j=1}^{n} h_{p(j)}(\xi_j)$$

Let's determine the first three orders ($m = 0, 1, 2$) to understand this definition (say two dimensional, thus $n = 2$):
Let $m = 0$, we have to determine all mappings p with $p \in G$ and $|p| = 0$ thus all $p : \{1, 2\} \to \mathbb{N}$ with $\sum_{k=1}^{2} p(k) = 0$. The latter means that $p(k) = 0$ for all $k \in \{1, 2\}$. There's one such mapping, it's the mapping $p_1 : \{1, 2\} \to \mathbb{N}$ given by $p_1(1) = 0$ and $p_1(2) = 0$. It follows that

$$h_{p_1}(\xi_1, \xi_2) = \prod_{j=1}^{2} h_{p_1(j)}(\xi_j) = h_0(\xi_1)h_0(\xi_2) = 1$$

Let $m = 1$, we have to determine all mappings p with $p \in G$ and $|p| = 1$ thus all $p : \{1, 2\} \to \mathbb{N}$ with $\sum_{k=1}^{2} p(k) = 1$. There are two such mappings, the first one is given by $p_2(1) = 1$, $p_2(2) = 0$ and the second one is given by $p_3(1) = 0$, $p_3(2) = 1$. It follows that

$$h_{p_2}(\xi_1, \xi_2) = \prod_{j=1}^{2} h_{p_2(j)}(\xi_j) = h_1(\xi_1)h_0(\xi_2) = \xi_1$$

and

$$h_{p_3}(\xi_1, \xi_2) = \prod_{j=1}^{2} h_{p_3(j)}(\xi_j) = h_0(\xi_1)h_1(\xi_2) = \xi_2$$

Let $m = 2$, then there are three mappings given by

$$h_{p_4}(\xi_1, \xi_2) = \xi_1^2 - 1$$
$$h_{p_5}(\xi_1, \xi_2) = \xi_1\xi_2$$
$$h_{p_6}(\xi_1, \xi_2) = \xi_2^2 - 1$$

In general there are $\binom{m+n-1}{n-1}$ mappings with $p \in G$ and $|p| = m$. Let $\{\xi_j\}_{j=1\ldots n}$ be a basis of a Gaussian Hilbert space H. It turns out that the set $\{h_p\}|_{p \in G}$ forms an orthogonal basis of $L^2(\Omega, \Sigma(H), \mathcal{P})$ with

$$(h_p(\xi_1, \ldots, \xi_n), h_q(\xi_1, \ldots, \xi_n)) = E[h_p(\xi_1, \ldots, \xi_n)h_q(\xi_1, \ldots, \xi_n)] = p!\ \delta_{p,q}$$

where $p! \equiv \prod_{k=1}^{n}(p(k))!$ and $\Sigma(H)$ the smallest σ-algebra on Ω such that the mappings $\{\xi_j\}_{j=1\ldots n}$ are all measurable.

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