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1. Introduction and summary

1.1. Landau's well-known inequality (cf.[5]) for twice differentiable functions may be put in the following form: if f and f" are bounded on $\mathbb{R}$ then $\|f'\| \le 2^{\frac{1}{2}}(\|f\|\|f''\|)^{\frac{1}{2}}$; here, and throughout this paper, $\|\cdot\|$ denotes the supremum norm. Landau's inequality is *best possible*, i.e., the constant $2^{\frac{1}{2}}$ cannot be replaced by a smaller one. Around 1939 Kolmogorov [4] obtained similar best possible inequalities connecting $\|f\|, \|f^{(n)}\|, \|f^{(k)}\|$ $(1 \le k \le n-1)$. The analogous problem for periodic functions has been dealt with by Northcott [7].

It is interesting to note that the extremal functions, i.e., the functions for which the inequalities above turn into equalities are the same for both problems; these extremal functions are the Euler splines. Cavaretta [1], who gave an elementary proof of Kolmogorov's inequalities by first establishing them for periodic functions, showed that Euler splines also maximize the functional $\|f^{(k+1)} + \alpha f^{(k)}\|$, for any $\alpha \in \mathbb{R}$ and for $0 \le k \le n-2$, on the set of functions f with prescribed upper bounds for $\|f\|$ and $\|f^{(n)}\|$ $(n \ge 2)$.

1.2. As the main result of this paper we show that the so-called Euler $\mathcal{L}$-splines are extremal with respect to a rather general class of differential operators defined on the set of periodic functions.

Preliminary material is collected in section 2. Section 3 contains a proof of the main result and an example.

2. Preliminary notions and results

2.1. By $W^{(n)}$ we denote the set of functions f having an absolutely continuous (n-1)-st derivative $f^{(n-1)}$ on every compact subinterval of $\mathbb{R}$ and a (Radon-Nikodym) derivative $f^{(n)}$ that is essentially bounded on $\mathbb{R}$, i.e., $f^{(n)} \in L_\infty(\mathbb{R})$. For a given period $T > 0$ the set $W_T^{(n)}$ is then defined by

$$W_T^{(n)} = \{f \in W^{(n)} \mid f(t+T) = f(t),\ t \in \mathbb{R}\}.$$

Let D be the ordinary differentiation operator and let p_n be a polynomial of degree n, then the corresponding differential operator of order n is denoted by $p_n(D)$, $D^0 = I$.

Let h be a positive number and let p_n be a monic polynomial of degree n. If a function s satisfies the conditions

$$(2.1)\qquad \begin{cases} s \in W^{(n)} \\ p_n(D)\, s(t) = -1 & (0 < t < h) \\ s(t+h) = -s(t) & (t \in \mathbb{R}), \end{cases}$$

then s is called an *Euler $\mathcal{L}$-spline* corresponding to the operator $p_n(D)$ and with mesh distance h. It can be shown that s is uniquely determined by (2.1) if p_n has only real zeros; in this case s will be denoted by $E(p_n,h,\cdot)$.

2.2. Let p_n $(n \geq 2)$ be a monic polynomial of degree n having only real zeros. Furthermore, let de function P_n be defined by means of its Fourier series with period T, i.e., let

$$(2.2) \qquad P_n(t) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} p_n^{-1}(i\omega j)\, e^{i\omega jt} \qquad (t \in \mathbb{R}) ,$$

where $\omega = 2\pi/T$. Then $P_n \in W_T^{(n-1)}$ and (cf. Ter Morsche [6,p. 137-138]) P_n can be written in the form

$$(2.3) \qquad P_n(t) = \frac{T}{2\pi i} \oint_C \frac{e^{tz}}{(1-e^{Tz})p_n(z)}\, dz \qquad (0 \leq t \leq T),$$

where C is a closed contour in the complex plane including the origin and the zeros of p_n, but excluding the points $z = i\omega j$ $(j = \pm 1, \pm 2, \ldots)$. It immediately follows from (2.3) that

$$(2.4) \qquad p_n(D)P_n(t) = -1 \qquad (0 < t < T).$$

Let p_k $(0 \leq k \leq n-2)$ be a monic polynomial of degree k that divides p_n. We now introduce $P_{n,k}$ defined by $P_{n,k} = p_k(D)P_n$; on account of (2.2), $P_{n,k}$ corresponds to $p_{n,k} := p_k^{-1} p_n$ in the same way as P_n corresponds to p_n in (2.2).

A representation formula for the elements of the set $W_T^{(n)}$ is given in the following lemma.

Lemma 2.1. If $f \in W_T^{(n)}$ then

$$(2.5) \qquad f(t) = T^{-1}\int_0^T f(\tau)d\tau + T^{-1}\int_0^T P_n(t-\tau)p_n(D)f(\tau)d\tau \qquad (t \in \mathbb{R}) .$$

For a proof of this lemma the reader is referred to Golomb [3] or Ter Morsche [6, Lemma 6.3.1].

2.3. In Section 3.1 we need an estimate on the number of zeros of various derivatives of P_n in the interval $(0,T]$. The following lemma is used for that purpose. Here $\mathrm{Ker}(p_n)$ denotes the kernel of $p_n(D)$, i.e., the set of real-valued functions f for which $p_n(D)f(t) = 0$ $(t \in \mathbb{R})$. By $Z_f(J)$ we denote the number of zeros of f in the set J, counting multiplicities.

Lemma 2.2. Let p_n be a monic polynomial of degree n having only real zeros, and let r be a nonnegative integer. Furthermore, let $f \not\equiv 0$ have the properties

$$(2.6) \qquad \begin{cases} \text{(i)} \quad f \in \mathrm{Ker}(p_n) \\ \\ \text{(ii)} \quad f^{(j)}(0) = f^{(j)}(T) \qquad (j = 0,1,\dots,n-r-1) . \end{cases}$$

Then

$$(2.7) \qquad Z_f((0,T]) \le \begin{cases} r-1 & (r \text{ odd}) \\ r & (r \text{ even}). \end{cases}$$

Proof. We distinguish between the cases $r \ge n$ and $0 \le r < n$. If $r \ge n$ then condition (ii) of (2.6) is void. Since p_n only has real zeros, a nontrivial function $f \in \mathrm{Ker}(p_n)$ has at most n-1 zeros in $\mathbb{R}$, and inequality (2.7) obviously holds. Now let $0 \le r < n$, and let $q \not\equiv 0$ be a continuously differentiable function satisfying $q(0) = q(T)$, $q'(0) = q'(T)$. Then for any $\lambda \in \mathbb{R}$

$$(2.8) \qquad Z_q((0,T]) \le Z_{q'-\lambda q}((0,T]) .$$

This inequality may be verified by writing

$$q'(t) - \lambda q(t) = e^{\lambda t} \frac{d}{dt}\left(e^{-\lambda t} q(t)\right)$$

and using Rolle's theorem. We note that $Z_q((0,T])$ is even if $q(0) \neq 0$. Denoting the zeros of p_n by $\alpha_1,\alpha_2,\ldots,\alpha_n$, we introduce the polynomials p_{r+1} and p_{n-r-1} defined by

$$p_{r+1}(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{r+1}) ,$$

$$p_{n-r-1}(x) = (x - \alpha_{r+2})(x - \alpha_{r+3})\ldots(x - \alpha_n) .$$

Then $g := p_{n-r-1}(D)f \in \mathrm{Ker}(p_{r+1})$ and in view of (ii) of (2.6) we conclude that $g(0) = g(T)$. We proceed by first assuming that $g \not\equiv 0$. As p_{r+1} has only real zeros it follows that $Z_g(J) \leq r$ for any set $J \subset \mathbb{R}$. Since

$$g(t) = e^{\alpha_{r+2}t} \frac{d}{dt} e^{-\alpha_{r+2}t} e^{\alpha_{r+3}t} \frac{d}{dt} e^{-\alpha_{r+3}t} \ldots e^{\alpha_n t} \frac{d}{dt} e^{-\alpha_n t} f(t),$$

repeated application of (2.8) yields

$$Z_f((0,T]) \leq Z_g((0,T]) .$$

Hence, $Z_f((0,T]) \leq r$. If $Z_f((0,T]) = r$ then obviously $Z_g((0,T]) = r$. It follows that $g(0) \neq 0$ and therefore that r is even, since otherwise one would have $Z_g((0,T]) > r$. This proves (2.7) in case $g \not\equiv 0$. It remains to consider $g \equiv 0$. Then $f \in \mathrm{Ker}(p_{n-r-1})$ and in view of (ii) of (2.6) f is periodic. If $p_{n-r-1}(0) \neq 0$ then $f \equiv 0$, contradicting the hypotheses of the lemma; however, if $p_{n-r-1}(0) = 0$ then f is a nonzero constant function for which (2.7) clearly holds. This proves the lemma. □

2.4. In order to formulate the next lemma we need the following definition.

<u>Definition 2.3.</u>

$$U = \{u \in L_\infty([0,T]) \,|\, \|u\| \leq 1, \int_0^T u(\tau)d\tau = 0\} .$$

Lemma 2.4. Let g be an arbitrary real-valued nonconstant analytic function defined on [0,T]. Then there is a uniquely determined real constant c_0 such that

$$(2.9)\qquad \max_{u\in U} \int_0^T g(\tau)u(\tau)d\tau = \int_0^T |g(\tau) - c_0|d\tau .$$

Moreover, functions $u \in U$ for which this maximum is attained are given by

$$(2.10)\qquad u(t) = \mathrm{sgn}(g(t) - c_0) \qquad \text{(a.e. on } [0,T]).$$

Proof. For every $u \in U$ and $c \in \mathbb{R}$ one has

$$\int_0^T g(\tau)u(\tau)d\tau = \int_0^T (g(\tau) - c)u(\tau)d\tau \le \int_0^T |g(\tau) - c|d\tau .$$

Hence

$$\int_0^T g(\tau)u(\tau)d\tau \le \min_{c\in\mathbb{R}} \int_0^T |g(\tau) - c|d\tau .$$

So the L_1-distance of g to the set of constant functions has to be determined. Since by assumption g is a real-valued nonconstant analytic function, it coincides with any constant c in at most finitely many points of [0,T]. According to a well-known characterization theorem for L_1-approximation (cf. Cheney [2,p.220]), the best approximation c_0 to g is uniquely determined by

$$(2.11)\qquad \int_0^T \mathrm{sgn}(g(\tau) - c_0)d\tau = 0.$$

Formula (2.9) now immediately follows by taking $u(t) = \mathrm{sgn}(g(t) - c_0)$. With respect to the second assertion of the lemma we note that for functions $u \in U$ the equality

$$\int_0^T (g(\tau) - c_0)u(\tau)d\tau = \int_0^T |g(\tau) - c_0|d\tau$$

holds if and only if u is given by (2.10). □

3. An extremal property of Euler $\mathcal{L}$-splines

3.1. Our main result is the following theorem.

Theorem 3.1. Let $p_n (n \geq 2)$ be a monic polynomial of degree n having only real zeros with $p_n(0) = 0$. Furthermore, let p_k $(0 \leq k \leq n-2)$ be a monic polynomial of degree k that divides p_n. Then the following two inequalities hold:

(i) if $p_k(0) = 0$ then for all $\alpha \in \mathbb{R}$ and all $f \in W_T^{(n)}$

$$\|p_k(D)(D+\alpha I)f\| \leq \|p_k(D)(D+\alpha I)E(p_n,T/2,\cdot)\| \; \|p_n(D)f\| ; \tag{3.1}$$

(ii) if $p_k(0) \neq 0$ then for all $f \in W_T^{(n)}$

$$\|p_k(D)Df\| \leq \|p_k(D)\, D\, E(p_n,T/2,\cdot)\| \; \|p_n(D)f\| . \tag{3.2}$$

Moreover, equality in (3.1) or (3.2) holds if and only if $\beta \in \mathbb{R}$ and $\xi \in (0,T]$ exist such that

$$f(t) = \beta E(p_n,T/2,t-\xi) \quad (t \in \mathbb{R}) .$$

Proof. Without loss of generality we may assume that $\|p_n(D)f\| \leq 1$. Accordingly, define

$$\overline{W}_T^{(n)} = \{f \in W_T^{(n)} \,|\, \|p_n(D)f\| \leq 1\}.$$

In order to prove (3.1) and (3.2) one has to determine

$$(3.3)\qquad \sup_{f\in \overline{W}_T^{(n)}} \| p_k(D)(D+\alpha I)f \| ,$$

with $\alpha = 0$ in case $p_k(0) \neq 0$. As the set $W_T^{(n)}$ is invariant under translation of arguments, (3.3) equals

$$(3.4)\qquad \sup_{f\in \overline{W}_T^{(n)}} |p_k(D)(D+\alpha I)f(T)| .$$

Applying $p_k(D)(D + \alpha I)$ to (2.5) and putting $t = T$, for any $f \in \overline{W}_T^{(n)}$ we obtain the relation

$$(3.5)\qquad p_k(D)(D+\alpha I)f(T) = T^{-1}\int_0^T G(T-\tau)p_n(D)f(\tau)d\tau ,$$

where G is given by (cf. p.3)

$$(3.6)\qquad G(t) = p_k(D)(D+\alpha I)P_n(t) = (D+\alpha I)P_{n,k}(t) .$$

Since $p_n(0) = 0$ one has $\int_0^T p_n(D)f(\tau)d\tau = 0$; this, together with $\|p_n(D)f\| \leq 1$, implies that $p_n(D)f \in U$. By (3.5) and on account of Definition 2.3 it follows that

$$(3.7)\qquad \sup_{f\in \overline{W}_T^{(n)}} \| p_k(D)(D+\alpha I)f \| = \max_{u\in U} T^{-1}\int_0^T G(T-\tau)u(\tau)d\tau .$$

Because of (2.4), G satisfies the differential equation

$$p_{n,k}(D)G(t) = -\alpha \qquad (0 < t < T)$$

and thus coincides with an analytic function on $(0,T)$. Moreover, G is not constant since (cf.(2.2)) otherwise $(i\omega j + \alpha)p_k(i\omega j)$ would be zero for all $j = 0,\pm 1,\pm 2,\ldots$, which cannot occur since by assumption $p_k \not\equiv 0$. Consequently, we may apply Lemma 2.4 to (3.7). This yields a constant

c_0 uniquely determined by (cf.(2.11))

$$\int_0^T \operatorname{sgn}(G(T-\tau) - c_0)\,d\tau = 0.$$

Let $H(t) := G(t) - c_0$, then H satisfies the differential equation

$$D\, p_{n,k}(D)H(t) = 0 \qquad (o < t < T)\ .$$

Moreover, $H^{(j)}(0) = H^{(j)}(T) \quad (j = 0,1,\ldots,n-k-3)$. In view of Lemma 2.2 one has $Z_H((0,T]) \leq 2$. Since $\int_0^T \operatorname{sgn} H(\tau)d\tau = 0$ it follows that either H has precisely one zero in (0,T) located at T/2, or H has precisely two zeros in (0,T) a distance T/2 apart. In any case H has equidistant zeros in $\mathbb{R}$ with distance T/2. These observations ascertain that a function $f \in \overline{W}_T^{(n)}$ yielding the supremum in (3.4) has the property $p_n(D)f(t) = \operatorname{sgn}(H(T-t))$. Moreover, any function yielding the supremum in (3.3) satisfies the differential equation

$$p_n(D)f(t) = \operatorname{sgn}(H(\eta - t)) \qquad (t \in \mathbb{R})$$

for some $\eta \in (0,T]$. Taking into account the definition of the Euler $\mathcal{L}$-splines (cf.p.2), we conclude that an extremal function f has the form

$$f(t) = \beta\, E(p_n, T/2, t-\xi)$$

for some $\beta \in \mathbb{R}$ and some $\xi \in (0,T]$, i.e., it is an appropriate multiple of an Euler $\mathcal{L}$-spline. This completes the proof of Theorem 3.1. □

<u>Remark.</u> If in case (ii) we take in particular $p_n(D) = D^n$ and $k = 0$, then (3.2) implies Northcott's theorem. We further note that results similar to Theorem 3.1 have been derived by Golomb [3] for specific subsets of

$W_T^{(n)}$ and for specific functionals.

3.2. As an application of Theorem 3.1 we consider the following example.

Example. Given $n \in \mathbb{N}$ and $\gamma > 0$ let

$$(3.8) \qquad p_{2n+1}(D) = D(D^2 - \gamma^2 I)(D^2-(2\gamma)^2 I)\dots(D^2 -(n\gamma)^2 I) .$$

According to (3.1) one has, taking $p_k(D) = D$ and $\alpha = 0$,

$$\| f'' \| \le \| E''(p_{2n+1}, T/2, \cdot) \| \, \| p_{2n+1}(D) f \| \qquad (f \in W_T^{(2n+1)}).$$

Applying Formula 3.2.30 in Ter Morsche [6, p.67], we obtain by elementary calculations

$$(3.9) \quad E(p_{2n+1}, T/2, t) = \frac{(-1)^{n+1}}{(n!)^2 \gamma^{2n}}(t-T/4) - \frac{2}{\gamma^{2n+1}} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{\sinh((t-T/4)k\gamma)}{\cosh(k\gamma T/4)} ,$$

where $0 \le t \le T/2$.

A careful count of the zeros of $E'''(p_{2n+1}, T/2, \cdot)$ shows that on $[0, T/2]$ this derivative only vanishes at the end-points of $[0, T/2]$. So $|E''(p_{2n+1}, T/2, \cdot)|$ attains its maximum at $t = 0$, and using (3.9) we get

$$(3.10) \qquad \| E''(p_{2n+1}, T/2, \cdot) \| = \frac{2}{\gamma^{2n-1}} \left| \sum_{k=1}^{\infty} \frac{(-1)^{n-k} k \tanh(k\gamma T/4)}{(n+k)!(n-k)!} \right| =$$

$$= \frac{1}{(2n)!\gamma^{2n-1}} \left| \sum_{k=0}^{2n} (-1)^k (n-k) \binom{2n}{k} \tanh((n-k)\gamma T/4) \right| .$$

As is apparent from (3.8) the polynomial case $p_{2n+1}(D) = D^{2n+1}$ is obtained by letting $\gamma \downarrow 0$. In order to evaluate (3.10) for $\gamma \downarrow 0$ we use the identities

$$(3.11) \qquad \sum_{k=0}^{2n} (-1)^k (n-k)^{2j} \binom{2n}{k} = (2n)! \, \delta_{j,n} \qquad (j = 0,1,2,\dots,n) ,$$

which are easily verified.

For small x let $\tanh x = \sum_{j=1}^{\infty} c_j x^{2j-1}$. Then for sufficiently small γ

$$\sum_{k=0}^{2n} (-1)^k (n-k)\binom{2n}{k} \tanh(n-k)\gamma T/4) = \sum_{j=1}^{\infty} c_j (\tfrac{T}{4})^{2j-1} \gamma^{2j-1} \sum_{k=0}^{2n} (-1)^k (n-k)^{2j} \binom{2n}{k}.$$

In view of (3.10) and (3.11) we conclude that

$$\lim_{\gamma \downarrow 0} \| E''(p_{2n+1}, T/2, \cdot) \| = |c_n| (T/4)^{2n-1} .$$

By the residue theorem

$$c_n = \frac{1}{2\pi i} \oint_C \frac{\tanh(z)}{z^{2n}} dz ,$$

C being a closed contour including $z = 0$, but excluding the poles of $\tanh(z)$. Since the sum of all residues of $\tanh(z)/z^{2n}$ is zero, it follows that

$$c_n = \frac{-2}{\pi^{2n}} \sum_{j=0}^{\infty} (j+\tfrac{1}{2})^{-2n} .$$

Consequently

$$\lim_{\gamma \downarrow 0} \| E''(p_{2n+1}, T/2, \cdot) \| = \frac{8}{T} (T/2\pi)^{2n} \sum_{j=0}^{\infty} (2j+1)^{-2n} .$$

Taking $T = 2\pi$ we obtain

$$\| f'' \| \le \frac{4}{\pi} \| f^{(2n+1)} \| \sum_{j=0}^{\infty} (2j+1)^{-2n} \qquad \left(f \in W_{2\pi}^{(2n+1)} \right) ,$$

which agrees with Northcott's theorem.

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