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Abstract

Monotone interpretations in the natural numbers provide a useful technique for proving termination of term rewriting systems. Termination proofs of this shape imply upper bounds on derivation lengths expressed in bounds on the interpretations. For a hierarchy of classes of interpretations we describe these upper bounds, among which the doubly exponential upper bound for polynomial interpretations, found by Hofbauer and Lautemann. By examples of term rewriting systems we show that all of these upper bounds are sharp in a natural sense. In particular we find all Ackermann functions as sharp upper bounds on derivation lengths, exhausting primitive recursive functions.

1 Introduction

A natural way to prove termination (strong normalization) of a term rewriting system (TRS) is by mapping terms to natural numbers in a compositional way, such that doing any rewrite step causes a decrease of the corresponding value. Since no infinite descending chains exist in the naturals, this proves that the TRS is terminating. Moreover, such a termination proof yields an upper bound on the length of any derivation starting with a particular term: the length is bounded by the interpretation of the starting term. A standard technique is the interpretation of the function symbols by polynomials, see [5, 1]. For polynomials it is well-known that the derivation length of a term is bounded by a function doubly exponential in the size of the term; this bound is sharp ([6, 4]). In this paper we give a simpler proof of this result.

However, we do not restrict to polynomials. The same doubly exponential upper bound holds for any interpretation for which there is a number a such that the diagonals of the interpretations of the function symbols are all bounded by $x \mapsto x^a$. In this paper this result is generalized as follows. Given a set of functions U, which we use as upper bound functions, we consider all TRS's that can be proved to terminate using interpretation

functions which are limited by some function in U. For these TRS's we derive a universal upper bound on their derivation lengths, the maximum number of rewrite steps that can be made, expressed in the height of the start term. For a hierarchy of interesting sets of these upper bound functions, upper bounds on the derivation lengths of the corresponding TRS's are obtained by applying this universal upper bound.

In this hierarchy all of these upper bounds on derivation lengths turn out to be sharp, which is shown by giving example TRS's and appropriate derivations. More precisely, for every set of functions U in the hierarchy we give a TRS having the following properties. On the one hand termination is proved by choosing interpretations not exceeding some u in U. On the other hand we construct derivations parametrized by a number n for which the length has the same order as the upper bound described above and the height of the start term is linear in n.

For example, for *elementary interpretations* as described in [7] we prove that the length of a derivation starting with a term of height n is bounded by $\exp^{c*n}(0)$, for some constant c, while by an example we show that this bound is indeed reached for some $c > 0$. Here exp is defined by $\exp^0(k) = k, \exp^{n+1}(k) = 2^{\exp^n(k)}$.

For any $i \geq 0$ we describe a set of upper bound functions such that the corresponding upper bound on the derivation length is $exp^i(n)$, where n is the height of the starting term. Similarly, for any i the function A_i occurs as an upper bound on derivation lengths, where A_i is the Ackermann function. Finally, we have a similar result in which the diagonal of the Ackermann function appears, which is not primitive recursive any more. For all of these cases we present TRS's and corresponding derivations proving sharpness of the corresponding upper bounds.

2   Termination by interpretation

Let $\mathcal{F}$ be a set of function symbols, each having some fixed arity ≥ 0, and let $\mathcal{X}$ be a set of variable symbols. The set of terms over $\mathcal{F}$ and $\mathcal{X}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Instead of $\mathcal{T}(\mathcal{F}, \emptyset)$ we write $\mathcal{T}(\mathcal{F})$, the set of *ground terms*, terms without variables. In this paper we assume that there is at least one function symbol with arity 0, hence $\mathcal{T}(\mathcal{F})$ is not empty. Investigating derivation lengths of terms, only ground terms will be considered. Changing a variable in a term into a constant symbol however, does not affect its height, nor its derivation tree. Therefore all our results for ground terms also hold for terms with variables.

In this paper, we will use an interpretation τ of a TRS R in the natural numbers $\mathbb{N}$ to prove termination of R. The precise definition is the following.

Definition 2.1 *Let R be a TRS over $\mathcal{F}$. Let $A = \{n \in \mathbb{N} | n \geq k\}$ for some $k \in \mathbb{N}$. For all $i \in \mathbb{N}$ let*

$$H_i = \{f : A^i \to A | f \text{ is strictly increasing in all } i \text{ coordinates}\}.$$

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Then an interpretation *for R is a function* $\tau : \mathcal{F} \to \cup_{i \in \mathbb{N}} H_i$ *such that* $\tau(F) \in H_{\mathrm{arity}(F)}$ *for all* $F \in \mathcal{F}$.

Often, we write F_τ for $\tau(F)$.

This τ induces a function $\mathrm{int}_\tau : \mathcal{T}(\mathcal{F}) \to \mathbb{N}$, which maps every ground term to a natural number. The int_τ of any ground term is recursively defined as follows:

- $\mathrm{int}_\tau(F) = F_\tau$ if F is a function symbol of arity 0

- $\mathrm{int}_\tau(F(t_1, \ldots, t_n)) = F_\tau(\mathrm{int}_\tau(t_1), \ldots, \mathrm{int}_\tau(t_n))$ if F has arity n

Definition 2.2 *An interpretation τ is compatible with a TRS R if for all rules $l \to r \in R$ and for all ground substitutions $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F})$:*

$$\mathrm{int}_\tau(\sigma(l)) > \mathrm{int}_\tau(\sigma(r))$$

If there is such a compatible τ, then R is terminating, see e.g. [9]. If we generalize the notion of interpretation to arbitrary well-founded sets, the converse also holds.

A typical example is the system

$$F(F(x,y),z) \to F(x,F(y,z)).$$

Choose $F_\tau(x,y) = 2x + y$ and $A = \{n \in \mathbb{N} | n \geq 1\}$. Clearly F_τ is strictly monotone in both coordinates, and

$$F_\tau(F_\tau(x,y),z) = 4x + 2y + z > 2x + 2y + z = F_\tau(x,F_\tau(y,z))$$

for all $x,y,z \in A$, proving termination.

Although there are examples of terminating TRS's that can not be proved terminating in this particular way (see [9]), in practice it is often successful. Moreover, Hofbauer ([3]) showed that for any TRS for which a termination proof can be given by a recursive path order, a corresponding interpretation in the natural numbers can be given, using primitive recursive interpretations.

The standard technique of polynomial interpretations, see [5, 1], is nothing else than this technique in which all interpretation functions are chosen to be polynomials.

It appears that such an interpretation in the naturals not only gives information on termination in itself, but also on maximum derivation lengths.

3   Derivation lengths from interpretation

In order to investigate derivation lengths of TRS's, it is convenient to introduce some appropriate definitions.

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Definition 3.1 *We define the height $|t|$ of a term $t \in T(\mathcal{F}, \mathcal{X})$ as follows:*

$$|t| = \begin{cases} 1 & \text{if } t \text{ is a constant or a variable} \\ 1 + \max\{|t_1|, \ldots, |t_n|\} & \text{if } t = F(t_1, \ldots, t_n) \end{cases}$$

Reduction steps of a TRS R are written as $\to_R$, or simply $\to$ if no confusion is possible. If t_1 reduces to t_2 in n steps (a derivation of length n), we write $t_1 \to^n t_2$.

We now define the derivation length of a term t as follows:

Definition 3.2 *Let R be a terminating TRS. Then the derivation length $\mathrm{dl}_R(t)$ of a term t is defined to be $\max\{n | \exists t' : t \to^n t'\}$*

This notion can be generalized to a derivation length of all terms of a height limited by some number n:

Definition 3.3 *Let R be a terminating TRS. Let n be some natural number. Then $\mathrm{Dl}_R(n)$ is defined to be*

$$\max\{\mathrm{dl}_R(t) \mid t \in T(\mathcal{F}) \wedge |t| \leq n\}.$$

So $\mathrm{dl}_R(t)$ is the maximum derivation length of a specific term t, whereas $\mathrm{Dl}_R(n)$ is the maximum derivation length over all terms of height of at most n.

Having a compatible interpretation for a given TRS R asserts the following:

Lemma 3.4 *For all $t \in T(\mathcal{F})$: $\mathrm{dl}_R(t) \leq \mathrm{int}_\tau(t)$.*

Proof

If $t \to t'$ then $\mathrm{int}_\tau(t) - \mathrm{int}_\tau(t') \geq 1$, because τ is a compatible interpretation in $\mathbb{N}$. And so inductively, if $t \to^n t'$, then $\mathrm{int}_\tau(t) - \mathrm{int}_\tau(t') \geq n$. Since $\mathrm{int}_\tau(t') \geq 0$, we get $n \leq \mathrm{int}_\tau(t)$, which yields the desired result.

End Proof

Therefore, an upper bound on the interpretation value of a term leads to an upper bound on its derivation length.

If we have an upper bound function on the interpretation functions of a TRS R, an upper bound on the interpretation value of all terms of a limited height can be computed. This then is also an upper bound on their derivation length, by lemma 3.4.

Since we will make use of the notion upper bound function more often, we have the following definition:

Definition 3.5 *Let a TRS R with function symbols $\mathcal{F}$ and a compatible interpretation τ of R be given. Then a function $u : \mathbb{N} \to \mathbb{N}$ is an upper bound function of τ if*

$$\exists x_0 : \forall x > x_0 : \forall F \in \mathcal{F} : F_\tau(x, \ldots, x) \leq u(x).$$

In the next theorem we will consider iterated functions. We will write this as f^n, meaning the function f applied n times. More precisely: $f^0 = \text{id}, f^{n+1} = f \circ f^n$.

Theorem 3.6 *Let τ be an interpretation for a TRS R. Let $c \in \mathbb{N}$ for which $c \geq F_\tau$ for all $F \in \mathcal{F}$ of arity 0, and let u be a monotone function satisfying*

$$\forall f \in \mathcal{F} : \forall x \geq c : F_\tau(x, \ldots, x) \leq u(x).$$

Then $\text{Dl}_R(n) \leq u^{n-1}(c)$.

Proof

By the definition of Dl_R we have to prove that for all terms $t : \text{dl}_R(t) \leq u^{|t|-1}(c)$. We have that $\text{dl}_R(t) \leq \text{int}_\tau(t)$, so it remains to prove that for all terms: $\text{int}_\tau(t) \leq u^{|t|-1}(c)$. This is done by induction on the structure of t.

Basic step: if t is a constant then

$$
\begin{aligned}
\text{int}_\tau(t) &\leq c \\
&= u^0(c) \\
&= u^{|t|-1}(c)
\end{aligned}
$$

Inductive step: if t is a function application then $t = F(t_1, \ldots, t_m)$, so

$$
\begin{aligned}
\text{int}_\tau(t) &= F_\tau(\text{int}_\tau(t_1), \ldots, \text{int}_\tau(t_m)) & \\
&\leq u(\max\{\text{int}_\tau(t_1), \ldots, \text{int}_\tau(t_m)\}) & \text{by definition 3.5, } F_\tau \text{ monotone} \\
&\leq u(\max\{u^{|t_1|-1}(c), \ldots, u^{|t_m|-1}(c)\}) & \text{by ind. hyp., } u \text{ monotone} \\
&= u(u^{\max\{|t_1|, \ldots, |t_m|\}-1}(c)) & u \text{ monotone} \\
&= u(u^{|t|-2}(c)) & \text{by definition of term height} \\
&= u^{|t|-1}(c) &
\end{aligned}
$$

End Proof

The requirement on u of being monotone is not essential, it only simpifies the proof. We shall only use monotone bound functions u.

4   A hierarchy of sharp derivation lengths

We now give a hierarchy of sets U of monotone upper bound functions and their derivation lengths, i.e. the maximal derivation lengths that can be reached by TRS's with a compatible interpretation having an upper bound function in U.

If we consider the polynomials for instance, there is no single polynomial function which is an upper bound function of all polynomials. For each polynomial p however, there is an $a \in \mathbb{N}$ such that $u_a(x) = x^a$ is an upper bound function of p. This is the reason why allowed interpretations are given by a set of upper bound functions rather than by a single upper bound function.

For each set of upper bound functions, a corresponding upper bound on the derivation length for all terms of some height up to n will be given, using theorem 3.6. Moreover, by giving derivations in example TRS's we shall show that the upper bounds on derivation lengths are all sharp. First we define what is meant by sharpness.

Definition 4.1 *We call a function $f : \mathbb{N} \to \mathbb{N}$ a sharp derivation length for a set U of monotone upper bound functions if:*

- *for each TRS R with a compatible interpretation having an upper bound function in U:*
$$\exists c, n_0 \in \mathbb{N} : \forall n > n_0 : \mathsf{Dl}_R(n) \leq f(c \cdot n)$$

- *and there is a TRS R with a compatible interpretation having an upper bound function in U such that*
$$\exists c, n_0 \in \mathbb{N} : \forall n > n_0 : f(n) \leq \mathsf{Dl}_R(c \cdot n)$$

The results for the considered sets of upper bound functions are summarized in the table below.

	U	a sharp derivation length for U
1	$\{x \mapsto ax \mid a \in \mathbb{N}\}$	$n \mapsto 2^n$
2	$\{x \mapsto x^a \mid a \in \mathbb{N}\}$	$n \mapsto 2^{2^n}$
3	$\{x \mapsto \lceil \exp^i(a \log^i x) \rceil \mid a \in \mathbb{N}\}$	$n \mapsto \exp^{i+1}(n)$
4	$\{x \mapsto \exp^a(x) \mid a \in \mathbb{N}\}$	$n \mapsto \exp^n(0)$
5	$\{x \mapsto A_i^a(x) \mid a \in \mathbb{N}\}$	$n \mapsto A_{i+1}(n)$
6	$\{x \mapsto DiagA^a(x) \mid a \in \mathbb{N}\}$	$n \mapsto DiagA^n(1)$

where

- $\exp(x) = 2^x$

- $\begin{aligned} A_0(x) &= 2x & \text{for } & x \geq 0 \\ A_i(0) &= 1 & \text{for } & i \geq 1 \\ A_i(x) &= A_{i-1}(A_i(x-1)) & \text{for } & i \geq 1, x \geq 1 \end{aligned}$

  (The Ackermann function)

- $DiagA(x) = A_x(x)$

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In the third and fifth row of the table, an extra index i occurs. In these cases we do not consider one single set U, but a sequence of sets, one set for each $i \in \mathbb{N}$. A corresponding sequence of sharp derivation lengths is given at the right hand, one for each $i \in \mathbb{N}$.

In row 5 of the table a sharp derivation length is given for each unary Ackermann function A_i. From recursion theory it is known that these functions A_i can be used to define a hierarchy of classes of functions, analogous to the Grzegorczyk hierarchy covering the primitive recursive functions ([8]). So we have given a sharp derivation length for the whole sequence of Ackermann functions which essentially spans the primitive recursive functions. A similar result without using Ackermann functions, independently found, was given in [3].

In row 6 the diagonal of the Ackermann function is used, which is not a primitive recursive function. This shows that the general upper bound on derivation lengths can also be sharp for non-primitive recursive functions.

As can easily be verified, row 1 of the table is a specific instance of row 3, namely for $i = 0$. In the same way, row 2 is an instance of row 3 for $i = 1$, and row 4 an instance of row 5 for $i = 1$.

So in order to prove the results of this table, it suffices to prove the sharp derivation lengths for the sets of row 3, 5, and 6. These proofs will be given successively in the next three subsections. Since the proofs follow the definition of a sharp upper bound, they all have the same structure.

4.1   Exponential derivation lengths

Proposition 4.2 *For all $i \in \mathbb{N}$, let $U_i = \{x \mapsto \lceil \exp^i(a \log^i x) \rceil \mid a \in \mathbb{N}\}$, and let $f_i(n) = \exp^{i+1}(n)$. Then, for all $i \in \mathbb{N}$, f_i is a sharp derivation length for U_i.*

Proof

Let $i \in \mathbb{N}$ be given. We check the two requirements of a sharp derivation length given in definition 4.1.

First it has to be proved that for each TRS R with a compatible interpretation having an upper bound function in $U_i : \exists c, n_0 \in \mathbb{N} : \forall n > n_0 :$

$$\mathsf{Dl}_R(n) \leq \exp^{i+1}(c \cdot n)$$

For all $i \in \mathbb{N}$, there is an $a \in \mathbb{N}$ such that $u(x) = \lceil \exp^i(a \log^i x) \rceil$ is an upper bound function of the interpretation of R. By theorem 3.6, $\mathsf{Dl}_R(n) \leq u^{n-1}(c_0)$ for some $c_0 \in \mathbb{N}$, so it remains to prove that $\exists c, n_0 : \forall n > n_0 : u^{n-1}(c_0) \leq \exp^{i+1}(c \cdot n)$.

We proceed by induction on n.

basic step: $n = 1$, then we have: $c_0 \leq \exp^{i+1}(c)$, which is true if $c > c_0$.

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inductive step: $n > 1$:

$$
\begin{aligned}
u^n(c_0) &= u(u^{n-1}(c_0)) \\
&\le u(\exp^{i+1}(cn)) && \text{(induction hyp., } u \text{ monotone)} \\
&\le \exp^i(a \log^i(\exp^{i+1}(cn))) + 1 && (\lceil x \rceil \le x + 1) \\
&= \exp^i(a \cdot 2^{cn}) + 1 && (\log^i \exp^i = \mathsf{id}) \\
&= \exp^i(2^{cn + \log a}) + 1 \\
&\le \exp^{i+1}(cn + \log a + 1) \\
&\le \exp^{i+1}(c(n+1)) && \text{(for } c \ge (\log a) + 1)
\end{aligned}
$$

Secondly, it has to be proved that there is a TRS R_i with a compatible interpretation having an upper bound function in U_i : $\exists c, n_0 \in \mathbb{N} : \forall n > n_0 : \mathsf{Dl}_{R_i}(c \cdot n) \ge \exp^{i+1}(n)$. Take R_0 to be

$$
D_1 D_0 x \;\rightarrow\; D_0 D_0 D_1 x
$$

For $i > 0$, R_i is the union of R_{i-1} and

$$
\begin{aligned}
D_{i+1} D_i x &\;\rightarrow\; D_i D_i D_{i+1} x \\
D_{i+1} D_{i-1} x &\;\rightarrow\; D_{i-1} x
\end{aligned}
$$

Take τ as follows:

$$
\begin{aligned}
\tau(D_0) &= x \mapsto x + 1 \\
\tau(D_{i+1}) &= x \mapsto \lceil \exp^i(3 \cdot log^i x) \rceil
\end{aligned}
$$

Using standard calculus it follows that τ is compatible with R_i and has an upper bound function in U_i.

For any i, n, we have the following derivation in R_i:

$$
D_{i+1}^n D_i D_{i-1} \cdots D_0 0 \rightarrow^* D_i^{2^n} D_{i+1}^n D_{i-1} \cdots D_0 0 \rightarrow^*
$$

$$
D_i^{2^n} D_{i-1} \cdots D_0 0 \rightarrow^* D_1^{\exp^i(n)} D_0 0 \rightarrow^* D_0^{\exp^{i+1}(n)} D_1^{\exp^i(n)} 0.
$$

The increase in the number of S-symbols shows that the derivation length exceeds $\exp^{i+1}(n)$, while the height of the starting term is linear in n. So $\mathsf{Dl}_{R_i}(c \cdot n) \ge \exp^{i+1}(n)$, which we had to prove.

End Proof

Note that for $i = 1$ the corresponding TRS has a polynomial interpretation and a doubly expontial derivation length. The proof we gave here is essentially simpler than the original one in [6, 4]. In particular in our system the observation that the derivation length is indeed doubly exponential is straightforward since there are no duplicating rules.

For $i > 1$ the proposition gives a hierarchy of interpretations that are greater than polynomials and smaller than exponential functions, each having sharp derivation lengths.

The system R_i is a pure string rewriting system. Of such a system we can take the reverse, i.e., all left hand sides and right hand sides are reversed as strings. The reversed system of R_i trivially has the same Dl as R_i itself, but the reversed system cannot be proved to be terminating in the style of section 2 by an interpretation in $\mathbb{N}$. It can be done by an interpretation in the well-founded set $\mathbb{N}^{i+2}$ with lexicographic order, but not in $\mathbb{N}^j$ for any $j < i + 2$. This is proved in [2].

4.2 The Ackermann function

Proposition 4.3 *Let A_i denote the Ackermann function as defined before. For all $i \in \mathbb{N}$, let $U_i = \{x \mapsto A_i^a(x) | a \in \mathbb{N}\}$ and let $f_i(n) = A_{i+1}(n)$. Then, for all $i \in \mathbb{N}$, f_i is a sharp derivation length for U_i.*

Proof

Let $i \in \mathbb{N}$ be given. Again we check the two requirements of a sharp derivation length given in definition 4.1.

First it has to be proved that for each TRS R with a compatible interpretation having an upper bound function in $U_i : \exists c, n_0 \in \mathbb{N} : \forall n > n_0 :$

$$\mathsf{Dl}_R(n) \leq A_{i+1}(c \cdot n)$$

For all $i \in \mathbb{N}$, there is an $a \in \mathbb{N}$ such that $u(x) = A_i^a(x)$ is an upper bound function of the interpretation of R. By theorem 3.6, $\mathsf{Dl}_R(n) \leq u^{n-1}(c_0)$ for some $c_0 \in \mathbb{N}$, so it remains to prove that $\exists c, n_0 : \forall n > n_0 : u^{n-1}(c_0) \leq A_{i+1}(c \cdot n)$.

We have:

$$
\begin{aligned}
(\lambda x. A_i^a(x))^{n-1}(c_0) \quad &= A_i^{a(n-1)}(c_0) \\
&\leq A_i^{a(n-1)+c_0}(1) \\
&= A_i^{c \cdot n}(1) \qquad \text{(choose } c \geq a + c_0) \\
&= A_{i+1}(c \cdot n)
\end{aligned}
$$

Secondly, for any i we have to give a suitable TRS R_i. These R_i are obtained by transforming the definition of the Ackermann function to rewrite format. Take R_0 to be

$$
\begin{aligned}
E_0 0 \quad &\to 0 \\
E_0 S x \quad &\to S S E_0 x
\end{aligned}
$$

and for $i > 0$, R_i is defined to be the union of R_{i-1} and

$$
\begin{aligned}
E_i 0 \quad &\to S 0 \\
E_i S x \quad &\to E_{i-1} E_i x.
\end{aligned}
$$

Take τ as follows:

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$$\tau(0) \quad = 2$$
$$\tau(S) \quad = x \mapsto x + 1$$
$$\tau(E_i) \quad = x \mapsto A_i(A_i(x))$$

One easily checks that τ is compatible with R_i and has an upper bound function in U_i. Now R_i is terminating since τ is a compatible interpretation, it is confluent since there are no critical pairs, and the ground normal forms are exactly the terms built from 0 and S. Further all rewrite rules are equalities if 0 and S are interpreted as zero and successor, and E_i is interpreted as A_i. So E_i computes the Ackermann function A_i. Hence for any n we have the derivation

$$E_i^n S0 \rightarrow^* S^{A_i^n(1)}(0).$$

The increase in the number of S-symbols shows that the derivation length exceeds $A_i^n(1)$. One easily shows by induction that $A_i^n(1) = A_{i+1}(n)$. Comparing the derivation length and the height of the starting term we conclude

$$\exists c, n_0 \in \mathbb{N} : \forall n > n_0 : A_{i+1}(n) \leq \mathsf{Dl}_{R_i}(c \cdot n),$$

which we had to prove.

End Proof

4.3 The diagonal of the Ackermann function

Proposition 4.4 *Let* $U = \{x \mapsto DiagA^a(x) | a \in \mathbb{N}\}$, *and let* $f(n) = DiagA^n(1)$. *Then* f *is a sharp derivation length for* U.

Proof

We will now check the two requirements of a sharp derivation length given in definition 4.1.

First it has to be proved that for each TRS R with a compatible interpretation having an upper bound function in U : $\exists c, n_0 \in \mathbb{N} : \forall n > n_0$:

$$\mathsf{Dl}_R(n) \leq DiagA^{c \cdot n}(1)$$

There is an $a \in \mathbb{N}$ such that $u(x) = DiagA^a(x)$ is an upper bound function of the interpretation of R. By theorem 3.6, $\mathsf{Dl}_R(n) \leq u^{n-1}(c_0)$ for some $c_0 \in \mathbb{N}$, so it remains to prove that $\exists c, n_0 : \forall n > n_0 : u^{n-1}(c_0) \leq DiagA^{c \cdot n}(1)$.

We have:

$$(\lambda x. DiagA^a(x))^{n-1}(c_0) \quad = DiagA^{a(n-1)}(c_0)$$
$$\leq DiagA^{a(n-1)+c_0}(1)$$
$$\leq DiagA^{c \cdot n}(1) \qquad \text{(choose } c \geq a + c_0)$$

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Secondly, it has to be proved that there is a TRS R with a compatible interpretation having an upper bound function in $U : \exists c, n_0 \in \mathbb{N} : \forall n > n_0 :$

$$DiagA^n(1) \leq \mathsf{DI}_R(c \cdot n)$$

Take R to be

$$
\begin{aligned}
F(0,0) &\rightarrow 0 \\
F(0,S(x)) &\rightarrow S(S(F(0,x))) \\
F(S(i),0) &\rightarrow S(0) \\
F(S(i),S(x)) &\rightarrow F(i,F(S(i),x))
\end{aligned}
$$

(F computes the binary Ackermann function A.)

Take τ as follows:

$$
\begin{aligned}
\tau(0) &= 2 \\
\tau(S) &= x \mapsto x + 1 \\
\tau(F) &= i, x \mapsto A_i(A_i(x))
\end{aligned}
$$

This τ is compatible with R and has an upper bound function in U.

For a given $n \geq 4$, take the start term t_n, where

$$
\begin{aligned}
t_1 &= 0 \\
t_2 &= S(0) \\
t_i &= F(t_{i-1}, t_{i-1}) \quad \text{if } i > 2
\end{aligned}
$$

Since $t_n \rightarrow^* S^{DiagA^{n-2}(1)}(0)$, the desired derivation length is achieved.

End Proof

5 Conclusions

In this paper we considered termination proofs of term rewriting systems by monotone interpretation in the natural numbers. A general upper bound on derivation lengths was derived in the following sense. Let a term rewriting system be given for which termination can be proved by a monotone interpretation in the natural numbers. Take one bound function such that for all operation symbols the corresponding interpretation function is smaller than the bound function. Then an upper bound on the length of a derivation can be expressed in the height of the start term and this bound function. A similar result can be found in [3].

Building on these result, we quantified upper bounds on derivation lengths for several cases. In any of these cases on the one hand we have a bunch of functions such that only interpretation functions are allowed that are bounded by one of the functions. For

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example, all polynomials are bounded by functions $x \mapsto x^a$. On the other hand we have a function f such that a derivation starting with a term of height $\leq n$ can not be longer than $f(n)$.

In any of these cases we proved that the derived bound on the derivation length is sharp in a natural sense: we gave examples of term rewriting systems and corresponding derivations in which the bound is indeed reached up to a constant. In the considered cases the bounds on the interpretation function are

- linear functions,

- polynomials,

- functions in between polynomials and exponentials,

- iterated exponential functions,

- the ith unary Ackermann function,

- the binary Ackermann function.

Except for the last case, all example systems are string rewriting systems.

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