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Citation for published version (APA):

Lith, van, B. S., Thijs Boonkkamp, ten, J. H. M., IJzerman, W. L., & Tukker, T. W. (2014). *Existence and uniqueness of solutions to Liouville's equation and the associated flow for Hamiltonians of bounded variation*. (CASA-report; Vol. 1434). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/2014

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 14-34
December 2014

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by

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ISSN: 0926-4507

Existence and uniqueness of solutions to Liouville's equation and the associated flow for Hamiltonians of bounded variation.

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December 17, 2014

Abstract

We prove existence and uniqueness for solutions to Liouville's equation for Hamiltonians of bounded variation. These solutions can be interpreted as the limit of a sequence generated by a series of smooth approximations to the Hamiltonian. This results in a converging sequence of approximations of solutions to Liouville's equation. As an added perk, our method allows us to prove a generalisation of Liouville's theorem for Hamiltonians of bounded variation. Furthermore, we prove there exists a unique flow solution to the Hamilton equations and show how this can be used to construct a solution to Liouville's equation.

Key words. partial differential equations, geometrical optics, Liouville's equation, flow.

1 Introduction

Hamiltonian systems are encountered in mechanics and optics. The Hamiltonian formulation of mechanics dates back to 1834 and is very well-known [1, 6]. The optical applications of Hamiltonian systems are less well-known, even though they pre-date their mechanical cousin by some six years [5]. Recently though, the concept of Hamiltonian optics and its phase space description has gained popularity in the lighting industry [13].

The mechanical Hamiltonian is an example of a separable Hamiltonian. It is given by

$$H_{\text{mech}}(t, \mathbf{q}, \mathbf{p}) := \frac{|\mathbf{p}|^2}{2m} + V(t, \mathbf{q}), \quad (1)$$

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where \mathbf{q} is the position vector, \mathbf{p} is the momentum vector, m is the mass of the particle and V is the potential. Hamiltonians of the form $H(t, \mathbf{q}, \mathbf{p}) = T(\mathbf{p}) + V(t, \mathbf{q})$ are called separable, since using these in the Hamilton equations yields a set of equations where the differential equation for \mathbf{q} depends only on \mathbf{p} and time and vice versa. Hauray showed existence and uniqueness of solutions for the mechanical Hamiltonian with force fields of bounded variation [7].

We are interested in a more general class of Hamiltonians, motivated by the optical Hamiltonian given by

$$H_{\text{opt}}(z, \mathbf{q}, \mathbf{p}) := -\sigma \sqrt{n^2(z, \mathbf{q}) - |\mathbf{p}|^2}, \quad (2)$$

where $\sigma \in \{-1, 0, 1\}$ is the index of direction (forward, perpendicular to the optical axis or backward respectively), n is the refractive index and z is the length down the optical axis [15]. Whereas mechanics describes the motion of particles, Hamiltonian optics describes the trajectories of rays on a hypothetical screen perpendicular to the optical axis, i.e., a plane in \mathbb{R}^3 satisfying $z = \text{const}$. In this setting, z takes the role of evolution coordinate instead of time.

Optics often deals with piecewise constant refractive index fields, such as lenses or mirrors [2]. This leads to the Hamiltonian having discontinuous jumps, where physical conservation laws then lead to the laws of refraction and specular reflection. In geometrical optics, Liouville's equation expresses the transport of energy and one can show that light rays transport energy. It would therefore make sense to simply define a solution to be constant along rays, which leads to a well-posed problem [8]. However, we intend to show that such a construction can be rigorously justified.

1.1 Outline

In §2, we start with the formulation of the problem and show that solving Liouville's equation is in some sense equivalent to solving the Hamilton equations for any initial condition. As we are interested in using Hamiltonians of bounded variation, we shall also give a short introduction to functions of bounded variation in §3. In §4, we prove the existence and uniqueness for solutions to Liouville's equation for Hamiltonians of bounded variation. Finally, we switch to the slightly more general problem of finding the flow defined by the Hamilton equations in §5. The flow, of course, supplies us with solutions to the Hamilton initial value problem.

2 Liouville's equation

Let $Q \subset \mathbb{R}^d$, $P \subset \mathbb{R}^d$ and $t > 0$. We define phase space \mathcal{P} as the product space $\mathcal{P} := Q \times P$. Liouville's equation described the evolution of some density $\rho : \mathbb{R}^+ \times \mathcal{P} \rightarrow \mathbb{R}$, it is a hyperbolic partial differential equation (PDE) on phase space given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial h}{\partial \mathbf{p}} \cdot \frac{\partial \rho}{\partial \mathbf{q}} - \frac{\partial h}{\partial \mathbf{q}} \cdot \frac{\partial \rho}{\partial \mathbf{p}} = 0, \quad (3)$$

where $h : \mathbb{R}^+ \times \mathcal{P} \rightarrow \mathbb{R}$ is the Hamiltonian, \mathbf{q} is the position vector and \mathbf{p} the momentum vector. In mechanics, ρ represents the particle density in phase space, while in geometric optics, ρ is an energy or power density. We shall write t as the evolution coordinate, hence for geometric optics applications one needs to transform ($t \rightarrow z$).

The operator $\frac{\partial}{\partial \mathbf{q}}$ is the gradient operator with respect to \mathbf{q} , thus $\frac{\partial}{\partial \mathbf{q}} := (\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_d})^T$. The operator $\frac{\partial}{\partial \mathbf{p}}$ is similarly defined, and the dot product is simply the Euclidean inner product. We shall also use the nabla operator

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial \mathbf{q}} \\ \frac{\partial}{\partial \mathbf{p}} \end{pmatrix}. \quad (4)$$

Furthermore, we shall denote \mathbf{E} to be the identity matrix in $\mathbb{R}^{d \times d}$, with which we construct

$$\Omega := \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix}. \quad (5)$$

Note that $\Omega^T = -\Omega = \Omega^{-1}$, it is sometimes referred to as the symplectic matrix. Clearly, we can then make our notation slightly more compact by realizing that

$$\Omega \nabla = \begin{pmatrix} \frac{\partial}{\partial \mathbf{p}} \\ -\frac{\partial}{\partial \mathbf{q}} \end{pmatrix}, \quad (6)$$

thus we are able to write Liouville's equation as

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot (\Omega \nabla h) = \frac{\partial \rho}{\partial t} - (\Omega \nabla \rho) \cdot \nabla h = 0. \quad (7)$$

Note that for any $f \in C^2(\mathbb{R}^{2d})$, we have

$$\nabla \cdot (\Omega \nabla f) = \sum_{i=1}^d \frac{\partial}{\partial q_i} \left(\frac{\partial f}{\partial p_i} \right) - \frac{\partial}{\partial p_i} \left(\frac{\partial f}{\partial q_i} \right) = 0, \quad (8)$$

where $\nabla \cdot$ is the divergence operator on phase space. We will, in some cases, also write $\mathbf{y} = (\mathbf{q}, \mathbf{p})$.

2.1 Solutions to Liouville's equation

A solution to Liouville's equation is defined in the following way.

Definition 2.1. *Let $I = [0, T]$, a weak solution to Liouville's equation $\rho \in L^\infty(I \times \mathcal{P})$ satisfies, for any $\varphi \in C_0^\infty(I \times \mathcal{P})$,*

$$\int_0^T \int_{\mathcal{P}} \rho \frac{\partial \varphi}{\partial t} \, dy \, dt - \int_0^T \int_{\mathcal{P}} \rho(t, \mathbf{y}) (\Omega \nabla \varphi(t, \mathbf{y})) \cdot dh(t, \mathbf{y}) \, dt = - \int_{\mathcal{P}} \rho_0(\mathbf{y}) \varphi(0, \mathbf{y}) \, dy, \quad (9)$$

where $dy = dq_1 \dots dq_d dp_1 \dots dp_d$, $dh(t, \mathbf{y}) = \nabla h(t, \mathbf{y}) dy$ if the gradient exists and ρ_0 is the initial condition.

The motivation of the definition of the weak solution is to integrate Liouville's equation with some smooth test function φ , integrate by parts and demand that this holds for any such test function. Thus, when some ρ satisfies Liouville's equation weakly, this is equal to saying that it satisfies Liouville's equation in a distributional sense, meaning almost everywhere in $I \times \mathcal{P}$.

Liouville's equation is, in some sense, equivalent with the Hamilton equations, which is a set of ordinary differential equations (ODEs), given by

$$\frac{d\mathbf{q}}{dt} = \frac{\partial h}{\partial \mathbf{p}}, \quad (10a)$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial h}{\partial \mathbf{q}}. \quad (10b)$$

Subject to a set of initial conditions, we shall also refer to (10a) - (10b) as the Hamilton initial value problem. Using the notation $\mathbf{y} = (\mathbf{q}, \mathbf{p})$ and our previous definitions, we can write the Hamilton equations as

$$\frac{d\mathbf{y}}{dt} = \Omega \nabla h(t, \mathbf{y}). \quad (11)$$

Before we show the relation between Liouville's equation and the Hamilton equations, we shall first give a brief treatment of ODE theory.

GENERAL REMARK. *We will routinely deal with functions that have different regularities in different variables. To simplify notation, we shall adopt the semi-colon notation for the target space, for instance $g \in L^1(\mathbb{R}^+; \mathbb{R}^m)$ is an integrable function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^m$. If the target space is unspecified, the default target space is \mathbb{R} . However, we may also have functions that map into function spaces, such as $f \in L^1(\mathbb{R}^+; C^\infty(\mathbb{R}^m))$, which is to say $f : \mathbb{R}^+ \rightarrow C^\infty(\mathbb{R}^m)$ integrable and $f(t) : \mathbb{R}^m \rightarrow \mathbb{R}$ infinitely differentiable for fixed $t \in \mathbb{R}^+$. In that case, we shall use $f(t, \mathbf{x}) = f(t)(\mathbf{x})$ as a shorthand notation. Thus, the statement $f \in L^1(\mathbb{R}^+; C^\infty(\mathbb{R}^m))$ can be read as a compact version of the statement that $f : \mathbb{R}^+ \times \mathbb{R}^m \rightarrow \mathbb{R}$ with $f(t, \cdot) \in C^\infty(\mathbb{R}^m)$ for fixed $t \in \mathbb{R}^+$ and $f(\cdot, \mathbf{x}) \in L^1(\mathbb{R}^+)$ for fixed $\mathbf{x} \in \mathbb{R}^m$.*

We shall first present some general and quite standard ODE theory. This is meant as an introduction to the classical solutions of Liouville's equation and the Hamilton IVP. We have taken the following theorem from [3].

Theorem 2.1. Existence and uniqueness for ODEs

Let $I = [0, T]$ and $\mathbf{f} \in L^1(I; W^{1, \infty}(\mathbb{R}^m; \mathbb{R}^m))$. Thus, $\mathbf{f} : I \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\mathbf{f}(\cdot, \mathbf{x}) = \mathbf{f}(\cdot)(\mathbf{x})$ integrable in t for fixed $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{f}(t, \cdot) = \mathbf{f}(t)(\cdot)$ Lipschitz continuous w.r.t. \mathbf{x} for fixed $t \in I$. Consider the IVP,

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad t \in I, \quad (12a)$$

$$\mathbf{x}(0) = \mathbf{x}_0. \quad (12b)$$

There exists a unique $\mathbf{x} : I \rightarrow \mathbb{R}^m$ that satisfies (12a) - (12b) almost everywhere. Furthermore, $\mathbf{x} \in AC(I; \mathbb{R}^m)$, which is to say \mathbf{x} is an absolutely continuous mapping from I to \mathbb{R}^m .

Corollary 2.1. Existence and uniqueness for the Hamilton IVP

Let $I = [0, T]$ and $h \in L^1(I; C^2(\mathcal{P}))$. Thus, $h : I \times \mathcal{P} \rightarrow \mathbb{R}$ and for fixed $t \in I$ we have $h(t, \cdot, \cdot) = h(t) \in C^2(\mathcal{P})$, hence $h(t) : \mathcal{P} \rightarrow \mathbb{R}$ is twice continuously differentiable. Furthermore, for fixed $(\mathbf{q}, \mathbf{p}) \in \mathcal{P}$, we have $h(\cdot, \mathbf{q}, \mathbf{p}) = h(\cdot)(\mathbf{q}, \mathbf{p}) \in L^1(I)$, thus $h(\cdot, \mathbf{q}, \mathbf{p}) : I \rightarrow \mathbb{R}$ is integrable. Then, there exists a unique solution to the Hamilton IVP.

Proof. Since h is twice continuously differentiable on phase space, we have that $\Omega \nabla h$ is continuously differentiable on phase space, which implies Lipschitz continuity. The conditions of Theorem 2.1 apply, where $m = 2d$ and hence, there exists a unique solution to the Hamilton IVP. \square

Whenever solutions of the Hamilton IVP are uniquely defined, we can find a solution using the method of characteristics (MOC) [4, 10]. The idea is to find a set of curves in $I \times \mathcal{P}$ such that the PDE reduces to a set of ODEs. Let us define $(\mathbf{q}, \mathbf{p}) : I \rightarrow \mathcal{P}$ as the solution to the Hamilton IVP and let us investigate the total derivative of the quantity $\rho^*(t) := \rho(t, \mathbf{q}(t), \mathbf{p}(t))$, i.e.,

$$\frac{d\rho^*}{dt} = \frac{\partial \rho}{\partial t} + \frac{d\mathbf{q}}{dt} \cdot \frac{\partial \rho}{\partial \mathbf{q}} + \frac{d\mathbf{p}}{dt} \cdot \frac{\partial \rho}{\partial \mathbf{p}} = 0, \quad (13)$$

where application of the Hamilton equations gives (3) and is therefore identically zero. Hence, we see that along curves defined by the Hamilton equations, the solution to Liouville's equation is constant, i.e., $\rho^*(t) = \text{const.}$, implying

$$\rho(t, \mathbf{q}(t), \mathbf{p}(t)) = \rho_0(\mathbf{q}(0), \mathbf{p}(0)), \quad (14)$$

where ρ_0 is the initial condition, $\rho(0, \mathbf{q}, \mathbf{p}) = \rho_0(\mathbf{q}, \mathbf{p})$. Therefore, we call solutions to the Hamilton IVP the characteristics of Liouville's equation. Along the characteristics, the solution to Liouville's equation is constant. Furthermore, if we can find the initial point $(\mathbf{q}(0), \mathbf{p}(0))$ corresponding to a characteristic going through any given $(\mathbf{q}, \mathbf{p}) \in \mathcal{P}$ at $t \in I$, the solution to Liouville's equation is completely determined. Thus, for fixed $t \in I$, we look for a characteristic satisfying the Hamilton equations, (10a) - (10b), and

$$(\mathbf{q}(t), \mathbf{p}(t)) = (\mathbf{q}, \mathbf{p}), \quad (15)$$

which exists and is unique by Corollary 2.1. Thus $(\mathbf{q}(0), \mathbf{p}(0))$ is a one-to-one function of the point (\mathbf{q}, \mathbf{p}) and t , and we can construct a solution to Liouville's equation by (14). However, the MOC can only be applied if the solution to Liouville's equation is sufficiently differentiable and the existence and uniqueness of the Hamilton IVP is guaranteed.

However, We wish to find solutions to (3) whenever the Hamiltonian is not smooth and classical solutions do not exist. However, we do not wish to extend the class of Hamiltonians as far as integrable functions, since we still need some notion of a gradient of the Hamiltonian. The proper space for the Hamiltonian is therefore the space of functions of bounded variation.

3 Functions of bounded variation

We shall now give the definition of functions of bounded variation and show some basic properties. There are several intuitive properties suggested by the name, such as no infinitely rapid oscillations and no infinitely high jump discontinuities.

Definition 3.1. Functions of bounded variation

Let $U \subset \mathbb{R}^m$, and let the space \mathcal{B} be defined as follows,

$$\mathcal{B} = \{ \varphi \in C_0^1(U; \mathbb{R}^m) \mid \|\varphi\|_{L^\infty(U)} = 1 \}. \quad (16)$$

For $u \in L^1(U)$, we shall denote the total variation of u by

$$\text{TV}(u) := \sup_{\varphi \in \mathcal{B}} \int_U u \nabla \cdot \varphi \, dx, \quad (17)$$

where $\nabla \cdot$ is the divergence operator in \mathbb{R}^m . Then u is said to be of bounded variation, or equivalently $u \in \text{BV}(U)$, if

$$u \in L^1(U) \quad \text{and} \quad \text{TV}(u) < \infty. \quad (18)$$

Furthermore, we endow the space $\text{BV}(U)$ with the semi-norm

$$\|u\|_{\text{BV}(U)} := \text{TV}(u). \quad (19)$$

We can also represent the gradient of u by a finite vector Radon measure du , such that

$$\int_U u(\nabla \cdot \varphi) \, dx = - \int_U \varphi \cdot du(\mathbf{x}), \quad \forall \varphi \in C_0^1(U; \mathbb{R}^m). \quad (20)$$

REMARK. We can also consider functions that are locally of bounded variation, denoted $\text{BV}_{\text{loc}}(U)$.

Functions of bounded variation are continuous almost everywhere, their gradients exist almost everywhere, but they still allow for discontinuities. A famous example of a function of bounded variation is the Heaviside function which has the Dirac- δ as its distributional derivative. Whenever a function has a (weak) gradient, the total variation is simply the absolute integral of the gradient. Hence, for any $g \in W_{\text{loc}}^{1,1}(U)$, we have

$$\|g\|_{\text{BV}_{\text{loc}}(U)} = \|\nabla g\|_{L_{\text{loc}}^1(U)}. \quad (21)$$

Functions of bounded variation can be approximated by smooth functions, for instance by mollification.

Definition 3.2. Mollification

Let $\phi \in C_0^\infty(\mathbb{R}^m)$, with $\phi \geq 0$ and $\text{supp } \phi \subset B_1(0)$, where $B_1(0)$ is the unit ball in \mathbb{R}^m . Furthermore, let ϕ satisfy

$$\int_{\mathbb{R}^m} \phi \, dx = 1. \quad (22)$$

Now define for any $\delta > 0$ the function

$$\phi_\delta(x) = \delta^{-m} \phi\left(\frac{\mathbf{x}}{\delta}\right). \quad (23)$$

The mollification of u with radius of mollification δ , denoted $\mathcal{M}_\delta u$, is then given by

$$(\mathcal{M}_\delta u)(\mathbf{x}) := \int_{\mathbb{R}^m} \phi_\delta(\mathbf{x} - \mathbf{y}) u(\mathbf{y}) \, d\mathbf{y}. \quad (24)$$

The function ϕ_δ is called the mollifier with radius δ .

An example of a mollifier is the bump function, i.e.,

$$\phi(\mathbf{x}) = \begin{cases} \exp\left(-\frac{1}{1-|\mathbf{x}|^2}\right), & \text{for } |\mathbf{x}| < 1, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

A few properties of mollification are that $\mathcal{M}_\delta u \in C_0^\infty(\mathbb{R}^m)$, and if $u \in L^p(U)$ with $U \subset \mathbb{R}^m$, then $\mathcal{M}_\delta u \rightarrow u$ in $L^p(U)$ as $\delta \rightarrow 0$. Furthermore, we have that

$$\|\mathcal{M}_\delta u\|_{L^p(U)} \leq \|\mathcal{M}_\delta u\|_{L^p(\mathbb{R}^m)} \leq \|u\|_{L^p(U)}. \quad (26)$$

Furthermore, looking at the definition of mollification, we can see that for any $f, g \in L^1_{\text{loc}}(U)$, we have

$$\int_{\mathbb{R}^m} f \mathcal{M}_\delta g \, dx = \int_{\mathbb{R}^m} g \mathcal{M}_\delta f \, dx. \quad (27)$$

The properties of mollification will prove very helpful. We have adapted the following theorem from [11] (see section 9.1.2).

Theorem 3.1. Approximation of $BV_{\text{loc}}(U)$ functions

Let $U \subset \mathbb{R}^m$ and let $u \in BV_{\text{loc}}(U)$. Then, for any $\varepsilon > 0$, there exists a $u_\varepsilon \in C^\infty(U)$ such that

$$\|u - u_\varepsilon\|_{L^1_{\text{loc}}(U)} \leq \varepsilon \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)} \leq \|u\|_{BV_{\text{loc}}(U)}, \quad (28)$$

with

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)} = \|u\|_{BV_{\text{loc}}(U)}. \quad (29)$$

If, in addition, $u \in L^1(U)$, then all these properties are globally valid on U .

REMARK. Using (21), we should note that we can rewrite (29) as

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{BV_{\text{loc}}(U)} = 0. \quad (30)$$

In other words, $u_\varepsilon \rightarrow u$ in $BV_{\text{loc}}(U)$.

Proof. 1. The properties in (28) follow directly from the properties of mollification. Since $\mathcal{M}_\delta u \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$, for every $\varepsilon > 0$ we can choose a $\delta > 0$ such that

$$\|\mathcal{M}_\delta u - u\|_{L^1_{\text{loc}}(\Omega)} \leq \varepsilon.$$

We then define $u_\varepsilon := \mathcal{M}_\delta u$.

2. Let us evaluate $\|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)}$, where the identity (21) yields

$$\|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)} = \|u_\varepsilon\|_{\text{BV}_{\text{loc}}(U)} = \sup_{\varphi \in \mathcal{B}} \int_U \nabla \cdot \varphi \mathcal{M}_\delta u \, dx.$$

Now, since mollification and differentiation are linear, we have due to (27) the following equality,

$$\|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)} = \sup_{\varphi \in \mathcal{B}} \int_U \nabla \cdot (\mathcal{M}_\delta \varphi) u \, dx, \quad (*)$$

where we do not need to extend the integral over the entire space \mathbb{R}^m due to the compact support of φ . Now, for any $\psi \in C^1_0(U; \mathbb{R}^m)$, we can find a $\phi \in \mathcal{B}$ so that $\psi = \|\psi\|_{L^\infty(U)} \phi$, from which it follows that

$$\sup_{\psi \in C^1_0(U; \mathbb{R}^m)} \int_U u \nabla \cdot \psi \, dx = \|\psi\|_{L^\infty(U)} \|u\|_{\text{BV}_{\text{loc}}(U)}.$$

If we apply this to (*), we find

$$\|\nabla u_\varepsilon\|_{L^1_{\text{loc}}(U)} = \|\mathcal{M}_\delta \varphi\|_{L^\infty(U)} \|u\|_{\text{BV}_{\text{loc}}(U)} \leq \|u\|_{\text{BV}_{\text{loc}}(U)},$$

by (26).

3. Finally, by the fact that $\mathcal{M}_\delta \varphi \rightarrow \varphi$ in $L^\infty(U)$ as $\delta \rightarrow 0$, we find (29). \square

Lemma 3.1. Integral Mean Value Theorem for functions in BV_{loc}

Let $U \subset \mathbb{R}^m$ be open and let $u \in \text{BV}_{\text{loc}}(U)$. Then, for almost every $\mathbf{y} \in U$, there exists a $U^* \subset U$ with $|U^*| > 0$ such that

$$u(\mathbf{y}) = \frac{1}{|U^*|} \int_{U^*} u \, dx. \quad (31)$$

Proof. Let $u_\varepsilon \in C^\infty(U)$ be the approximation to u from Theorem 3.1. Fix $\mathbf{y} \in U$. Due to the Integral Mean Value Theorem, there exists a U^* with $|U^*| > 0$ such that

$$u_\varepsilon(\mathbf{y}) = \frac{1}{|U^*|} \int_{U^*} u_\varepsilon \, dx.$$

Let us now define

$$\bar{u} := \frac{1}{|U^*|} \int_{U^*} u \, dx.$$

We take the absolute value of the difference and see that

$$|u_\varepsilon(\mathbf{y}) - \bar{u}| = \frac{1}{|U^*|} \left| \int_{U^*} u_\varepsilon - u \, dx \right| \leq \frac{1}{|U^*|} \int_{U^*} |u_\varepsilon - u| \, dx = \frac{1}{|U^*|} \|u_\varepsilon - u\|_{L^1(U^*)}.$$

Since $U^* \subset U$, we have

$$|u_\varepsilon(\mathbf{y}) - \bar{u}| \leq \frac{1}{|U^*|} \|u_\varepsilon - u\|_{L^1_{\text{loc}}(U)} \leq \frac{\varepsilon}{|U^*|}.$$

We apply Theorem 3.1 to see that $u_\varepsilon \rightarrow u$ in $L^1_{\text{loc}}(U)$, meaning $u_\varepsilon(\mathbf{y}) \rightarrow u(\mathbf{y})$ for almost every $\mathbf{y} \in U$. Hence, letting $\varepsilon \rightarrow 0$ completes the proof. \square

Corollary 3.1. *Let $I = [0, T]$ and $u \in L^1(I; BV_{\text{loc}}(U))$, then for almost every $t \in I$ there exists a $U^*(t) \subset U$ such that u satisfies*

$$u(t, \mathbf{y}) = \frac{1}{|U^*(t)|} \int_{U^*(t)} u(t, \mathbf{x}) \, dx, \quad (32)$$

for almost every $t \in I$ and almost every $\mathbf{y} \in U$. Furthermore, $t \mapsto \frac{1}{|U^*(t)|}$ is in $L^\infty(I)$.

Proof. Fix $t \in I$ and apply Lemma 3.1, so that we obtain (32). Note that $\mathbf{y} \in U$ is fixed. Observe now that the left-hand side of (32) is in $L^1(I)$ while the right-hand side is also in $L^1(I)$. Thus, for the equality to hold for almost every $t \in I$, we must have that $t \mapsto \frac{1}{|U^*(t)|}$ is in $L^\infty(I)$. \square

REMARK. *The conclusion that $t \mapsto \frac{1}{|U^*(t)|}$ is in $L^\infty(I)$ is equivalent with stating that $|U^*(t)| > 0$ for almost every $t \in I$.*

4 Existence and uniqueness of solutions to Liouville's equation

With the definition of bounded variation and Lemma 3.1 in place, we are in a position to reinterpret the definition of weak solutions to Liouville's equation. In fact, looking back at Definition 2.1 and (9) in particular, we see that we only need to allow for measure-valued gradients of the Hamiltonian. Or, more precisely, we allow the Hamiltonian to be of bounded variation. However, before we prove existence and uniqueness of solutions to Liouville's equation, we will need Liouville's Theorem, which is an application of Liouville's formula [9].

Theorem 4.1. Liouville's Theorem

Let $I = [0, T]$ and $h \in L^1(I; C^2(\mathcal{P}))$, then the flow generated by the Hamilton equations preserves volume. More specifically, let J be the Jacobian determinant of the mapping $\mathbf{y}(0) \mapsto \mathbf{y}(t)$,

$$J(t) := \det \left(\frac{\partial \mathbf{y}(t)}{\partial \mathbf{y}(0)} \right), \quad (33)$$

then $J(t) = 1$ for almost every $t \in I$.

Theorem 4.2. Existence and uniqueness for Liouville's equation

Let $I = [0, T]$ and let $h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$ and let $\rho_0 \in L^\infty(\mathcal{P})$, then there exists a unique solution to Liouville's equation with $\rho \in L^\infty(I \times \mathcal{P})$. Furthermore, let $\{h_\varepsilon\}_{\varepsilon>0} \subset L^1(I; C^\infty(\mathcal{P}))$ be a sequence of smooth approximations such that $h_\varepsilon \rightarrow h$ in $L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$. Let ρ_ε be the solution to Liouville's equation with Hamiltonian h_ε . Then, in fact, $\rho_\varepsilon \rightarrow \rho$ in $L^\infty(I \times \mathcal{P})$.

Proof. 1. For fixed $\varepsilon > 0$, h_ε satisfies the conditions of Liouville's Theorem, therefore we have

$$\int_{\mathcal{P}} (\rho_\varepsilon(t))^r dy = \int_{\mathcal{P}} (\rho_0)^r dy,$$

for any $r \in \mathbb{R}$, since the transformation $\mathbf{y}(0) \mapsto \mathbf{y}(t)$ has unit Jacobian determinant by Liouville's Theorem 4.1. We take the $\frac{1}{r}$ th power on both sides and let $r \rightarrow \infty$ to find

$$\|\rho_\varepsilon(t)\|_{L^\infty(\mathcal{P})} = \|\rho_0\|_{L^\infty(\mathcal{P})},$$

which holds for almost all $t \in I$, allowing us to take the supremum, yielding

$$\|\rho_\varepsilon\|_{L^\infty(I \times \mathcal{P})} = \|\rho_0\|_{L^\infty(\mathcal{P})}. \quad (*)$$

Note that this equality does not depend on ε .

2. Let us define $F : L^\infty(I \times \mathcal{P}) \times L^1(I; \text{BV}_{\text{loc}}(\mathcal{P})) \times C_0^\infty(I \times \mathcal{P}) \rightarrow \mathbb{R}$, given by

$$F[\rho, h, \varphi] := \int_0^T \int_{\mathcal{P}} \rho \frac{\partial \varphi}{\partial t} dy dt - \int_0^T \int_{\mathcal{P}} \rho (\boldsymbol{\Omega} \nabla \varphi) \cdot dh dt + \int_{\mathcal{P}} \rho_0 \varphi(0) dy, \quad (**)$$

where $dy = dq_1 \dots dq_d dp_1 \dots dp_d$ and $dh = \nabla h dy$, wherever ∇h exists. Clearly, the relation $F[\rho, h, \varphi] = 0$ for all $\varphi \in C_0^\infty(I \times \mathcal{P})$ defines a weak solution to Liouville's equation, so that for every pair $(\rho_\varepsilon, h_\varepsilon)$, we have

$$F[\rho_\varepsilon, h_\varepsilon, \varphi] = 0,$$

for all $\varphi \in C_0^\infty(I \times \mathcal{P})$, since ρ_ε satisfies Liouville's equation in a classical sense with h_ε as the Hamiltonian. Furthermore, we can estimate

$$\begin{aligned} F[\rho, h, \varphi] &\leq \|\rho\|_{L^\infty(I \times \mathcal{P})} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1(I \times \mathcal{P})} \\ &\quad + \|\rho\|_{L^\infty(I \times \mathcal{P})} \int_0^T \|\nabla \varphi(t)\|_{L^\infty(\mathcal{P})} \|h(t)\|_{\text{BV}_{\text{loc}}(\mathcal{P})} dt, \\ &\quad + \|\rho_0\|_{L^\infty(\mathcal{P})} \|\varphi(0)\|_{L^1(\mathcal{P})}, \end{aligned}$$

where the adjective ‘loc’ in the total variation of h comes from the compact support of φ . Applying Hölder’s inequality to the time integral finally gives us,

$$\begin{aligned} F[\rho, h, \varphi] &\leq \|\rho\|_{L^\infty(I \times \mathcal{P})} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1(I \times \mathcal{P})} \\ &\quad + \|\rho\|_{L^\infty(I \times \mathcal{P})} \|\nabla \varphi(t)\|_{L^\infty(I \times \mathcal{P})} \|h\|_{L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))}, \\ &\quad + \|\rho_0\|_{L^\infty(\mathcal{P})} \|\varphi(0)\|_{L^1(\mathcal{P})}, \end{aligned}$$

which shows us that F is well defined and bounded. Furthermore, by choosing $(\rho, h) = (\rho_\varepsilon, h_\varepsilon)$, we see that (*) gives us

$$\begin{aligned} F[\rho_\varepsilon, h_\varepsilon, \varphi] &\leq \|\rho_0\|_{L^\infty(\mathcal{P})} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1(I \times \mathcal{P})} \\ &\quad + \|\rho_0\|_{L^\infty(\mathcal{P})} \|\nabla \varphi\|_{L^\infty(I \times \mathcal{P})} \|h\|_{L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))} \\ &\quad + \|\rho_0\|_{L^\infty(\mathcal{P})} \|\varphi(0)\|_{L^1(\mathcal{P})}, \end{aligned}$$

which is uniformly bounded in ε .

3. Now, $F[\rho_\varepsilon, h_\varepsilon, \varphi]$ is uniformly bounded for every $\varphi \in C_0^\infty(I \times \mathcal{P})$ with the bound depending only on φ . Hence, there exists a subsequence such that the pair $(\rho_{\varepsilon_k}, h_{\varepsilon_k})_{k \geq 1}$ converges weakly in $L^\infty(I \times \mathcal{P}) \times L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$. However, by construction we have that $h_\varepsilon \rightarrow h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$. Thus, we find that there exists a subsequence $\{\rho_{\varepsilon_k}\}_{k \geq 1}$ which converges weakly in $L^\infty(I \times \mathcal{P})$ and let us call the limit $\rho \in L^\infty(I \times \mathcal{P})$. In view of (*) holding for any $\varepsilon > 0$, we furthermore assert that for each $k \geq 1$ fixed, we have that ρ_{ε_k} satisfies (*). Letting $k \rightarrow \infty$, we find that the weak limit ρ also satisfies (*).

4. Suppose there are two such limits to this smoothing and limiting procedure, $\tilde{\rho}_1$ and $\tilde{\rho}_2$, for the same initial conditions. Hence, both $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are the limits of a sequence of solutions generated by the sequence $\{h_\varepsilon\}_{\varepsilon > 0} \subset L^1(I; C^\infty(\mathcal{P}))$. Next, we subtract the two solutions such that $\tilde{\rho}_1 - \tilde{\rho}_2$ has initial condition identically zero. We apply (*) and we see that

$$\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{L^\infty(I \times \mathcal{P})} = 0,$$

hence $\tilde{\rho}_1 = \tilde{\rho}_2$ almost everywhere on $I \times \mathcal{P}$. We conclude that this limiting procedure provides us with a unique candidate solution to Liouville’s equation.

5. Let ρ_1 and ρ_2 satisfy Liouville’s equation for the same Hamiltonian $h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$ and the same initial conditions. Define $\rho := \rho_1 - \rho_2$ and let $\rho_\delta := \mathcal{M}_\delta \rho$ be mollified, thus ρ_δ is a smooth approximation of ρ . Note that both ρ and ρ_δ satisfy Liouville’s equation almost everywhere in $I \times \mathcal{P}$ for zero initial condition. Fix some $s \in I$ and we investigate

$$0 = \int_0^s \int_{\mathcal{P}} \rho \frac{\partial \varphi}{\partial t} dy dt - \int_0^s \int_{\mathcal{P}} \rho (\Omega \nabla \varphi) \cdot dh(t, \mathbf{y}) dt, \quad \forall \varphi \in C_0^\infty(I \times \mathcal{P}).$$

Next, we pick $\varphi = \mathcal{M}_\delta \psi$, with $\psi \in C_0^\infty(I \times \mathcal{P})$. The properties of mollification, in particular (27), give us that

$$0 = \int_0^s \int_{\mathcal{P}} \rho_\delta \frac{\partial \psi}{\partial t} dy dt - \int_0^s \int_{\mathcal{P}} \rho_\delta (\boldsymbol{\Omega} \nabla \psi) \cdot dh(t, \mathbf{y}) dt,$$

which holds for all $\psi \in C_0^\infty(I \times \mathcal{P})$. Integrating by parts, we obtain

$$\int_{\mathcal{P}} \rho_\delta(s) \psi(s) dx = \int_0^s \int_{\mathcal{P}} \psi \frac{\partial \rho_\delta}{\partial t} dy dt + \int_0^s \int_{\mathcal{P}} \psi (\boldsymbol{\Omega} \nabla \rho_\delta) \cdot dh(t, \mathbf{y}) dt,$$

which also holds for all $\psi \in C_0^\infty(I \times \mathcal{P})$. Next, we note that ρ_δ is a weak solution to Liouville's equation with zero initial condition. Therefore it satisfies Liouville's equation almost everywhere in $I \times \mathcal{P}$, giving us

$$\int_{\mathcal{P}} \rho_\delta(s) \psi(s) dx = 0, \quad \forall \psi \in C_0^\infty(I \times \mathcal{P}),$$

where $s \in I$ is also arbitrary. We move the mollifier back to ψ , leaving us with

$$\int_{\mathcal{P}} \rho(s) \varphi(s) dx = 0, \quad \forall s \in I, \varphi \in C_0^\infty(I \times \mathcal{P}),$$

which tells us that $\rho = \rho_1 - \rho_2$ is identically zero almost everywhere on $I \times \mathcal{P}$. \square

Theorem 4.2 gives us existence and uniqueness of solutions to Liouville's equation for Hamiltonians in $L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$. Furthermore, this solution can be constructed as the limit of a sequence generated by Liouville's equation for closer and closer smooth approximations to the Hamiltonian. This has a further interesting consequence which we show in the following Corollary.

Corollary 4.1. A generalization of Liouville's Theorem

Let $I = [0, T]$ and let $h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$, then the flow generated by the Hamilton equations preserves volume in phase space \mathcal{P} . In other words, let J be the Jacobi matrix defined in (33), then instead of having $J = 1$ for almost every $t \in I$, we now have $J = 1$ almost everywhere in $I \times \mathcal{P}$.

Proof. Let the sequence $\{\rho_\varepsilon\}_{\varepsilon \geq 0}$ be the sequence of which the existence is asserted in Theorem 4.2. Applying Liouville's Theorem 4.1, we find

$$\int_{\mathcal{P}} \rho_\varepsilon(t, \mathbf{y}) dy = \int_{\mathcal{P}} \rho_0(\mathbf{y}) dy, \quad \forall t \in I,$$

where $dy = dq_1 dq_2 \dots dq_d dp_1 dp_2 \dots dp_d$ is the volume element in phase space. Observe that the left-hand side depends on ε whereas the right-hand side does not. Thus, we let $\varepsilon \rightarrow 0$ and we deduce the same equality for the limit ρ . Finally, since this holds for all initial conditions $\rho_0 \in L^\infty(\mathcal{P})$, we find that $J = 1$ almost everywhere in $I \times \mathcal{P}$. \square

REMARK. The conclusion of a volume-preserving flow can also be expressed as the Hamilton equations generating a weakly divergence free velocity field. Let us take a test function $\varphi \in L^1(I; C_0^2(\mathcal{P}))$, yielding

$$\int_{\mathcal{P}} \nabla \varphi \cdot (\Omega dh(x)) = - \int_{\mathcal{P}} (\Omega \nabla \varphi) \cdot dh(x) = \int_{\mathcal{P}} \nabla \cdot (\Omega \nabla \varphi) h dx = 0, \quad (34)$$

by (8), holding almost everywhere in I . Hence, the velocity field generated by the Hamilton equations is weakly divergence free, provided we allow for twice continuously differentiable test functions.

One important thing to note about Corollary 4.1 is that it does not prove that Hamiltonian evolution is symplectic. A symplectic transformation does preserve volume on phase space, but not every volume preserving transformation is symplectic. We can, for instance, have a given volume broken up into two unconnected volumes, such as occurs due to a refractive surface in geometric optics. One part of the volume may be refracted while the other can be reflected. In such a case, the flow may be locally symplectic [12], as either reflection or refraction by itself defines a symplectic transformation.

5 Existence of solutions to the Hamilton IVP

Now that we have existence and uniqueness for Liouville's equation, we can ask if this carries over to the Hamilton IVP. In the case of twice continuously differentiable Hamiltonians, the answer is clearly yes. However, when classical solutions do not exist, the answer is less straightforward. However, by using the concept of an almost everywhere flow, we are able to resolve the matter with a positive answer. We shall first examine the existence of solutions to the Hamilton IVP.

Lemma 5.1. *Let $I = [0, T]$, $U \subset \mathbb{R}^m$, $\mathbf{g} \in L^1(I; U)$ and $u \in L^1(I; \text{BV}_{\text{loc}}(U))$. Thus, $u : I \rightarrow \text{BV}_{\text{loc}}(U)$ integrable, while for fixed $t \in I$, $u(t) : U \rightarrow \mathbb{R}$ with bounded variation. Or, in other words, we have that for fixed $\mathbf{x} \in U$, $u(\cdot, \mathbf{x})$ is integrable and for fixed $t \in I$, $u(t, \cdot)$ is of bounded variation. Let $\{u_\varepsilon\}_{\varepsilon > 0}$ be the smooth approximations from Theorem 3.1 and let*

$$\mathbf{f}_\varepsilon(t) := \int_0^t \nabla u_\varepsilon(s, \mathbf{g}(s)) ds, \quad (35)$$

then $\mathbf{f}_\varepsilon \rightarrow \mathbf{f}$ in $\text{BV}(I; U) \cap L^\infty(I; U)$. Furthermore, we have the following bound

$$\|\mathbf{f}\|_{\text{BV}(I; U)} + \|\mathbf{f}\|_{L^\infty(I; U)} \leq C \|u\|_{L^1(I; \text{BV}_{\text{loc}}(U))}, \quad (36)$$

where $C > 0$ does not depend on \mathbf{g} .

Proof. 1. We apply the Integral Mean Value Theorem to (35) to find that

$$\mathbf{f}_\varepsilon(t) = \int_0^t \frac{1}{|U^*(s)|} \int_{U^*(s)} \nabla u_\varepsilon(s, \mathbf{x}) \, dx \, ds,$$

where $U^*(s) \subset U$ such that $\mathbf{g}(s) \in U^*(s)$ for almost every $s \in I$. The argument can be broken down into four steps. First, we fix some $s \in [0, t]$. Second, we define $\mathbf{y} := \mathbf{g}(s)$, which is then also a fixed element in U . Third, we apply the Integral Mean Value Theorem to ∇u_ε for \mathbf{y} , this is allowed since u_ε is infinitely differentiable. Finally, the fourth step is conclude that this must hold for all $s \in [0, t]$. Hence, we have

$$\mathbf{f}'_\varepsilon(t) = \frac{1}{|U^*(t)|} \int_{U^*(t)} \nabla u_\varepsilon(t, \mathbf{x}) \, dx, \quad (*)$$

where we have for every $\varepsilon > 0$ that $\mathbf{f}_\varepsilon \in W^{1,1}(I; U)$. This follows from the fact that $s \mapsto |U^*(s)| \in L^\infty(I)$ according to Lemma 3.1, while the spatial integral of ∇u_ε is integrable in time. Thus, the product is an integrable function and therefore we find that \mathbf{f}'_ε is integrable.

2. We compute the total variation of \mathbf{f}_ε using (21), yielding

$$\|\mathbf{f}_\varepsilon\|_{\text{BV}(I; U)} = \int_0^T |\mathbf{f}'_\varepsilon(s)| \, ds,$$

where \mathcal{B} is defined in Definition 3.1. We now use (*) to find that

$$\|\mathbf{f}_\varepsilon\|_{\text{BV}(I; U)} \leq \int_0^T \frac{1}{|U^*(s)|} \int_{U^*(s)} |\nabla u_\varepsilon(s, \mathbf{x})| \, dx \, ds \leq C \|u\|_{L^1(I; \text{BV}_{\text{loc}}(U))}, \quad (**)$$

where $C > 0$ may depend on \mathbf{g} . However, Corollary 3.1 holds for any $s \in I$ and any $\mathbf{x} \in U$, thus there exists a $\delta > 0$ such that $|U^*(s)| \geq \delta$ for all $s \in I$ and all $\mathbf{x} \in U$, regardless of \mathbf{g} . Therefore, we define $C := \frac{1}{\delta}$ and see that the bound (**) is independent of \mathbf{g} . Note also that the right-hand side of (**) does not depend on ε , allowing us to take the limit $\varepsilon \rightarrow 0$, from which we see that $\mathbf{f} \in \text{BV}(I; U)$.

3. From (*), we can furthermore find

$$|\mathbf{f}_\varepsilon(t)| \leq C \int_0^t \|\nabla u_\varepsilon(s)\|_{L^1_{\text{loc}}(U)} \, ds,$$

where $C = \frac{1}{\delta}$ again. Applying the bound from Theorem 3.1, we obtain

$$|\mathbf{f}_\varepsilon(t)| \leq C \|u\|_{L^1(I; \text{BV}_{\text{loc}}(U))},$$

which is independent of time, allowing us to take the supremum, giving us that $\mathbf{f}_\varepsilon \in L^\infty(I; U)$. We note furthermore that the bound is uniform in ε , therefore also $\mathbf{f} \in L^\infty(I; U)$. Choosing $C = \frac{2}{\delta}$ in (36) completes the proof. \square

REMARK. Abusing notation slightly, we can write for \mathbf{f} from Lemma 5.1,

$$\mathbf{f}(t) := \int_0^t \nabla u(s, \mathbf{g}(s)) \, ds, \quad (37)$$

which shows that such integrals are, if not technically correct, bounded and therefore well defined.

Corollary 5.1. Existence of solutions to the Hamilton IVP

Let $I = [0, T]$ and $h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$, then there exists a solution to the Hamilton IVP $\mathbf{y} \in \text{BV}(I; \mathcal{P}) \cap L^\infty(I; \mathcal{P})$, which satisfies the integral equation

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_0^t \Omega \nabla h(s, \mathbf{y}(s)) \, ds, \quad (38)$$

where $\mathbf{y}_0 \in \mathcal{P}$ is the initial condition.

Proof. 1. Let us define the Picard iterations

$$\mathbf{y}^{k+1}(t) = \mathbf{y}_0 + \int_0^t \Omega \nabla h(s, \mathbf{y}^k(s)) \, ds, \quad k \geq 1, \quad (*)$$

and $\mathbf{y}^0(t) := \mathbf{y}_0$ for all $t \in I$. By Lemma 5.1, these integrals are well defined for every $k \geq 1$ and, in fact, $\mathbf{y}^k \in \text{BV}(I; \mathcal{P}) \cap L^\infty(I; \mathcal{P})$ for all $k \geq 1$.

2. Lemma 5.1 also asserts that $\{\mathbf{y}^k\}_{k \geq 1}$ is a uniformly bounded sequence. Therefore, there exists a subsequence $\{\mathbf{y}^{k_l}\}_{l \geq 1}$ which converges in $\text{BV}(I; \mathcal{P}) \cap L^\infty(I; \mathcal{P})$. By construction, the limit, which we denote by \mathbf{y} , satisfies (38) and thus satisfies the Hamilton IVP for almost every $t \in I$. \square

Now that we have shown existence of solutions to the Hamilton IVP, we are in a position to show the existence and uniqueness of a flow generated by the Hamiltonian velocity field. We shall first recall the definition of a flow, afterwards showing the existence and uniqueness of such a flow.

Definition 5.1. (Almost Everywhere) Flow

Let $I = [0, T]$ and $U \subset \mathbb{R}^m$, a function $\Phi : I \times U \rightarrow U$ is called a flow if it satisfies the following two properties,

$$\Phi_0 \mathbf{x} = \mathbf{x}, \quad (39a)$$

$$\Phi_t \Phi_s \mathbf{x} = \Phi_{s+t} \mathbf{x}, \quad (39b)$$

where $\Phi_t \mathbf{x} := \Phi(t, \mathbf{x})$. Furthermore, a flow is said to be an almost everywhere flow if these conditions hold in a distributional sense, meaning for almost every $t, s \in I$ and almost every $\mathbf{x} \in U$.

Theorem 5.1. Flow representation

Let $I = [0, T]$, $\mathcal{P}_T := I \times \mathcal{P}$ and $h \in L^1(I; \text{BV}_{\text{loc}}(\mathcal{P}))$, then there exists a unique flow $\Phi : I \times \mathcal{P}_T \rightarrow \mathcal{P}_T$, such that the solution to Liouville's equation is given by

$$\rho(\Phi_t(0, \mathbf{y})) = \rho_0(\mathbf{y}), \quad (40)$$

in a distributional sense. Furthermore, Φ is the unique flow solution to the Hamilton equations.

Proof. 1. Let us denote $\mathbf{x} = (t, \mathbf{y})$ and let us define $F \in \text{BV}_{\text{loc}}(\mathcal{P}_T)$, up to a constant, by

$$dF(\mathbf{x}) := \begin{pmatrix} dt \\ \Omega dh(\mathbf{x}) \end{pmatrix}, \quad (41)$$

where $dh(\mathbf{x}) = \nabla h(t, \mathbf{y}) dy$, wherever the gradient exists. Fix $\mathbf{x}_0 = (t_0, \mathbf{y}_0)$, we now look for a flow solution $\Phi : I \times \mathcal{P}_T \rightarrow \mathcal{P}_T$ which satisfies

$$\frac{d}{dt} \Phi_t \mathbf{x}_0 = \begin{pmatrix} 1 \\ \Omega \nabla h(\Phi_t \mathbf{x}_0) \end{pmatrix}, \quad (42)$$

in a distributional sense. Abusing notation slightly, we have that $\Phi_t \mathbf{x}_0$ satisfies

$$\Phi_t \mathbf{x}_0 = \mathbf{x}_0 + \int_0^t \square F(\Phi_s \mathbf{x}_0) ds, \quad (*)$$

where we have defined

$$\square := \begin{pmatrix} \frac{\partial}{\partial t} \\ \nabla \end{pmatrix},$$

which is the gradient operator in \mathcal{P}_T . By Corollary 5.1, there exists a Φ that satisfies (*).

2. Choosing $\mathbf{x}_0 = (0, \mathbf{y}_0)$, we can represent the solution to Liouville's equation as

$$\rho(\Phi_t \mathbf{x}_0) = \rho(\mathbf{x}_0) = \rho_0(\mathbf{y}_0), \quad (**)$$

which becomes evident if we differentiate with respect to time. Adopting the operator \square , we find

$$\frac{d}{dt} \rho(\Phi_t \mathbf{x}_0) = \square \rho(\Phi_t \mathbf{x}_0) \cdot \begin{pmatrix} \frac{d}{dt} \Phi_t \mathbf{x}_0 \end{pmatrix} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot (\Omega \nabla h) = 0,$$

where Liouville's equation is to be evaluated at $\mathbf{x} = \Phi_t \mathbf{x}_0$, and holds in a distributional sense. Hence, (**) also holds in a distributional sense, meaning for almost every $\mathbf{y}_0 \in \mathcal{P}$ and almost every $t \in I$. However, since ρ is the unique solution to Liouville's equation for a given $\rho_0 \in L^\infty(\mathcal{P})$, we see that Φ is uniquely defined by (**) almost everywhere.

3. Pick $\mathbf{x}_0 = \Phi_s \mathbf{x}$ in (**), applying (**) several times yields

$$\rho(\Phi_t \Phi_s \mathbf{x}) = \rho(\Phi_s \mathbf{x}) = \rho(\mathbf{x}) = \rho(\Phi_{s+t} \mathbf{x}),$$

which must hold in a distributional sense and furthermore for all possible initial conditions $\rho_0 \in L^\infty(\mathcal{P})$, from which we find that

$$\Phi_t \Phi_s \mathbf{x} = \Phi_{s+t} \mathbf{x},$$

which also holds in a distributional sense. Together with the initial condition $\Phi_0 \mathbf{x} = \mathbf{x}$ from (*), we conclude that Φ is indeed an almost everywhere flow. \square

We can interpret Theorem 5.1 as a generalisation of the MOC. In fact, it shows us that if we can find the flow Φ , the solution to Liouville's equation is given by (40). By Corollary 5.1, the characteristics can also be found by a limiting procedure much like the solutions to Liouville's equation. Thus, also the construction by the MOC converges in some sense to a limit.

If we happen to know the flow, we can also extract the characteristics which then satisfy a Hamilton IVP. For almost every $\mathbf{y}_0 \in \mathcal{P}$, we can define $\mathbf{y} : I \rightarrow \mathcal{P}$ by

$$(t, \mathbf{y}(t)) = \Phi_t(0, \mathbf{y}_0), \quad (43)$$

which is a solution to the Hamilton IVP for initial condition \mathbf{y}_0 . Note however that, since Φ is an almost everywhere flow, there is a null-set of initial conditions where uniqueness is not guaranteed. In geometrical optics, such a null-set is easily found by considering light rays incident on a refracting plane under the critical angle. Such rays will refract to run parallel to and along the refracting surface. Hence, every point along the ray is an intersection point of the ray and the plane, thereby losing uniqueness.

Interpreting Theorem 5.1 as a generalisation of the MOC, and invoking (43), it follows that for almost every $t \in I$ and almost every $\mathbf{y}_0 \in \mathcal{P}$, we have

$$\rho(t, \mathbf{y}(t)) = \rho_0(\mathbf{y}_0). \quad (44)$$

Let us say that at some $\tau \in I$, which we assume to be a regular point, the characteristic \mathbf{y} encounters a discontinuity in h . We then have that

$$\rho(\tau^+, \mathbf{y}(\tau^+)) = \rho(\tau^-, \mathbf{y}(\tau^-)), \quad (45)$$

where the pluses and minuses are the one-sided limits. Thus, by Theorem 5.1, we have justified the construction of solutions to Liouville's equation by use of discontinuous characteristics. In another work [14], we have developed a numerical solver for Liouville's equation based on (45).

6 Conclusion

We have shown existence and uniqueness for solutions to Liouville's equation for Hamiltonians with bounded variation. Furthermore, we can interpret these solutions as the limit of a sequence of solutions generated by smooth Hamiltonians. We have also shown that this implies an almost everywhere flow solution to the Hamilton equations. Thus, as the name suggests, there is a null-set of initial conditions which does not yield a unique solution to the Hamilton IVP.

As a bonus, we were also able to give a generalisation of the MOC and show that Hamiltonian flow on phase space is volume preserving even for Hamiltonians with bounded variation. In another work, we have applied this generalisation of the MOC to develop a novel scheme for the numerical resolution of Liouville's equation.

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