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Optimal and heuristic repairable stocking and expediting in a fluctuating demand environment

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Abstract

We consider a single stock point for a repairable item. The repairable item is a critical component that is used in a fleet of technical systems such as trains, planes or manufacturing equipment. A number of spare repairables is purchased at the same time as the technical systems they support. Demand for those items is a Markov modulated Poisson process of which the underlying Markov process can be observed. Backorders occur when demand for a ready-for-use item cannot be fulfilled immediately. Since backorders render a system unavailable for use, there is a penalty per backorder per unit time. Upon failure, defective items are sent to a repair shop that offers the possibility of expediting repair. Expedited repairs have shorter lead times than regular repairs but are also more costly. For this system, two important decisions have to be taken: How many spare repairables to purchase initially and when to expedite repairs. We formulate the decision to use regular or expedited repair as a Markov decision process and characterize the optimal repair expediting policy for the infinite horizon average and discounted cost criteria. We find that the optimal policy may take two forms. The first form is to never expedite repair. The second form is a type of threshold policy. We provide necessary and sufficient closed-form conditions that determine what form is optimal. We also propose a heuristic repair expediting policy which we call the world driven threshold (WDT) policy. This policy is optimal in special cases and shares essential characteristics with the optimal policy otherwise. Because of its simpler structure, the WDT policy is fit for use in practice. We show how to compute optimal repairable stocking decisions in combination with either the optimal or a good WDT expediting policy. In a numerical study, we show that the WDT heuristic performs very close to optimal with an optimality gap below 0.76% for all instances in our test bed. We also compare it to more naive heuristics that do not explicitly use information regarding demand fluctuations and find that the WDT heuristic outperforms these naive heuristics by 11.85% on average and as much as 63.67% in some cases. This shows there is great value in leveraging knowledge about demand fluctuations in making repair expediting decisions.

Keywords: expediting, Markov decision process, optimal policies, heuristic policies

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1. Introduction

Both service and manufacturing industries depend on the availability of expensive equipment to deliver their products. Examples of such equipment include aircraft, rolling stock and manufacturing equipment. When this equipment is not working, the primary processes of their owners come to an immediate stop. To reduce the downtime of equipment, companies stock critical components such that the equipment can be returned to an operational state quickly by replacing a defective component with a ready-for-use component. Many components represent a significant financial investment and so they are repaired rather than discarded after a defect occurs. Consider for example, aircraft engines, bogies, or lenses for wafer steppers; these are components of aircraft, rolling stock, and integrated circuit manufacturing equipment, respectively, and their prices range from several hundreds of thousands up to tens of millions of dollars. These expensive components are very specific to the equipment they service. Consequently, the best time for companies to buy these components is at the same time as when the original equipment is purchased, because, at this time, it is possible to negotiate reasonable prices. In the literature, this is often referred to as the initial spare parts supply problem and it occurs in many different environments (e.g. Rustenburg et al., 2001; Pérès and Grenouilleau, 2002). Later in time, such components often have to be custom made and prices are very steep, if the component can be purchased at all. An aggravating factor is that demand intensity for these components typically fluctuates over time, reflecting the fluctuating need for maintenance over time. Companies anticipate these demand fluctuations by leveraging the possibility of expediting the repair of defective components, rather than buying new components. Expediting repair comes at a price, either because an external repair shop charges more for expedited repairs or because an internal repair shop can only handle a limited amount of expedited repairs. In the latter case, the cost of expediting can be thought of as a Lagrange multiplier that enforces a constraint on the number of expedited repairs that can be requested per time unit. In this situation, the model in the present paper serves as a building block for a multi-item model.

Companies that operate in the environment described above face two major decisions related to their inventory control, one at the tactical level, and another at the operational level:

1. How many repairable spare parts should the firm buy? (tactical)
2. When should the firm request that the repair of a part is expedited? (operational)

We refer to the first decision as the dimensioning decision and to the second as the expediting decision. The spare repairables are usually purchased at the same time as the technical systems that they support. After this time, the repairables are either no longer available in the market or prohibitively expensive. Thus the decision to buy repairables is a tactical decision that occurs one time only. The S spare repairables that are purchased at the time of the acquisition of the technical system are also called the *turn-around stock*. After this initial tactical decision there is an operational recurring decision to either expedite or not expedite the repair of a spare part each time a demand occurs. These decisions should take demand fluctuations as well as current inventory levels into account. The model in this paper is intended to aid both the dimensioning and the expediting decision. Especially for the dimensioning decision, it is important to consider the fact that expediting will occur later at the operational level.

We study the decision problem of the previous paragraph via a stochastic inventory model for repairable items. In this model a defective item is replaced with a ready-for-use item and sent to a repair shop immediately after the defect occurs. At this point in time, the inventory manager is faced with the decision to either expedite or not expedite the repair of the part. Expediting repair is more costly but has a shorter lead time. This expediting decision is informed by knowledge about the fluctuation of demand intensity over time.

Our model runs in continuous time, and demand for the component is a Markov modulated Poisson process (MMPP). The state of the Markov chain that drives the demand process can be observed directly and is used to inform the expediting decision. This demand model is quite rich and can serve to model such divers things as economic conditions, seasons of a year, the degradation of a fleet of equipment, and knowledge about the maintenance program of equipment (Song and Zipkin, 1993). It has also been observed empirically that demand for repairable spare parts behaves as a non-stationary Poisson process (Slay and Sherbrooke, 1988). In any case, the MMPP offers the flexibility to model both stationary and non-stationary demand processes and so it can be used to model a wide variety of demand models. In particular it offers the possibility to model demand fluctuations.

We assume that inventory is replenished by an $(S-1, S)$-policy, meaning that each defective item is sent immediately to the repair shop. This replenishment policy is often used in practice and it is optimal when there are no economies of scale in replenishment. We model the expedited lead time as being deterministic and the regular lead time as being the sum of the expedited lead time and several exponential phases, the passing of which is monitored. Modeling lead time as such is a convenient device to investigate the value of tracking order progress information and the effect of different lead time distributions. (Gaukler et al., 2008, use a very similar model of order progress information.) Many lead time distributions can be modeled quite closely by this device and in particular deterministic lead times can be approximated arbitrarily closely by letting the number of exponential phases approach infinity.

The main contributions of this paper are the following: Firstly, for the described setting, we characterize the optimal repair expediting policy for the infinite horizon average and discounted cost criteria by formulating the problem as a Markov decision process. We find that the optimal policy may take two forms. The first form is simply to never expedite repair. The second form is a state dependent threshold policy, where the threshold depends on both the state of the modulating chain of demand and the pipeline of repair orders. We also provide monotonicity results for the threshold as a function of the pipeline of repair orders. We give closed-form conditions that determine which of the two forms is optimal. In analyzing the optimal policy, we confirm a conjecture of Song and Zipkin (2009) that the expediting policy they propose is optimal for some special cases.

Secondly, we show that for the joint problem of determining the turn-around stock and the expediting policy, the cost function is submodular with respect to the turn-around stock and the expediting thresholds. This means that the possibility to expedite repairs can act as a substitute for holding turn-around stock. We also show how to optimally solve the joint problem of determining the turn-around stock and the expediting policy.

Thirdly, we propose a heuristic that is computationally efficient, and is shown to perform well compared to the optimal solution. In this heuristic, we replace the optimal expediting policy with a parameterized threshold policy that shares important monotony properties with the optimal expediting policy. The thresholds depend on the available knowledge about the fluctuation of demand. Borrowing the terminology of Zipkin (2000), we call this policy the world driven threshold (WDT) policy. In a numerical study involving a large test bed, this heuristic has an average and maximum optimality gap of 0.15% and 0.76%, respectively.

Finally, we investigate the value of anticipating demand fluctuations by comparing optimal joint stocking and expediting policy optimization against naive heuristics that do not explicitly model demand fluctuations, or that separate the stocking and expediting policy decisions. These naive heuristics have optimality gaps of 11.85% on average and range up to 63.67% in our numerical study. The comparison with these naive heuristics show that:

1. There is great value in leveraging knowledge about demand fluctuations in making repair expediting decisions.

2. Fluctuations of demand and the possibility to anticipate these through expediting repairs should be considered explicitly in sizing the turn-around stock and can lead to substantial savings.

This paper is organized as follows. In §2, we review relevant literature and position our contribution with respect to existing results. The model is described in §3 and analyzed exactly in §4. In general, the exact analysis leads to algorithms that suffer from the curse of dimensionality. Therefore, in §5, we study a heuristic informed by our exact analysis that is computationally tractable. In §6, we provide numerical results on the performance of the heuristic we propose and we investigate the value of anticipating demand fluctuations through the joint optimization of the turn-around stock and expediting policy. Concluding remarks are provided in §7.

2. Literature review

Our model is situated at the intersection of two streams of literature. The first one deals with sizing the turn-around stock of repairable item inventories and the second one with expediting, or inventory models with two (or more) supply modes.

An important characteristic of repairable item inventories is that inventory is replenished by repairing defective items. Repairable item inventory systems thus form a closed loop system that implicitly dictates base-stock levels. Often, the number of supported assets is large and the demand process is assumed to be independent of the number of outstanding orders. A small stream of literature considers situations where the number of supported technical systems is low and so the number of outstanding orders will affect the demand process (e.g. Gross and Ince, 1978). We assume that demand for the repairable is not affected by the number of outstanding repair orders. This is in line with the modeling assumptions of most of the repairable item inventory literature that was started with the METRIC model introduced by Sherbrooke (1968). Most of the important results in this stream of literature have been consolidated in the books by Sherbrooke (2004) and Muckstadt (2005). This paper adds to the literature on repairable item inventories by studying what happens when it is possible to expedite the repair of a defect part, and in particular if this flexibility can be used to respond to a fluctuating demand environment. In doing this, we relax the commonly held assumption that demand is a stationary Poisson process. Our assumption of a Markov modulated Poisson process is more in line with empirical findings (Slay and Sherbrooke, 1988). Verrijdt et al. (1998) already studied simple heuristics for the case that demand is a stationary Poisson process and emergency and regular repair lead times are both exponentially distributed. We relax the assumptions that the demand process is stationary and consider a more general lead time structure. Furthermore, we study optimal solutions as well as a new heuristic informed by the structure of the optimal solution. We also remark that expediting repair is not the same as shipping a ready-for-use part from a different stocking location which is commonly known as an emergency shipment (e.g. Alfredsson and Verrijdt, 1999).

Inventory models with multiple supply modes have been reviewed by Minner (2003). Here we review the important and more recent results. Most authors consider a *periodic review* setting where the regular and expedited lead time differ by a single period and find that a base-stock policy is optimal for both the regular and expedited supply modes (e.g. Fukuda, 1964). When the lead time of the regular and expedited supply modes differ by more than a single period, optimal policies do not exhibit simple structure and depend on the entire vector of outstanding orders (e.g. Whittmore and Saunders, 1977; Feng et al., 2006). As a result, recent research considers heuristic policies for the control of dual supply systems, the most notable of these being the dual-index policy and variations thereof (Veeraraghavan and Scheller-Wolf, 2008; Sheopuri et al., 2010; Arts et al., 2011). Under the dual-index policy, a regular and emergency inventory position are tracked separately, and both are kept at or above their respective order-up-to levels.

As opposed to the above mentioned papers, Moinzadeh and Schmidt (1991) consider a system running in *continuous time* facing Poisson demand with deterministic emergency and regular replenishment lead times. They show how to evaluate a given dual-index policy, although the name was not coined at the time, and the structure was not recognized as such. Song and Zipkin (2009) reinterpret the model of

Moinzadeh and Schmidt (1991) revealing the simple structure of the policy and show how the performance of any such policy can be evaluated in closed form using an equivalence to a special type of queueing network that has a product form solution. This equivalence also allows them to consider very general lead time structures. Verrijdt et al. (1998) consider a similar system in the context of repairable items. In their model, the regular and expedited supply/repair modes have independent exponentially distributed lead times. They consider a different policy where repair is expedited when the inventory on hand drops below a certain critical level.

While two different heuristic expediting policies have been suggested in the literature by Moinzadeh and Schmidt (1991) and Song and Zipkin (2009), and Verrijdt et al. (1998), the optimal expediting policy has not yet been investigated. Song and Zipkin (2009) conjecture that their policy is optimal in some special cases. In this paper, we analyze the optimal repair expediting policy in the case of deterministic expedited repair lead times and stochastic regular repair lead times. As it turns out, the form suggested by Moinzadeh and Schmidt (1991) and Song and Zipkin (2009) is optimal in the special case that the regular repair lead time has a shifted exponential distribution and demand is a Poisson process. For more general lead time structures and demand processes, the optimal policy is a generalization of this policy. We note that Song and Zipkin (2009) also considered Markov modulated Poisson demand as an extension, but their expediting policy does not depend on the state of modulating chain of demand.

3. Model formulation

Our model supports two decisions: (i) How to dimension the turn-around stock S and (ii) what expediting policy to follow. The two decisions we consider in this paper live in different time scales. For the analysis, we will use a nested procedure that determines the optimal expediting policy for a given turn-around stock, and use this to determine the optimal turn-around stock. Below we give an integrated description of the model. In §3.1, we discuss the main assumptions of the model and their justifications.

We consider a repairable item stock-point operated in continuous time with an infinite planning horizon $[0,\infty)$. The stock-point faces Markov modulated Poisson demand, i.e., demand is a Poisson process whose intensity varies with the state of an exogenous Markov process $Y(t)$. The Markov process $Y(t)$ is irreducible and has a finite state space $\Theta=\{1,...,|\Theta|\}$ with generator matrix $\mathbf{Q}$ whose elements we denote by q_{ij}. For notational convenience, we also define $q_i=-q_{ii}$ and $q_{\max}=\max_{i\in\Theta} q_i$. When $Y(t)=y$, the intensity of Poisson demand is given by $\lambda_y\geq 0$; $\boldsymbol{\lambda}=(\lambda_1,...,\lambda_{|\Theta|})$, $\lambda_y>0$ for at least one $y\in\Theta$. For convenience, we also define $\lambda_{\max}=\max_{y\in\Theta}\lambda_y$. We denote demand in the time interval $(t_1,t_2]$ given $Y(t_1)=y$ as $D^y_{t_1,t_2}$. Note that $Y(t_1)$ provides information about the distribution of demand in the interval $(t_1,t_2]$, $t_2>t_1$. We assume that $Y(t)$ can be observed by the decision maker and so it provides a form of aggregated advance demand information.

The size of the turn-around stock, $S\in\mathbb{N}_0$, of the repairable is determined at time $t=0$ and cannot

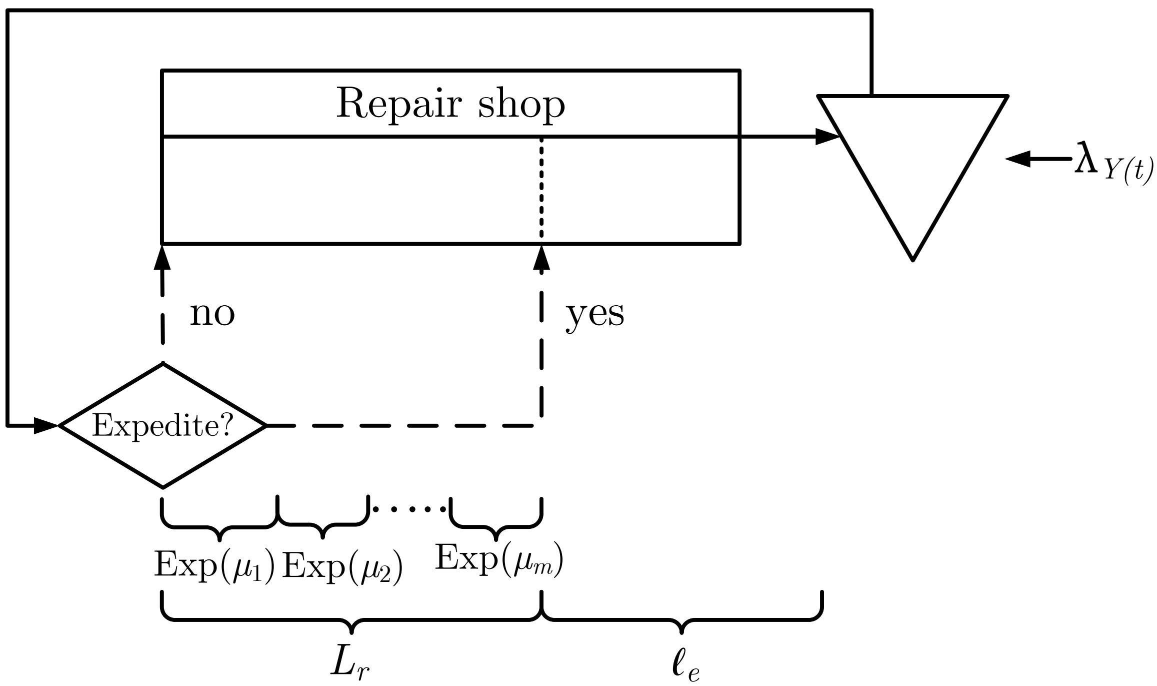

Figure 1: Repairable item inventory system with the possibility to expedite repair

be adapted afterwards. We assume that failed parts can always be repaired (no condemnation) and that defective parts are sent to the repair shop immediately, i.e., we use an $(S-1, S)$ replenishment policy.

There exists a regular and an expedited repair option. The expedited repair lead time, ℓ_e, is deterministic. The expedited repair lead time may represent things such as the transport time and the repair time or a lead time agreed upon with an external company that provides emergency repair service. We also refer to using the expedited repair mode as expediting repair.

The regular repair lead time consists of the emergency repair lead time ℓ_e, and a random component of length L_r, with mean $\mathbb{E}[L_r] < \infty$. We shall also refer to L_r as the *additional regular repair lead time.* L_r is used to model such things as the time that a part waits for resources to become available in the repair shop or the lead time difference between regular and emergency repair lead times as contracted with an external repair shop. We assume that this additional time is distributed as the sum of m exponential phases, with mean $1/\mu_i$ for the i-th exponential phase. We also let $\mu_{\max} = \max_{i\in\{1,\ldots,m\}} \mu_i$. The inventory manager can observe the pipeline of outstanding orders and so she knows how many phases each part in the pipeline has completed. In particular, the inventory manager knows when the last phase (m) is completed and the remaining lead time of a regular order is ℓ_e. A graphical representation of the system under study is given in Figure 1.

Turn-around stock holding (and depreciation) costs are incurred with a constant rate $h > 0$ for all repairable spare parts, regardless of where they are in the supply chain. Repair expediting costs per item are $c_e > 0$, i.e., c_e represents the cost difference between using the regular and emergency repair modes. A penalty cost rate $p > 0$ per item short per time unit for the repairable item inventory is also charged (backordering). We are interested in minimizing the long run average cost rate by (i) deciding on the turn-around stock, S, to be purchased at time $t = 0$ and (ii) implementing a repair expediting policy. (As an extension, we shall also consider minimizing total discounted costs for the expediting policy in §4.1.4.)

3.1 Main assumptions and justifications

In the model of §3, some assumptions require either a practical or analytical justification. Here we list the main assumptions and their justifications.

- The turn-around stock S is determined at time $t = 0$, and remains fixed after that: Because repairables are specific to the capital asset that they support, they are only produced in small series when the capital asset is produced. After the particular capital asset is no longer produced, the repairable is either no longer available or has to be custom made against a steep price. Thus, for the user of the capital asset, it is most economical to purchase all spare repairables jointly with the asset they support.

- We consider an infinite planning horizon. The lifetime of repairables considered in the model is as long as the life cycle of the assets they support which is typically several decades. This is long compared to other time characteristics in the problem such as lead times which are typically measured in weeks, and justifies using infinite horizon models.

- Demand is a Markov modulated Poisson process: In spare parts literature, the Poisson demand model is perhaps the most common (e.g. Sherbrooke, 2004; Muckstadt, 2005). For relatively short periods of time, this demand model is often sufficiently accurate, and Markov modulated Poisson demand can handle Poisson demand as a special case. For longer periods of time, the demand intensity for repairables may be affected by things such as weather conditions (increased wear) and periodic inspections. Slay and Sherbrooke (1988) observe that demand for aircraft components behaves as a Poisson process for which the rate varies over time. There are many reasons for this behavior such as weather, asset loading, and the fact that many capital assets undergo one or more major revisions during their lifetime. During these revision periods, demand for repairables peaks, as inspections reveal latent failures. Often, the exact timing of revision periods is uncertain when the asset is acquired. The Markov modulated Poisson process offers the flexibility to model these and many other demand scenarios.

- Repair of a part is always possible (no condemnation): Under normal operations, the expensive repairables considered in our model only fail permanently in case of industrial accidents. Generally, the probability of this happening is negligible.

- The additional regular repair lead time, L_r, can be modeled by a sum of exponential phases, and phase transitions can be observed. This assumption may appear to be quite strong. However, we think of the m exponential phases of L_r primarily as a device to model order progress information. A special case occurs as $m \to \infty$ as this will approach deterministic replenishment lead times and order progress is known exactly. We also note that if the first two moments of L_r are known (or estimated from data) and satisfy $c^2_{L_r} = \mathbf{Var}[L_r]/\mathbb{E}^2[L_r] \leq 1$, then m and μ_i, $i = 1, ..., m$ can be chosen so as to match these moments. Such a fitting procedure will require that $m \geq \lfloor 1/c^2_{L_r} + 1 \rfloor$ (Aldous and Shepp, 1987). It is evident that as $c^2_{L_r}$ decreases, more information is available on when a repairable completes its additional regular repair lead time. Under the present model, this is naturally matched by increasing m. Therefore, we may think of the parameter m as a modeling device that conveys how closely one tracks, or is able to track, the progress of repairables through the replenishment pipeline. In particular, as $m \to \infty$, the regular replenishment lead time approaches a deterministic lead time and order progress is known exactly. Thus, this assumption allows us to gain insight on the added value of being able to track repairable order progress carefully. In §6, we show that this added value is small. Therefore, we believe it is unnecessary to refine the model of the additional regular repair lead time and order progress.

4. Exact Analysis

The analysis of the model benefits from first considering the optimization of the expediting policy separately. That is, we use a nested procedure. Therefore in §4.1, we consider our model where the turn-around stock, S, is fixed, and focus on finding an optimal expediting policy. We call this problem $\mathfrak{M}(S)$. After this, we turn our attention to the *joint* problem of sizing the turn-around stock and determining an expediting policy in §4.2.

4.1 Expediting policy optimization

In this subsection, we consider the problem of finding optimal repair expediting policies for fixed S, Problem $\mathfrak{M}(S)$). Since the holding costs depend linearly on S only, we need not consider holding cost in finding an optimal expediting policy for a fixed S. We make several steps in our analysis. First, we give the state space description and give closed form conditions under which the state space can be truncated to yield a finite state space for the purpose of finding average optimal expediting policies. We also show that when these conditions do not hold, the optimal policy is to never expedite repair. After that, we formulate a finite horizon finite state space Markov decision process. The average optimal expediting policy is characterized in §4.1.3 and the infinite horizon discounted version in §4.1.4.

4.1.1 State space description

Let $X_i(t)$ denote the number of items in regular repair at time t that are in the i-th phase of their additional repair lead time ($i = 1, ..., m$), and let $\mathbf{X}(t) = (X_1(t), ..., X_m(t))$. The following observation shows that $\mathbf{X}(t)$ and $Y(t)$ contain all the information needed to make expediting decisions. Let $c_p(x, y)$ denote the expected penalty cost rate at time $t+\ell_e$ conditional on $\sum_{i=1}^m X_i(t) = \mathbf{X}(t)\mathbf{e}^{\mathrm{T}} = x$ and $Y(t) = y$; $c_p : \mathbb{N}_0 \times \Theta \to \mathbb{R}$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbf{e} = (1, 1, \ldots, 1)$). To find $c_p(x, y)$, note that $S - \mathbf{X}(t)\mathbf{e}^{\mathrm{T}} = S - x$ represents the number of parts that are on hand at the stockpoint at time t or will arrive at the stockpoint before time $t+\ell_e$. Thus the expected number of backorders at time $t+\ell_e$ given $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}} = x$ and $Y(t) = y$ is $\mathbb{E}\left[\left(D^{Y(t)}_{t,t+\ell_e} - (S - \mathbf{X}(t)\mathbf{e}^{\mathrm{T}})\right)^+ \,\middle|\, \mathbf{X}(t)\mathbf{e}^{\mathrm{T}} = x, Y(t) = y\right]$. From this it is easily verified that

$$
\begin{aligned}
c_p(x, y) &= p\,\mathbb{E}\left[\left(D^{Y(t)}_{t,t+\ell_e} - \left(S - \mathbf{X}(t)\mathbf{e}^{\mathrm{T}}\right)\right)^+ \,\middle|\, \mathbf{X}(t)\mathbf{e}^{\mathrm{T}} = x, Y(t) = y\right] \\
&= p \sum_{k=S-x}^{\infty} (k - (S - x))\,\mathbb{P}\left\{D^{y}_{t,t+\ell_e} = k\right\}. \qquad (1)
\end{aligned}
$$

When convenient, we also use the notation $c_p(x, y|S)$ for $c_p(x, y)$ to make the dependence on S explicit. We note that to use (1), one must be able to evaluate $\mathbb{P}\left\{D^{y}_{t,t+\ell_e} = k\right\}$. This can be done numerically by inverting the generating function of $\mathbb{P}\left\{D^{y}_{t,t+\ell_e} = k \,\middle|\, Y(t+\ell_e) = y'\right\}$ which is given in the form of a matrix exponential (e.g. Fischer and Meier-Hellstern, 1992) and then un-conditioning on the event $Y(t+\ell_e) = y'$. We relegate further details of this to appendix A. Next, we note that whenever an item fails at time t, and is not expedited, $X_1(t)$ increases by one. Thus, $\mathbf{X}(t)$ and $Y(t)$ contain all information needed to do cost accounting, and, in particular, to make optimal expediting decisions.

Let Δ denote the difference operator with respect to the first argument of a function, i.e., $\Delta c_p(x, y) = c_p(x+1, y) - c_p(x, y)$. The following lemma establishes some useful properties of c_p. The proof of Lemma 1, in Appendix B.1, is similar to the proof of these same properties for the cost function of a news-vendor problem.

Lemma 1. *$c_p(x, y)$ has the following properties:*

(i) *$c_p(x + 1, y) \geq c_p(x, y)$ for all $x \in \mathbb{N}_0$ and $y \in \Theta$, i.e., c_p is non-decreasing in x.*

(ii) *$\Delta c_p(x + 1, y) \geq \Delta c_p(x, y)$ for all $x \in \mathbb{N}_0$ and $y \in \Theta$, i.e., c_p is convex in x.*

(iii) *$\Delta c_p(x, y) \leq p$ for all x and $y \in \Theta$ and $\Delta c_p(x, y) = p$ for all $x \geq S$ and $y \in \Theta$.*

(iv) *$\Delta c_p(x, y \mid S) \geq \Delta c_p(x, y \mid S + 1)$ for all $x \in \mathbb{N}_0$, $y \in \Theta$ and $S \in \mathbb{N}_0$, i.e., c_p is submodular with respect to x and S.*

Proposition 1 below allows us to truncate the relevant state space if $c_e < p\mathbb{E}[L_r]$, and fully characterizes an optimal expediting policy if $c_e \geq p\mathbb{E}[L_r]$. Proposition 1 can be understood intuitively by making the following casual observation: Whenever a repair is expedited, this may avert a backorder at most for the additional regular repair lead time L_r. Thus, when the cost of expediting repair is more than or equal to the expected backorder cost over the additional regular repair lead time, expediting is never beneficial. Conversely, if expediting is cheaper than the cost of a backorder over the expected additional regular lead time, then expediting is almost certainly beneficial if the number of parts already in repair that will not arrive within the expedited lead time is sufficiently large.

Proposition 1. *For the infinite horizon, average cost criterion, the following statements hold:*

(i) If $c_e \geq p\mathbb{E}[L_r]$ then it is optimal to never expedite repair.

(ii) If $c_e < p\mathbb{E}[L_r]$ then there is an $M \in \mathbb{N}$ such that whenever $\mathbf{X}(t)\mathbf{e}^T \geq M$ it is optimal to expedite repair upon failure of a part.

Proof. Here we prove part (i). The proof of part (ii) is in the appendix; that proof is more subtle, involving the verification that several limits exist, but is based on a similar idea.

The proof is based on showing that any policy that expedites in some state when $c_e \geq p\mathbb{E}[L_r]$ can be improved by a policy that is identical except that it does not expedite in that state. Let π denote an arbitrary policy that expedites for some state $(\mathbf{x}, y)$. Suppose now that at time t', the process is in state $(\mathbf{x}, y)$ and a demand occurs. Let $(\mathbf{X}(t), Y(t))$ denote the process under policy π. Next we construct a coupled process, $(\mathbf{X}'(t), Y(t))$, that is identical to $(\mathbf{X}(t), Y(t))$ except that the failed part arriving at time t' is *not* expedited. Let $\widetilde{\mathbf{X}}(t)$ denote the evolution of the part expedited at time t' by policy π through the pipeline, i.e., $\widetilde{\mathbf{X}}(t) = \mathbf{e}_i$ if the part sent to regular repair at time t' has completed its first $i-1$ phases of the additional regular repair lead time at time t, and $\widetilde{\mathbf{X}}(t) = \mathbf{0}$ if the part has completed its additional regular repair lead time. ($\mathbf{e}_i$ is the i-th unit vector with dimension m.) With this notation, we can write $\mathbf{X}'(t) = \mathbf{X}(t) + \widetilde{\mathbf{X}}(t)$. Now let $T_r = \inf\{t - t' | \widetilde{\mathbf{X}}(t) = \mathbf{0},\ t \geq t'\}$ and note that $T_r \stackrel{d}{=} L_r$, where $\stackrel{d}{=}$ denotes equality in distribution. By construction, any cost difference between the processes $(\mathbf{X}'(t), Y(t))$ and $(\mathbf{X}(t), Y(t))$ must occur in the interval $[t', t' + T_r)$, because these processes are identical outside that interval. In $[t', t' + T_r)$, $\mathbf{X}(t)$ incurs exactly c_e more emergency repair costs due to the part expedited at time t', and $\mathbf{X}'(t)$ incurs more penalty costs because $\mathbf{X}'\mathbf{e}^{\mathrm{T}} = \mathbf{X}(t)\mathbf{e}^{\mathrm{T}} + 1$ for $t \in [t', t' + T_r)$. The expected cost difference between the processes $(\mathbf{X}(t), Y(t))$ and $(\mathbf{X}'(t), Y(t))$ thus satisfies:

$$\begin{aligned} c_e - \mathbb{E}_{T_r}\left\{\mathbb{E}_{(\mathbf{X}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}(t)\mathbf{e}^{\mathrm{T}}, Y(t)) \,\middle|\, T_r\right]\right\} &\geq c_e - \mathbb{E}_{T_r}[pT_r] \\ &= c_e - p\mathbb{E}[L_r] \geq 0 \end{aligned} \tag{2}$$

where the first inequality holds by lemma 1 (iii). Thus we see that when $c_e \geq pE[L_r]$, any policy π that expedites for some states, can be improved (in the weak sense) by changing the decisions to not expedite in those states. This implies that when $c_e \geq p\mathbb{E}[L_r]$, the policy to never expedite is optimal. □

Proposition 1 has an important implication: When $c_e < p\mathbb{E}[L_r]$, there is a finite M such that it is optimal to expedite repair in all states $(\mathbf{x}, y)$ such that $\mathbf{x}\mathbf{e}^{\mathrm{T}} \geq M$. We can limit ourselves to such policies, and then all states with $\mathbf{x}\mathbf{e}^{\mathrm{T}} \geq M$ are transient. Consequently, for the purpose of finding average optimal policies, we may restrict the state space of $(\mathbf{X}(t), Y(t))$ to the finite set $\mathcal{S} = \{(\mathbf{x}, y) \in \mathbb{N}_0^m \times \Theta | \mathbf{x}\mathbf{e}^{\mathrm{T}} \leq M\}$ for some $M \in \mathbb{N}$. We remark that in the proof of Proposition 1 (ii), it is shown how such an M can be found.

4.1.2 MDP formulation with bounded transition rates

In this section, we consider the model $\mathfrak{M}(S)$ with $c_e < p\mathbb{E}[L_r]$, and state space $\mathcal{S} = \{(\mathbf{x}, y) \in \mathbb{N}_0^m \times \Theta | \mathbf{x}\mathbf{e}^{\mathrm{T}} \leq M\}$, where M is chosen such that it is optimal to expedite whenever $\mathbf{X}\mathbf{e}^{\mathrm{T}} \geq M$. (By Proposition 1, such a finite $M \in \mathbb{N}$ exists.) With a slight abuse of notation, we term the problem of finding an optimal policy for this model as $\mathfrak{M}(S, M)$. In this finite state space, transition rates are bounded and so we can apply the technique of uniformization to transform the problem of finding an optimal expediting policy to discrete time.

Remark 1. Without Proposition 1, uniformization would not have been possible. Thus, Proposition 1, not only facilitates the computation of optimal policies, but is also essential in establishing the structure of optimal policies using an inductive approach based on the dynamic programming recursion. $\diamondsuit$

In each state $(\mathbf{x}, y)$, we take a decision as to whether we expedite the repair of a part if the next event happens to be the arrival of a defective part. We let 1 denote the decision to expedite if a part arrives and let 0 be the decision to not expedite if a part arrives. Thus the action space in state $(\mathbf{x}, y)$ is $\mathcal{A}(\mathbf{x}, y) = \{0, 1\}$ when $\mathbf{x}\mathbf{e}^{\mathrm{T}} < M$ and $\mathcal{A}(\mathbf{x}, y) = \{1\}$ otherwise. Observe that if we take a decision 1 in some state of the system, this does not necessarily imply we will expedite some part, because the next event in the systems may not be the arrival of a defective part.

As uniform transition rate for this MDP, we choose $\Lambda = \lambda_{\max} + M \sum_{i=1}^{m} \mu_i + q_{\max}$. Let $p((\mathbf{x}', y') | (\mathbf{x}, y), a)$ denote the transition probability from state $(\mathbf{x}, y) \in \mathcal{S}$ to $(\mathbf{x}', y') \in \mathcal{S}$ when action $a \in \mathcal{A}(\mathbf{x}, y)$ is taken and note that the time between transitions has an exponential distribution with mean $1/\Lambda$. Without loss of generality, we rescale time such that $\Lambda = 1$. Then we have:

$$p((\mathbf{x}', y')|(\mathbf{x}, y), a) = \begin{cases} \lambda_y, & \text{if } \mathbf{x}' = \mathbf{x} + \mathbf{e}_1,\ y' = y,\ a = 0; \\ x_m\mu_m, & \text{if } \mathbf{x}' = \mathbf{x} - \mathbf{e}_m,\ y' = y,\ a \in \{0,1\}; \\ x_i\mu_i, & \text{if } \mathbf{x}' = \mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1},\ y' = y, \\ & a \in \{0,1\},\ i = 1, ..., m-1; \\ q_{y,y'}, & \text{if } \mathbf{x}' = \mathbf{x},\ y' \neq y,\ a \in \{0,1\}; \\ \sum_{i=1}^{m}(M - x_i)\mu_i + & \\ \quad q_{\max} - q_y + \lambda_{\max} - \lambda_y, & \text{if } (\mathbf{x}', y') = (\mathbf{x}, y),\ a = 0; \\ \sum_{i=1}^{m}(M - x_i)\mu_i + & \\ \quad q_{\max} - q_y + \lambda_{\max}, & \text{if } (\mathbf{x}', y') = (\mathbf{x}, y) \text{ and } a = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $\mathbf{e}_i$ is the i-th unit vector in dimension m. Regardless of the decision taken, between transitions, an expected penalty cost of $c_p(\mathbf{x}\mathbf{e}^{\mathrm{T}}, y)$ is incurred. Additionally, a cost of c_e is incurred if an arriving defective part is rejected from the system.

Now let $V_n(\mathbf{x}, y)$ denote the optimal total cost function when in state $(\mathbf{x}, y)$ and having n transitions to go and define $V_0(\mathbf{x}, y) \equiv 0$. The finite horizon dynamic programming recursion (Bellman equation) is given by

$$\begin{aligned} V_{n+1}(\mathbf{x}, y) = {} & c_p(\mathbf{x}\mathbf{e}^{\mathrm{T}}, y) + \lambda_y \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\mathrm{T}} < M\}} \min\{c_e + V_n(\mathbf{x}, y), V_n(\mathbf{x} + \mathbf{e}_1, y)\} \\ & + \lambda_y \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\mathrm{T}} = M\}}(c_e + V_n(\mathbf{x}, y)) + \textstyle\sum_{i=1}^{m-1} x_i\mu_i V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y) \\ & + x_m\mu_m V_n(\mathbf{x} - \mathbf{e}_m, y) + \textstyle\sum_{i=1}^{m}(M - x_i)\mu_i V_n(\mathbf{x}, y) \\ & + \textstyle\sum_{y' \in \Theta \setminus \{y\}} q_{yy'} V_n(\mathbf{x}, y') + (q_{\max} - q_y + \lambda_{\max} - \lambda_y) V_n(\mathbf{x}, y), \end{aligned} \tag{4}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

Remark 2. Note that an alternate uniformization constant is given by $\Lambda' = \lambda_{\max} + M\mu_{\max} + q_{\max}$, where $\mu_{\max} = \max_{i \in \{1,\dots,m\}} \mu_i$. This uniformization constant is smaller, and therefore more suitable if the dynamic programming recursion is used in value iteration algorithms. With this uniformization constant the MDP recursion becomes (we again scale time such that $\Lambda' = 1$):

$$\begin{aligned} V_{n+1}(\mathbf{x}, y) = {} & c_p(\mathbf{x}\mathbf{e}^{\mathrm{T}}, y) + \lambda_y \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\mathrm{T}} < M\}} \min\{c_e + V_n(\mathbf{x}, y), V_n(\mathbf{x} + \mathbf{e}_1, y)\} \\ & + \lambda_y \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\mathrm{T}} = M\}}(c_e + V_n(\mathbf{x}, y)) \\ & + \textstyle\sum_{i=1}^{m-1} x_i\mu_i V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y) + x_m\mu_m V_n(\mathbf{x} - \mathbf{e}_m, y) \\ & + \textstyle\sum_{i=1}^{m} x_i(\mu_{\max} - \mu_i) V_n(\mathbf{x}, y) + \left(M - \textstyle\sum_{i=1}^{m} x_i\right) \mu_{\max} V_n(\mathbf{x}, y) \\ & + \textstyle\sum_{y' \in \Theta \setminus \{y\}} q_{yy'} V_n(\mathbf{x}, y') + (q_{\max} - q_y + \lambda_{\max} - \lambda_y) V_n(\mathbf{x}, y). \end{aligned} \tag{5}$$

In this section, we will work with the formulation with the computationally 'less efficient' Λ so that we can reuse some results in the literature to prove structural properties of optimal policies. In the numerical section, §6, we use the formulation in because a smaller uniformization constant leads to quicker convergence of value iteration algorithms (e.g Kulkarni, 1999). ◊

To analyze the value function $V_n(\mathbf{x}, y)$ in §4.1.3, we employ the event based dynamic programming approach introduced by Koole (1998, 2006). To this end, let $\mathcal{V}$ denote the set of all functions $v : \mathcal{S} \to \mathbb{R}$ and let $f, f_1, ..., f_{m+2} \in \mathcal{V}$. We define the following operators $\mathbb{T}_{\text{cost}}, \mathbb{T}_{\text{AC}(i)}, \mathbb{T}_{\text{TD}(i)}, \mathbb{T}_{\text{D}(i)}, \mathbb{T}_{\text{env}} : \mathcal{V} \to \mathcal{V}$, $\mathbb{T}_{\text{unif}} : \mathcal{V}^{m+2} \to \mathcal{V}$.

$$\mathbb{T}_{\text{cost}} f(\mathbf{x}, y) = c_p(\mathbf{x}\mathbf{e}^{\text{T}}, y) + f(\mathbf{x}, y) \tag{6}$$

$$\begin{aligned}\mathbb{T}_{\text{AC}(i)} f(\mathbf{x}, y) = {} & \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\text{T}}<M\}} \min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_i, y)\} \\ & + \mathbf{1}_{\{\mathbf{x}\mathbf{e}^{\text{T}}=M\}} (c_e + f(\mathbf{x}, y))\end{aligned} \tag{7}$$

$$\mathbb{T}_{\text{TD}(i)} f(\mathbf{x}, y) = \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y) + \frac{M - x_i}{M} f(\mathbf{x}, y) \tag{8}$$

$$\mathbb{T}_{\text{D}(i)} f(\mathbf{x}, y) = \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i, y) + \frac{M - x_i}{M} f(\mathbf{x}, y) \tag{9}$$

$$\mathbb{T}_{\text{env}} f(\mathbf{x}, y) = \sum_{y \in \Theta \setminus \{y\}} q_{yy'} f(\mathbf{x}, y') + (q_{\max} - q_y + \lambda_{\max} - \lambda_y) f(\mathbf{x}, y) \tag{10}$$

$$\mathbb{T}_{\text{unif}}(f_1, \cdots, f_{m+2})(\mathbf{x}, y) = \lambda_y f_1(\mathbf{x}, y) + \sum_{i=1}^{m} M \mu_i f_{i+1}(\mathbf{x}, y) + f_{m+2}(\mathbf{x}, y) \tag{11}$$

These operators are variations to operators defined by Koole (1998, 2004, 2006) and are originally intended to model various common queueing mechanisms such as arrival control ($\mathbb{T}_{\text{AC}(i)}$), transfer departures from multi-server tandem queues ($\mathbb{T}_{\text{TD}(i)}$), and departures from multi-server queues ($\mathbb{T}_{\text{D}(i)}$), while the operators $\mathbb{T}_{\text{cost}} f(\mathbf{x}, y)$, $\mathbb{T}_{\text{env}}$ and $\mathbb{T}_{\text{unif}}$ are mainly convenient for bookkeeping. The Bellman recursion for our MDP, (4), can now be written succinctly as

$$\begin{aligned}V_{n+1}(\mathbf{x}, y) = \mathbb{T}_{\text{cost}} \mathbb{T}_{\text{unif}} \big[& \mathbb{T}_{\text{AC}(1)} V_n(\mathbf{x}, y), \mathbb{T}_{\text{TD}(1)} V_n(\mathbf{x}, y), \cdots, \mathbb{T}_{\text{TD}(m-1)} V_n(\mathbf{x}, y), \\ & \mathbb{T}_{\text{D}(m)} V_n(\mathbf{x}, y), \mathbb{T}_{\text{env}} V_n(\mathbf{x}, y) \big].\end{aligned} \tag{12}$$

This formulation of the MDP recursion is convenient because the propagation of value function properties over n can be analyzed through the propagation properties of operators, for which results are available in literature.

We remark that the operators used to rewrite the MDP recursion reveal that the MDP we are dealing with is equivalent to an admission control problem for a tandem line of ample exponential server queues. A similar equivalence is exploited by Song and Zipkin (2009) in finding effective means to evaluate heuristic policies.

4.1.3 Average optimal expediting policies

To characterize average optimal policies, we study properties of the value function and how these properties propagate through recursion (12). We define the first order difference operator with respect to x_i,

Δ_i, as $\Delta_i f(\mathbf{x}, y) = f(\mathbf{x}+\mathbf{e}_i, y) - f(\mathbf{x}, y)$. We distinguish the following subsets of $\mathcal{V}$:

$$\mathcal{I}(i) = \{f \in \mathcal{V} | f(\mathbf{x}, y) \leq f(\mathbf{x}+\mathbf{e}_i, y)\} \tag{13}$$

$$\mathcal{C}(i) = \{f \in \mathcal{V} | \Delta_i f(\mathbf{x}, y) \leq \Delta_i f(\mathbf{x}+\mathbf{e}_i, y)\} \tag{14}$$

$$\mathcal{UI} = \{f \in \mathcal{V} | f(\mathbf{x}+\mathbf{e}_{i+1}, y) \leq f(\mathbf{x}+\mathbf{e}_i, y), i = 1, ..., m-1\} \tag{15}$$

$$\mathcal{SM}(i,j) = \{f \in \mathcal{V} | \Delta_i f(\mathbf{x}, y) \leq \Delta_i f(\mathbf{x}+\mathbf{e}_j, y)\}. \tag{16}$$

In (13)-(16), it is understood that the inequalities that characterize each set must hold when the arguments on both sides of the inequality exist in $\mathcal{S}$. $\mathcal{I}(i)$ and $\mathcal{C}(i)$ are the sets of non-decreasing and convex functions with respect to x_i respectively. $\mathcal{UI}$ is the set of upstream increasing functions as introduced in Koole (2004) and renamed in Koole (2006). The set $\mathcal{SM}(i,j)$ consists of functions with a specific supermodularity property. Finally, define $\mathcal{F}$ as

$$\mathcal{F} = \left(\bigcap_{i=1}^{m} \mathcal{I}(i)\right) \cap \left(\bigcap_{j=2}^{m} \mathcal{SM}(1,j)\right) \cap \mathcal{UI} \cap \mathcal{C}(1). \tag{17}$$

Lemma 2. *The following statements hold:*

(i) The function $g \in \mathcal{V}$ defined by $g(\mathbf{x}, y) = c_p(\mathbf{x}\mathbf{e}^T, y)$ is a member of $\mathcal{F}$, i.e., $g \in \mathcal{F}$.

(ii) If $f \in \mathcal{F}$ then $\mathbb{T}_{\text{cost}} f(\mathbf{x}, y), \mathbb{T}_{\text{AC}(1)} f(\mathbf{x}, y), \mathbb{T}_{\text{D}(m)} f(\mathbf{x}, y), \mathbb{T}_{\text{env}} f(\mathbf{x}, y) \in \mathcal{F}$ and $\mathbb{T}_{\text{TD}(i)} f(\mathbf{x}, y) \in \mathcal{F}$ for $i = 1, ..., m-1$.

(iii) If $f_j \in \mathcal{F}$ for $j = 1, ..., m+2$, then $\mathbb{T}_{\text{unif}}(f_1, ..., f_{m+2})(\mathbf{x}, y) \in \mathcal{F}$.

The proof of this lemma is in Appendix B.3. The properties of functions in $\mathcal{V}$ that are shown to propagate through operators (6)-(11) in Lemma 2, imply structure on the optimal policy. To state the next lemma, we introduce some notation. Let $\mathbf{x}^{(-1)}$ denote the vector $\mathbf{x}$ with its first component set to 0, i.e., $\mathbf{x}^{(-1)} = (0, x_2, \dots, x_m)$. The next lemma explains how the optimal policy at transition epoch n is related to properties of V_{n-1}.

Lemma 3. *If $V_{n-1} \in \mathcal{F}$, then, at transition epoch n, there are state dependent thresholds $T_n\left(\mathbf{x}^{(-1)}, y\right)$ such that it is optimal to expedite the repair of an arriving part at transition epoch n if $X_1(t_n) \geq T_n\left(\mathbf{X}^{(-1)}(t_n), Y(t_n)\right)$, where t_n is the time corresponding to transition epoch n. Furthermore the thresholds $T_n\left(\mathbf{x}^{(-1)}, y\right)$ satisfy the following monotonicity property: $\Delta_i T_n\left(\mathbf{x}^{(-1)}, y\right) \leq 0$, for $i = 2, ..., m$.*

Proof. Let $V_{n-1} \in \mathcal{F}$. The fact that a state dependent threshold policy is optimal at decision epoch n follows immediately from the fact that $V_{n-1} \in \mathcal{C}(1)$ and Koole (2006), Theorem 8.1. Because $V_{n-1} \in \bigcap_{i=2}^{m} \mathcal{SM}(1,i)$, this threshold is non-increasing in x_2 to x_m, again by Koole (2006), Theorem 8.1. This can be written as $T_n(\mathbf{x}+\mathbf{e}_i, y) \leq T_n(\mathbf{x}, y)$ for $i = 2, ..., m$, and by subtracting $T_n(\mathbf{x}, y)$ from both sides we obtain $\Delta_i T_n(\mathbf{x}, y) \leq 0$. □

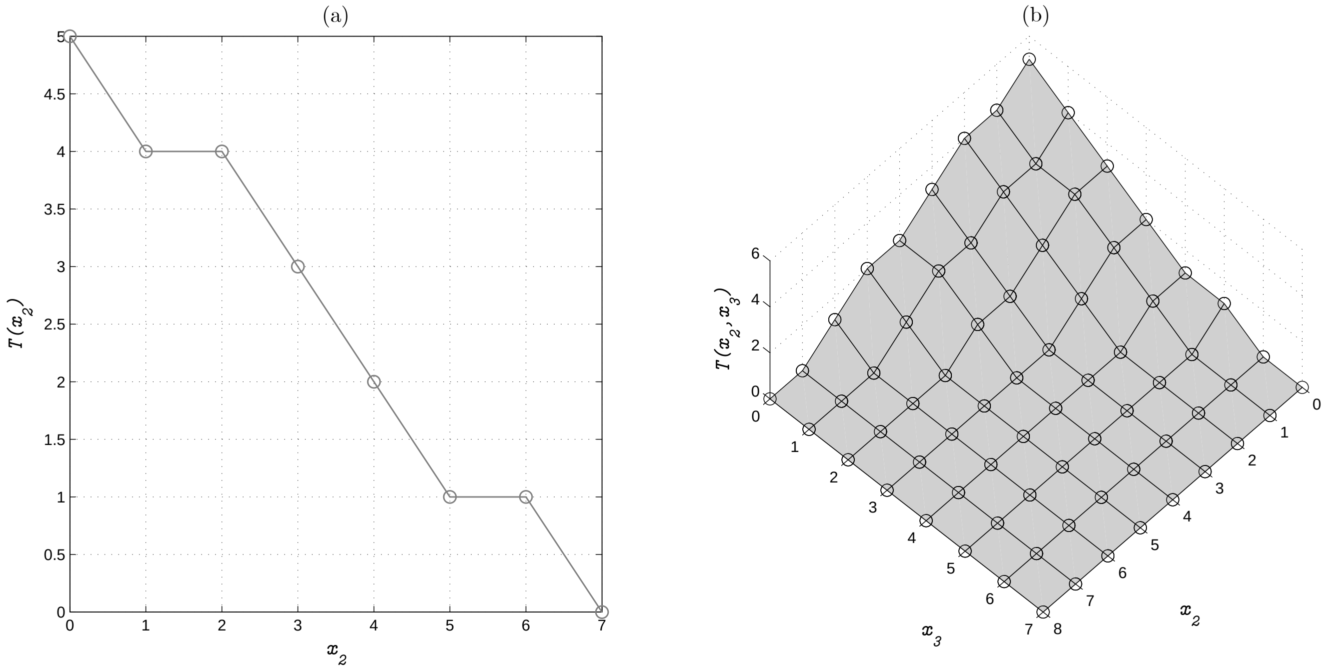

Figure 2: Part (a) shows the state dependent threshold for $n = 593$ in the case where L_r has an Erlang(2) distribution ($m = 2$). Part (b) shows the state dependent thresholds for $n = 1513$ when L_r is Erlang(3) distributed ($m = 3$). Both cases are based on the problem instance with $|\Theta| = 1$, $\lambda_1 = 1$, $\mathbb{E}[L_r] = 4$, $\ell_e = 2$, $c_e = 8$, $p = 10$, and $S = 8$. In both cases, n coincides with the iteration in which average optimal policies are found within some precision.

An alternative interpretation of Lemma 3 is that the optimal policy at transition epoch n (under the stated condition) is a switching curve between expediting and not expediting repair. This switching curve is decreasing in x_i for $i = 2, ..., m$. Figure 2 illustrates two such switching curves. In part (a) of the figure, for a given x_2, it is optimal to expedite repair if x_1 is on or above the shown line. In part (b) of the figure, for given (x_2, x_3) it is optimal to expedite repair if x_1 is on or above the shown surface.

The policy described in Lemma 3 can also be reinterpreted as a state dependent expedite-up-to policy. To see this, define $IP_e(t) = S - \mathbf{X}(t)\mathbf{e}^{\mathrm{T}}$, and note that this can be interpreted as the expedited inventory position: on-hand inventory minus backorders plus outstanding orders arriving within the expedited lead time ℓ_e. The optimal policy is now to expedite parts to retain $IP_e(t_n)$ at or above the level $S - T_n\left(\mathbf{X}^{(-1)}(t_n), Y(t_n)\right)$. Thus the resulting policy is a state dependent version of the dual-index policy (Veeraraghavan and Scheller-Wolf, 2008; Arts et al., 2011, consider state independent dual-index policies), where regular and emergency inventory positions are both kept at or above their order-up-to levels. Note however, that the regular order-up-to level was assumed to be S from the start as we are dealing with a closed loop system. Without this fixed base-stock level, a state dependent dual-index replenishment policy need not be optimal.

The main result of this section is that average optimal policies also have the structure described in

Lemma 3.

Theorem 1. *Consider the model $\mathfrak{M}(S)$. If $c_e \geq p\mathbb{E}[L_r]$, then it is average optimal to never expedite repair. If $c_e < p\mathbb{E}[L_r]$, then there are state dependent threshold levels $T\left(\mathbf{x}^{(-1)}, y\right) \in \mathbb{N}_0$ such that it is average optimal to expedite the repair of an arriving defective part at time t if $X_1(t) \geq T\left(\mathbf{X}^{(-1)}(t), Y(t)\right)$. Furthermore these threshold levels $T\left(\mathbf{x}^{(-1)}, y\right)$ satisfy the property in Lemma 3, i.e., $\Delta_i T\left(\mathbf{x}^{(-1)}, y\right) \leq 0$ for $i = 2, ..., m$.*

Proof. The first part of the theorem is simply a restatement of part (ii) of Proposition 1. If $c_e < p\mathbb{E}[L_r]$, we may apply Proposition 1.(ii) to truncate the state space at a finite M and optimal solutions to $\mathfrak{M}(S, M)$ will coincide to optimal solutions to $\mathfrak{M}(S)$ provided M is sufficiently large. Therefore, let us consider $\mathfrak{M}(S, M)$. Observe that state $(\mathbf{0}, y)$ for any $y \in \Theta$ is reachable from any other state for any policy, so this MDP is unichain. Furthermore, since under any policy, there are transitions from state $(\mathbf{0}, y)$ to itself, this MDP is aperiodic. By Theorem 8.5.4 of Puterman (1994), $\max_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y)) - \min_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y))$ converges to the optimal average costs as $n \to \infty$. Now for each n, a policy of the form described in Lemma 3 is optimal because $V_0 \in \mathcal{F}$ and so, by induction using Lemma 2, so are V_n for $n \in \mathbb{N}$. Finally since both the state and action space of this MDP are finite, there are finitely many policies that satisfy Lemma 3, and at least one of them will be found infinitely often throughout recursion (12). Such a repair expediting policy is average optimal. □

Theorem 1 also answers a question and conjecture posed by Song and Zipkin (2009, p. 371): "Are there any systems for which some policy of the form above is in fact optimal?". The policy Song and Zipkin (2009) propose is exactly the policy described in Theorem 1 for the special case that $m = 1$. For $m \geq 2$ one obtains a generalized form of this policy.

4.1.4 Infinite horizon discounted optimal expediting policies

The same policy structure results hold for the case where we are interested in the infinite horizon discounted cost criterion. Let $\beta > 0$ be the discount rate. Proposition 1 continues to hold with $p\mathbb{E}[L_r]$ replaced by the expected discounted penalty costs over an interval of length L_r:

$$\mathbb{E}_{L_r}\left[\int_0^{L_r} pe^{-\beta t} dt\right] = \frac{p}{\beta} - \frac{p}{\beta}\mathbb{E}\left[e^{-\beta L_r}\right].$$

This holds in general for all non-negative distributions that might model L_r. In our particular model, the Laplace-Stieltjes transform of L_r is given by:

$$\mathbb{E}\left[e^{-\beta L_r}\right] = \prod_{i=1}^{m} \frac{\mu_i}{\mu_i + \beta}.$$

The MDP recursion can be written in exactly the same manner as before except that $\mathbb{T}_{\text{cost}}$, needs to be changed to $\mathbb{T}^{\beta}_{\text{cost}} : \mathcal{V} \to \mathcal{V}$ with

$$\mathbb{T}^{\beta}_{\text{cost}} f(\mathbf{x}, y) = \frac{c_p(\mathbf{x}\mathbf{e}^{\mathrm{T}}, y)}{\Lambda + \beta} + \frac{\Lambda}{\Lambda + \beta} f(\mathbf{x}, y).$$

It is readily verified that $\mathbb{T}^{\beta}_{\text{cost}}$ propagates the same properties as $\mathbb{T}_{\text{cost}}$, that is, if $f \in \mathcal{F}$ then also $\mathbb{T}^{\beta}_{\text{cost}} f(\mathbf{x}, y) \in \mathcal{F}$. With this change, it is easy to verify that Theorem 1 still holds, again with $p\mathbb{E}[L_r]$ changed to $\frac{p}{\beta} - \frac{p}{\beta}\mathbb{E}\left[e^{-\beta L_r}\right]$.

Theorem 2. *Consider the infinite horizon discounted cost criterion for model $\mathfrak{M}(S)$ with discount rate β. If $c_e \geq \frac{p}{\beta} - \frac{p}{\beta}\mathbb{E}\left[e^{-\beta L_r}\right] = \frac{p}{\beta} - \frac{p}{\beta}\prod_{i=1}^{m} \frac{\mu_i}{\mu_i + \beta}$, then it is β-discounted optimal to never expedite repair. If $c_e < \frac{p}{\beta} - \frac{p}{\beta}\mathbb{E}\left[e^{-\beta L_r}\right] = \frac{p}{\beta} - \frac{p}{\beta}\prod_{i=1}^{m} \frac{\mu_i}{\mu_i + \beta}$, then there are state dependent threshold levels $T\left(\mathbf{x}^{(-1)}, y\right) \in \mathbb{N}_0$ such that it is β-discounted optimal to expedite repair at time t if $X_1(t) \geq T\left(\mathbf{X}^{(-1)}(t), Y(t)\right)$. Furthermore these threshold levels $T\left(\mathbf{x}^{(-1)}, y\right)$ satisfy the property in Lemma 3, i.e., $\Delta_i T\left(\mathbf{x}^{(-1)}, y\right) \leq 0$ for $i = 2, ..., m$.*

4.2 Turn-around stock optimization

In the analysis in the previous sections, we have considered problem $\mathfrak{M}(S)$, i.e., we have considered S to be a fixed constant that was determined at $t = 0$, and have focussed on using the expedition decision to minimize expedition and backorder penalty costs. Now, we focus on the joint optimization of the turn-around stock S and the expediting policy. For brevity of exposition, we only discuss the average cost criterion in this section.

To facilitate presentation, we first present some notation: We let $C(S)$ denote the optimal average expediting and backorder penalty costs per time unit for a turn-around stock of size S, i.e., $C(S) = \lim_{n\to\infty} V_n(\mathbf{0}, 1)/n$ is the optimal cost associated with $\mathfrak{M}(S)$. Furthermore, we let $C_{\text{tot}}(S) := hS + C(S)$ denote the total cost rate associated with a turn-around stock of S if an optimal repair expediting policy is used.

Whenever we drop the time index of a stochastic process, we are referring to the process in steady state, e.g. $\mathbb{P}\{Y = y\} = \lim_{t\to\infty} \mathbb{P}\{Y(t) = y\}$. We let the random variable $D(L)$ denote demand in an interval of length $L \geq 0$ when the modulating chain of demand is in steady state, i.e.,

$$\mathbb{P}\{D(L) \leq k\} = \sum_{y\in\Theta} \mathbb{P}\{Y = y\}\mathbb{P}\left\{D^{y}_{t,t+L} \leq k\right\}.$$

A lower bound of $C_{\text{tot}}(S)$ is given by the average holding and backorder penalty cost rates of the system with turn-around stock S under the feasible policy of expediting everything against zero expediting cost:

$$C_{LB}(S) := hS + p\mathbb{E}\left[(D(\ell_e) - S)^{+}\right].$$

When we do include the expediting costs, we obtain an upper bound for $C_{\text{tot}}(S)$:

$$C_{UB}(S) := C_{LB}(S) + c_e\bar{\lambda}.$$

Here, $\bar{\lambda} = \sum_{y\in\Theta} \lambda_y \mathbb{P}\{Y = y\}$ is the long run average demand per time period. Let $S^* := \operatorname{argmin}_{S\in\mathbb{N}_0} C_{\text{tot}}(S)$ denote the optimal turn-around stock. An upper bound to $C_{\text{tot}}(S^*)$ is obtained by minimizing $C_{UB}(S)$. The S that minimizes $C_{UB}(S)$ (as well as $C_{LB}(S)$) can be easily found as $C_{UB}(S)$ is convex. We denote this minimizer $\hat{S}$ and it is the smallest integer that satisfies the newsvendor inequality

$$\mathbb{P}\left\{D(\ell_e) \le \hat{S}\right\} \ge \frac{p-h}{p}. \tag{18}$$

Since $C_{LB}(S)$ is convex, it is easy to find the greatest $S \le \hat{S}$ and smallest $S \ge \hat{S}$ such that $C_{LB}(S) \ge C_{UB}(\hat{S})$. This will provide lower and upper bounds respectively on S^*.

Proposition 2. *The optimal turn-around stock, S^*, that minimizes $C_{tot}(S)$ is bounded as $S_{LB} \le S^* < S_{UB}$, where S_{LB} and S_{UB} are given by*

$$S_{LB} = \max\left\{\{0\} \cup \min\{x \in \mathbb{N}_0, x \le \hat{S} : C_{LB}(x) \ge C_{UB}(\hat{S})\}\right\} \tag{19}$$

$$S_{UB} = \min\{x \in \mathbb{N}_0, x \ge \hat{S} : C_{LB}(x) \ge C_{UB}(\hat{S})\}, \tag{20}$$

Furthermore, if $C(S) \le h$ for some $S \in \mathbb{N}_0$, then $S^* \le S$.

Proof. The bounds established by S_{LB} and S_{UB} follow directly from the analysis preceding Proposition 2. To verify the last statement, observe that $C(S)$ is non-negative and decreasing and that $\Delta C_{\text{tot}}(S) = h + \Delta C(S)$. Combining these facts implies that if $C(S) \le h$, then $\Delta C_{\text{tot}}(S) \ge 0$ and so $S^* < S$. □

Proposition 2 gives us bounds that can be used to minimize $C_{\text{tot}}(S)$ by enumeration.

A natural question is whether $C(S)$ is convex in S as this would make optimization easy. Unfortunately $C(S)$ is not convex in S as can be verified by considering the problem instance with Poisson demand with rate $\lambda = 10$, $\ell_e = 1$, $\mathbb{E}[L_r] = 3$, $m = 1$, $p = 10$ and $c_e = 15$. In this case, $C(S)$ can be obtained exactly without dynamic programming using the results in Moinzadeh and Schmidt (1991) and Song and Zipkin (2009). (In §5.1.1, we also show how to compute $C(S)$ without dynamic programming for this special case.) For this instance it can be verified that $C(20) - 2C(19) + C(18) \le -0.03$, showing that $C(S)$ is not convex in general; see also Figure 3. The non-convexity of $C(S)$ does affect the unimodality of $C_{\text{tot}}(S)$ but only in rather extreme cases. Figure 4 presents such a case. In §6 we present numerical work for instances as they are typically encountered in practice. For all these instances, $C(S)$ is convex and $C_{\text{tot}}(S)$ is unimodal. In fact, a cursory look at Figures 3 and 4.a does not immediately reveal that $C(S)$ and $C_{\text{tot}}(S)$ are not convex. This is typical for all counterexamples we have found.

4.2.1 Trading off safety stock and safety time

In this subsection, we formally show that as the turn-around stock increases, the need for expediting decreases and vice versa. To formalize this, we need some additional notation. Let $\mathcal{W}$ be set of all functions $w : \mathcal{S} \times \mathbb{N}_0 \to \mathbb{R}$. The extra argument corresponds to the turn-around stock. We make the

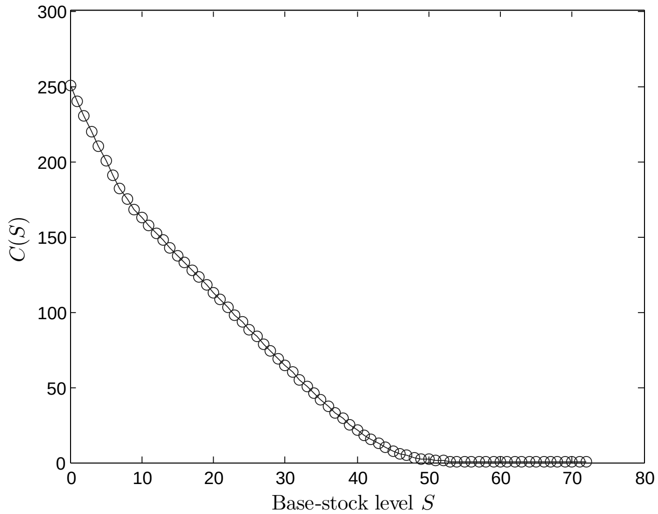

Figure 3: Consider the problem instance with Poisson demand with rate $\lambda = 10$, $\ell_e = 1$, $m = 1$, $\mathbb{E}[L_r] = 3$, $p = 10$ and $c_e = 15$. This figure shows the optimal cost for expediting and backordering as a function of the turn-around stock $C(S)$. Note in particular the non-convexity around $S = 19$.

dependence of $V_n(\mathbf{x}, y)$ on S explicit by writing $V_n(\mathbf{x}, y, S)$. Note that operators (6)-(11) are also mappings from $\mathcal{W} \to \mathcal{W}$. We start with a sub-modularity of $V_n(\mathbf{x}, y, S)$ with respect to x_1 and S.

Lemma 4. *For all $n \in \mathbb{N}_0$, V_n is submodular with respect to x_1 and S, i.e., for all $(x, y) \in \mathcal{S}$, and $S < S_{UB}$,*

$$\Delta_1 V_n(\mathbf{x}, y, S) \geq \Delta_1 V_n(\mathbf{x}, y, S+1) \tag{21}$$

where $\mathcal{S}$ is chosen with an appropriate M that is common for all $S < S_{LB}$

The proof of this lemma is in the appendix and follows an inductive approach. Now we can formally state that as the turn-around stock increases, the need for expediting decreases and vice versa.

Proposition 3. *Let $T_S\left(\mathbf{x}^{(-1)}, y\right)$ denote the expediting threshold that is average optimal under a turn-around stock level of S at $(\mathbf{x}, y) \in \mathcal{S}$. Then $T_S\left(\mathbf{x}^{(-1)}, y\right) \leq T_{S+1}\left(\mathbf{x}^{(-1)}, y\right)$ for all $(\mathbf{x}, y) \in \mathcal{S}$ and $S \in \mathbb{N}_0$, that is, when the turn-around stock increases, the need for expediting decreases.*

Proof. Because the expediting decision is taken whenever $\min\{V_n(\mathbf{x}, y, S)+c_e, V_n(\mathbf{x}+\mathbf{e1}, y, S)\} = V_n(\mathbf{x}, y)+c_e$, which is equivalent to $\Delta_1 V_n(\mathbf{x}, y, S) \geq c_e$, it is clear that Lemma 4 implies the result. $\square$

The interpretation of Proposition 3 is that the possibility to expedite acts as kind of safety time, that can be used in stead of (safety) stock to lower the risk of backorders.

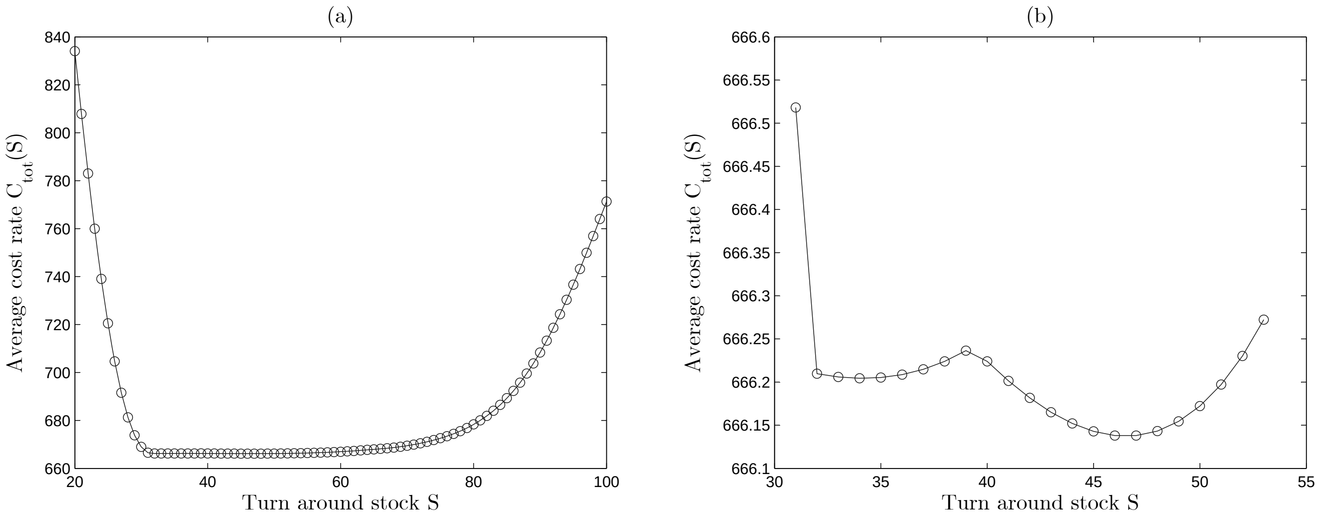

Figure 4: Consider the problem instance with Poisson demand with rate $\lambda = 3.43$, $\ell_e = 8$, $m = 1$, $\mathbb{E}[L_r] = 15$, $p = 37.2$ and $c_e = 116$. This figure shows $C_{\text{tot}}(S)$ for this instance. Although the fact that $C_{\text{tot}}(S)$ is not unimodal is not immediately apparent from sub-figure (a), it is apparent from sub-figure (b). Any small deviation of any of the problem parameters will make $C_{\text{tot}}(S)$ unimodal.

5. E-WDT Heuristic

In the previous section, we have analyzed an exact solution to our problem. However, finding the optimal solution involves solving an MDP that suffers from the curse of dimensionality for each S between S_{LB} and S_{UB}. Furthermore, the optimal expediting policy is rather intricate, depending on the entire vector of repair that will not arrive within the expedited lead time. In this section, we describe a heuristic for our model that involves an expediting policy that is much easier to interpret and that does not impose the same computational burden. We call this heuristic the E-WDT heuristic for reasons that will become clear later. This section is organized in the same fashion as the previous section: First we discuss heuristic expediting policies in §5.1 and then we discuss the heuristic optimization of the turn-around stock in §5.2.

5.1 World driven threshold policies

Computing the state dependent optimal threshold levels quickly becomes computationally prohibitive as m increases. A plausible heuristic policy is to aggregate all orders in $\mathbf{X}(t)$ and to put a threshold expediting level, $T(y)$, on their sum $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}}$. This threshold will then only depend on $Y(t)$ and so, borrowing the terminology of Zipkin (2000), we call such a policy a world driven threshold (WDT) policy. It is readily verified that the WDT policy satisfies the monotonicity property in Theorem 1 that $\Delta_i T\left(\mathbf{x}^{(-1)}, y\right) \leq 0$. Indeed, observe that the thresholds $(T^{WDT}(\mathbf{x}, y))$ of a WDT policy satisfy:

$$\Delta_i T^{WDT}(\mathbf{x}, y) = \begin{cases} -1, & \text{if } T(\mathbf{x}, y) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

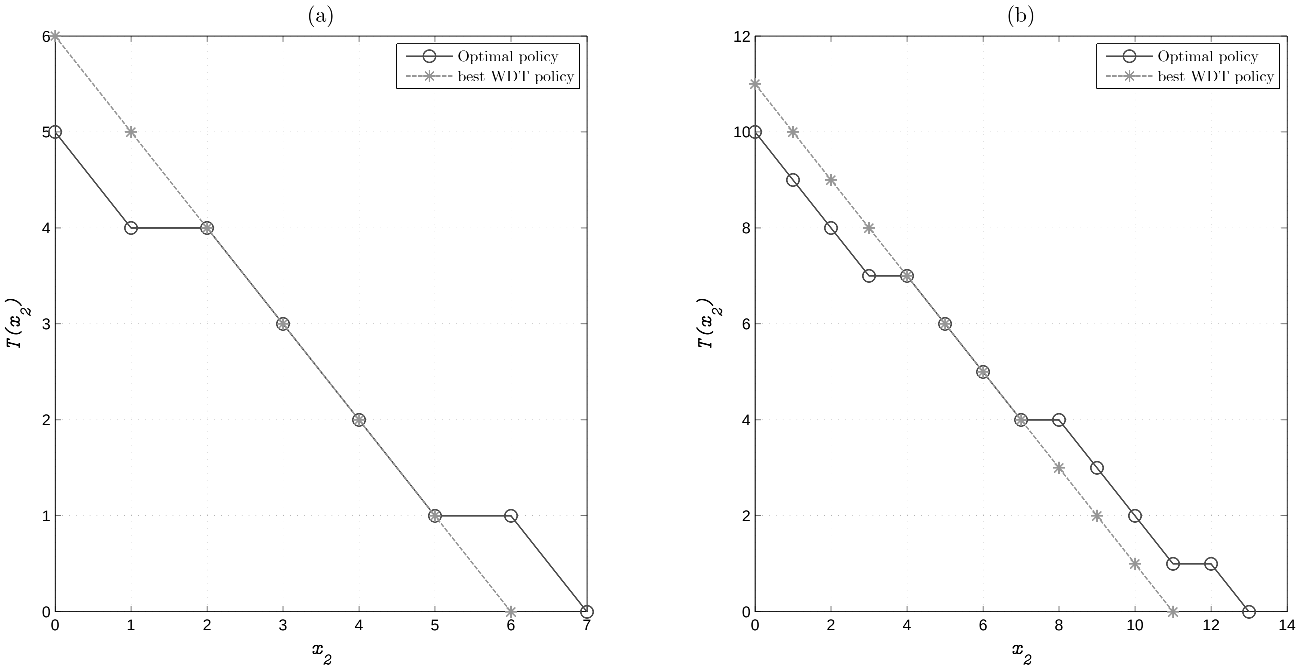

Figure 5: Optimal and heuristic policies. Part (a) shows the optimal state dependent threshold in conjunction with the best WDT policy for the case with $\lambda = 1$, $\mathbb{E}[L_r] = 4$, $\ell_e = 2$, $p = 10$, $c_e = 8$, $S = 8$ and $m = 2$. Part (b) shows the optimal state dependent threshold for in conjunction with the best heuristic policy for the same case except $S = 12$.

This is shown graphically in Figure 5, where the optimal thresholds are shown with the best WDT thresholds. As before, the most convenient way to interpret Figure 5 is to think of it as a switching curve: If x_1 is on or above the shown line for some x_2, then expedite the repair, otherwise do not expedite repair.

For $m > 1$, finding the best WDT policy is about as difficult as finding an optimal policy since the stationary distribution of $\mathbf{X}(t)$ under such a policy still requires the evaluation of an $m + 1$ dimensional Markov chain. A notable exception, that we discuss in §5.1.1, occurs when demand is a stationary Poisson process, i.e., $|\Theta| = 1$.

In general, for $|\Theta| > 1$ and $c_e < p\mathbb{E}[L_r]$, we propose the following heuristic way of finding a good WDT policy. In stead of working with the $(m+1)$-dimensional space, move to two-dimensional space by approximating L_r by a single exponential phase with the same mean $\mu_1 = \mu = 1/E[L_r]$. Then we are left with a two-dimensional space for which we can easily solve the resulting MDP to optimality using any common algorithm to solve finite state and action space MDPs such as value iteration, policy iteration or linear programming.

The WDT policies that result from this procedure are not necessarily optimal within the class of WDT policies and the computed cost is not exact but an approximation. Since the system under study is equivalent to a type of ample server queue, we may expect this approximation to be quite accurate. In the next subsection, we show that this approach is exact for Poisson demand and in §6, we provide numerical evidence that WDT policies that are found in this manner perform exceptionally well compared to optimal policies under Markov modulated Poisson demand. Furthermore, the cost approximations of this method are also very accurate.

5.1.1 Special case: Poisson demand

Now we briefly consider the evaluation of WDT policies for the special case where $|\Theta| = 1$, and we are dealing with stationary Poisson demand. In this case, the evaluation of a WDT policy can be done exactly for any distribution of L_r in closed form using the results of Song and Zipkin (2009). (In this context, it might be appropriate to refer to a WDT policy, simply as a threshold policy. For convenience, we use the name WDT policy also in this context.) Alternatively, one may simply observe that $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}}$, under such a policy, has the same stationary distribution as the number of customers in a $M/G/c/c$ queue, where the number of servers c is set equal to the threshold T and the service time is distributed as L_r. In this equivalence, a customer being blocked from the queue because all T servers are busy corresponds to a repair being expedited because there are T or more parts that will not arrive within ℓ_e. The average expediting and backorder penalty cost rate for such a policy with threshold level T and base-stock level S, $C(T|S)$ is therefore given by:

$$C(T|S) = \lambda c_e B(T, \lambda\mathbb{E}[L_r]) + \sum_{x=0}^{T} c_p(x|S) \frac{(\lambda\mathbb{E}[L_r])^x/x!}{\sum_{k=0}^{T}(\lambda\mathbb{E}[L_r])^k/k!} \tag{22}$$

where $B(c,\rho) = \frac{\rho^c/c!}{\sum_{k=0}^{c}\rho^k/k!}$ is the Erlang loss function with c servers and traffic intensity ρ, λ is the intensity of the Poisson demand process, and $c_p(x|S) = c_p(x,1|S)$. Expression (22) also reveals that the performance of a WDT policy is insensitive to the distribution of L_r for the special case of Poisson demand. This insensitivity does not hold for Markov modulated Poisson demand. In the numerical section however, we provide evidence that the performance evaluation of a WDT policy is nearly insensitive to the exact distribution of L_r for Markov modulated Poisson demand processes.

5.2 Heuristic optimization of the turn-around stock: The E-WDT heuristic

In §5.1, we discussed a heuristic to obtain a good WDT policy for a given turn-around stock S, namely by finding the optimal expediting policy after replacing L_r by a single exponential phase with the same mean. (This is of course exact if L_r happens to be exponentially distributed.) The heuristic for joint optimization is based on this idea.

Let $C_{\mathrm{E}}(S)$ denote the optimal expediting and penalty cost rate when L_r has an exponential distribution with mean μ^{-1} and the turn-around stock is S. (The E stands for exponential distribution, as L_r has an exponential distribution in C_{E}.) The heuristic we propose is to minimize $C_{\text{E-tot}}(S) = hS + C_{\mathrm{E}}(S)$ with $\mu^{-1} = \mathbb{E}[L_r]$ using a greedy algorithm such as golden section search. We call this heuristic the

E-WDT heuristic. (E stands for exponential and WDT stands for world driven threshold.) There is no guarantee that a greedy search of $C_{\text{E-tot}}(S)$ yields the global optimum of $C_{\text{E-tot}}(S)$ because $C_{\text{E}}(S)$ need not be convex as shown in §4.2. We let S_{E} denote the base-stock level that is found by applying the E-WDT heuristic and $T_{\text{E}}(y)$ denote the corresponding expediting thresholds that depend only on $y \in \Theta$.

We emphasize that in general $C_{\text{E}}(S_{\text{E}})$ does not represent the real backordering and expediting cost of applying the WDT policy with $T_{\text{E}}(y)$, because these costs are not insensitive to the distribution of L_r unless demand is a Poisson process. Let $C_{\text{R}}(S)$ denote the real backordering and expediting costs when applying the WDT policy found by assuming L_r is exponential to the real system where L_r is not necessarily exponential. Similarly, let $C_{\text{R-tot}}(S) = hS + C_{\text{R}}(S)$. We provide numerical evidence in §6.3 that $C_{\text{E}}(S_{\text{E}}) \approx C_{\text{R}}(S_{\text{E}})$.

6. Numerical results

The numerical section is divided in four subsections. In §6.1, we present the test bed that is used for all numerical experiments. In §6.2, we benchmark the performance of the WDT policy described in §5.1 against the optimal expediting policy for fixed turn-around stock. We benchmark the performance of the E-WDT heuristic against optimal joint optimization of turn-around stock and expediting policy in §6.3. This section also investigates how accurate the performance estimates are when L_r is approximated by an exponential distribution. Finally in 6.4, we present results that shed light on the value of leveraging the possibility to expedite repair in anticipating demand fluctuations. We do this in a setting with exponential lead times where WDT expediting policy is optimal. We compare with two naive heuristics that assume demand is a Poisson process. We also compare the E-WDT heuristic to a heuristic that determines the size of the turn-around stock and expediting policy separately.

6.1 Test bed and set-up

For all experiments, the backorder penalty cost is fixed at $p = 10$. The other problem parameters are varied as a full factorial experiment. The expected additional regular repair lead time was either low or high, $\mathbb{E}[L_r] \in \{2, 4\}$, and takes on an Erlang distribution, i.e., $\mu_1 = \mu_2 = \ldots = \mu_m$. The level of detail with which order progress is tracked, as modeled by m, is varied between 1 and 6, depending on what is computationally feasible. (For example the computational burden is higher when demand is a MMPP as opposed to a stationary Poisson process.) The parameter m is shown when results are presented so that it is always clear exactly how m was varied. The expedited repair lead time is either low or high, $\ell_e \in \{1, 2\}$. By Proposition 1, we know that expediting is only useful when $c_e < p\mathbb{E}[L_r]$. Therefore, we chose $c_e = \nu p\mathbb{E}[L_r]$ for $\nu \in \{0.2, 0.4\}$.

Demand is a stationary Poisson process or a MMPP. The long run average demand intensity $\bar{\lambda} \in \{1, 2\}$. For the Markov modulated Poisson demand, we use two basic 'modulating processes' that we refer to as the cyclic and erratic MMPP respectively. They are specified by the generator matrices and intensity vectors

$$\mathbf{Q}_{\text{cyclic}} = \cdot \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\lambda}_{\text{cyclic}} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} & 1 \end{pmatrix}$$

$$\mathbf{Q}_{\text{erratic}} = \cdot \begin{pmatrix} -\frac{3}{2} & 1 & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{2}{5} & -\frac{4}{5} \end{pmatrix}, \qquad \boldsymbol{\lambda}_{\text{erratic}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 2 \end{pmatrix}$$

It is readily verified that both these MMPPs have a long run average demand of 1 per time unit. Therefore, by multiplying $\boldsymbol{\lambda}$ by $\bar{\lambda}$ we obtain a MMPP with a long run average demand of $\bar{\lambda}$ per time unit. Secondly, we scale how quickly the modulating chain of demand evolves by pre-multiplying the generator of $Y(t)$, $\mathbf{Q}$, by a scalar $\bar{q}$. For our experiment, $\bar{q} \in \left\{\frac{1}{20}, \frac{1}{10}\right\}$ so that the demand environment fluctuates either quickly or slowly relative to the replenishment lead times. Note that $\bar{q}$ does not affect the stationary distribution of the modulating chain of demand, and so it does not affect the long run average demand per time unit. However, it does affect the variability of demand over any finite time horizon.

In §6.2, we investigate the performance of the WDT policy for fixed turn-around stock. In this section, the fixed turn-around stock is set as

$$S := \left\lceil \lambda(\mathbb{E}[L_r] + \ell_e) + k\sqrt{\lambda(\mathbb{E}[L_r] + \ell_e)} \right\rceil$$

with $k \in \{0, 1, 2\}$. ($\lceil x \rceil$ denotes x rounded up to the nearest integer.) We refer to k as the safety factor and turn-around stock is tight for $k = 0$ up to ample for $k = 2$.

In §6.3, we optimize the expediting policy jointly with the turn-around stock. In this section, the test bed has two levels of holding cost, $h \in \left\{\frac{1}{2}, 1\right\}$.

A summary of the test bed is given in Table 1.

In the numerical experiments, we use value iteration to determine $C(S)$, $C_{\text{E}}(S)$, and $C_{\text{R}}(S)$. (We used Bellman equation (2) with the smaller uniformization constant in our algorithm). All value iteration algorithms are implemented in C and the value iteration is terminated when the relative error is less then 10^{-4}, i.e., value iteration to find optimal policies stops after $n + 1$ iterations if

$$\frac{\max_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y)) - \min_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y))}{\frac{1}{2}[\max_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y)) + \min_{(\mathbf{x},y)\in\mathcal{S}}(V_{n+1}(\mathbf{x}, y) - V_n(\mathbf{x}, y))]} < 10^{-4}.$$

To evaluate a given WDT policy exactly when $m \neq 1$ (i.e. evaluate $C_R(S)$), we use value iteration with the same stopping criterion.

Parameter	#	Values
Long run average demand per period ($\bar{\lambda}$)	2	1,2
Expected additional regular lead time ($\mathbb{E}[L_r]$)	2	2,4
Expedited lead time (ℓ_e)	2	1,2
Per unit expediting costs (c_e)	2	$0.2 \cdot p\mathbb{E}[L_r]$, $0.4 \cdot p\mathbb{E}[L_r]$
Holding costs per time unit (h)	2	0.5,1
Average transition rate of modulating chain ($\bar{q}$)	2	0.1, 0.05
Turn-around stock safety factor (k)	3	0,1,2
Backorder penalty costs (p)	1	10
Basic demand process type	3	Poisson, MMPP-erratic, MMPP-cyclic

Table 1: Description of instances in our test bed

6.2 Performance of the WDT policy for fixed turn-around stock

In this section, we investigate how the WDT policy performs relative to the optimal expediting policy for fixed turn-around stock. The turn-around stock is fixed so that the expediting policy can be studied in isolation from the stocking decision. To this end, we investigate the relative difference of the WDT obtained by assuming L_r is exponential with respect to the optimal expediting policy. Formally this is defined as:

$$\delta_C = \frac{C_{\mathrm{E}}(S) - C(S)}{C(S)} \cdot 100\%,$$

Tables 2 and 3 show the average and maximum optimality gaps for this situation over the test bed in §6.1. In both cases, the optimality gaps increase as m increases, meaning that order progress information does have added value, especially for more predictable demand (Poisson demand and cyclic MMPP demand). When demand is not very predictable (erratic MMPP demand), the variance of demand increases and so do the costs. The value of order progress information remains relatively steady and so the relative value of this information decreases. However, the magnitude of all optimality gaps is small especially considering the fact that turn-around stock holding cost hS is not included in the costs. The larger gaps occur when k is large (Table 4), and consequently the turn-around stock is large. The reason for this is that when the turn-around stock is large, expediting is rarely necessary and backorders seldom occur so that penalty and expediting costs are small. In this situation, small absolute deviations from optimality can constitute large relative deviations.

The computation times for determining an optimal policy are in the order of a week when $m = 6$ and demand is a Poisson process on a machine with 2.4 GHz CPU and 4 GB of RAM. For $m = 6$ and MMPP demand, computation was no longer practical and so these results are missing from Tables 2 and 3. Throughout, when '-' appears in a table, it indicates that computation was not feasible for these instances.

Table 2: Average optimality gaps δ_C in backorder and expediting costs for fixed turn-around stocks

$\bar{q}$	NA	0.1	0.05	0.1	0.05	
m	Poisson	cyclic MMPP		erratic MMPP		AVG
2	0.52%	0.45%	0.43%	0.16%	0.13%	0.34%
3	0.92%	0.80%	0.76%	0.32%	0.26%	0.61%
4	1.22%	1.07%	1.02%	0.43%	0.36%	0.82%
5	1.45%	1.28%	1.22%	0.53%	0.44%	0.98%
6	1.64%	-	-	-	-	1.64%
AVG	1.15%	0.90%	0.86%	0.36%	0.30%	0.88%

Table 3: Maximum optimality gaps δ_C in backorder and expediting costs for fixed turn-around stocks

$\bar{q}$	NA	0.1	0.05	0.1	0.05	
m	Poisson	cyclic MMPP		erratic MMPP		MAX
2	1.92%	1.35%	1.62%	1.38%	0.84%	1.92%
3	2.76%	2.10%	2.39%	2.03%	1.24%	2.76%
4	3.38%	2.63%	2.91%	2.49%	1.58%	3.38%
5	3.86%	3.03%	3.29%	2.84%	1.89%	3.86%
6	4.25%	-	-	-	-	4.25%
MAX	4.25%	3.03%	3.29%	2.84%	1.89%	4.25%

Table 4: Average and maximum optimality gaps δ_C for different fixed turn-around stock sizes

k	0	1	2
AVG	0.42%	0.83%	0.96%
MAX	2.59%	4.25%	3.29%

6.3 Performance of the E-WDT heuristic

Now we consider the joint optimization of expediting policy and turn-around stock. In this situation the optimality gap is defined as

$$\delta_{C_{\text{tot}}} = \frac{C_{\text{R-tot}}(S_{\text{E}}) - C_{\text{tot}}(S^*)}{C_{\text{tot}}(S^*)} \cdot 100\%,$$

where $S_{\text{E}} = \text{argmin}_{S \in \mathbb{N}_0} C_{\text{E-tot}}(S)$ and $S^* = \text{argmin}_{S \in \mathbb{N}_0} C_{\text{tot}}(S)$. Tables 5-6 show that the average and maximum optimality gaps $\delta_{C_{\text{tot}}}$ have the same trends as in the case where optimization over S is not included. However, since the costs of holding repairables is now included, the optimality gaps are very small, never exceeding 0.76%. In all but 35 out of 480 instances, S^* and S_{E} coincide, and the absolute difference is never more than 1.

Table 5: Average optimality gaps $\delta_{C_{\text{tot}}}$ when optimization over S is included

$\bar{q}$	NA	0.1	0.05	0.1	0.05	
m	Poisson	cyclic MMPP		erratic MMPP		AVG
2	0.08%	0.09%	0.09%	0.08%	0.07%	0.08%
3	0.14%	0.17%	0.16%	0.16%	0.13%	0.15%
4	0.19%	0.23%	0.21%	0.21%	0.17%	0.20%
AVG	0.14%	0.17%	0.15%	0.15%	0.12%	0.15%

Table 6: Maximum optimality gaps $\delta_{C_{\text{tot}}}$ when optimization over S is included

$\bar{q}$	NA	0.1	0.05	0.1	0.05	
m	Poisson	cyclic MMPP		erratic MMPP		MAX
2	0.22%	0.23%	0.29%	0.39%	0.23%	0.39%
3	0.36%	0.34%	0.43%	0.63%	0.31%	0.63%
4	0.46%	0.44%	0.53%	0.76%	0.41%	0.76%
MAX	0.46%	0.44%	0.53%	0.76%	0.41%	0.76%

Recall that $C_{\text{R-tot}}(S)(S) \neq C_{\text{E-tot}}(S)$, because the function $C_{\text{E-tot}}(S)$ assumes that L_r has an exponential distribution. Now we investigate how closely $C_{\text{E-tot}}(S)$ approximates $C_{\text{R-tot}}(S)$ by looking at the relative error

$$\epsilon_{\text{E}} = \frac{C_{\text{E-tot}}(S_{\text{E}}) - C_{\text{R-tot}}(S_{\text{E}})}{C_{\text{R-tot}}(S_{\text{E}})} \cdot 100\%.$$

The relative error ϵ_{E} is always positive and the averages and maxima are shown in Tables 7 and 8. This observation can be explained by observing that the variability of the exponential distribution is higher than that of the Erlang distribution. Since lead time variability generally degrades performance, we should expect that ϵ_{E} is generally positive.

Note that the approximation errors are larger than the optimality gaps shown in Tables 5 and 6, but still very acceptable. This is an important observation: *The insensitivity of the WDT policies with regard to the distribution of L_r is not only in performance evaluation, but even more so in policy optimality.* Furthermore, this approximation leads to a slight overestimation of the real costs which, from the managers perspective, is usually a safer deviation than an underestimation.

Table 7: Average error ϵ_{E} made by approximating L_r as having an exponential distribution when optimization over S is included

$\bar{q}$	0.1	0.05	0.1	0.05	
m	cyclic MMPP		erratic MMPP		AVG
2	0.46%	0.26%	0.89%	0.52%	0.53%
3	0.63%	0.35%	1.23%	0.71%	0.73%
4	0.73%	0.40%	1.42%	0.81%	0.84%
AVG	0.61%	0.33%	1.18%	0.68%	0.70%

Table 8: Maximum error ϵ_{E} made by approximating L_r as having an exponential distribution when optimization over S is included

$\bar{q}$	0.1	0.05	0.1	0.05	
m	cyclic MMPP		erratic MMPP		MAX
2	1.13%	0.66%	1.70%	1.01%	1.70%
3	1.57%	0.89%	2.38%	1.39%	2.38%
4	1.81%	1.02%	2.76%	1.60%	2.76%
MAX	1.81%	1.02%	2.76%	1.60%	2.76%

6.4 Value of anticipating demand fluctuations

In this section, we discuss three simple heuristics that either ignore the fact that demand is a Markov modulated Poisson process, or that separate the expediting policy and turn-around stock sizing decisions. Thus these heuristics fail to anticipate demand fluctuations.

In the context of repairables, the Poisson process has traditionally been used to model demand (Muckstadt, 2005; Sherbrooke, 2004). Our experience and that of Slay and Sherbrooke (1988) indicates that this model is usually accurate for short periods of time (say up to several lead times) but is not accurate for extended periods of time as demand intensity fluctuates. This effect is captured in the present model by using the MMPP to model demand. Nevertheless, it is convenient to use the Poisson demand model as the evaluation $C_{\text{E-tot}}(S)$ can be done exactly in closed form using (22). Consequently, an easy heuristic is to use the Poisson demand model with demand intensity either equal to the long run average demand or, to be on the safe side, equal to the demand intensity in peak periods, $\lambda_{\max}$. We refer to these two heuristics as POIS-AVG and POIS-MAX respectively when average and peak demand intensities are used.

Another common approach is to use a traditional news-vendor inventory model without expediting to determine the turn-around stock. A simple approach would be to select S to minimize

$$hS + p\mathbb{E}\left[\left(D^{Y}_{t,t+\ell_e+\mathbb{E}[L_r]} - S\right)^{+}\right]. \tag{23}$$

The S that minimizes (23) is the smallest integer that satisfies the newsvendor inequality

$$\sum_{y\in\Theta} \mathbb{P}\left\{D^{y}_{t,t+\ell_e+\mathbb{E}[L_r]} \leq S\right\} \mathbb{P}\{Y = y\} \geq \frac{p-h}{p}, \tag{24}$$

and we denote this minimizer by S_{NVF}. (NVF is short for newsvendor fractile.) After determining S_{NVF} using (24), we determine the optimal expediting policy for S_{NVF}. We refer to this heuristic as NVF-D_∞.

We consider the case that L_r has an exponential distribution so that the optimal expediting policy is a WDT policy. The POIS-AVG and POIS-MAX coincide with the optimal and E-WDT solution when demand is a Poisson process for all the instances in our test bed. Therefore, Tables 9 and 10 show the average and maximum optimality gaps only for the cases that demand is an MMPP for all three naive heuristics: POIS-AVG, POIS-MAX and NVF-D_∞.

Table 9: Average optimality gaps of naive heuristics

$\bar{q}$	0.1	0.05	0.1	0.05	
Heuristic	cyclic MMPP		erratic MMPP		AVG
POIS-AVG	2.41%	3.04%	19.48%	23.26%	12.05%
POIS-MAX	12.07%	11.32%	12.64%	11.17%	11.80%
NVF-D_∞	11.61%	10.99%	11.93%	12.26%	11.69%
AVG	8.70%	8.45%	14.68%	15.56%	11.85%

When demand is relatively steady, as is the case in the cyclic MMPP demand proces, POIS-AVG does not perform very bad, but when demand is more erratic, the performance deteriorates dramatically with optimality gaps up to 63.67%. The POIS-MAX policy avoids these extreme optimality gaps (although 24.21% is still quite substantial), but this occurs at the expense of cost performance when demand follows the more moderate cyclic MMPP process. The NVF-D_∞ solution performs similarly for all demand scenario's. In general, however all naive heuristics performs quite poorly with an average optimality gap

Table 10: Maximum optimality gaps for naive heuristics

$\bar{q}$	0.1	0.05	0.1	0.05	
Heuristic	cyclic MMPP		erratic MMPP		MAX
POIS-AVG	6.14%	8.07%	51.78%	63.67%	63.67%
POIS-MAX	15.38%	15.35%	24.21%	23.43%	24.21%
NVF-D_∞	29.44%	29.40%	29.72%	34.15%	34.15%
MAX	29.44%	29.40%	51.78%	63.67%	63.67%

of more than 11.69% for each heuristic and maximum optimality gaps above 24.21% for each heuristic. Consequently, we conclude that

1. There is great value in leveraging knowledge about demand fluctuations, as contained in $Y(t)$ for making repair expediting decisions.

2. Fluctuations of demand and the possibility to anticipate these through expediting repairs should be considered explicitly in sizing the turn-around stock and can lead to substantial savings.

7. Conclusion

In this paper, we have considered the joint problem of finding the best turn-around stock and expediting policy for repairables that experience fluctuating demand. With regard to expediting policies, we have characterized the structure of optimal policies, confirming a conjecture by Song and Zipkin (2009). Since computing optimal expediting policies suffers from the curse of dimensionality, we proposed the use of WDT policies. These policies have an intuitive appeal and share important monotonicity properties with optimal policies.

We have shown that the joint problem can be solved to optimality, even though it is not convex in general. We have also shown that the possibility to expedite repair can be used as a substitute for inventory in buffering uncertainty in demand. Solving the full problem to optimality suffers from the curse of dimensionality so we proposed the E-WDT heuristic that inherits all the structural results of optimal solutions. In a numerical study, we have shown that the E-WDT heuristic performs very close to optimal with an optimality gap of 0.15% on average and 0.76% at most across our test bed.

Finally, we investigated the value of anticipating demand fluctuations by proper joint optimization of the turn-around stock and expediting policy by comparing the E-WDT heuristic with more naive heuristics that do not anticipate demand fluctuations or that separate the stocking and expediting problems. With optimality gaps of 11.85% on average and of at most 63.67%, we have shown that

1. There is great value in leveraging knowledge about demand fluctuations when making repair expediting decisions.

2. Fluctuations of demand and the possibility to anticipate these through expediting repairs should be considered explicitly in sizing the turn-around stock and can lead to substantial savings.

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A. Determining $\mathbb{P}\{D^y_{t,t+\ell_e} = k\}$

In this appendix, we show how $\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}$ can be determined numerically. To this end, let $p_{y,y'}(k,\ell_e) = \mathbb{P}\left\{D^y_{t,t+\ell_e} = k | Y(t+\ell_e) = y'\right\}$ be the (y,y')-entry of the matrix $\mathbf{P}(k,\ell_e)$. Then the matrix generating function $\widetilde{\mathbf{P}}(z,\ell_e) = \sum_{k=0}^{\infty} \mathbf{P}(k,\ell_e) z^k$ satisfies (e.g Fischer and Meier-Hellstern, 1992):

$$\widetilde{\mathbf{P}}(z,\ell_e) = \exp\left([\mathbf{Q} - (1-z)\,\mathrm{diag}(\boldsymbol{\lambda})]\ell_e\right).$$

A plethora of numerical methods to compute the matrix exponential are discussed in Moler and Van Loan (2003). For the numerical work in this paper, we use the scaling and squaring algorithm with a Padé approximation. The probabilities $\mathbb{P}\left\{D^y_{t,t+\ell_e} = k | Y(t+\ell_e) = y'\right\}$ can be obtained from $\widetilde{\mathbf{P}}(z,\ell_e)$ by numerical inversion using the LATTICE-POISSON algorithm of Abate and Whitt (1992) which uses the approximation

$$\begin{aligned}&\mathbb{P}\left\{D^y_{t,t+\ell_e} = k | Y(t+\ell_e) = y'\right\} \\ &\quad\approx \frac{1}{2kr^k}\left\{\widetilde{\mathbf{P}}(r,\ell_e) + (-1)^k \widetilde{\mathbf{P}}(-r,\ell_e) + 2\sum_{n=1}^{k-1}(-1)^n \,\mathrm{Re}(\widetilde{\mathbf{P}}(r\exp(n\pi\mathrm{i}/k),\ell_e))\right\},\end{aligned}$$

where $\mathrm{i} = \sqrt{-1}$, $0 < r < 1$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x. The absolute error in this approximation is bounded by $\frac{r^{2k}}{1-r^{2k}}$ and so by choosing $r = 10^{-\gamma/(2k)}$, we obtain an accuracy of approximately $10^{-\gamma}$. Then the needed probability, $\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}$, can be found by un-conditioning:

$$\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} = \sum_{y' \in \Theta} \mathbb{P}\left\{D^y_{t,t+\ell_e} = k | Y(t+\ell_e) = y'\right\} \mathbb{P}\{Y(t+\ell_e) = y' | Y(t) = y\}$$

The probabilities $\mathbb{P}\{Y(t+\ell_e) = y' | Y(t) = y\}$ are found from the transient analysis of $Y(t)$. In particular, if we let $r_{y,y'} = \mathbb{P}\{Y(t+\ell_e) = y' | Y(t) = y\}$ be the (y,y')-th element of the matrix $\mathbf{R}(\ell_e)$, then $\mathbf{R}(\ell_e) = \exp(\ell_e \mathbf{Q})$.

B. Proofs

B.1 Proof of Lemma 1

Proof. The proof is a direct proof. Note that $\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} = 0$ for $k < 0$. For part (i) we have:

$$\begin{aligned}
\Delta c_p(x,y) &= c_p(x+1,y) - c_p(x,y) \\
&= p\left[\sum_{k=S-(x+1)}^{\infty}(k-S+x+1)\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right. \\
&\qquad \left.- \sum_{k=S-x}^{\infty}(k-S+x)\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right] \\
&= p\left[\sum_{k=S-x-1}^{\infty}(k-S+x)\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} + \sum_{k=S-x-1}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right. \\
&\qquad \left.- \sum_{k=S-x}^{\infty}(k-S+x)\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right] \\
&= p\sum_{k=S-x}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} \geq 0.
\end{aligned} \tag{25}$$

For part (ii) we have:

$$\begin{aligned}
\Delta^2 c_p(x,y) &= \Delta c_p(x+1,y) - \Delta c_p(x,y) \\
&= p\left[\sum_{k=S-(x+1)}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} - \sum_{k=S-x}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right] \\
&= p\mathbb{P}\left\{D^y_{t,t+\ell_e} = S-x-1\right\} \geq 0.
\end{aligned} \tag{26}$$

Part (iii) follows immediately from (25) and noting that $\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} = 0$ for $k < 0$.

Finally for part (iv), we can write using (25)

$$\begin{aligned}
&\Delta c_p(x,y|S) - \Delta c_p(x,y|S+1) \\
&\qquad = p\left[\sum_{k=S-x}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\} - \sum_{k=S+1-x}^{\infty}\mathbb{P}\left\{D^y_{t,t+\ell_e} = k\right\}\right] \\
&\qquad = p\mathbb{P}\left\{D^y_{t,t+\ell_e} = S-x\right\} \geq 0
\end{aligned} \tag{27}$$

□

B.2 Proof of Proposition 1 (ii)

Proof. Here we present the proof of part (ii), the proof of part (i) is in the main text. The proof is based on constructing two coupled processes and showing that expediting repair is expected to dominate using regular repair when there are many parts still undergoing additional regular repair.

Let $\varepsilon = (p\mathbb{E}[L_r] - c_e)/3 > 0$. We denote the probability density function of L_r as f_{L_r} and fix $\alpha < \infty$ to verify

$$\int_{t=\alpha}^{\infty} t f_{L_r}(t_r)dt \leq \varepsilon/p. \tag{28}$$

Such an α exists because $tf_{L_r}(t) > 0$ for $t \in (0,\infty)$ so that $\int_{t=\alpha}^{\infty} tf_{L_r}(t)dt$ is strictly decreasing in α and furthermore $\lim_{\alpha\to\infty}\int_{t=\alpha}^{\infty} tf_{L_r}(t)dt = 0$. Let E_μ denote an exponential random variable with mean μ^{-1}. We fix an integer M' to verify

$$\mathbb{P}\left\{E_{\mu_m} < \alpha\right\}^{M'}\mathbb{E}[L_r] \leq \varepsilon. \tag{29}$$

Such an $M' \in \mathbb{N}$ exists because $\alpha < \infty$ and so $\mathbb{P}\{E_{\mu_m} < \alpha\} < 1$.

Now we consider an arbitrary policy π that does *not* expedite when $\mathbf{x}\mathbf{e}^{\mathrm{T}} \geq S + M' = M$ for some $(\mathbf{x}, y) \in \mathcal{S}$. Consider an arbitrary moment in time, t', when a failed part arrives to the system and $\sum_{i=1}^{m} X_i(t') \geq S + M' = M$ and policy π stipulates that the part should *not* be expedited. Denote this process $\mathbf{X}^{\pi}(t)$. We let $\widetilde{\mathbf{X}}(t)$ denote the evolution of the part sent to regular repair at time t' by policy π, so $\widetilde{\mathbf{X}}(t) = \mathbf{e}_i$ if the part sent to repair at time t' has completed its first $i-1$ phases of the additional regular repair, and $\widetilde{\mathbf{X}}(t) = \mathbf{0}$ if the part has completed its additional regular repair lead time. Next, we consider an alternate process which is identical to $\mathbf{X}^{\pi}(t)$ except that it does expedite the unit arriving at t'. We denote this process $\mathbf{X}^{e}(t)$, and formally define it as $\mathbf{X}^{e}(t) = \mathbf{X}^{\pi}(t) - \widetilde{\mathbf{X}}(t)$. We let $T_r = \inf\{t - t' | \widetilde{\mathbf{X}}(t) = \mathbf{0}, t \geq t'\}$ and note that $T_r \overset{d}{=} L_r$.

Analogous to the proof of part (i), $\mathbf{X}^{\pi}(t)\mathbf{e}^{\mathrm{T}} = \mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} + 1$ for $t \in [t', t' + T_r)$, and $\mathbf{X}^{\pi}(t) = \mathbf{X}^{e}(t)$ for $t \geq t' + T_r$. Also both processes make exactly the same expediting decisions for all $t > t'$. Thus any cost differences between $\mathbf{X}^{e}(t)$ and $\mathbf{X}^{\pi}(t)$ occur in the time interval $[t', t' + T_r)$. Denote the expectation of this cost difference Ξ. Then we have:

$$
\begin{aligned}
\Xi &= c_e - \mathbb{E}_{T_r}\left\{\mathbb{E}_{(\mathbf{X}^{e}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}, Y(t))dt \,\middle|\, T_r\right]\right\} \\
&= c_e - \int_{t_r=0}^{\infty} \mathbb{E}_{(\mathbf{X}^{e}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}, Y(t))dt \,\middle|\, T_r = t_r\right] f_{L_r}(t_r)dt_r \\
&= c_e - \int_{t_r=0}^{\infty} \mathbb{E}_{(\mathbf{X}^{e}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}, Y(t))dt \,\middle|\, T_r = t_r,\, \mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + T_r)\right] \\
&\quad \times \mathbb{P}\{\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + t_r)\} f_{L_r}(t_r)dt_r \\
&\quad - \int_{t_r=0}^{\infty} \mathbb{E}_{(\mathbf{X}^{e}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}, Y(t))dt \,\middle|\, T_r = t_r,\, \mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} < S \text{ for some } t \in (t', t' + T_r)\right] \\
&\quad \times \mathbb{P}\{\mathbf{X}_{e}(t)\mathbf{e}^{\mathrm{T}} < S \text{ for some } t \in (t', t' + t_r)\} f_{L_r}(t_r)dt_r \\
&\leq c_e - \int_{t_r=0}^{\infty} \mathbb{E}_{(\mathbf{X}^{e}(t),Y(t))}\left[\int_{t=t'}^{t'+T_r} \Delta c_p(\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}, Y(t))dt \,\middle|\, T_r = t_r,\, \mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + T_r)\right] \\
&\quad \times \mathbb{P}\{\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + t_r)\} f_{L_r}(t_r)dt_r \\
&= c_e - \int_{t_r=0}^{\infty} p t_r f_{L_r}(t_r) \mathbb{P}\{\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + t_r)\} dt_r
\end{aligned} \tag{30}
$$

The third equality is obtained by conditioning on whether or not $\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}}$ stays above S on the interval $[t', t' + T_r)$. The first inequality follows from dropping the last term and the last equality follows from Lemma 1 (iii).

Next we observe that $\mathbb{P}\{\mathbf{X}^{e}(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t' + t_r)\}$ is bounded below by the probability that there are fewer than M' parts already in additional regular repair at time t', finish additional regular repair before $t' + t_r$. Since the remaining time in regular repair for any of these parts is at least an E_{μ_m} random variable (by the lack of memory property), we conclude that

$$\mathbb{P}\{\mathbf{X}^e(t)\mathbf{e}^{\mathrm{T}} \geq S \text{ for all } t \in (t', t'+t_r)\} \geq 1 - \mathbb{P}\{E_{\mu_m} < t_r\}^{M'}. \tag{31}$$

Now continuing from (30) and using (31) we obtain:

$$\begin{aligned}
\Xi &\leq c_e - \int_{t_r=0}^{\infty} p t_r f_{L_r}(t_r)\left(1 - \mathbb{P}\{E_{\mu_m} < t_r\}^{M'}\right) dt_r \\
&= c_e - p\mathbb{E}[L_r] + \int_{t_r=0}^{\infty} p t_r f_{L_r}(t_r)\mathbb{P}\{E_{\mu_m} < t_r\}^{M'} dt_r \\
&= c_e - p\mathbb{E}[L_r] + \int_{t_r=0}^{\alpha} p t_r f_{L_r}(t_r)\mathbb{P}\{E_{\mu_m} < t_r\}^{M'} dt_r \\
&\quad + \int_{t_r=\alpha}^{\infty} p t_r f_{L_r}(t_r)\mathbb{P}\{E_{\mu_m} < t_r\}^{M'} dt_r \\
&\leq c_e - p\mathbb{E}[L_r] + \mathbb{P}\{E_{\mu_m} < \alpha\}^{M'} \int_{t_r=0}^{\alpha} p t_r f_{L_r}(t_r) dt_r + \int_{t_r=\alpha}^{\infty} p t_r f_{L_r}(t_r) dt_r \qquad (32) \\
&\leq c_e - p\mathbb{E}[L_r] + \mathbb{P}\{E_{\mu_m} < \alpha\}^{M'}\mathbb{E}[L_r] + \int_{t_r=\alpha}^{\infty} p t_r f_{L_r}(t_r) dt_r \\
&\leq -3\varepsilon + \varepsilon + \varepsilon = -\varepsilon < 0. \qquad (33)
\end{aligned}$$

Inequality (32) follows because $\mathbb{P}\{E_{\mu_m} < t_r\}$ is increasing in t_r and the final inequalities follow from the choice of ε, α and M'. Since $\Xi < 0$, we conclude that the expected cost of process $\mathbf{X}^\pi(t)$ is greater than the cost of $\mathbf{X}^e(t)$. Thus, we have shown that any policy that does not expedite when $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}} \geq M$ and $c_e < p\mathbb{E}[L_r]$ can be strictly improved by expediting whenever $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}} \geq M$. That is, if $c_e < p\mathbb{E}[L_r]$, then there is a $M \in \mathbb{N}$ such that whenever $\mathbf{X}(t)\mathbf{e}^{\mathrm{T}} \geq M$ it is optimal to expedite. □

B.3 Proof of Lemma 2

To facilitate the presentation of the proof we introduce the following shorthand:

$$\mathcal{I} = \bigcap_{i=1}^{m} \mathcal{I}(i), \quad \mathcal{SM}(1) = \bigcap_{j=2}^{m} \mathcal{SM}(1,j)$$

Furthermore, when $f \in \mathcal{X}$ implies $\mathbb{T}_{\mathcal{Y}} f(\mathbf{x}, y) \in \mathcal{X}$, we say that $\mathbb{T}_{\mathcal{Y}}$ propagates $\mathcal{X}$.

Proof. Part (i) of the lemma can be verified directly by using Lemma 1.

For part (ii) we consider each operator separately. Let $f \in \mathcal{F}$. For operator $\mathbb{T}_{\mathrm{cost}}$ the results hold because of part (i) of this lemma and Theorem 7.1 in Koole (2006). For $\mathbb{T}_{\mathrm{env}}$ the result hold trivially as this operator produces linear combinations of functions in $\mathcal{F}$.

By Theorems 7.3 and 7.4 of Koole (2006) we have that $\mathbb{T}_{\mathrm{TD}(i)}$ propagates $\mathcal{F}$ for $i = 1, ..., m-1$ and $\mathbb{T}_{\mathrm{D}(m)}$ propagates $\mathcal{F}$.

For $\mathbb{T}_{\mathrm{AC}(1)}$, the inequalities that characterize $\mathcal{I} \cap \mathcal{UI}$ are propagated whenever $\mathbf{x}\mathbf{e}^{\mathrm{T}} < M - 1$ by Theorem 7.2 of Koole (2006). When $\mathbf{x}\mathbf{e}^{\mathrm{T}} = M - 1$ we have for $i = 1, \ldots, m$:

$$\begin{aligned}
\Delta_i \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) &= c_e + f(\mathbf{x} + \mathbf{e}_i, y) - \min(c_e + f(\mathbf{x}, y), f(\mathbf{x} + \mathbf{e}_1, y)) \\
&\geq c_e + f(\mathbf{x} + \mathbf{e}_i, y) - c_e - f(\mathbf{x}, y) \\
&= f(\mathbf{x} + \mathbf{e}_i, y) - f(\mathbf{x}, y) \geq 0, \qquad (34)
\end{aligned}$$

where the second inequality holds because $f \in \mathcal{I}$. This shows $\mathbb{T}_{\mathrm{AC}(1)}$ propagates $\mathcal{I}$. Similarly, and again for $\mathbf{x}\mathbf{e}^{\mathrm{T}} = M - 1$, we find for $i = 1, \dots, m-1$:

$$\mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}+\mathbf{e}_i, y) - \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}+\mathbf{e}_{i+1}, y) = c_e + f(\mathbf{x}+\mathbf{e}_i, y) - c_e - f(\mathbf{x}+\mathbf{e}_{i+1}, y) \geq 0, \tag{35}$$

where the inequality holds because $f \in \mathcal{UI}$. Thus we have shown that $\mathbb{T}_{\mathrm{AC}(1)}$ propagates $\mathcal{UI}$. (Recall that the case $\mathbf{x}\mathbf{e}^{\mathrm{T}} = M$ need not be considered because, in this case, the inequalities do not exist in $\mathcal{S}$. A similar observation will hold for the other inequalities in $\mathcal{F}$.) Also by Theorem 7.2 of Koole (2006), for all $\mathbf{x}$ that satisfy $\mathbf{x}\mathbf{e}^{\mathrm{T}} < M - 2$ it holds that $\mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) \in \mathcal{C}(1) \cap \mathcal{SM}(1)$. Consider the case that $\mathbf{x}\mathbf{e}^{\mathrm{T}} = M - 2$. To show $\mathbb{T}_{\mathrm{AC}(1)}$ preserves convexity, we consider three cases:

(a) Case: $\min\{c_e + f(\mathbf{x}+\mathbf{e}_1, y), f(\mathbf{x}+2\mathbf{e}_1, y)\} = f(\mathbf{x}+2\mathbf{e}_1, y)$. This case implies that $c_e \geq f(\mathbf{x}+2\mathbf{e}_1, y) - f(\mathbf{x}+\mathbf{e}_1, y)$ and furthermore as $f \in \mathcal{C}(1)$ we have $\min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} = f(\mathbf{x}+\mathbf{e}_1, y)$. Thus we have:

$$\begin{aligned}\Delta_1^2 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) &= c_e + f(\mathbf{x}+2\mathbf{e}_1, y) - 2f(\mathbf{x}+2\mathbf{e}_1, y) + f(\mathbf{x}+\mathbf{e}_1, y) \\ &= c_e - f(\mathbf{x}+2\mathbf{e}_1, y) + f(\mathbf{x}+\mathbf{e}_1, y) \geq 0. \end{aligned} \tag{36}$$

The inequality holds because $c_e \geq f(\mathbf{x}+2\mathbf{e}_1, y) - f(\mathbf{x}+\mathbf{e}_1, y)$.

(b) Case: $\min\{c_e + f(\mathbf{x}+\mathbf{e}_1, y), f(\mathbf{x}+2\mathbf{e}_1, y)\} = c_e + f(\mathbf{x}+\mathbf{e}_1, y)$ and $\min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} = c_e + f(\mathbf{x}, y)$. Now we have

$$\begin{aligned}\Delta_1^2 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) &= c_e + f(\mathbf{x}+2\mathbf{e}_1, y) - 2c_e - 2f(\mathbf{x}+\mathbf{e}_1, y) + c_e + f(\mathbf{x}, y) \\ &= f(\mathbf{x}+2\mathbf{e}_1, y) - 2f(\mathbf{x}+\mathbf{e}_1, y) + f(\mathbf{x}, y) \geq 0, \end{aligned} \tag{37}$$

where the inequality holds because $f \in \mathcal{C}(1)$.

(c) Case: $\min\{c_e + f(\mathbf{x}+\mathbf{e}_1, y), f(\mathbf{x}+2\mathbf{e}_1, y)\} = c_e + f(\mathbf{x}+\mathbf{e}_1, y)$ and $\min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} = f(\mathbf{x}+\mathbf{e}_1, y)$. Now we have:

$$\begin{aligned}\Delta_1^2 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) &= c_e + f(\mathbf{x}+2\mathbf{e}_1, y) - 2c_e - 2f(\mathbf{x}+\mathbf{e}_1) + f(\mathbf{x}+\mathbf{e}_1, y) \\ &= f(\mathbf{x}+2\mathbf{e}_1, y) - f(\mathbf{x}+\mathbf{e}_1, y) - c_e \geq 0. \end{aligned} \tag{38}$$

The inequality holds because the case implies that $c_e \leq f(\mathbf{x}+2\mathbf{e}_1, y) - f(\mathbf{x}+\mathbf{e}_1, y)$.

Thus we have shown that $\mathbb{T}_{\mathrm{AC}(1)}$ propagates $\mathcal{C}(1)$ if $f \in \mathcal{F}$. To show $\mathbb{T}_{\mathrm{AC}(1)}$ also propagates $\mathcal{SM}(1)$, we distinguish 2 cases.

(a) Case: $\min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} = c_e + f(\mathbf{x}, y)$. In this case we have

$$\begin{aligned}&\Delta_1 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}+\mathbf{e}_j, y) - \Delta_1 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) \\ &\quad = c_e + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y) - \min\{c_e + f(\mathbf{x}+\mathbf{e}_j, y), f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y)\} \\ &\qquad - \min\{c_e + f(\mathbf{x}+\mathbf{e}_1, y), f(\mathbf{x}+2\mathbf{e}_1)\} + \min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} \\ &\quad \geq 2c_e + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j) - 2c_e - f(\mathbf{x}+\mathbf{e}_j, y) - f(\mathbf{x}+\mathbf{e}_1, y) + f(\mathbf{x}, y) \\ &\quad = f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j) - f(\mathbf{x}+\mathbf{e}_j, y) - f(\mathbf{x}+\mathbf{e}_1, y) + f(\mathbf{x}, y) \geq 0. \end{aligned} \tag{39}$$

The second inequality holds because $f \in \mathcal{SM}(1, j)$.

(b) Case: $\min(c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)) = f(\mathbf{x}+\mathbf{e}_1, y)$ Now we find that:

$$\begin{aligned}
&\Delta_1 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}+\mathbf{e}_j, y) - \Delta_1 \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y) \\
&\quad = c_e + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y) - \min\{c_e + f(\mathbf{x}+\mathbf{e}_j, y), f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y)\} \\
&\quad\quad - \min\{c_e + f(\mathbf{x}+\mathbf{e}_1, y), f(\mathbf{x}+2\mathbf{e}_1)\} + \min\{c_e + f(\mathbf{x}, y), f(\mathbf{x}+\mathbf{e}_1, y)\} \\
&\quad \geq c_e + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y) - f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_j, y) \\
&\quad\quad - c_e - f(\mathbf{x}+\mathbf{e}_1, y) + f(\mathbf{x}+\mathbf{e}_1, y) = 0.
\end{aligned} \tag{40}$$

Thus we have shown that $\mathbb{T}_{\mathrm{AC}(1)}$ propagates $\mathcal{SM}(1)$ if $f \in \mathcal{F}$.

Part (iii) holds trivially as $\mathbb{T}_{\mathrm{unif}}$ produces linear combinations of functions in $\mathcal{F}$. □

B.4 Proof of Lemma 4

Proof. The property in (21) is submodularity (c.f. Altman and Koole, 1998). In this proof, we actually prove a slightly stronger property, namely that (21) also holds with Δ_1 replaced by Δ_i for $i = 1, \ldots, m$. We define $\mathcal{SB}$ as:

$$\mathcal{SB} = \{f \in \mathcal{W} | \Delta_i f(\mathbf{x}, y, S) \geq \Delta_i f(\mathbf{x}, y, S+1) \text{ for } i = 1, \ldots, m\}.$$

Because of (12) and the fact that $V_0(\mathbf{x}, y, S) \in \mathcal{SB}$, we only need to show that if $f \in \mathcal{SB}$ and $f_j \in \mathcal{SB}$ for $j = 1, \ldots, m+2$, then also

$$\begin{aligned}
&\mathbb{T}_{\mathrm{cost}} f(\mathbf{x}, y, S), \mathbb{T}_{\mathrm{AC}(1)} f(\mathbf{x}, y, S) \mathbb{T}_{\mathrm{D}(m)} f(\mathbf{x}, y, S), \mathbb{T}_{\mathrm{env}} f(\mathbf{x}, y, S) \in \mathcal{SB} \\
&\mathbb{T}_{\mathrm{TD}(j)} f(\mathbf{x}, y, S) \in \mathcal{SB} \text{ for } j = 1, \ldots, m-1 \text{ and} \\
&\mathbb{T}_{\mathrm{unif}}(f_1, \ldots f_{m+2})(\mathbf{x}, y, S) \in \mathcal{SB}.
\end{aligned}$$

For $\mathbb{T}_{\mathrm{cost}}$, this follows from Lemma 1 (iv) and Theorem 7.1 of Koole (2006). For $\mathbb{T}_{\mathrm{env}}$ and $\mathbb{T}_{\mathrm{unif}}$ this follows because these operators take linear combinations of functions in $\mathcal{SB}$. For $\mathbb{T}_{\mathrm{AC}(1)}$ and $\mathbb{T}_{\mathrm{D}(m)}$ this follows from Theorems 7.2 and 7.3 of Koole (2006) respectively. For $\mathbb{T}_{\mathrm{TD}(j)}$, we distinguish two cases:

(a) Case: $j \neq i$. We have

$$\begin{aligned}
&\Delta_i \mathbb{T}_{\mathrm{TD}(j)} f(\mathbf{x}, y, S) - \Delta_i \mathbb{T}_{\mathrm{TD}(j)} f(\mathbf{x}, y, S+1) \\
&\quad = \frac{x_j}{M} f(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S) + \frac{M-x_j}{M} f(\mathbf{x}+\mathbf{e}_i, y, S) \\
&\quad\quad - \frac{x_j}{M} f(\mathbf{x}-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S) - \frac{M-x_j}{M} f(\mathbf{x}, y, S) \\
&\quad\quad - \frac{x_j}{M} f(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S+1) - \frac{M-x_j}{M} f(\mathbf{x}+\mathbf{e}_i, y, S+1) \\
&\quad\quad + \frac{x_j}{M} f(\mathbf{x}-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S+1) + \frac{M-x_j}{M} f(\mathbf{x}, y, S+1) \\
&\quad = \frac{x_j}{M} \left(\Delta_i f(\mathbf{x}-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S) - \Delta_i f(\mathbf{x}-\mathbf{e}_j+\mathbf{e}_{j+1}, y, S+1)\right) \\
&\quad\quad \frac{M-x_j}{M} \left(\Delta_i f(\mathbf{x}, y, S) - \Delta_i f(\mathbf{x}, y, S+1)\right) \geq 0
\end{aligned}$$

The inequality holds because $f \in \mathcal{SB}$.

(b) Case $j = i$. We have

$$\begin{aligned}
&\Delta_i \mathbb{T}_{\mathrm{TD}(i)} f(\mathbf{x}, y, S) - \Delta_i \mathbb{T}_{\mathrm{TD}(i)} f(\mathbf{x}, y, S+1) \\
&= \frac{x_i+1}{M} f(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S) + \frac{M - x_i - 1}{M} f(\mathbf{x} + \mathbf{e}_i, y, S) \\
&\quad - \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S) - \frac{M - x_i}{M} f(\mathbf{x}, y, S) \\
&\quad - \frac{x_i+1}{M} f(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S+1) - \frac{M - x_i - 1}{M} f(\mathbf{x} + \mathbf{e}_i, y, S+1) \\
&\quad + \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S+1) + \frac{M - x_i}{M} f(\mathbf{x}, y, S+1) \\
&\geq \frac{x_i}{M} f(\mathbf{x} + \mathbf{e}_{i+1}, y, S) + \frac{M - x_i - 1}{M} f(\mathbf{x} + \mathbf{e}_i, y, S) \\
&\quad - \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S) - \frac{M - x_i - 1}{M} f(\mathbf{x}, y, S) \\
&\quad - \frac{x_i}{M} f(\mathbf{x} + \mathbf{e}_{i+1}, y, S+1) - \frac{M - x_i - 1}{M} f(\mathbf{x} + \mathbf{e}_i, y, S+1) \\
&\quad + \frac{x_i}{M} f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S+1) + \frac{M - x_i - 1}{M} f(\mathbf{x}, y, S+1) \\
&= \frac{x_i}{M} \left(\Delta_i f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S) - \Delta_i f(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}, y, S+1) \right) \\
&\quad \frac{M - x_i - 1}{M} \left(\Delta_i f(\mathbf{x}, y, S) - \Delta_i f(\mathbf{x}, y, S+1) \right) \geq 0
\end{aligned}$$

The first inequality follows by adding $\frac{1}{M}(\Delta_{i+1} f(\mathbf{x}, y, S+1) - \Delta_{i+1} f(\mathbf{x}, y, S))$ which is less than 0 because $f \in \mathcal{SB}$. (Note that $\Delta_{i+1} f(\mathbf{x}, y, S)$ is well defined here because because $j < m$ and so $i + 1 \leq m$ because $i = j$ by assumption.) The final inequality also follows from the induction hypothesis.

□

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