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No acute tetrahedron is an 8-reptile

Herman Haverkort*

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Abstract

An r -gentiling is a dissection of a shape into $r \geq 2$ parts which are all similar to the original shape. An r -reptiling is an r -gentiling of which all parts are mutually congruent. This article shows that no acute tetrahedron is an r -gentile or r -reptile for any $r < 9$, by showing that no acute spherical diangle can be dissected into less than nine acute spherical triangles.

Introduction

We call a geometric figure, that is, a set of points in Euclidean space, T an r -gentile if T admits an r -gentiling, that is, a subdivision of T into $r \geq 2$ figures (*tiles*) T_1, \dots, T_r , such that each of the figures T_1, \dots, T_r is similar to T . In other words, T is an r -gentile if we can tile it with r smaller copies of itself. This generalizes the concept of *reptiles*, coined by Golomb [4]: a figure T is an r -reptile if T admits an r -reptiling, that is, a subdivision of T into $r \geq 2$ figures T_1, \dots, T_r , such that each of the figures T_1, \dots, T_r is similar to T and all figures T_1, \dots, T_r are mutually congruent. In other words, T is a r -reptile if we can tile it with r equally large smaller copies of itself. Interest in reptile tetrahedra (or triangles, for that matter) exists, among other reasons, because of their application in meshes for scientific computing [1, 10]. In this realm certain reptile-based techniques are well-developed in two dimensions [1], but three-dimensional space poses great challenges [6].

It is known what triangles are r -reptiles [12] and r -gentiles [3, 9] for what r . However, for tetrahedra the situation is much less clear; in fact the identification of reptile and gentile tetrahedra and tetrahedra that tile space has been a long-standing open problem. The shape of a tetrahedron has five degrees of freedom and these have not been fully explored yet. Matoušek and Safernová argued that r -reptilings with tetrahedra exist if and only if r is a cube number [11]. In particular, it is known that all so-called *Hill tetrahedra* (attributed to Hill [8] by Hertel [7] and Matoušek and Safernová [11]) are 8-reptiles. It has been conjectured that the Hill tetrahedra are the only reptile tetrahedra [7], but this conjecture is false: two non-Hill tetrahedra are known that have been recognized as 8-reptiles by Liu and Joe [10]. To the best of our knowledge, the Hill tetrahedra and the two non-Hill tetrahedra from Liu and Joe are the only tetrahedra known to be reptiles, but there might be others. This paper provides a small contribution to the answer of the question: exactly what tetrahedra are reptiles?

In mesh construction applications one typically needs to enforce certain quality constraints on the mesh elements. This has motivated studies into acute triangles [5] and *acute tetrahedra* [2]:

*Dept. of Mathematics and Computer Science, Eindhoven University of Technology, the Netherlands, cs.herman@haverkort.net

Definition 1. *A tetrahedron is acute if each pair of its facets has a dihedral angle strictly less than $\pi/2$.*

All facets of an acute tetrahedron are acute triangles themselves (Eppstein et al. [2], Lemma 2). The Hill tetrahedra, as well as the two non-Hill tetrahedra from Liu and Joe, all have right dihedral angles. Thus, no acute reptile tetrahedra are known.

Results

In this article we will prove the following statement, which may serve as evidence that acute reptile tetrahedra are probably hard to find, if they exist at all:

Theorem 1. *Let T be an acute tetrahedron subdivided into $r \geq 2$ acute tetrahedra T_1, \dots, T_r . If the diameter (longest edge) of each tetrahedron T_i is smaller than the diameter (longest edge) of T , then $r \geq 9$.*

In particular we get:

Corollary 1. *No acute tetrahedron is an r -gentile for any $r < 9$.*

With the result from Matoušek and Safernová that r -reptile tetrahedra can only exist when r is a cube number [11], we get:

Corollary 2. *No acute tetrahedron is an r -reptile for any $r < 27$.*

The proof

Note that if a tetrahedron T is subdivided into tetrahedra T_1, \dots, T_r with smaller diameter than T , then at least one tetrahedron T_i , for some $i \in \{1, \dots, r\}$, must have a vertex v on the longest edge of T . For the proof of Theorem 1 we analyse \mathcal{S}_v , the subdivision of an infinitesimal sphere around v that is induced by the facets of T and T_1, \dots, T_r . In such a subdivision, we find:

- faces: each face is either a spherical triangle, corresponding to a tetrahedron T_i of which v is a vertex, or a spherical diangle (also called lune), corresponding to a tetrahedron that has v on the interior of an edge;
- edges: the edges of \mathcal{S}_v are segments of great circles and correspond to facets of T_1, \dots, T_r that contain v ; the angle between two adjacent edges on a face of \mathcal{S}_v corresponds to the dihedral angle of the corresponding facets of a tetrahedron T_i .
- vertices: each vertex of \mathcal{S}_v corresponds to an edge of a tetrahedron T_i that contains v .

Thus, \mathcal{S}_v consists of a spherical diangle D corresponding to T , subdivided into a number of spherical triangles, and possibly some spherical diangles, that correspond to the tetrahedra from T_1, \dots, T_r that touch v . Below we will see that \mathcal{S}_v must contain at least nine faces (not counting the outer face, that is, the complement of D), which proves Theorem 1.

In what follows, when we talk about diangles and triangles, we will mean *acute, spherical* diangles and *acute, spherical* triangles on a sphere with radius 1. Note that the faces are diangles or triangles in the geometric sense, but they may have more than two or three vertices on their boundary. More precisely, a diangle or triangle has, respectively, exactly two or three vertices, called *corners*, where its boundary has an acute angle, and possibly a number of other vertices where its boundary has a straight angle. A chain of edges of a diangle or triangle from one

corner to the next is called a *side*. Note that \mathcal{S}_v contains at least one triangle, since v is a vertex of at least one tetrahedron T_i . Therefore, in what follows we consider a subdivision \mathcal{S} of a diangle D into a number of diangles and triangles, among which at least one triangle. We call such subdivisions *valid*. Henceforth, we will assume that \mathcal{S} has the smallest number of faces out of all possible valid subdivisions of all possible diangles D . Our goal is now to prove that \mathcal{S} contains at least 9 faces.

Lemma 1. *Each face of \mathcal{S} is a triangle.*

Proof. If \mathcal{S} would have any diangular face F , it must have the same corners as D , because the corners of any diangle must be an antipodal pair and there is only one antipodal pair within D . The removal of F would separate \mathcal{S} into at most two diangular components with the same corners as D . By construction, at least one of these components contains a triangle. That component would then constitute a valid subdivision that has fewer faces than \mathcal{S} , contradicting our choice of \mathcal{S} . \square

In \mathcal{S} , we distinguish *boundary vertices* (vertices on the boundary of D) and *interior vertices* (vertices in the interior of D). Among the boundary vertices, we distinguish *poles* (the corners of D) and *side vertices* (the remaining boundary vertices). Among the interior vertices we distinguish *full vertices* and *hanging vertices*: a vertex v is a full vertex if it is a corner of each face incident on v ; a vertex v is a hanging vertex if it is a non-corner vertex of one of the faces incident on v .

We will now derive a few properties of \mathcal{S} from the acuteness of its angles.

Lemma 2. *Each side of each face of \mathcal{S} has length strictly less than $\pi/2$.*

Proof. Consider any face F of \mathcal{S} . Let a be the length of a particular side of F , let α be the angle in the opposite corner of F , and let β and γ be the angles in the other two corners of F . Since F is acute, the sines and cosines of α, β and γ are all positive. By the supplementary cosine rule ([13], Art. 47) we have $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$, so $\cos a = (\cos \alpha + \cos \beta \cos \gamma) / (\sin \beta \sin \gamma) > 0$. It follows that $a < \pi/2$. \square

Lemma 3. *There are at least four side vertices: two on each side of D .*

Proof. For the sake of contradiction, suppose one side of D contains only one side vertex. Then this side would consist of two edges, at least one of which has length at least $\pi/2$, contradicting Lemma 2. \square

Lemma 4. *Each pole is incident on at least two edges.*

Each hanging vertex and each side vertex is incident on at least four edges.

Each full vertex is incident on at least five edges.

Proof. The poles are incident on at least two edges by definition. If a side vertex or a hanging vertex would be incident on only three edges, then two of these edges make a straight angle on one side, while the third edge divides the straight angle on the other side. Thus, at least one of the angles that results from this division would be non-acute. If a full vertex would be incident on at most four edges, then at least two of those edges must make an angle of at least $2\pi/4 = \pi/2$ on their common face, again contradicting the assumption that all faces are acute. \square

Now we can combine these properties with Euler's formula and find:

Lemma 5. *The number of faces equals $2f + h + s$, where $f \geq 2$ is the number of full vertices, h is the number of hanging vertices, and $s \geq 4$ is the number of side vertices.*

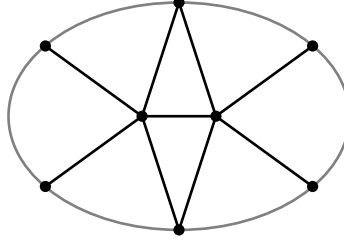


Figure 1: In grey: the boundary of D sketched as a smooth loop, without indicating the location of the poles. In black: the only network of edges that subdivides D into at most eight faces and complies with Lemma 5. However, this contradicts Lemma 4.

Proof. Let $v = f + h + s + 2$ be the number of vertices, let e be the number of edges and r be the number of triangles of \mathcal{S} . By Lemma 1 all faces are triangles, so by Euler's formula we have $v + r = f + h + s + 2 + r = e + 1$, hence $2e = 2f + 2h + 2s + 2 + 2r$. We say that a hanging vertex is *owned* by the triangle of which it is a non-corner vertex; each hanging vertex is owned by exactly one triangle. If we add up the edges of all triangles, we count, for each triangle, three edges plus the number of hanging vertices it owns, making $3r + h$ in total. This counts all edges double, except the $s + 2$ edges on the boundary, which are counted only once. Therefore we have $2e - s - 2 = 3r + h$. Hence we have $2e = 3r + h + s + 2 = 2f + 2h + 2s + 2 + 2r$, which solves to $r = 2f + h + s$.

Thus we have $2e = 3r + h + s + 2 = 6f + 4h + 4s + 2$. By Lemma 4 we also have $2e \geq 5f + 4h + 4s + 4$, so $6f + 4h + 4s + 2 \geq 5f + 4h + 4s + 4$, which solves to $f \geq 2$.

The condition $s \geq 4$ is given by Lemma 3. □

Lemma 6. *The number of faces of \mathcal{S} is at least nine.*

Proof. Suppose, for the sake of contradiction, that \mathcal{S} has at most eight faces. Then, by Lemma 5, we have four side vertices, two full vertices, and no hanging vertices. Each of the two interior vertices is incident on five faces, so in order to have at most eight faces in total, the interior vertices must share two of their incident faces, see Figure 1. Adding further edges in the interior of D is not possible, as this would increase the number of faces. Thus there are at least six boundary vertices, four of which are incident on only one interior edge and therefore on only three edges in total, and at least two of these must be side vertices. This contradicts Lemma 4. □

This concludes the proof of Theorem 1.

Bold conjectures

The proof of Theorem 1 is based on the following observation: any dissection of an acute spherical diangle into acute spherical triangles requires at least nine triangles. I do not know whether this bound is tight. It seems that a dissection into ten triangles is easy to achieve: in Figure 1, add an edge connecting the two side vertices on the left, add an edge connecting the two side vertices on the right, place the poles on the left and the right end of the figure, and distort the construction to make all angles acute. So far, I have not been able to find a solution with nine triangles, so I conjecture:

Conjecture 1. *Any dissection of an acute spherical diangle into acute spherical triangles requires at least ten triangles.*

Note that proving this would immediately strengthen Corollary 1 to: no acute tetrahedron is an r -gentile for any $r < 10$.

Now let b be this lower bound on the required number of triangles, proven to be at least 9 and conjectured to be 10. Suppose we can prove Conjecture 1, then how could we go about improving Corollary 2? A first step could be the following. If T is an acute tetrahedron with diameter d , and we can identify three segments x , y and z of length $d/3$ on the edges of T , then the arguments presented in this paper tell us that any 27-reptiling of T must contain at least b tiles that intersect x , b tiles that intersect y , and b tiles that intersect z . If additionally, one can ensure that x , y and z lie at distance more than $d/3$ from each other, these three sets of b tiles must be mutually disjoint, so there must be at least $3b$ tiles in total. With $b = 10$, this would contradict the existence of a 27-reptiling and thus improve Corollary 2 to: no acute tetrahedron is an r -reptile for any $r < 64$.

However, given that no acute tetrahedron can be an r -reptile or r -gentile for small values of r (and given, in general, that the past hundred years did not turn up any reptile tetrahedra without right dihedral angles), we may rather restate the obvious:

Conjecture 2. *There are no reptile acute tetrahedra.*

A stronger conjecture would be:

Conjecture 3. *There are no gentile acute tetrahedra.*

We conclude with an even bolder conjecture:

Conjecture 4. *There are no gentile tetrahedra that do not have a dihedral angle of exactly $\pi/2$.*

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