On the Riemannian rationale for diffusion tensor imaging

On the Riemannian Rationale for Diffusion Tensor Imaging

Andrea Fuster, Tom Dela Haije, and Luc Florack

Imaging Science & Technology Eindhoven (IST/e), Eindhoven University of Technology, The Netherlands

Abstract—One of the approaches in the analysis of brain diffusion MRI data is to consider white matter as a Riemannian manifold, with a metric given by the inverse of the diffusion tensor. Such a metric is used for white matter tractography and connectivity analysis. Although this choice of metric is heuristically justified it has not been derived from first principles. We propose a modification of the metric tensor motivated by the underlying mathematics of diffusion.

I. Introduction

A possible approach to study white matter from diffusion MRI is to consider a geometric framework in which quantities of interest, such as connectivity measures, are derived from a Riemannian metric. In this way white matter is represented as a Riemannian manifold, and candidate neural fibres are postulated to coincide with geodesic curves. The common choice in the literature is to consider a Riemannian metric given by the inverse of the diffusion tensor D [1], [2]. The intuitive idea behind this choice is to transform anisotropic diffusion in Euclidean space to (free) Brownian diffusion in a curved Riemannian space. Brownian motion is characterized by an infinitesimal diffusion generator $\mathcal{L}$ which can be written as $\mathcal{L} = \triangle_{\text{LB}}$, with $\triangle_{\text{LB}}$ the Laplace-Beltrami operator for the appropriate metric tensor g [3]. Such a generator is called an intrinsic Laplacian. However, the usual choice of metric from an anisotropic diffusion generator, $g = D^{-1}$, does not lead to Brownian motion in the corresponding curved space. We propose a slight modification of the Riemannian metric in order to accomplish this. The question of how to choose an appropriate metric has already been addressed in [4], [5].

II. Discrepancy

Inhomogeneous anisotropic diffusion is commonly described by the generator

$$\mathcal{L} = \partial_i(D^{ij}\partial_j) = D^{ij}\partial_i\partial_j + (\partial_j D^{ij})\partial_i \qquad (1)$$

where $i, j = 1, 2, 3$, D^{ij} is the diffusion tensor, $\partial_i = \partial/\partial x^i$, and in which we use Einstein's summation convention. A Riemannian metric $g_{ij} = D_{ij}$ can be introduced, where D_{ij} is the inverse diffusion tensor.

The generator (1) can then be expressed as

$$\mathcal{L} = \triangle_g - \sqrt{d}\,\left(\partial_j\frac{1}{\sqrt{d}}\right)D^{ij}\partial_i \qquad (2)$$

where d is the determinant of the diffusion tensor D^{ij} and $\triangle_g$ is the Laplace-Beltrami operator

$$\triangle_g = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\,g^{ij}\partial_i) \qquad (3)$$

Here, $g = \det g_{ij}$. In our case, $g_{ij} = D_{ij}$, we have

$$\triangle_g = D^{ij}\partial_i\partial_j + \sqrt{d}\,\partial_j\left(\frac{1}{\sqrt{d}}D^{ij}\right)\partial_i \qquad (4)$$

From Eq. (2) we see that the usual identification $g = D^{-1}$ does not lead to Brownian motion on the manifold (M, g) since the diffusion generator $\mathcal{L}$ is not an intrinsic Laplacian. This is only the case when the second term on the right-hand side of Eq. (2) vanishes, which occurs for $d = \det D^{ij}$ constant. Clearly, this cannot be assumed in general.

III. Proposal

Consider now the diffusion generator given by

$$\tilde{\mathcal{L}} = d^{-1}\mathcal{L} = d^{-1}D^{ij}\partial_i\partial_j + d^{-1}(\partial_j D^{ij})\partial_i \qquad (5)$$

where we use the same notation as in section II. Again a Riemannian metric can be introduced, namely, $\tilde{g}_{ij} = dD_{ij}$. It can be shown that

$$\tilde{\mathcal{L}} = \triangle_{\tilde{g}} \qquad (6)$$

The generator (5) is therefore an intrinsic Laplacian, and the proposed choice of metric results in Brownian motion on the manifold $(M, \tilde{g})$.

IV. Discussion

We propose a new Riemannian metric in the context of diffusion tensor imaging, motivated by first principles. In future work experiments will be performed to assess whether our modified metric leads to improved results for tractography and connectivity analysis in comparison to the usual choice of metric. It would also be very interesting to clarify the relation to other modified Riemannian metrics, such as the one in [4].

References

[1] L. ODonnell, S. Haker, and C.-F. Westin, "New approaches to estimation of white matter connectivity in diffusion tensor MRI: elliptic PDEs and geodesics in a tensor-warped space," in *Medical Image Computing and Computer-Assisted Intervention MICCAI 2002*, T. Dohi and R. Kikinis, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, vol. 2488, pp. 459–466.

[2] C. Lenglet, R. Deriche, and O. Faugeras, "Inferring white matter geometry from diffusion tensor MRI: application to connectivity mapping," in *in Lecture Notes in Computer Science, T. Pajdla and.* Springer, 2004, pp. 127–140.

[3] M. C. de Lara, "Geometric and symmetry properties of a nondegenerate diffusion process," *The Annals of Probability*, vol. 23, no. 4, pp. 1557–1604, Oct. 1995.

[4] X. Hao, R. Whitaker, and P. Fletcher, "Adaptive Riemannian metrics for improved geodesic tracking of white matter," in *Information Processing in Medical Imaging*, 2011, p. 1324.

[5] S. Jbabdi, P. Bellec, R. Toro, J. Daunizeau, M. Plgrini-Issac, and H. Benali, "Accurate anisotropic fast marching for diffusion-based geodesic tractography," *International Journal of Biomedical Imaging*, vol. 2008, pp. 1–12, 2008.