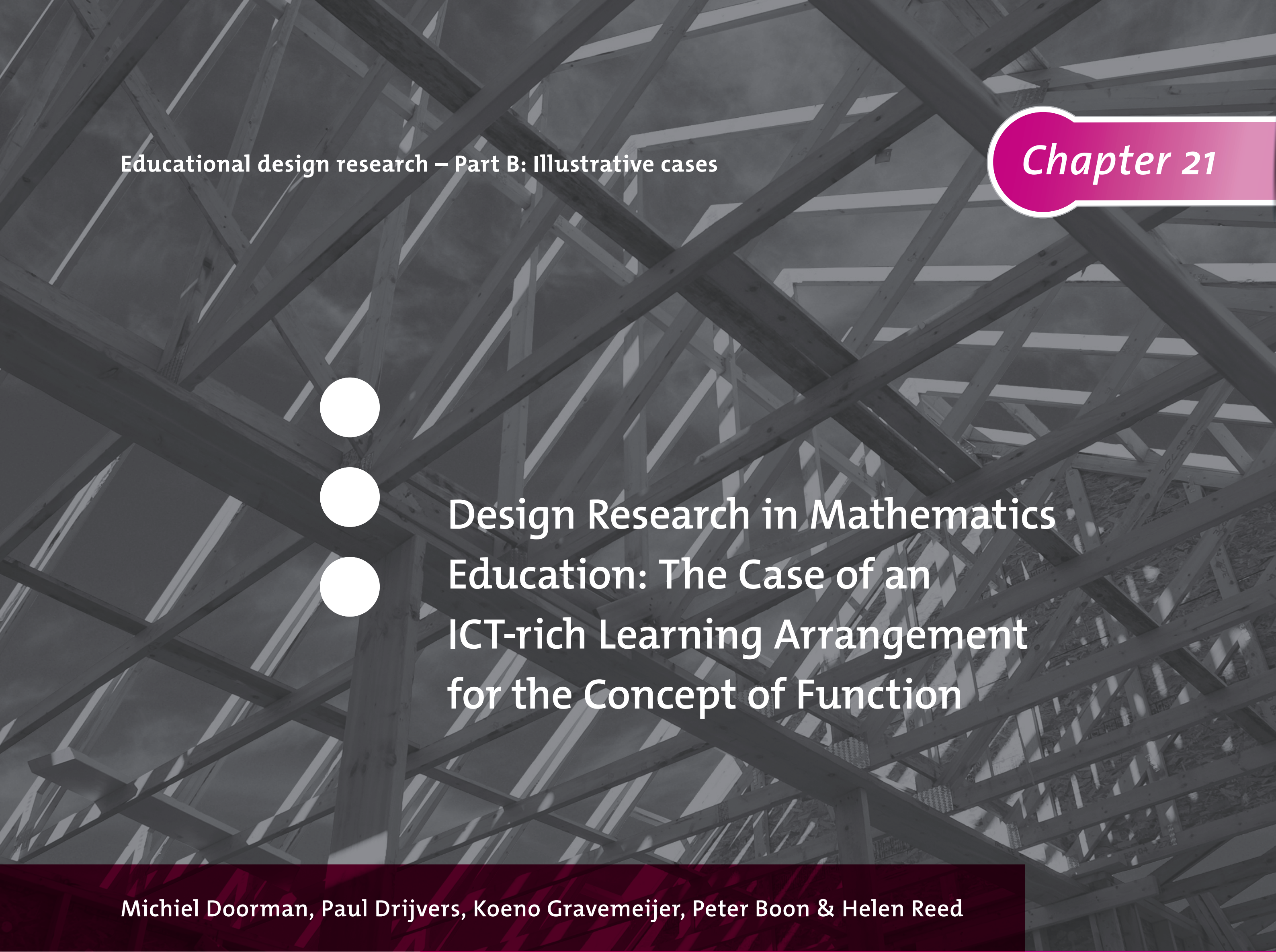
Design research in mathematics education : the case of an ict-rich learning arrangement for the concept of function



Chapter 21

Design Research in Mathematics Education: The Case of an ICT-rich Learning Arrangement for the Concept of Function

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21. Design research in mathematics education: The case of an ICT-rich learning arrangement for the concept of function[1]

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Abstract

The concept of function is a central but difficult topic in secondary school mathematics curricula, which encompasses a transition from an operational to a structural view. The question in this paper is how to design and evaluate a technology-rich learning arrangement that may foster this transition. With domain-specific pedagogical knowledge on the learning of function as a starting point, and the notions of emergent modeling and instrumentation as design heuristics, such a learning arrangement was designed for grade 8 students and field tested. The results suggest that these design heuristics provide fruitful guidelines for the design of both a hypothetical learning trajectory and concrete tasks, and can be generalized to other design processes.

1. Introduction

The function concept is a central but difficult topic in secondary school mathematics curricula (Akkus, Hand, & Seymour, 2008; Ponce, 2007). Functions have different facets, and to make students perceive these as facets of the same mathematical concept is a pedagogical and didactical challenge. In lower secondary grades, functions mainly have an operational character, and are calculation 'engines' that process input values into output values. In higher grades, functions have the character of an object with various properties. The transition from functions as calculation operations to functions as objects is fundamental for conceptual understanding in mathematics.

Can computer tools help to foster this transition? Computer tools offer opportunities for mediating the learning activities in which students engage (Sfard & McClain, 2002). The availability of sophisticated computer tools for mathematics education, however, also raises questions. Representations (e.g., formulas and graphs) and techniques (e.g., rewriting equations) in such computer environments may signify mathematical concepts that are still to be constructed by the students. As experts, we see the mathematics in the tool use, but does the learner see this too? This inherent circularity is known as the learning paradox (Bereiter, 1985; Gravemeijer, Lehrer, Van Oers, & Verschaffel, 2002; Van den Heuvel-Panhuizen, 2003).

To design a learning arrangement that avoids the learning paradox in using ICT-tools and fosters the learning of the concept of function - and the transition from an operational to a structural view in particular - is the central issue in this chapter. In a design research study we

[1] This contribution is based on the following publication with permission of the editor: Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: from repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243-1267.

investigate the question of how a learning arrangement with computer tools can foster the transition from an operational to a structural understanding of functions.

2. Theoretical framework

The theoretical framework that guides the design in this study includes (1) domain-specific knowledge on the concept of mathematical function; (2) the notion of emergent modeling; (3) theories on tool use and its instrumentation.

The concept of mathematical function

The teaching and learning of the concept of mathematical function is a widely researched topic (e.g., see; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ainley, Bills, & Wilson, 2005; Oehrtman, Carlson, & Thompson, 2008). A key issue in these studies is what Sfard calls the two sided nature of functions (Sfard, 1991, 1992). Her theory on the dual nature of mathematical conceptions identifies operational and structural concepts, the first concerning mathematical processes and the latter mathematical objects. In Sfard's view, the operational and structural conceptions are complementary. Historically, the operational aspect preceded the structural aspect, and the same might hold for individual learning processes, because the structural approach is more abstract than the operational. Dichotomies related to Sfard's operational and structural aspects have been brought afore by other researchers such as Dubinsky (1991) and Tall and Thomas (1991).
Based on an analysis of the available literature on this issue of operational conceptions preceding structural conceptions, the following sequence of three interrelated aspects of the function concept are distinguished:

1. The function as an *input-output assignment*
   The function as an input-output assignment that guides the stepwise calculation of an output value for a given input value. For example, we can think of a function that converts an amount of dollars into an amount of euros, or a temperature in degrees Fahrenheit into degrees Celsius. Often, this view on function is seen as the starting point for students. An appropriate symbolic representation for this function view is an input-calculation-output chain. Figure 1 shows such a chain for calculation temperature in degrees Celsius from temperature in degrees Fahrenheit.

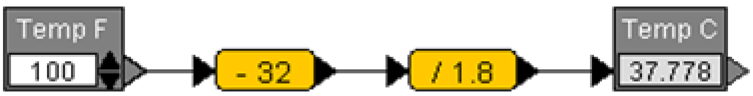

Figure 1: Input-output chain for temperature conversion

2. The function as a *dynamic process of co-variation*
   The second aspect of the function concept concerns the notion that an independent variable, while running through its domain, causes the dependent variable to move through a set of possible outcome values. The dependent variable co-varies with the independent. Helpful representations for studying co-variation are tables and graphs, which can be scrolled through or traced. Figure 2 shows such a table for the temperature conversion example.

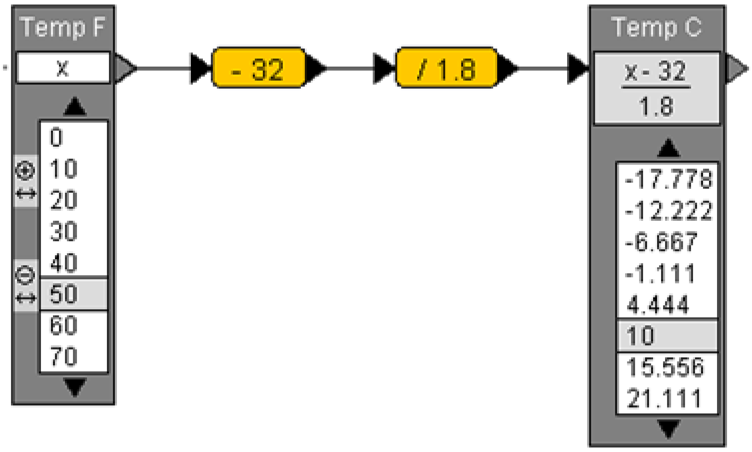

Figure 2: Table for temperature conversion

3. The function as a *mathematical object*
   A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, phrases, each of which providing a different view on the same object. The more structural view on functions includes families of functions, function comparison, and later on function differentiation or integration. For the object view on function, it is important to be able to see the connections between the different function representations. Figure 3 combines different representations for the temperature conversion example.

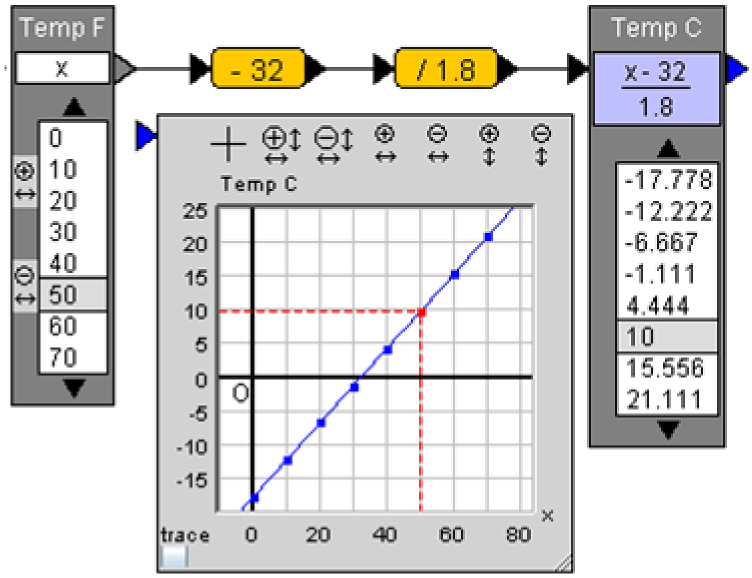

Figure 3: The different representations of the temperature conversion function

These three aspects of function reflect the operational- structural dimension as identified by Sfard (1991): the function as an input - output assignment reflects an operational conception and the function as a mathematical object involves a structural view, whereas the function as a dynamic co-variation process reflects an intermediate conception.

In line with this conceptual analysis, which is further elaborated in Doorman, Drijvers, Gravemeijer, Reed, and Boon (2012, see also Gravemeijer, Doorman & Drijvers, 2010), we want to design a learning arrangement for the topic of function that fosters the transition from an input-output conception to a more versatile view including a structural conception. This is the first design criterion. In the envisioned learning arrangement we implement computer tools with representations such as arrows, tables, formulas and graphs that initially may have hardly any meaning for students. The challenge for us - as designers - is to find ways in which these representations become useful models for the students. The notion of emergent modeling offers an orientation for addressing this challenge.

Emergent modeling

Emergent modeling is a design heuristic which originated in the framework of the theory of realistic mathematics education (RME) (Gravemeijer, 1999, 2007). RME builds on Freudenthal's (1991) image of mathematics as 'a human activity': while engaging in mathematics as an activity, students should be supported in re-creating or reinventing mathematics. The emergent modeling design heuristic is a means for organizing such a *reinvention process*. The starting point is an activity of modeling problems that are experientially real to the students. Models initially are context-specific and refer to realistic or paradigmatic situations. Then, while the students gather more experience with similar problems, attention shifts towards the mathematical relations involved. This enables students to use the model in a different manner: it derives its meaning from mathematical relations and becomes a base for more formal mathematical reasoning: a *model of* informal mathematical activity develops into a *model for* more formal mathematical reasoning (e.g., Gravemeijer, 1999; Cobb, 2002; Doorman & Gravemeijer, 2009). The learning paradox described in the introduction can be circumvented by designing a chain of activities that allow for a process of emergent modeling that fosters a thinking process in which symbolizations and meaning co-evolve (Cobb, 2002; Meira, 1995). This approach is consistent with conceptual change literature, which argues that students must first explore a domain and recognize limitations or a need for more sophisticated tools, before they are ready to learn complex concepts (Lehrer & Schauble, 2002).

In this study, the notion of emergent modeling is an important design heuristic. It stresses the importance of starting with problems that offer opportunities to develop situation-specific reasoning and tentative representations for organizing repeated calculations and that have the potential to develop into more sophisticated - mathematical - tools and concepts (Gravemeijer, 2007). Taking this process of emerging models seriously asks for a design in which models *for* emerge in a natural way from models *of*, that are rooted in suitable contexts. This is a second design challenge for the learning arrangement.

Tools and instrumentation

Emergent modeling as a design heuristic does not particularly focus on the use of computer tools and the relationship between computer techniques, paper-and-pencil techniques, and conceptual understanding. Drawing upon Vygotsky's view on the dialectic relation between tool use and cultural practices, we consider it important to understand and carefully plan the role of tools in a learning process (Vygotsky, 1986; Wertsch, 1998). As Hoyles and Noss (2003) point out, tool characteristics do matter in the sense that their visualizations and the available techniques affect student learning.

Instrumentation theory was developed to address the problems that may arise when one starts to use a ready-made computer tool and explains the importance of aligning techniques that emerge in contextual problems with the techniques available in the computer tool. The theory focuses on the mediating role of tools by stressing the co-emergence of tool techniques and meaning in a process of *instrumental genesis* (Artigue, 2002; Trouche, 2004; Drijvers & Trouche, 2008). Instrumental genesis comprises the development of cognitive schemes containing conceptual understanding and techniques for using a tool for a specific type of task. The resulting instrument integrates the tool and mental schemes. A bilateral relationship between the tool and the user exists: while the student's knowledge guides the way the tool is used and in a sense shapes the tool--the affordances and constraints of the tool influence the student's problem-solving strategies and the corresponding emergent conceptions.

In the case of functions, computer tools offer opportunities to dynamically and flexibly deal with different representations such as tables, graphs, and formulas. This may help students to overcome the difficulty of construing and integrating the operational and structural aspects of the function concept and its different representations. Computer tools can support students in exploring dependency relationships and investigating the dynamics of co-variation. The third design challenge, therefore, is to shape these opportunities, while ensuring that the techniques in the computer tool match with the targeted conceptual development.

3. Design

The study aims at designing and evaluating an ICT-rich learning arrangement for grade 8 students (age group 13-14 year) in the Netherlands,

1. which fosters the transition from an input-output conception to a structural conception
2. in which 'models *for'* emerge in a natural way from 'models *of'*, that are rooted in suitable starting points
3. and which benefits from the opportunities ICT offers, while ensuring that the techniques in the computer tool match with the targeted conceptual development.

As a first design step, a hypothetical learning trajectory (Simon, 1995) was designed based on the theoretical framework and the literature review. In this trajectory description, problem situations, models, ICT techniques and targeted conceptual development were summarized and outlined.

Next, the design research was carried out in a cyclic process of three design cycles, one pilot and two full cycles, each lasting for about one year. The learning arrangement includes (a) a computer tool called AlgebraArrows, (b) a student textbook with both paper and pencil and computer tasks, (c) a teacher guide describing the various activities and their possible orchestrations, and (d) a computer- and written test [2].

The *computer tool* is an applet called AlgebraArrows, which allows for the construction and use of chains of operations (so-called arrow chains) and provides options for creating tables, graphs and formulae and for scrolling and tracing. The applet is embedded in an electronic learning environment in which tasks are provided and through which the teacher can monitor students' responses (Boon, 2009). The AlgebraArrows window is embedded in the learning environment (see Figure 4). Figures 1, 2 and 3 were also made with this applet.

AlgebraArrows is meant to support the transition from operational understanding to structural understanding the concept of function in the following manner. First, students construct input-output chains of operations to carry out calculations. The chains can be applied to single numerical values as well as to variables. Gradually, the student activities focus on the investigation of dynamic input-output dependencies. The computer tool's slider bar and table

[2] The trajectory description and the learning arrangement are available through http://www.fisme.science.uu.nl/tooluse/en/ . The digital part of it can be found at http://www.fisme.uu.nl/dwo/demo/en/

scrolling options, in conjunction with the tasks, invite the students' development of a dynamic notion of a variable (see Figure 4). Finally, the students investigate families of functions, with their properties and representations, to develop a structural view of function. For instance, the table representation is initially understood as a tool for organizing corresponding input-output values in a contextual problem on mobile phone offers. Next, the applet's table option can be used as a tool for scrolling and zooming in on or out of these values, e.g., to analyze growth behavior or to find break-even points. This imagery of table and actions initially signifies repeated calculations and emerges into a model for a structural understanding of dependency relationships. In practice, these design considerations led to changes in the applet, such as the addition of zooming tools in the table option, similar to the zooming tools in graphs.
The tool techniques are meant to relate to paper-and-pencil techniques, and as such instrumental genesis should be natural and should contribute to the targeted cognitive development.

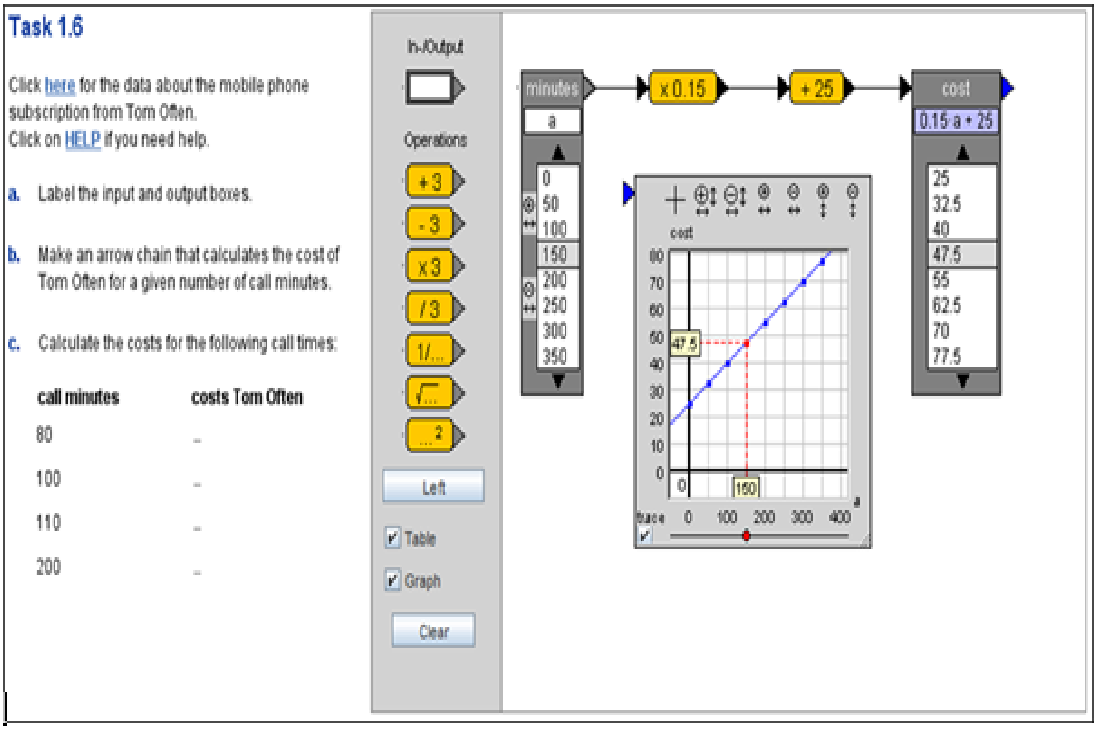

Figure 4: The computer tool AlgebraArrows embedded in the learning environment

The *student textbook* describes the activities for eight lessons. It starts with three preliminary paper-and-pencil group activities to create an exploratory orientation on the topic, to create the need for organizing series of calculations and to introduce the computer tool. In addition, these activities are designed to avoid discrepancies between tool techniques and conventional paper-and-pencil techniques and to anticipate instrumental genesis through the use of representations and techniques that will be used in the computer tool. The three successive activities are:

- Calculating the area of a flexible quadrilateral, which students explore using a paper model, and which then leads to the introduction of a variable (the height of a parallelogram) and a calculation procedure or a formula for the area.
- Comparing two cell phone offers, with a focus on break-even points which creates the need for a function-concept (Ainley, Bills, & Wilson, 2005; Küchemann, 1981).

- Calculating the braking distance of a scooter for various speeds. It includes a table-representation and is intended to create the need for graphing a trend for doing predictions.

During a classroom discussion of students' solutions orchestrated by the teacher (see below), an arrow chain is supposed to emerge as a useful way to investigate input-output relationships. The arrangement continues with two computer lessons. During the computer activities, the focus shifts from solving specific situations towards the investigation of dependency relationships by means of break-even problems that ask for the generation and comparison of tables of input- and output values, and moving around in a space of possible values. According to the emergent modeling heuristic, a shift is expected to take place from sub-models that signify repeated input-output calculations to models signifying dependency-relationships, co-variation and function object properties. The arrangement continues with a reflective lesson in which whole-class discussions and demonstrations align computer techniques with paper-and-pencil techniques, a computer lesson with applications, and a closing lesson for summarizing results and creating consensus on representations and functional reasoning.

The *teacher guide* describes the different activities and their possible orchestrations (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). It suggests different classroom arrangements, such as small group work, poster presentations, computer activities in pairs, and whole class discussions on the results, supported by projections of computer work (Stein, Engle, Smith, & Hughes, 2008). This variety of classroom arrangements is expected to foster social interaction, to promote the articulation of tool use and to enhance reflection and generalization.

The final part of the learning arrangement consists of a *computer test* and a *written test*. This mixed media assessment reflects the arrangement as a whole. The written test captures transfer of learning from computer tool to conventional methods and contains questions about break-even points, and computer tasks where students could use the applet AlgebraArrows[3]. Test items in the two tests are comparable in length and difficulty and have similar scoring instructions.

4. Research methods

To evaluate the design and to understand why the particular instructional setting potentially supports learning, design research was carried out (Gravemeijer & Cobb, 2006, 2013). The main hypothesis to be investigated is that the learning environment indeed fosters a transition from an operational view to a structural view, as reflected in the students' reasoning and in their way of using the computer tool. We briefly now describe the teaching experiments and data collection, and the data analysis.

Teaching experiments and data collection

After a first small-scale pilot study in one class, two successive teaching experiments were conducted in grade 8, with mid and high-achieving 13-14 year old students, for investigating their learning processes in relation to the learning arrangement. Each round of experiments took eight 50-minute lessons. The first was conducted with three classes from three different schools. The second was conducted on a larger scale. In this paper, we focus on screencast video data from pairs of students in the first teaching experiment. These students were selected by their teachers based upon criteria we provided: an average level of mathematical performance, a communicative attitude, i.e., willing to explain their reasoning and solution strategies, and audible articulation. The qualitative findings are triangulated with quantitative

[3] The full computer test and written test can be found at http://www.fisme.uu.nl/tooluse/en/

posttest data from 155 students from five classes from two different schools in the second experiment, for which the most complete data were available.
In both teaching experiments, whole class teaching sessions, group work and work in pairs were videotaped in two of the participating classes and, in each of these classes, screencast-audio videos of two pairs of students working with the computer tool were collected. In addition, students' answers and results to the written and the computer test administered at the end of the experiment were collected.
Between the teaching experiments, the learning arrangement was improved. The overall outline of the three different aspects of the function concept was not changed but fine-tuning the model-*of* to model-*for* development and the relationship between tool techniques, paper-and-pencil work and mathematical thinking was needed. In order to better investigate the development of the function concept, we placed a stronger emphasis on break-even points as a motivation for investigating the dynamics of relationships. In addition, to establish to what extent the students' final performance depended on the use of the available computer tool and to assess the transfer of learning, both a written test and a computer test were added to the end of the learning arrangement.

Analysis

The data analyses started with organization, annotation and description of the data with software for qualitative data analysis (ATLAS.ti; see van Nes & Doorman, 2010). Initially, the tasks in the learning arrangement served as the unit of analysis. Factual codes (task number, student names, et cetera) were used to organize and document the data. Events in class videos and screencast videos that were notable from the perspective of the research question were transcribed and discussed in the research team.
In analyzing the first teaching experiment, we constructed a storyline as a reconstruction of the students' learning process. The qualitative data sources were discussed with two external experts to identify illuminating examples of the resultant learning process. This discussion resulted in recognizing a shift in the students' reasoning with functions. Next, illuminating and representative examples for important steps in the shift were identified. A code book was set up. Next, we distinguished solution strategies used on two similar computer tasks on break-even points that were designed for studying the development of students' reasoning with the computer tool. One task was situated at the beginning of the computer lessons and the other at the end. The strategies led to the construction of codes with respect to the use of representations in the computer tool. These codes reflected the different aspects of the function concepts, the types of tool use, and the position in the hypothetical learning trajectory. In a later phase, these codes were also used for the quantitative analyses of students' answers to the paper-and-pencil activity concerning the arrow chain (the booklets of 5 students were missing), the screenshots of the answers to the two computer tasks, and the screenshots of students' final work on the two computer tasks. A second researcher coded 55 out of 306 items (18%). Good inter-rater-reliability was achieved (Cohen's Kappa = 0.79). A paired *t*-test was used to compare the results on the written test and the computer test.

5. Results

The results of the study are presented in the form of a storyline of students' learning process in relation with the learning environment, as reflected in their reasoning and in their way of using the computer tool. This storyline offers opportunities to trace the development of our interpretations and understanding of these learning processes. This approach is linked to Smaling's (1987) view on `trackability' and to Freudenthal's notion of 'justification' (Freudenthal, 1991). The storyline is illustrated with examples of student work and is also supported by the quantitative findings.

The initial cell phone offer task: organizing calculations

The results of the initial open-ended group activities of the first teaching experiment show a variety of solution strategies. In Figure 5 (top) the cell phone offer task is shown. The students' posters below are illustrative for their attempts to organize the situations mathematically, i.e., to organize repeated calculations, construct variables and use various representations. In the poster on the left, students organize their repeated calculations by systematically writing them in a list, which resembles an input-output relationship. This helped them to see the pattern in the calculation, and to apply this pattern to a new input value. In the right poster, students use formulae to describe their repeated calculations. Although not in conventional form, the formulae show the identification of the two variables of the dependency relationship (*m* for minutes and *b* for costs). The repeated calculations reflect how the operational aspects precede the structural aspects of functions (Sfard, 1991). From an emergent modelling perspective, the context apparently provides a suitable starting point.

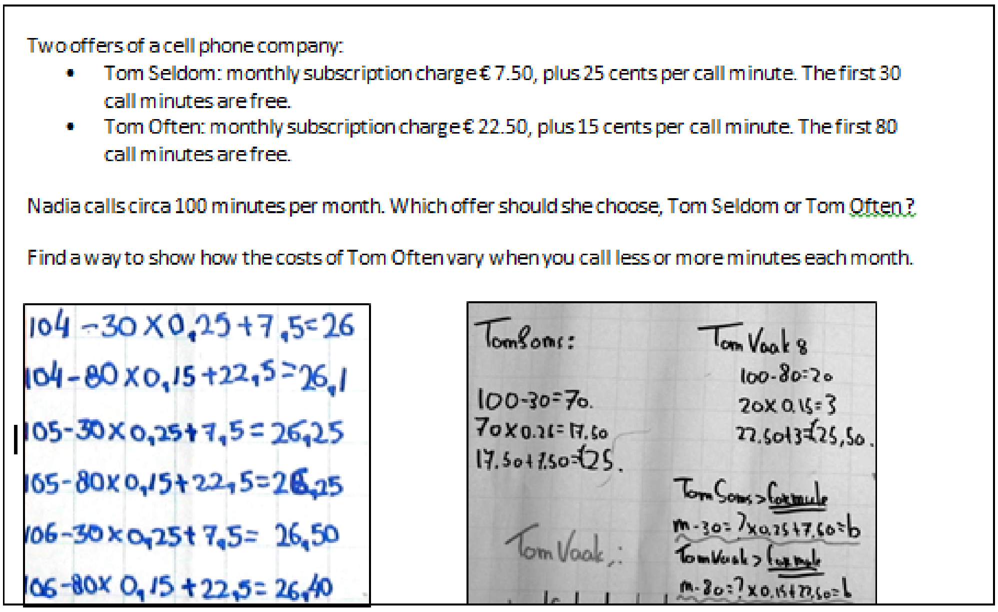

Figure 5: Cell phone offer task and students' poster

Introducing arrow chains: visualize functions

In a whole-class discussion, the posters helped the teacher to evoke the need for determining variables, dependency relationships and more efficient notations and tools for finding break-even points. The teacher exploited the students' strategies by linking their initial ideas to the intended tool use. A new mathematical goal emerged: repeated calculations are time-consuming, what is the general pattern in the calculation procedures? How can this pattern be described to calculate results 'automatically'? How could a 'calculator' help in comparing cell phone offers? A calculation recipe comprising a fixed chain of operations emerged. This arrow chain is the central representation of the computer tool. In the last activity before the computer lessons, the students were asked to draw calculation chains for the two cell phone offers. Figure 6 shows an illustrative answer to this task.

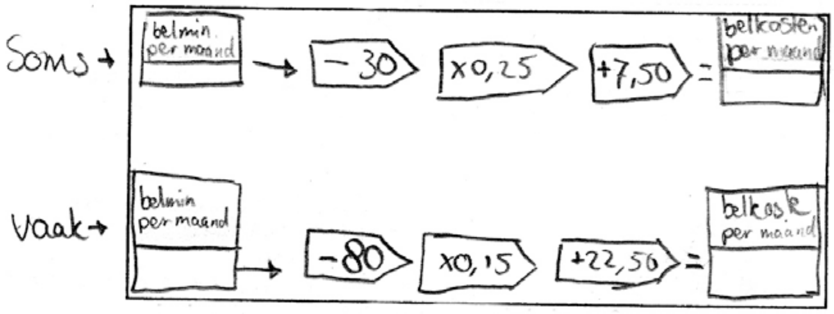

Figure 6: Written calculation chains for cell phone offers (called 'Soms' and 'Vaak').

The arrow chain as a model of repeated calculations appeared to be grounded in students' previous activities and could begin to function as a mathematical model *for* (in terms of the theory of emergent modeling) reasoning about dependency relationships. This assumption is backed up with an analysis of the results of student booklets from the second experiment. The analysis shows that 124 out of 150 students (83%) drew a similar chain while another 8 (5%) drew this chain, but used it only for one specific calculation and did not label input and output boxes. From an instrumental genesis perspective, the tool techniques were prepared for by paper-and-pencil work.

Using the computer tool to create arrow chains

The third lesson - the first computer lesson - started with some introductory activities and then continued with the cell phone problem (Figures 5 and 6). The task was to determine, with the computer tool at hand, when it is advisable to change from one offer to the other. Two students, Lily and Rosy, worked together on this task. After reading the problem on the computer screen, they started to construct arrow chains with the computer tool.

[Rosy drags an input box into the drawing area. The box is connected to the operation - 80 and that one is connected to x 0.15.]
Lily: And that added to the fixed costs.
[R agrees and connects the chain to + 22.5. Finally, they connect the chain to an output box.]
L: Well, when you phone for 100 minutes….
[enters 100 as input in both chains. TomSeldom (Soms) is cheaper than the other offer.].
R: …. Well, maybe 50.
[L enters 50 as input for both chains. They look at the results and are still not satisfied. They try some more input values. Finally, they enter 200 in the input box of both chains. For the first time TomOften (Vaak) is cheaper than the other offer. They are satisfied with this result and proceed to the next task.]

The vignette illustrates how the arrow chain has become a means to organize the situation and the calculation procedure. Lily and Rosy built the chain from the input box, adding operations and finally connecting an output box. This construction signifies their previous calculations. They focused on specific cases by entering respectively 100, 50 … and finally 200 for comparing the differences between the two offers. The arrow chain is used for repeated calculations. The tool supports the construction of these chains, and the chain becomes a means for analyzing and discussing relationships and successive operations, as a sequel to merely solving repeated calculations. The tool technique clearly is linked to the students' paper-and-pencil experience and supports their thinking.

Whole-class discussion capitalizing on hands-on experience

The start of the second computer lesson (the fourth in the learning arrangement), consisted of a teacher-guided classroom discussion of students' computer work with the computer tool connected to a data projector. Topics of discussion were the possibility to label input and output boxes and to use tables. These tool techniques were already used by some of the students, and now became 'institutionalized' through the whole class discussion. The teacher started the discussion with an arrow chain for one of the cell phone companies:

Teacher: I heard different ways to find out how much I have to pay, how the amount changes, how can you demonstrate that? [Silence] For example, if I call 10 minutes. How can I find out how much I pay for 10 minutes?
Student 1: Put 10 in the input box.
Teacher enters 10 into the input box. This gives an odd result (first 30 free minutes are subtracted). The result is discussed and the teacher varies the input by entering some more values.
Teacher: Suppose I want to know the output from many input values, what more can I use?
Student 1: The table.
Teacher clicks the table tool and shows how you can scroll through the input and output values.
Teacher: Does anyone know another way to show how the output values change for different input values?
Student 2: With a graph…

After this suggestion the teacher opened the graph tool. She demonstrated how to connect an arrow chain to the graph window. Together with the students she investigated the options to trace a graph and to zoom in and out. In this way, while demonstrating the tool techniques, the teacher discussed the dependency relationships and the ways tables and graphs can be used to analyze their dynamics. In the previous computer activities, Lily and Rosy did sometimes click for a table or graph, but had not used it for scrolling or tracing values.
This observation shows how the teacher used the computer tool to create whole-class consensus on how to use it for investigating dependency relationships. Both the techniques and related concepts were part of the discussion. She did not show how to use these features for finding break-even points. That was still a task for the students.

A different view on function

During the third computer lesson (the sixth lesson of the learning arrangement), we observed that the strategy of Lily and Rosy for using the computer tool had changed while solving a task on two offers by contractors called Pieters and Tweehoog (Figure 7).

> To get some jobs done in the house we can choose from two contractors:
> Contractor "Pieters" charges 92 start costs and an hourly rate of 30.
> Contractor "Tweehoog" charges 45 start costs and an hourly rate of 32,75.
> You have a job of 9 hours. How much cheaper is Tweehoog than Pieters?
> After how many hours of work is contractor Pieters cheaper than Tweehoog?

Figure 7: Handyman task offers

The task in Figure 7 is quite similar to the cell phone activity (Figure 5). However, the way in which these students analyzed the problem situation, phrased the structure of the solution procedure and used the representations in the tool changed: AlgebraArrows was now used to investigate the dynamics of relationships, rather than for case-by-case calculations. Figure 8 shows some subsequent phases in these students' work.

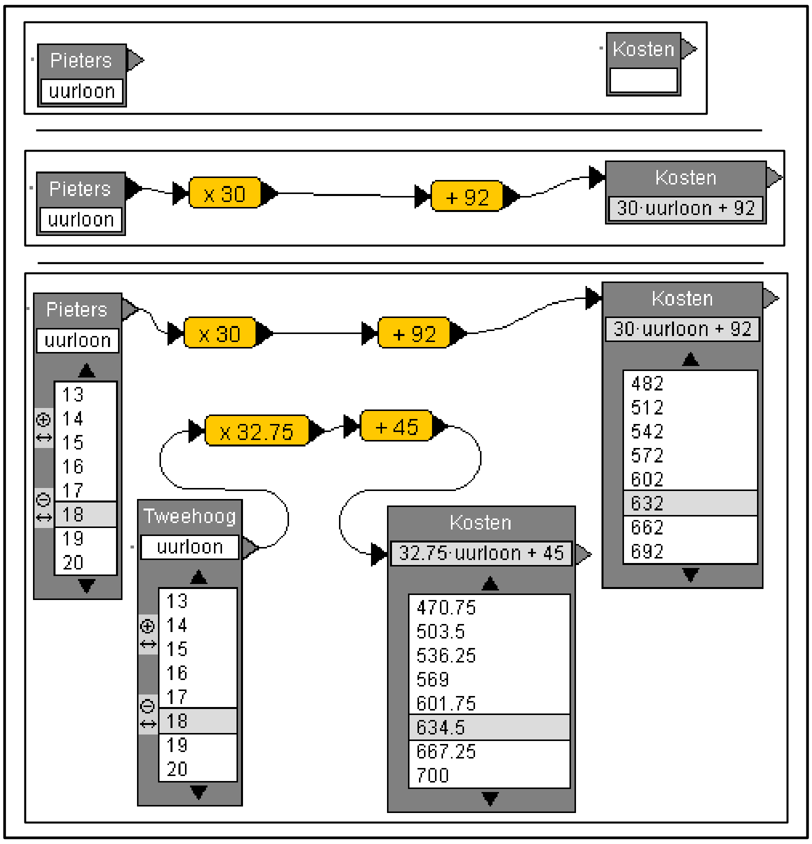

Figure 8: Different phases in students' work

First, the students identified the variables by creating labeled input- and output boxes for the first arrow chain, with 'uurloon' meaning hourly rate and 'Kosten' for costs (Figure 8 top). Next, they filled in the operations and completed the chain (Figure 8 middle). Then they constructed the second chain for the Tweehoog company in similar way. Finally, they added tables, and scrolled through them to find the break-even point for 18 hours (Figure 8 bottom part). This behavior illustrates a different view on, and use of, the arrow chain compared to the initial operational left-to-right construction of a chain for repeated calculations. More specifically, the vignette illustrates how the arrow chain had become a tool for analyzing dependency relationships. Lily and Rosy organized the dependency relationship by identifying, positioning and labeling boxes for input and output variables and filling the gap between them with operations. This construction signifies an understanding of the problem as a question about two dependency relationships. For investigating the dynamics, they successfully operated the table tool to zoom in and out for finding the break-even point. Corresponding to this new view on function, the technique of constructing an arrow chain changed, now starting with labeled input and output boxes rather than with input and then operations. This suggests a structural view on function.

Triangulating case study findings with computer output

The second teaching experiment was used to quantitatively verify the hypothesis on the transformation of the view on function during the computer activities. We found that for the initial task 130 out of 155 students (84%) used the tool only for calculating successive input-output values, while in the task at the end, 89 out of 152 students (58%) used the tool for structuring and investigating the dynamics of the relationships.

Whereas the students' initial technique with the applet suggested an operational view on functions, this more advanced technique reflects a transition towards a gradually developing structural view. In other words, the shift in the use of the tool reflects the development from viewing calculation recipes as tools for repeated calculations (as processes) towards reasoning with structural characteristics of dependency relationships (as objects).

To investigate whether this transition also encompassed students' written work during the eight lessons, we compared the scores of the computer test and the written final test. A paired *t*-test showed no significant difference (p=0.20) between the final scores on the written test and the computer test. The Pearson correlation coefficient of 0.38 between these scores was moderate (p=0.001). This suggests that students who improved their reasoning through the described shift evoked by the use of the computer tool were able to transfer learning to a more conventional paper-and-pencil test.

6. Conclusion and discussion

Conclusion

In the introduction, we raised the question of how a learning arrangement with computer tools can foster the transition from an operational understanding to a structural understanding of functions. From the data analysis, we see the following characteristics of the learning arrangement as decisive for fostering a transition from a calculation understanding to working with correspondence and co-variation:

- The three initial open-ended problems and the poster activity helped students in coming to see input-output structures in problems about dependency relationships and to use representations to explore them.
- By enabling students to design and use arrow chains as a means of support for reasoning about calculation procedure that bear meaning in everyday-life phenomena, the computer tool supported the students in developing the notion of a chain of operations.
- By generating the results of a series of calculations for a variety of input values, the computer tool strengthened the students' notion of a function as a calculation procedure that transforms input values into output values.
- By generating output values for series of input values, by generating tables of input and output values, and by enabling the students to move up and down the values in these tables, the computer tool supported the students in developing a dynamic notion of a variable that can move in a space of possible values, and the corresponding idea of co-variation.
- By displaying arrow chains, tables and graphs the computer tool offers representations, which the students could construe as affordances to start treating functions as objects before they had become objects for them. In this way, the learning paradox may be circumvented.
- By discussing students' work with the tool and showing specific features that some of the students discovered, the teacher appeared successful in supporting most students in using the table representation.

We conclude that the learning arrangement with a computer tool helped students to overcome the difficulty of integrating operational and structural aspects of the concept function and supported explorative activities for investigating the dynamics of co-variation, even though the students didn't reach the full level of "function as a mathematical object" but just began to make the transition towards this level of understanding. Further advances in this direction would encompass, for example, more different types of functions and operations on functions such as composition and multiplication.

Discussion

Before discussing the design heuristics of this study, we first address its limitations. The sequence of eight lessons fostered conceptual development in the domain of functions. However, the final test data do not give insight in whether *all* students made a similar step towards a dual conceptualization, with procedural and structural views, of functions. Students' work during the computer lessons revealed that the final screenshots of an activity sometimes are only approximate representations of the students' reasoning. Students can be close to a good answer and then delete everything as a result of a sudden doubt. Final screenshots don't capture this entire reasoning process. Additional process information is needed for a full understanding of students' conceptual development in relation to their tool techniques, although a balance must be found between the extensiveness and the manageability of the data collection.

As a final issue, we mention the generalizability from design-based case studies (Yin, 2003). We agree with Plomp and Nieveen (2009) that the results of design research have an analytical generalizability and a replication logic: the findings in this case may not be directly generalizable, but the design experiment may be treated as a paradigm case (Gravemeijer & Cobb, 2006, see also this volume), which offers domain-specific design heuristics that can be extrapolated to similar design studies. The extensive description of the interventions in Doorman, Drijvers, Gravemeijer, Boon, & Reed (2012) is intended to enable researchers to reenact the experiment in other settings as a way to evaluate our findings and to contribute to the development of a more comprehensive theory.

In retrospect, how do we view the three theoretically based design heuristics described in section 2? First, we used domain-specific theories on the acquisition of the concept of function, and on the transition from an operational to a structural conception in particular. This design heuristic was fruitful for outlining the learning arrangement. It had an important impact on setting up a hypothetical learning trajectory and on the design of the tasks. This particular design heuristic can be generalized into an overarching one, which is that - before engaging in design - a deep analysis of the topic is needed to be able to identify its learning obstacles and pedagogical challenges and to outline a learning trajectory.

A second design heuristic was the notion of emergent modeling, and its focus on the shift from model-*of* to model-*for*. This heuristic not only guided the design of the learning arrangement, but the students' cognitive development, and their ways of using the tool initially as a tool for calculations and later as a tool for reasoning with function, reflects this shift. As such, this was a suitable design heuristic.

Third, instrumentation theory focuses on the interrelated development of techniques and concepts. As a design heuristic, it fostered a careful integration of computer activities and paper-and-pencil activities, which prevented discrepancies between tool techniques and conventional methods. The process of instrumental genesis was enhanced by the alternation between classroom discussion, small group activities and computer activities.

Altogether, while outlining the hypothetical learning trajectory and designing the tasks the three design heuristics provided important and concrete guidelines. Having a set of such design heuristics is indispensable, but in the meanwhile does not guarantee a successful design: it is not just knowledge of appropriate design heuristics, but their application in different domain-specific situations that forms the heart of the ‘art’ of the designer.

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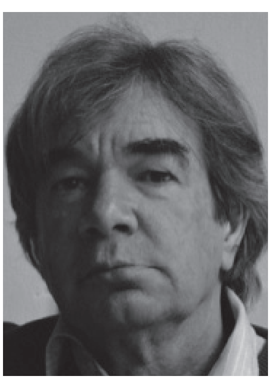

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