The mathematics of french fries

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The Mathematics of French Fries

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Abstract

Although the act of cutting a single potato (*Solanum tuberosum*) into french fries may appear to be trivial, the questions concerning the efficiency of this process on an industrial scale are quite daunting. Therefore, many producers are looking for a rigorous method to evaluate the market potential of a given potato crop by predicting the number and parameters of the fries that can be cut from it. Applying the methods of geometry and numerical analysis our group was able to propose several algorithms that can be directly incorporated into the existing production process.

Keywords: French fries, geometry, cutting, Finite Fry Method, simulations, histograms

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1 Introduction

The HZPC Holland B.V. company is a major international supplier of potatoes. The question posed by HZPC could be summarized as follows:

Question: *Given a set of potato tubers with approximate information about the size of each tuber and the desired cross-section of a fry, how many french fries of various lengths and textural quality can be obtained from this set, and what will be the volume fraction of waste?*

During the SWI week our group was able to solve the geometrical part of the problem. Whereas, the question concerning the textural quality remains partially open. Below we describe two methods to cut tubers into virtual fries and estimate their number, their length distribution, and the amount of waste. The analytical approach provides with the formula for the volume of each tuber based on the best elliptic approximation of the surface and with a quick and dirty way to compute the fry length histogram from experimental data. The second approach, which we call the Finite Fry Method (FFM), has been implemented as two separate numerical algorithms: one that is tuned to work with the currently available experimental data, and the other that allows generating tubers of arbitrary shape and perform Monte-Carlo studies on the obtained ensemble. Finally, we have developed an analytical procedure to interpolate the DMC quality trait distribution sampled in a small number of points onto the whole volume of the tuber.

2 Geometrical modeling

The available information about the shape of each tuber consists of its maximal extent (*length*) and several measurements in other directions. Namely, in nine equidistant planes orthogonal to the length direction the measurements of the *width* and *height* are available, two in each plane. However, although it is known that the directions of the width and height are mutually orthogonal and consistent over all nine planes, the measurements do not provide with the actual coordinates of the boundary points.

This limited information allows constructing only a simplified geometrical model of a tuber. We consider the intersection of the tuber boundary surface with each of the nine cross-sectional measurement planes to be a concentric but not necessarily a confocal ellipse with known semi-axes.

Computing the volume of a tuber requires interpolating its boundary surface between the aforementioned cross-sectional ellipses. Depending on the chosen interpolation model the surface between the planes may or may not have an elliptic cross-section. Without losing much precision we assume that *all* cross-sections of the tuber in the direction orthogonal to its length are indeed elliptic. The advantage of this assumption is the ability to compute the

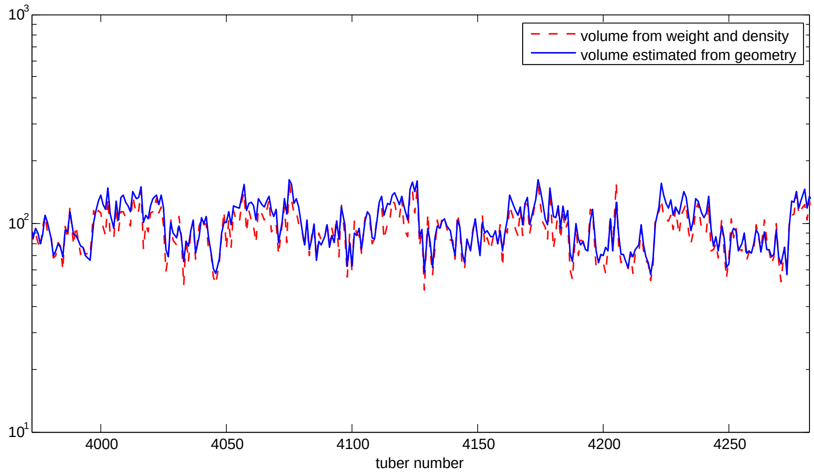

Figure 1: Calculated volume of the geometric model compared with the volume estimated from the weight and density of tubers.

volume of a tuber explicitly as a sum of the volumes of eight twisted elliptic cylinders defined by the measurement planes plus the volume of two caps that can be modeled as elliptic cones. The result of computing the volume via this approach is shown in Figure 1 where it is compared with the volume estimate based on the weight and density of a tuber.

Further improvement of this geometric model may consist in a better description of the tuber caps, for example, as elliptic paraboloids instead of elliptic cones.

3 Cutting potatoes into fries

With the model of the tuber boundary surface at hand one can proceed 'cutting' the potato into fries and estimating the waste – fries that do not conform to the industry standards.

Let the z-axis be along the length direction. Introduce a uniform two-dimensional Cartesian grid in the horizontal (x, y)-plane with the grid step h – the chosen width of a fry ($h = 6, 7, 8, 9, 10, 12, 14,$ or 16 mm). Cut the tuber into fries numerically by constructing finite elements that extend along the z-axis with their vertical edges coinciding with the grid points of the previously defined horizontal grid. The surface of the tuber is approximated

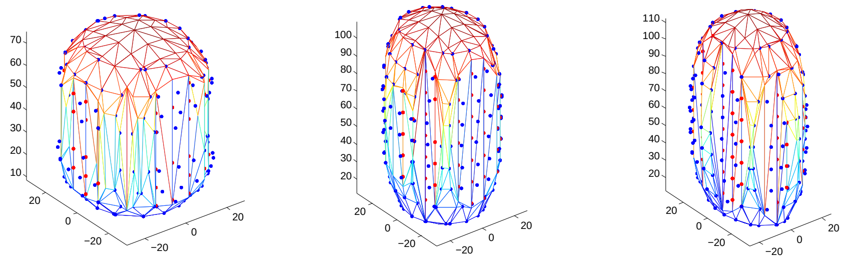

Figure 2: Three tubers from the $40 - 50$ mm class (experimental data) cut into fries of width $h = 6$ mm. Only acceptable fries are indicated.

in a piecewise linear fashion at this stage. Discard all fries that do not conform to the industry standard and compute the waste.

The result of the application of this numerical algorithm, which we call the Finite Fry Method (FFM), to the first three tubers in the $40 - 50$ mm width class is shown in Figure 2.

Although, we were able to run this algorithm on as many as 11333 tubers, the discretization of a large number of tubers into fries may take significant time (about an hour on a laptop). Therefore, we have also developed an approximate but simple method to estimate the number of fries that can be cut from each tuber. The main idea is to count only the fries that have a square cross-section discarding all wedge-shaped fries that are cut from the sides of a tuber. This method detects the smallest of the cross-sectional ellipses of each tuber, and is extremely fast (seconds) and surprisingly accurate (see Figure 3).

4 Modelling variations in tuber geometry

Another realization of the finite-fry method (FFM) was developed to relax the current experimentally imposed limitations on the tuber geometry. Namely, as was mentioned above, the measurements did not provide with the actual coordinates of the boundary points and forced us to assume the concentric elliptical geometry when working with the data. In reality, however, tubers can have very asymmetric shapes. In this second realization of the FFM we have implemented a more systematic finite-element approach and have simulated a large statistical ensemble of asymmetric tubers.

As before, the potato shape is determined by a number of slices with elliptic contours orthogonal to its maximal (length, vertical) dimension. However,

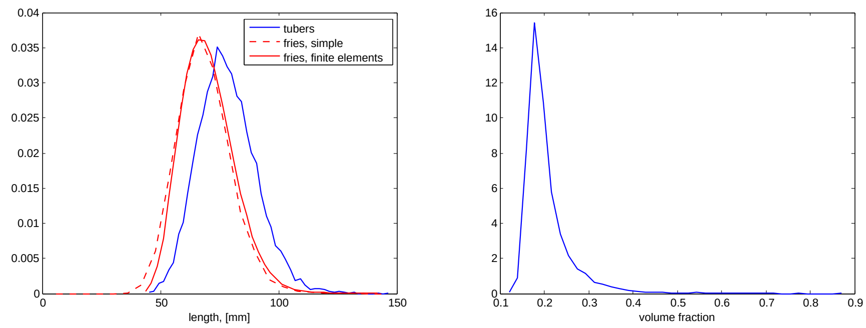

Figure 3: Left: Distribution of lengths of all acceptable 6 mm wide fries that can be cut out of 11333 tubers from the $40-50$ mm class calculated using a simple geometrical estimate and a more exact numerical procedure. Also shown is the distribution of the tuber length within the class. Right: Distribution of the volume fraction of waste.

this time not only the dimensions and orientation of the ellipses are allowed to vary but also their centres are allowed to shift off the length axis. In Figure 4 we show a schematic representation of tubers defined by a set of algorithmically generated ellipses.

To get the number of fries, we use a background (horizontal) square domain with the width and height chosen in such a way that all tubers of the simulated ensemble can be projected onto this domain. This domain is referred to as the *reference surface*. This area is divided into squares of equal size, which we call elements. Each elliptic slice is projected onto the reference area so that each square fits either completely, or partially, or not at all in the projected slice.

To make the whole procedure more systematic we construct an ordered list containing the indexes of all points in groups of four – the vertices of each element. Such lists are usually referred to as the *topology* of the grid. This list makes it easy to determine whether a fry belongs to a projected ellipse by testing for each vertex and each element whether the coordinates (x, y) of the vertex satisfy

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1.$$

If that is the case, then the vertex point and thus the edge of the fry is inside the tuber. If an element, which represents the cross-section of a fry, fits either

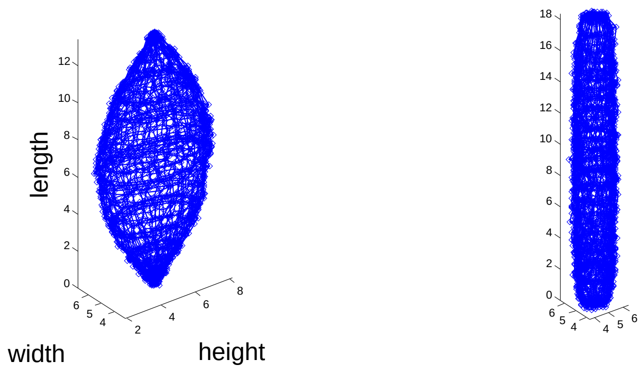

Figure 4: Two examples (Case I and Case II) of computer-generated tubers defined by ellipses with variable length, width, and center offset. Here, the length and width are deterministic functions, whereas the coordinates of the centres are taken from a normal distribution with zero mean and the standard deviation of 10 mm

partially or entirely within the projection of the slice onto the reference area, then this element is interpreted as a 'fry'.

Among all the fries we determine the list of *accepted fries* by accepting a fry if the intersection of the corresponding element with the projected ellipse has an area of at least $0.7\,A$, where A denotes the element area. A schematic of accepted and rejected fries over a single slice is given in Figure 5. This procedure is repeated for all the elliptic slices of the tuber.

The length of a fry corresponding to a certain element is determined by adding all the distances between adjacent slices that have yielded an acceptable intersection area with this element during the previous step. Due to the allowed variations in the tuber geometry some fries may have to be cut into pieces. In such cases each piece is considered to be a separate fry. As before, fries are only accepted if their length exceeds 40 mm. All the acceptance criteria for the fries can be changed in the model whenever necessary.

In the Monte-Carlo simulations, the tuber length, the centres, and the orientation and length of the semi-axes of the elliptic cross-sections are all assumed to be random normally distributed variables with adjustable means and variances. Furthermore, the mean width and height of each ellipse change according to a prescribed function in the length direction. We note that this function can be adjusted to model any desired shape.

Similarly to Figure 3 we present the results by comparing the distribution of the length of tubers (which is an input parameter) to the distribution of the

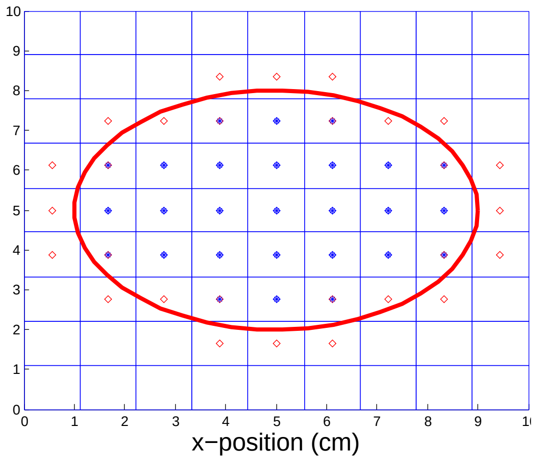

Figure 5: A schematic representation of accepted fries over an elliptic slice of a tuber. The red and blue dots, respectively, indicate the centres of the fries that are rejected and accepted

length of accepted fries (output parameter). As a tuber shape, we consider a mean tuber length of $L = 150$ mm, with the standard deviation of 35 mm, that is $L \sim \mathcal{N}(150, 35^2)$. The elliptic projections were simulated according to the formula:

$$\left(\frac{x - x_c}{a(z)}\right)^2 + \left(\frac{y - y_c}{b(z)}\right)^2 = 1,$$

where z is the vertical coordinate and $x_c, y_c \sim \mathcal{N}(0, 10^2)$ are the center coordinates taken from a normal distribution. We present the results for the following two shape functions:

$$\mathbb{E}\{a(z)\} = \frac{4}{L} z(L - z) = \mathbb{E}\{b(z)\}, \quad \text{Case I;} \tag{1}$$

$$\mathbb{E}\{a(z)\} = \left(\frac{4}{L} z(L - z)\right)^{\frac{1}{8}} = \mathbb{E}\{b(z)\}, \quad \text{Case II;} \tag{2}$$

where $\mathbb{E}\{\dots\}$ denotes the expectation used in the Monte-Carlo simulations. Note that we also set $a = b$ in the current simulations but we have the flexibility to use any value.

The results are presented in Figure 6. It can be seen that, in agreement with the experimental results of Figure 3, the average length of a fry is lower

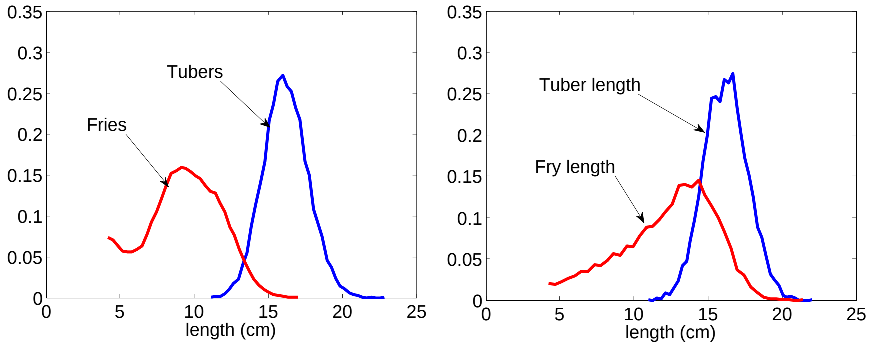

Figure 6: Statistical distribution of the tuber and fry lengths for two classes of tuber shapes, see (1)–(2).

than the average tuber length. This is a direct consequence of the large number of fries that is cut from the part of the tuber that is further away from the main axis. Along the main axis the length of fries is (approximately) equal to the tuber length, whereas at positions away from the main axis the fries are shorter and more numerous. This causes the relatively large number of fries with a length of 40 mm in the first case.

While the tuber shape is more ellipsoidal in the first case, in the second case the shape is more cylindrical (see Figure 4). Therefore, in the second case, the number of small fries is smaller. Thus, the proposed mathematical models can be used to quantify the observations. In particular, it is interesting to observe that the tuber geometry determines the shape of the probability density function.

5 Fry texture and the DMC quality trait

Another question posed by the HZPC company was to investigate the texture of fries after frying and the so-called DMC quality trait of raw fries. The DMC trait is currently measured in just a few points over the volume of some of the tubers. To interpolate the very sparse measurements of the DMC so that they can be used to determine the DMC of each individual fry we have introduced and computed the distance between an inner point of an elliptic slice and its bounding ellipse. This distance is then inserted into a parametric expression approximating the more recent fine-scale measurements of the DMC:

$$\phi(x, y) = \phi_c - \alpha(\mathrm{dist}((x, y); \Gamma))^2, \tag{3}$$

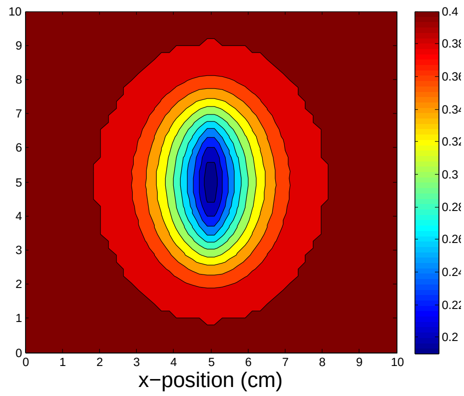

Figure 7: The distribution of the DMC quality trait over an elliptic slice.

where (x, y), ϕ_c, α and Γ, respectively, represent any point within the ellipse, the DMC content near the tuber surface (the maximum), and two tunable constants. The spatial distribution of DMC is denoted by ϕ. The proposed function is shown in Figure 7. We note that all tunable parameters can be easily determined by an optimization algorithm. In the current simulation, $\phi_0 = 0.4$, and $\alpha = 0.025$. We did not move further into this direction during the SWI-week.

6 Conclusions

The problem posed by the HZPC company was to find a mathematical technique that could help evaluate the market potential of a given set of potato tubers. For the particular French fries market the main quality factor is the amount and type of fries that can be cut from the tubers. As tubers vary in shape and size even after having been sorted into several size-related classes, the best approach appears to be the analysis of a histogram depicting the distribution of length among the fries. In order to obtain such a histogram we have developed an approximate and fast analytical procedure and a more exact numerical technique, which both cut tubers into virtual fries with a chosen cross-section. The advantage of the numerical approach stems from the fact that it allows investigating the influence of rather arbitrary shape perturbations on the histogram by running a Monte-Carlo simulation with a computer generated set of tubers. Both methods were tested on the provided experimental data and showed similar results. The analytical expression for the volume of each tuber based on the available spatial measurements is in good agreement with the volume computed from the weight and density data.

The second part of the question concerned the distribution of the DMC quality trait inside the tubers and its relation to the texture of French fries. In this respect we have proposed a parametric expression that can be fit to the available sparse experimental data in order to interpolate the DMC density over the fry volume. Armed with this DMC trait distribution one could extend the histogram approach by showing the statistics of the DMC for each fry length.