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Network-Based Dissolution

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Abstract

We introduce a graph-theoretic dissolution model that applies to a number of redistribution scenarios such as gerrymandering in political districting or work balancing in an online situation. The central aspect of our model is the deletion of certain vertices and the redistribution of their loads to neighboring vertices in a perfectly balanced way.

We investigate how the underlying graph structure, the pre-knowledge of which vertices should be deleted, and the relation between old and new vertex loads influence the computational complexity of the underlying graph problems. Our results establish a clear borderline between tractable and intractable cases.

1   Introduction

Motivated by applications in areas like political redistricting, economization, and distributed systems, we introduce a class of graph modification problems that we call *network-based dissolution*. We are given an undirected graph where each vertex carries a load consisting of discrete entities (voters, tasks, data). These loads are *balanced*: all vertices carry the same load. Now a certain number of vertices has to be *dissolved*, that is, they are to be deleted from the graph and their loads are to be redistributed among their neighbors such that afterwards all loads are balanced again.

Indeed, our dissolution problem comes in two flavors called DISSOLUTION and BIASED DISSOLUTION. DISSOLUTION is the basic version, as described in the preceding paragraph. BIASED DISSOLUTION is a variant that is motivated by gerrymandering in the context of political districting. It is centered around a bipartisan scenario with two types A and B of discrete entities. The goal is to find a redistribution that maximizes the number of vertices in which the A-entities form a majority. See Section 2 for a formal definition of these models.

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Our main focus lies in analyzing the computational complexity of network-based dissolution problems, and in getting a good understanding of polynomial-time solvable and NP-hard cases.

Three application scenarios. We discuss three example scenarios in some detail. The first and third example relate to BIASED DISSOLUTION, while the second example is closer to DISSOLUTION.

Our first example comes from political districting, the process of setting electoral districts. Let us consider a situation with two political parties (A and B) and an electorate of voters that each support either A or B. The electorate is currently divided into n districts, each consisting of precisely s individual voters. A district is won by the party that receives the majority of votes in this district. The local government performs an electoral reform that reduces the number of districts, and the local governor (from party A) is in charge of the redistricting process. His goal is of course to let party A win as many districts as possible while dissolving some districts and moving their voters to adjacent districts. All resulting new districts should have equal sizes s_{new} (where $s_{\text{new}} > s$). In the *network-based dissolution* model, the districts and their neighborhoods are represented by an undirected graph: vertices represent districts and edges indicate that two districts are adjacent.

Our second example concerns economization in a fairly general form. Let us consider a company with n employees, each producing s units of a desirabe good during an eight-hour working day; for concreteness, let us say that each employee proves s theorems per working day. Now due to the increasing support of automatic theorem provers, each employee is suddenly able to prove s_{new} theorems per day ($s_{\text{new}} > s$). Hence, without lowering the total number of proved theorems per day, some employees may be moved to a special task force for improving automatic theorem provers: this will secure the company's future competitiveness in proving theorems, without decreasing the overall theorem output. By company regulations, all theorem proving employees have to be treated equally and should have identical workloads. In the *network-based dissolution* model, employees correspond to vertices. Employees in the special task force are dissolved and disappear from the scene of action; their workload is to be taken over by neighboring employees who are comparable in qualification and research interests.

Our third and last example concerns storage updates in parallel or distributed systems. Let us consider a distributed storage array consisting of n storage nodes, each having a capacity of s storage units, of which some space is free. As the prices on cheap hard disk space are rapdily decreasing, the operators want to upgrade the storage capacity of some nodes and to deactivate other nodes for saving energy and cost. As their distributed storage concept takes full advantage only in case all nodes have equal capacity, they want to upgrade all (non-deactivated) nodes to the same capacity s_{new} and move capacities from deactivated nodes to non-deactivated neighboring nodes. In the resulting system, every non-deactivated node should only use half of its storage capacity.

Related work. We are not aware of any previous work on our network-based dissolution problem. Our main inspiration comes from the area of political districting, and in particular from gerrymandering [15, 19, 20] and from supervised regionalization methods [9]. Of course, graph-theoretic models have been employed before for political districting; for instance Mehrota et al. [17] draws a connection to graph partitioning, and Duque [8] and Maravalle and Simeone [16] use graphs to model geographic information in the regionalization problem. These models are tailored towards very specific applications and are mainly used for the purpose of developing efficient heuristic algorithms, often relying on mathematical programming techniques. The computational hardness of districting problems has been known for many years [1].

Remark on nomenclature. For the ease of presentation, throughout the paper we will adopt a political districting point of view on network-based dissolution: the words districts and vertices are used interchangeably, and the entities in districts are referred to as voters or supporters.

Contributions and organization of this paper. We propose two simple models DISSOLUTION and BIASED DISSOLUTION for network-based dissolution (Section 2). In the main body of the paper, we provide a variety of computational tractability and intractability results for both models. Furthermore, we investigate how the structure of the underlying graphs or an in-advance fixing of vertices to be dissolved influence the computational complexity (mainly in terms of polynomial-time solvability versus NP-hard cases).

- First of all, network flow techniques show that BIASED DISSOLUTION is polynomially solvable if the set of districts to be dissolved and the set of districts to be won are both specified as part of the input. The general version is NP-hard for every fixed $s \geq 3$. These results are given in Section 3.

- Section 4 presents a complexity dichotomy for DISSOLUTION and BIASED DISSOLUTION with respect to the old district size s and the increase Δ_s in district size (= difference between new and old district size). DISSOLUTION is polynomially solvable for $s = \Delta_s$, and BIASED DISSOLUTION is polynomially solvable for $s = \Delta_s = 1$; all other cases are NP-hard.

- Section 5 analyzes the complexity of DISSOLUTION and BIASED DISSOLUTION for various specially structured graphs, including planar graphs (hard), cliques (easy), and graphs of bounded treewidth (easy).

2 Formal setting

We start by introducing notation and formal definitions of the technical terms that we use throughout the paper.

Graphs. We consider simple undirected graphs $G = (V, E)$, where V is a set of $|V| = n$ vertices and $E \subseteq \binom{V}{2}$ is a set of $|E| = m$ edges. For a given graph G, we denote by $V(G)$ the set of vertices and by $E(G)$ the set of edges of G. For a vertex $v \in V$, we denote by $N(v) := \{u \in V \mid \{u, v\} \in E\}$ the *(open) neighborhood* of v, that is, all vertices that are connected to v by an edge. For a subset $E' \subseteq E$ of edges, the graph $G[E] := (V, E')$ is called a *subgraph* of G. For a vertex subset $V' \subseteq V$, the *induced* subgraph $G[V']$ of G is defined as $G[V'] := (V', E \cap \binom{V'}{2})$. A *t-star* is a graph $K_{1,t}$ on $t+1$ vertices where exactly one vertex v is connected to all other t vertices by an edge. The vertex v is called the *center* of the star. A *t-star partition* of G is a partition $\{V_1, \ldots, V_{n/(t+1)}\}$ of the vertex set V into subsets of size $t + 1$ such that each $G[V_i]$ contains a t-star as a subgraph. Note that a 1-star partition is also called a *perfect matching*.

Networks and flows. A flow network I^* consists of a directed graph $G^* = (V^*, E^*)$ where V^* is the set of nodes and E^* is a set of directed edges (arcs) over V^*, an arc capacity function $c^* : E^* \to \mathbb{R}_0$, and two distinguished nodes $\sigma, \tau \in V^*$ denoted as the source and the target of the network. A (σ, τ)-flow $f : E^* \to \mathbb{R}_0$ is an arc value function with $f(u, v) \geq 0$ for all $(u, v) \in E^*$ such that (1) the capacity constraint is fulfilled, i.e. $f(u, v) \leq c(u, v), \forall (u, v) \in E^*$, and (2) the conservation property is satisfied, i.e. $\sum_{(u,v) \in E^*} f(u, v) = \sum_{(v,u) \in E^*} f(v, u)$, $\forall u \in V^* \setminus \{\sigma, \tau\}$. We call f *integer* if all its values are integers. The *value* of f is defined as $\sum_{(\sigma,u) \in E^*} f(\sigma, u)$.

Dissolutions. Let $G = (V, E)$ be an undirected graph representing n districts. Let $s, \Delta_s \in \mathbb{N}^+$ be the *district size* and *district size increase*, respectively. For a subset $V' \subseteq V$ of districts, let $Z(V', G) = \{(x, y) \mid x \in V' \wedge y \in V(G) \setminus V' \wedge \{x, y\} \in E(G)\}$ be the set of pairs of districts in V' and their neighbors that are not in V'. The central notion for our studies is that of a *dissolution*, which basically describes a valid movement of voters from dissolved districts into remaining districts. The formal definition is the following:

Definition 1 (Dissolution)**.** Let $G = (V, E)$ be an undirected graph and let $D \subset V$ be a subset of *districts to dissolve* and $z \colon Z(D, G) \to \{0, \ldots, s\}$ be a function that describes how many voters shall be *moved* from one district to its non-dissolved neighbors. Then, (D, z) is called an (s, Δ_s)-*dissolution* for G if

a) no voter remains in any dissolved district:

$$\forall v' \in D : \sum_{(v',v) \in Z(D,G)} z(v', v) = s, \text{ and}$$

b) the size of all remaining (non-dissolved) districts increases by Δ_s:

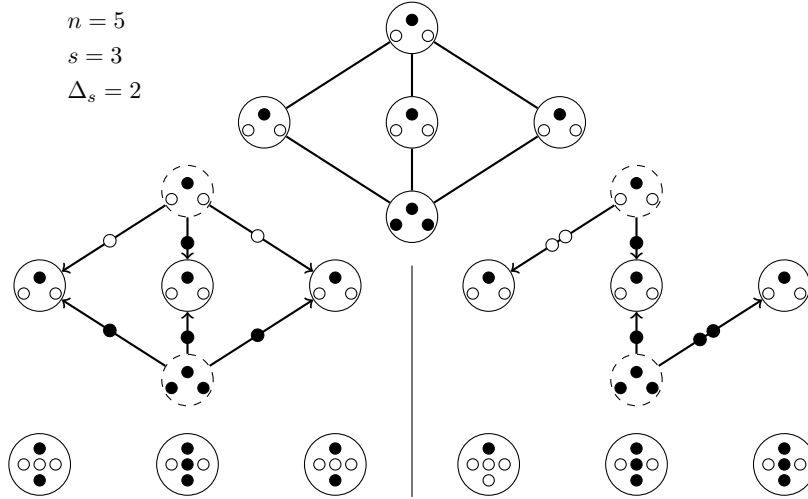$$\forall v \in V \setminus D : \sum_{(v',v) \in Z(D,G)} z(v', v) = \Delta_s.$$

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Figure 1: An illustration of a 1-biased $(3, 2)$-dissolution (left) and a 2-biased $(3, 2)$-dissolution (right). Black circles represent A-supporters while white circles represent B-supporters. The figure on the top shows the neighborhood graph of five districts, each district consisting of three voters. The task is to dissolve two districts such that each remaining district contains five voters. The figures in the middle show two possible realizations of dissolutions. The figures on the bottom show the two corresponding outcomes. The arrows point from the districts to be dissolved to the "goal districts" and the black/white circle labels on the arrows indicate which kind of voters are moved along the arrows.

Throughout this work, we use $s_{\text{new}} := s + \Delta_s$ to denote the new district size, $d := |D| = |V| \cdot \Delta_s / s_{\text{new}}$ to denote the number of dissolved districts, and $r := |V| - d$ to denote the number of remaining, non-dissolved districts.

We write *dissolution* instead of (s, Δ_s)-dissolution when s and Δ_s are clear from the context. By definition, a dissolution only ensures that the numbers of voters moving between districts fulfill the given constraints on the district sizes, that is, the size of each remaining district increases by Δ_s. For instance, the two pictures in the middle illustrated in Figure 1 indicated two possible $(3, 2)$-dissolutions.

Motivated from the social choice context, we additionally assume that each voter supports one of two parties A and B. We then seek a dissolution such that the number of remaining districts won by party A is maximized. Here, a district is won by the party that is supported by a strict majority of the voters inside the district. This yields the notion of a *biased dissolution*, which is defined as follows:

Definition 2 (Biased dissolution)**.** Let G be an undirected graph and let $\alpha \colon V(G) \to \{0, \ldots, s\}$ be an *A-supporter distribution*, where $\alpha(v)$ denotes the

number of A-supporters in district $v \in V$. Let (D, z) be an (s, Δ_s)-dissolution for G. Let $r_\alpha \in \mathbb{N}$ be the minimum number of districts that party A shall win after the dissolution and $z_\alpha \colon Z(D, G) \to \{0, \ldots, s\}$ be an *A-supporter movement*, where $z_\alpha(v', v)$ denotes the number of A-supporters moving from district v' to district v. Finally, let $R_\alpha \subseteq V(G) \setminus D$ be a size-r_α subset of districts. Then, $(D, z, z_\alpha, R_\alpha)$ is called an r_α-*biased* (s, Δ_s)-*dissolution* for (G, α) if and only if

c) a district cannot receive more A-supporters from a dissolved district than the total number of voters it receives from that district:

$$\forall (v', v) \in Z(D, G) : z_\alpha(v', v) \leq z(v', v),$$

d) no A-supporters remain in any dissolved district:

$$\forall v' \in D : \sum_{(v', v) \in Z(D, G)} z_\alpha(v', v) = \alpha(v'), \text{ and}$$

e) each district in R_α has a strict majority of A-supporters:

$$\forall v \in R_\alpha : \alpha(v) + \sum_{(v', v) \in Z(D, G)} z_\alpha(v', v) > \frac{s + \Delta_s}{2}.$$

We also say that a district *wins* if it has a strict majority of A-supporters and *loses* otherwise.

Figure 1 shows two biased dissolutions: one with $r_\alpha = 1$ and the other one with $r_\alpha = 2$. We are now ready to formally state the definitions of the two dissolution problems that we discuss in this work:

DISSOLUTION
Input: An undirected graph $G = (V, E)$ and positive integers s and Δ_s.
Question: Is there an (s, Δ_s)-dissolution for G?

BIASED DISSOLUTION
Input: An undirected graph $G = (V, E)$, positive integers s, Δ_s, r_α, and an A-supporter distribution $\alpha : V \to \{0, \ldots, s\}$.
Question: Is there an r_α-biased (s, Δ_s)-dissolution for (G, α)?

Note that DISSOLUTION is equivalent to BIASED DISSOLUTION with $r_\alpha = 0$. As we will see later, both DISSOLUTION and BIASED DISSOLUTION are NP-hard in general. In this work, we additionally look into special cases of our dissolution problems and investigate where the intractability results lie.

3 Partially known dissolutions

In this section, we investigate some relevant special cases of our (in general) NP-hard dissolution problems. These include situations where the districts to be dissolved or to be won are fixed in advance. We find out that BIASED DISSOLUTION is only polynomial-time solvable if both are fixed, and NP-hard otherwise.

3.1 Fixed set D of dissolved districts and fixed set R_α of winning districts

Sometimes, the districts that are to be dissolved and the districts that are to win are not arbitrary but already determined beforehand. For this case, we show that BIASED DISSOLUTION can be modelled as a network flow problem which can be solved in polynomial time.

Theorem 1. *Let $I = (G = (V, E), s, \Delta_s, r_\alpha, \alpha)$ be a BIASED DISSOLUTION instance, and let $D, R_\alpha \subset V$ be two disjoint subsets of districts. The problem of deciding whether (G, α) admits an r_α-biased (s, Δ_s)-dissolution in which D is the set of dissolved districts and in which all districts in R_α are won can be reduced in linear time to a maximum flow problem with $2|V| + 2$ nodes, $2|V| + 3|E|$ arcs, and maximum arc capacity $\max(s, \Delta_s)$.*

Proof. Denote the set of remaining districts by R, that is, $R = V \setminus D$. With $R_\alpha \subseteq R$ given beforehand, we can calculate how many A-supporters a district $v \in R_\alpha$ needs from its neighboring dissolved district $w \in D \cap N(v)$ in order to win after the dissolution. With D given beforehand, we can even construct a flow network where there are two nodes corresponding to each district (denoted as a *dissolved* (or a *non-dissolved*) node if the corresponding district is dissolved (or non-dissolved)) and appropriately add arcs from dissolved nodes to non-dissolved nodes. The capacities of these arcs model the movement of A-supporters from the districts in D to the districts in R that are necessary for a district $v \in R_\alpha$ to win.

To this end, we first assume that our given neighborhood graph G is bipartite with the two disjoint sets D and R since only edges between D and R may be taken into account for the dissolution. Second, we observe that in order to let a district $v \in V \setminus D$ win after the dissolution, v needs at least $\min\{0, \lceil (s_{\text{new}}+1)/2 \rceil - \alpha(v)\}$ additional A-supporters. Hence, we compute a "demand" function $\kappa : R \to \{0, \ldots, \lceil (s_{\text{new}}+1)/2 \rceil\}$ for each non-dissolved district v by $\kappa(v) = \min\{0, \lceil (s_{\text{new}}+1)/2 \rceil - \alpha(v)\}$ if $w \in R_\alpha$ and $\kappa(v) = 0$ otherwise.

We construct a flow network $I^* = (G^* = (V^*, E^*), c^*, \sigma, \tau)$ for our input instance I. The node set V^* in G^* consists of a source node σ, a target node τ, and two nodes u_i and $\overline{u_i}$ for each district $v_i \in V$. We say u_i and $\overline{u_i}$ correspond to district v_i. In total, V^* has $2|V| + 2$ nodes. Also see Figure 2 for an illustration.

The arcs in E^* are divided into three layers: i) arcs from the source node to all dissolved nodes, ii) arcs from the dissolved nodes to some non-dissolved nodes, and iii) arcs from all non-dissolved nodes to the target node.

In layer i), for each dissolved district $v_i \in D$, add to E^* two arcs (σ, u_i) and $(\sigma, \overline{u_i})$ with capacities $c^*(\sigma, u_i) = \alpha(v_i)$ and $c^*(\sigma, \overline{u_i}) = s - \alpha(v_i)$. In layer ii), for each dissolved district $v_i \in D$ and for each $v_j \in N(v_i)$ of its non-dissolved neighbors, add to E^* three arcs (u_i, u_j), $(u_i, \overline{u_j})$, and $(\overline{u_i}, \overline{u_j})$ with capacities $c^*(u_i, u_j) = c^*(u_i, \overline{u_j}) = \alpha(v_i)$ and $c^*(\overline{u_i}, \overline{u_j}) = s - \alpha(v_i)$. In layer iii), for each non-dissolved district $v_j \in R$, add to E^* two arcs (u_j, τ) and $(\overline{u_j}, \tau)$ with capacities $c^*(u_j, \tau) = \kappa(v_j)$ and $c^*(\overline{u_j}, \tau) = \Delta_s - \kappa(v_j)$. This completes the description of the flow network construction.
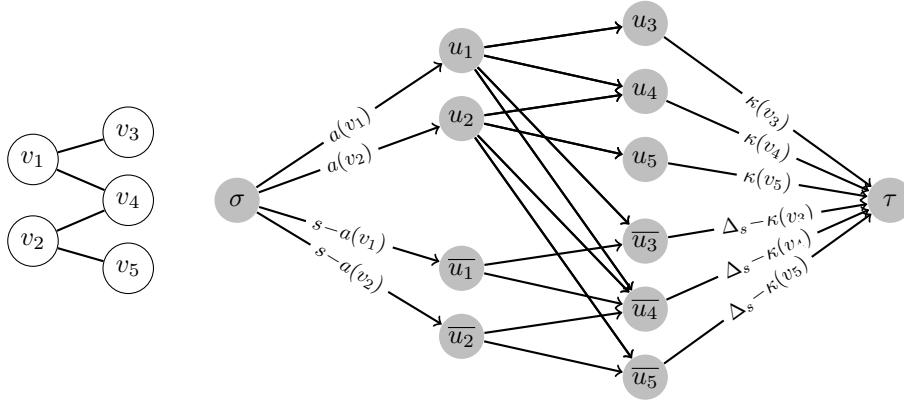
Figure 2: An illustration of the network flow construction. Left: the graph G of an instance of BIASED DISSOLUTION with $D = \{v_1, v_2\}$. Right: the corresponding network flow. The capacities of the arcs from dissolved nodes to non-dissolved nodes are omitted for the sake of brevity.
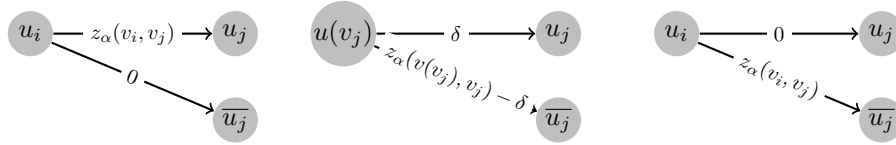

Figure 3: Three cases of setting the flow values for arcs towards u_j and $\overline{u_j}$ in layer ii), where v_j is a non-dissolved vertex. Left: u_i corresponds to a vertex $v_i \in M(v_j)$; middle: $u(v_j)$; right: u_i corresponds to a vertex $v_i \in N(v_j) \setminus (M(v_j) \cup \{v(v_j)\})$.

We show that there is an r_α-biased (s, Δ_s)-dissolution $(D, z, z_\alpha, R_\alpha)$ for (G, α) if and only if the constructed flow network I^* has a (σ, τ)-flow of value $s \cdot |D|$.

For the "only if" part, suppose that there is a dissolution $(D, z, z_\alpha, R_\alpha)$ for (G, α). Construct a (σ, τ)-flow $f : E^* \to \mathbb{R}$ by first defining $f(\sigma, u_i) := c^*(\sigma, u_i)$ where u_i corresponds to a dissolved district and defining $f(u_j, \tau) := c^*(u_j, \tau)$ where u_j corresponds to a non-dissolved district. It remains to define the flow values for the arcs in layer ii). First, for each $v_i \in D$ and for each $v_j \in N(v_i)$ of its non-dissolved neighbors, define $f(\overline{u_i}, \overline{v_j}) := z(v_i, v_j) - z_\alpha(v_i, v_j)$. Let $v_j \in R$ be a non-dissolved district. Let $M(v_j) \subseteq N(v_j)$ be a subset of v_j's dissolved neighbors with the following two properties:

i) $\sum\limits_{x \in M(v_j)} z_\alpha(x, v_j) \leq \kappa(v_j)$ and

ii) $\forall v \in N(v_j) \setminus M(v_j) : z_\alpha(v, v_j) + \sum\limits_{x \in M(v_j)} z_\alpha(x, v_j) > \kappa(v_j)$.

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Now, for each $v_i \in M(v_j)$, define $f(u_i, u_j) := z_\alpha(v_i, v_j)$ and $f(u_i, \overline{u_j}) := 0$ (see the left picture in Figure 3). Let $v(v_j) \in N(v_j) \setminus M(v_j)$ be an arbitrary (but fixed) dissolved neighbor of v_j that is not in $M(v_j)$ and let $u(v_j)$ and $\overline{u(v_j)}$ be the corresponding nodes in the flow network. Note that such a neighbor $v(v_j)$ needs not to exist when $M(v_j) = N(v_j)$, but if it exists, then $z_\alpha(v(v_j), v_j) + \sum_{x \in M(v_j)} z_\alpha(x, v_j) > \kappa(v_j)$.

Let $\delta = \max\{0, \kappa(v_j) - \sum_{x \in M(v_j)} z_\alpha(x, v_j)\}$. Define $f(u(v_j), u_j) = \delta$ and $f(\overline{u(v_j)}, u_j) = z_\alpha(v(v_j), v_j) - \delta$ (see the middle picture in Figure 3). Finally, for each $v_i \in N(v_j) \setminus (M(v_j) \cup \{v(v_j)\})$, define $f(u_i, u_j) := 0$ and $f(u_i, \overline{u_j}) := z_\alpha(v_i, v_j)$ (see the right picture in Figure 3).

Now, observe that if the constructed (σ, τ)-flow f is valid, then it has value $\sum_{(s,x) \in E^*} f(s, x) = s \cdot |D|$ because $(D, z, z_\alpha, R_\alpha)$ is a biased dissolution. It remains to show that f is valid. It is easy to verify that the flow value of each arc does not exceed its capacity. To verify the conservation property, let $v_i \in D$ be a dissolved district. By the construction of f (Figure 3), it holds that $f(u_i, u_j) + f(u_i, \overline{u_j}) = z_\alpha(v_i, v_j)$ for all $v_j \in N(v_i)$. Since $(D, z, z_\alpha, R_\alpha)$ is a biased dissolution for (G, α), it must, by Property d), also hold that

$$\sum_{(x,u_i) \in E^*} f(x, u_i) = f(\sigma, u_i) = \sum_{v_j \in N(v_i)} z_\alpha(v_i, v_j) = \sum_{v_j \in N(v_i)} f(u_i, u_j) + f(u_i, \overline{u_j}) = \sum_{(u_i, x) \in E^*} f(u_i, x).$$

Analogously, the conservation property for node $\overline{u_i}$ holds because of Property a) and d).

As for nodes corresponding to non-dissolved districts, let $v_j \in R$ be a non-dissolved district. Let $V(v_j)$, $v(v_j)$, and $u(v_j)$ be computed as described above. Then,

$$\sum_{(u_i, u_j) \in E^*} f(u_i, u_j) = f(u(v_j), u_j) + \sum_{v_i \in M(v_j)} f(u_i, u_j) + \sum_{v_{i'} \in W(v_j)} f(u_{i'}, u_j),$$

where $W(v_j) = N(v_j) \setminus (M(v_j) \cup \{v(v_j)\})$. Analogously, the conservation law for node $\overline{u_j}$ can be shown due to Properties b) and e).

For the "if" part, suppose that f is a (σ, τ)-flow for I^* with value $s \cdot |D|$. Let $z_\alpha : Z(D, G) \to \{0, \ldots, s\}$ and $z : Z(D, G) \to \{0, \ldots, s\}$ be two functions with values $z_\alpha(v_i, v_j) = f(u_i, u_j) + f(u_i, \overline{u_j})$ and $z(v_i, v_j) = z_\alpha(u_i, u_j) + f(\overline{u_i}, \overline{u_j})$. One can verify that $(D, z, z_\alpha, R_\alpha)$ is an r_α-biased (s, Δ_s)-dissolution for (G, α). $\square$

With the help of the above flow network construction we can design a polynomial time algorithm solving BIASED DISSOLUTION when the number of districts is a constant.

Proposition 1. *Any instance* $((V, E), s, \Delta_s, \alpha)$ *of* BIASED DISSOLUTION *can be solved in time* $O(3^{|V|} \cdot (\max(s, \Delta_s) \cdot |V| \cdot |E| + |V|^3))$.

Proof. Since each district will either be dissolved, won, or lost, there are at most $3^{|V|}$ different ways to fix the roles of all $|V|$ districts. For each of these fixtures, we can construct a flow network with $O(|V|)$ nodes and maximum

capacity $\max(s, \Delta_s)$ in $O(\max(s, \Delta_s) \cdot |V| \cdot |E|)$ time and compute the maximum flow (Theorem 1) to solve BIASED DISSOLUTION. Hence, by using an $O(|V|^3)$ time maximum flow algorithm we solve BIASED DISSOLUTION in $O(3^{|V|}(\max(s, \Delta_s) \cdot |V| \cdot |E| + |V|^3))$ time. $\qquad \square$

3.2 Fixed set D of dissolved districts

For the case that only the set D of dissolved districts is given beforehand, the remaining task is to decide how many A-supporters are moved to a certain non-dissolved district. However, as we will see in Section 4.2 from the hardness construction for Theorem 2 for the case of $s \geq 3$, it is already determined which districts are to be dissolved. Hence, already knowing which districts are to be dissolved beforehand does not help in attacking the NP-hardness of BIASED DISSOLUTION.

Proposition 2. BIASED DISSOLUTION *is NP-hard for every fixed $s \geq 3$, even if the set D of dissolved districts is known in advance.*

As for DISSOLUTION which is the special case of BIASED DISSOLUTION with $R_\alpha = \emptyset$, with D given beforehand, we can utilize the flow network approach behind Theorem 1 to solve it in polynomial time (see Lemma 1).

3.3 Fixed set R_α of winning districts

Since DISSOLUTION is the special case of BIASED DISSOLUTION with $r_\alpha = 0$ (which implies $R_\alpha = \emptyset$) and since DISSOLUTION is NP-hard for the case of $s \neq \Delta_s$ (Theorem 2), we obtain the following.

Proposition 3. BIASED DISSOLUTION *is NP-hard, even if the set R_α of districts required to be won is empty.*

4 Complexity dichotomy with respect to district sizes

In this section, we study the computational complexity of DISSOLUTION and BIASED DISSOLUTION with respect to the ratio of the two integers: old district size s and district size increase Δ_s. We show that DISSOLUTION is polynomial-time solvable if $s = \Delta_s$, and NP-complete otherwise (Theorem 2). BIASED DISSOLUTION is only polynomial-time solvable if $s = \Delta_s = 1$ and NP-complete otherwise (Theorem 3). We start by showing some useful structural observations for dissolutions in Section 4.1 before we come to the results for DISSOLUTION in Section 4.2 and for BIASED DISSOLUTION in Section 4.3.

4.1 Structural properties

If the districts to dissolve are fixed, then DISSOLUTION turns into a simple transportation problem (see Theorem 1). The following lemma shows that the
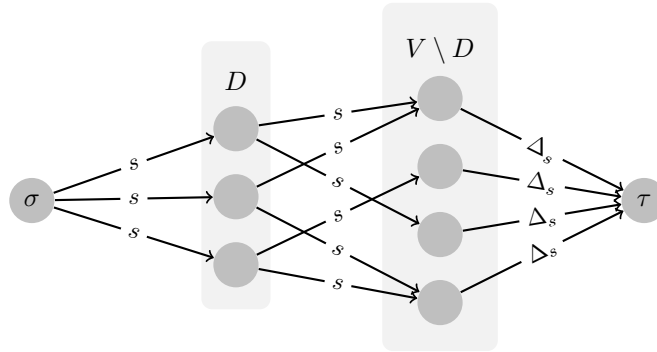
Figure 4: Flow network for DISSOLUTION when the set D of districts to dissolve is known.

number of nodes and arcs in the corresponding network is polynomially bounded and that the capacity values used are either s or Δ_s.

Lemma 1. *Let $G = (V, E)$ be a graph and let $D \subset V$. If there exists an (s, Δ_s)-dissolution (D, z) for G, then it can be found by computing the maximum flow in a network with $|V| + 2$ nodes and $|E| + 2|V|$ arcs where all capacities are either s or Δ_s.*

Proof. If the districts to dissolve are known and we only search for a dissolution (or $r_\alpha = 0$), then the flow network used to compute a dissolution from the proof of Theorem 1 basically reduces to a much simpler flow network. We can assume that $R_\alpha = \emptyset$, remove all arcs with capacity zero, and finally also remove nodes without a directed path to the sink.

We have a source σ and a sink τ and two additional layers of nodes: the first layer contains one node for each vertex from D and the second layer contains one node for each vertex from $V \setminus D$. There is an arc from the source σ to each node in the first layer with capacity s and an arc from each node in the second layer to the sink τ with capacity Δ_s. Finally, there is an arc of capacity s from a node in the first layer to a node in the second layer if and only if the corresponding vertices in the neighborhood graph G are adjacent. See Figure 4 for an illustration. □

If $\Delta_s = 1$, then each non-dissolved district receives exactly one addition voter from one of its neighboring districts. Each dissolved district has to move exactly one voter each to s of its neighboring districts. Hence, it is easy to see that a graph has an $(s, 1)$-dissolution if and only if it has an s-star partition.

Using the flow construction from Lemma 1, we can even show that this equivalence to star partition generalizes to the case that s is any multiple of Δ_s.

Lemma 2. *There exists a $(t \cdot \Delta_s, \Delta_s)$-dissolution for an undirected graph G, if and only if G has a t-star partition.*

Proof. If G can be partitioned into t-stars, then it is easy to see that there is a $(t \cdot \Delta_s, \Delta_s)$-dissolution for G: Let $C = \{c_1, \dots, c_d\} \subset V$ be the set of t-star centers and let $L_i \subset V, 1 \leq i \leq d$, be the set of leaves of the i-th star. Define function $z : Z(C, G) \to \{0, \dots, t \cdot \Delta_s\}$ such that for all $(c_i, l) \in Z(C, G)$, $z(c_i, l) := \Delta_s$ if $l \in L_i$ and $z(c_i, l) := 0$ otherwise. Obviously, (C, z) is a $(t \cdot \Delta_s, \Delta_s)$-dissolution for G.

Now, let (D, z) be a $(t \cdot \Delta_s, \Delta_s)$-dissolution for G. We show that G can be partitioned into t-stars with D being the t-star centers. To this end, consider the network flow constructed in Lemma 1 and modify the network as follows. For each arc, divide its capacity by Δ_s. Clearly, if there is a flow with value $|D| \cdot t \cdot \Delta_s = |V \setminus D| \cdot \Delta_s$, then the modified network has a flow with value $|D| \cdot t = |V \setminus D|$. As all capacities are integers, there exists a maximum integer flow f. Hence, a partition of G into t-stars consists of one star for each $v_i \in D$ such that v_i is the star center connected to its leaves $L_i = \{u \mid f(v_i, u) = 1\}$. $\square$

As a third property, we observe a symmetry concerning the district size s and the district size increase Δ_s in the sense that exchanging their values yields an equivalent instance of DISSOLUTION. Intuitively, the idea behind the following lemma is that the roles of dissolved and non-dissolved districts in a given (s, Δ_s)-dissolution can in fact be exchanged by "reversing" the movement of voters to obtain a (Δ_s, s)-dissolution.

Lemma 3. *There exists an (s, Δ_s)-dissolution for an undirected graph G, if and only if there exists a (Δ_s, s)-dissolution for G.*

Proof. Let (D, z) be an (s, Δ_s)-dissolution for G. Then, $(V(G) \setminus D, z')$ with $z'(x, y) = z(y, x)$ is a (Δ_s, s)-dissolution for G: The domain of z' is correct: $Z(V(G) \setminus D, G) = \{(x, y) \mid x \in V(G) \setminus D \wedge y \in V(G) \setminus (V(G) \setminus D) \wedge \{x, y\} \in E(G))\} = \{(x, y) \mid x \in V(G) \setminus D \wedge y \in D \wedge \{x, y\} \in E(G))\}$. Let us check whether $(V(G) \setminus D, z')$ fulfills all properties from Definition 1. Property a is fulfilled for $(V(G) \setminus D, z')$ if and only if Property b is fulfilled for (D, z) and Property b is fulfilled for $(V(G) \setminus D, z')$ if and only if Property a is fulfilled for (D, z). $\square$

4.2 Complexity dichotomy for Dissolution

In this subsection, we show a P vs. NP dichotomy of DISSOLUTION with respect to the district size s and the size increase Δ_s. Using Lemma 2, we can show that finding an (s, s)-dissolution essentially corresponds to finding a perfect matching and can thus be done in polynomial time.

Lemma 4. *If $s = \Delta_s$, then DISSOLUTION can be solved by computing a perfect matching in graph G.*

Proof. Let $I = (G, s, \Delta_s)$ be a DISSOLUTION instance with $\Delta_s = s$. Set $t := s/\Delta_s = 1$. Lemma 2 implies that I is a yes-instance if and only if G has a t-star partition. A t-star partition with $t = 1$ is indeed a perfect matching. $\square$
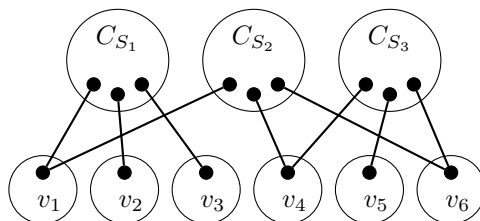
Figure 5: The constructed instance for $t = 3$.

If $s \neq \Delta_s$, then DISSOLUTION becomes NP-hard. We can use a result from the field of number theory to encode the NP-complete EXACT COVER BY t-SETS problem into our dissolution problem.

EXACT COVER BY t-SETS
Input: A finite set X and a collection $\mathcal{C}$ of subsets of X of size t.
Question: Is there a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ that partitions X, that is, each element of X is contained in exactly one subset in $\mathcal{C}'$?

Now, let us briefly recall some prerequisite from elementary number theory.

Lemma 5 (Bézout's identity). *Let a and b be two positive integers and let g be their greatest common divisor. Then, there exist two integers x and y with $ax + by = g$.*

Moreover, x and y can be computed in polynomial time using the extended Euclidean algorithm [5, Section 31.2]. Indeed, we can infer from Lemma 5 that any two integers x' and y' with $x' = ix + jb/g$ and $y' = iy - ja/g$ for some $i, j \in \mathbb{Z}$ satisfy $ax' + by' = ig$. We will make use of this fact several times in the NP-hardness proof of the following theorem.

Theorem 2. *If $s = \Delta_s$, DISSOLUTION is solvable in $O(n^\omega)$ time (where ω is the matrix multiplication exponent); otherwise the problem is NP-complete.*

Proof. First, Lemma 4 implies that an (s, s)-dissolution corresponds to a perfect matching in G, which can be computed in $O(n^\omega)$ time, where ω is the smallest exponent such that matrix multiplication can be computed in $O(n^\omega)$ time. Currently, the smallest known upper bound of ω is 2.3727 [21].

For the case $s \neq \Delta_s$, we show that DISSOLUTION is NP-complete if $s > \Delta_s$. Due to Lemma 3, this also transfers to the cases where $s < \Delta_s$. First, given a DISSOLUTION instance (G, s, Δ_s) and a function $z : Z(D, G) \to \{0, \dots, s\}$ where $D \subset V(G)$, one can check in polynomial time whether (D, z) is an (s, Δ_s)-dissolution. Thus, DISSOLUTION is in NP.

To show the NP-hardness result, we give a reduction from the NP-complete EXACT COVER BY t-SETS problem [13] for $t := (s + \Delta_s)/g > 2$, where $g := \gcd(s, \Delta_s) \leq \Delta_s$ is the greatest common divisor of s and Δ_s.

Given an Exact Cover by t-Sets instance $(X, \mathcal{C})$, we construct a Dissolution instance (G, s, Δ_s) with a neighborhood graph $G = (V, E)$ defined as follows: For each element $u \in X$, add a clique C_u of properly chosen size q to G and let v_u denote an arbitrary fixed vertex in C_u. For each subset $S \in \mathcal{C}$, add a clique C_S of properly chosen size $r \geq t$ to G and connect each v_u for $u \in S$ to a unique vertex in C_S.

Next, we explain how to choose the values of q and r. We set $q = x_q + y_q$, where $x_q \geq 0$ and $y_q \geq 0$ are integers satisfying $x_q s - y_q \Delta_s = g$. Such integers exist by our preliminary discussion (Lemma 5). The intuition behind is as follows: Dissolving x_q districts in C_u and moving the voters to y_q districts in C_u creates an overflow of exactly g voters that have to move out of C_u. Notice that the only way to move voters into or out of C_u is via district v_u. Moreover, in any dissolution, exactly x_q districts in C_u are dissolved because dissolving more districts leads to an overflow of at least $g + s + \Delta_s > s$ voters, which is more than v_u can move, whereas dissolving less districts yields a demand of at least $s + \Delta_s - g > \Delta_s$ voters, which is more than v_u can receive. Thus, v_u must be dissolved since there is an overflow of g voters to move out of C_u and this can only be done via district v_u.

The value of $r \geq t$ is chosen in such a way that, for each $S \in \mathcal{C}$ and each $u \in S$, it is possible to move g voters from v_u to C_S (recall that v_u must be dissolved). In other words, we require C_S to be able to receive in total $t \cdot g = s + \Delta_s$ voters in at least t non-dissolved districts. Thus, we set $r := x_r + y_r$, where $x_r \geq 0$ and $y_r \geq t$ are integers satisfying $x_r s - y_r \Delta_s = -(s + \Delta_s)$. Again, since $-(s + \Delta_s)$ is divisible by g, such integers exist by our preliminary discussion. It is thus possible to dissolve x_r districts in C_S moving the voters to the remaining y_r districts in C_S such that we end up with a demand of $s + \Delta_s$ voters in C_S. Note that the only other possibility is to dissolve $x_r + 1$ districts in C_S in order to end up with a demand of zero voters. In this case, no voters of any other districts connected to C_S can move to C_S. By the construction of C_u above, it is clear that it is also not possible to move any voters out of C_S because no v_u can receive voters in any dissolution. Thus, for any dissolution, it holds that either all or none of the v_u connected to some C_S move g voters to C_S. Figure 5 shows an example of the constructed neighborhood graph for $t = 3$.

The proof of correctness is as follows. Suppose $(X, \mathcal{C})$ is a yes-instance, that is, there exists a partition $\mathcal{C}' \subseteq \mathcal{C}$ of X. We can thus dissolve x_q districts in each C_u (including v_u) and move the voters such that all y_q non-dissolved districts receive exactly Δ_s voters. This is always possible since C_u is a clique. If we do so, then, by construction, g voters have to move out of each v_u. Since $\mathcal{C}'$ partitions X, each $u \in X$ is contained in exactly one subset $S \in \mathcal{C}'$. We can thus move the g voters from each v_u to C_S. Now, for each $S \in \mathcal{C}'$, we dissolve any x_r districts that are not adjacent to any v_u and for the subsets in $\mathcal{C} \setminus \mathcal{C}'$, we simply dissolve arbitrary $x_r + 1$ districts in the corresponding cliques. By the above discussion of the construction, we know that this in fact yields an (s, Δ_s)-dissolution. Hence, (G, s, Δ_s) is a yes-instance.

Now assume that there exists an (s, Δ_s)-dissolution for (G, s, Δ_s). As we have already seen in the above discussion, any (s, Δ_s)-dissolution generates an

overflow of g voters in each C_u that has to be moved over v_u to some district in C_S. Furthermore, each C_S either receives g voters from all its adjacent v_u or no voters at all. Therefore, the subsets S corresponding to cliques C_S that receive $t \cdot g$ voters form a partition of X, showing that $(X, \mathcal{C})$ is a yes-instance. $\qquad\square$

4.3 Complexity of Biased Dissolution

Since DISSOLUTION is a special case of BIASED DISSOLUTION, the NP-hardness results for $s \neq \Delta_s$ transfer to BIASED DISSOLUTION. It remains to see whether BIASED DISSOLUTION remains polynomial-time solvable when $s = \Delta_s$. Interestingly, this is true for $s = \Delta_s = 1$, but BIASED DISSOLUTION becomes NP-hard when $s = \Delta_s \geq 2$.

First, we introduce a notion called "edge set" for a given dissolution (D, z) of a given graph G which will be used in several proofs. Let $E_z \subseteq E(G)$ contain all edges $\{x, y\}$ with $(x, y) \in Z(D, G)$ and $z(x, y) > 0$. Then, we call E_z the *edge set used by* the dissolution (D, z).

The following lemma shows that finding an r_α-biased $(1, 1)$-dissolution essentially corresponds to finding a maximum weighted perfect matching.

Lemma 6. *Let* $(G = (V, E), s = 1, \Delta_s = 1, r_\alpha, \alpha)$ *be a* BIASED DISSOLUTION *instance. There exists a* r_α-*biased* $(1, 1)$-*dissolution for* (G, α) *if and only if there exists a perfect matching of weight at least* r_α *in* (G, w) *with* $w(\{x, y\}) := 1$ *if* $\alpha(x) = \alpha(y) = 1$ *and* $w(\{x, y\}) := 0$ *otherwise.*

Proof. "$\Rightarrow$": Let $(D, z, z_\alpha, R_\alpha)$ be an r_α-biased $(1, 1)$-dissolution for (G, α). Then, the edge set $E_z \subseteq E$ used by $(D, z, z_\alpha, R_\alpha)$ partitions G into 1-stars or in other words, E_z is a perfect matching for G (see Lemma 2). Note that a non-dissolved district can only win if it already contains an A-supporter and receives one additional A-supporter. By the construction of w, this implies that the weight of each edge that connects a winning district is one (i.e. $\forall e \in E_z :$ $e \cap R_\alpha \neq \emptyset \leftrightarrow w(e) = 1$). Since $|R_\alpha| \geq r_\alpha$, the perfect matching E_z has weight at least r_α.

"$\Leftarrow$": Let $E' \subseteq E$ be a perfect matching of weight at least r_α. By the construction of w, E' must contain at least r_α edges each of which has weight one. Then, we construct an r_α-biased $(1, 1)$-dissolution $(D, z, z_\alpha, R_\alpha)$ as follows. For each edge $\{x, y\} \in E'$, arbitrarily add one of its endpoints, say x, to D and set $z(x, y) := 1$. Further, if $\alpha(x) = 1$, then set $z_\alpha(x, y) := 1$. If $w(\{x, y\}) = 1$ meaning that the district corresponding to x and y have an A-supporter each, then add y to R_α since y wins after the dissolution. Finally, $|R_\alpha| \geq r_\alpha$ since $|E'| \geq r_\alpha$. $\qquad\square$

As we have already seen from Lemma 4, the edge set used by a $(1, 1)$-dissolution is a perfect matching. This was useful to find polynomial-time algorithm solving BIASED DISSOLUTION, since even maximum weighted perfect matchings can be computed very efficiently. Can we find similar useful characterizations for (s, s)-dissolutions with $s > 1$?
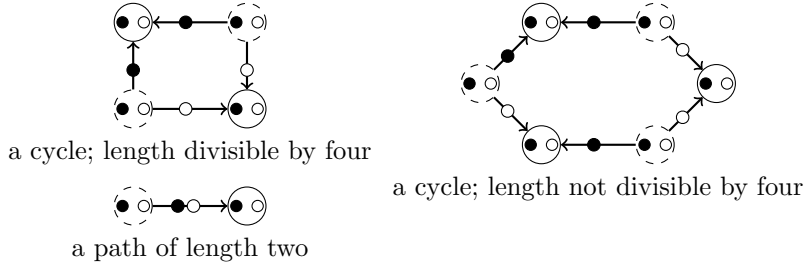
a cycle; length divisible by four

a cycle; length not divisible by four

a path of length two

Figure 6: Graphs induced by edge sets used by an r_α-biased $(2,2)$-dissolution. In order to have a majority of A-supporters (black dots) in at least half of the new districts, each component must be a cycle whose length is divisible by four.

Already for $(2,2)$-dissolutions, a characterization by the edge set used is not as compact as for $(1,1)$-dissolutions: The edge set used by a $(2,2)$-dissolution for some graph G corresponds to a partition of the graph into disjoint cycles of even length and disjoint paths of length two. For the case of r_α-biased $(2,2)$-dissolution one would at least need some weights and it is not clear how to find such a partition efficiently. However, by appropriately setting α and r_α we can enforce that the edge set used by any r_α-biased $(2,2)$-dissolution only induces cycles of lengths divisible by four: We let each district have one A-supporter and one B-supporter (i.e. $\alpha : V \to \{1\}$ for each district v) and let $r_\alpha := |V(G)|/4$. Doing this we end up with a restricted two-factor problem which is already studied in the literature [14].

A two-factor of a graph $G = (V, E)$ is a subset of edges $E' \subseteq E$ such that each vertex in $G' := (V, E')$ has degree exactly two, that is, G' only consists of disjoint cycles.

Lemma 7. *Let $G = (V, E)$ be an undirected graph with $4q$ vertices ($q \in \mathbb{N}$). Then G has a two-factor E' whose cycle lengths are all multiples of four if and only if (G, α) admits a q-biased $(2,2)$-dissolution where $\alpha(v) = 1$ for all $v \in V$.*

Proof. "$\Rightarrow$": Let $E' \subseteq E$ be an edge subset such that each vertex in $G' := (V, E')$ has degree two and G' consists of disjoint cycles of lengths divisible by four. We now construct a q-biased $(2,2)$-dissolution $(D, z, z_\alpha, R_\alpha)$ for (G, α). To this end, we start with $D := \emptyset$, $R_\alpha = \emptyset$, and we do the following for each cycle $c_1 c_2 \ldots_{4l} c_1, l \geq 1$. For each number i with $1 \leq i \leq 2l$, add c_{2i} to D, set $z(c_{2i}, c_{2i-1}) := z(c_{2i}, c_{(2i+1) \mod 4l}) := 1$. For each $1 \leq i \leq l$, we set $z_\alpha(c_{4i-2}, c_{4i-3}) := 1$, $z_\alpha(c_{4i-2}, c_{4i-1}) := 0$, $z_\alpha(c_{4i}, c_{(4i+1) \mod 4l}) := 1$, and $z_\alpha(c_{4i}, c_{4i-1}) := 0$. It is easy to verify that $(D, z, z_\alpha, R_\alpha)$ is indeed a q-biased $(2,2)$-dissolution.

"$\Leftarrow$": Let $(D, z, z_\alpha, R_\alpha)$ be a q-biased $(2,2)$-dissolution for (G, α). Furthermore, let E_z denote the edge set used by $(D, z, z_\alpha, R_\alpha)$. Each component C in $G[E_z]$ is either a path of length two or a cycle of even length and consists of exactly $|V(C)|/2$ dissolved and $|V(C)|/2$ non-dissolved districts. Since each

non-dissolved district needs at least two A-supporters in order to win and only $|V(C)|/2$ A-supporters can be moved from the $|V(C)|/2$ dissolved districts, at most $|V(C)|/4$ districts can win. With $r_\alpha = q$, this implies that in total exactly q districts must win. This can only succeed if each component C is a cycle of length which is divisible by four (also see Figure 6 for an illustration). □

Now, we are ready to show that BIASED DISSOLUTION is NP-complete even for constant values of s and Δ_s, except if $s = \Delta_s = 1$ where it is solvable in polynomial time.

Theorem 3. BIASED DISSOLUTION *on graphs* $G = (V, E)$ *can be solved in* $O(|V|(|E| + |V|\log|V|))$ *time if* $s = \Delta_s = 1$; *otherwise it is NP-complete for any constant value* $s = \Delta_s \geq 2$.

Proof. For $s = \Delta_s = 1$, the problem reduces to computing a maximum weight perfect matching (see Lemma 6). This can be done in $O(|V|(|E| + |V|\log|V|))$ time [12].

It is easy to see that BIASED DISSOLUTION is in NP for $s = \Delta_s \geq 2$. Now we show the NP-hardness. For $s = \Delta_s = 2$, observe that Lemma 7 implicitly gives a polynomial-time reduction from the L-RESTRICTED TWO FACTOR problem to BIASED DISSOLUTION where $L \subseteq \{3, \ldots, |V|\}$.

L-RESTRICTED TWO FACTOR
Input: A graph $G = (V, E)$.
Question: Is there a two factor $E' \subseteq E$ such that the number of vertices in each component in (V, E') belongs to L?

Two-factors of graphs are computable in polynomial time [10]. However, L-RESTRICTED TWO FACTOR is NP-hard if $(\{3, 4, \ldots, |V|\} \setminus L) \nsubseteq \{3, 4\}$ [14]. By Lemma 7, $(G = (V, E), L)$ with $|V| = 4q$ and $L = \{4, 8, \ldots, q\}$ is a yes-instance of L-RESTRICTED TWO FACTOR if and only if $(G, 2, 2, q, \alpha)$ with $\alpha(v) = 1$ for all $v \in V$ is a yes-instance of BIASED DISSOLUTION. Since $(\{3, 4, \ldots, |V|\} \setminus \{4, 8, \ldots, 4q\}) \nsubseteq \{3, 4\}$ for all $q > 1$, it follows that BIASED DISSOLUTION is NP-hard when $s = \Delta_s = 2$.

For $s = \Delta_s \geq 3$, we show NP-hardness by a polynomial-time reduction from the NP-complete EXACT COVER BY t-SETS problem for $t \geq 3$ (see the corresponding definition in Section 4.2). Given an EXACT COVER BY t-SETS instance $(X, \mathcal{C})$ with $|X| = t \cdot q$ elements and $m := |\mathcal{C}|$ we construct a BIASED DISSOLUTION instance $(G = (V, E), t, t, r_\alpha, \alpha)$.

To construct graph G, we use the so-called t-*elements gadgets*. An t-elements gadget consists of a t-star where each leaf has an additional degree-one neighbor. We call the degree-t vertex *center district*, the original star leaves *inner districts*, and the additional degree one vertices *element districts*. A 3-element gadget is illustrated on the left hand side in Figure 7. Now, we add to the graph G the following: i) q t-elements gadgets (we arbitrarily identify each element $x \in X$ with exactly one of the $(q \cdot t)$ *element districts*; denoted as v_x in the following), ii) a *set district* v_Y, for each subset $Y \in \mathcal{C}$, and iii) $m - q$ *dummy districts*.
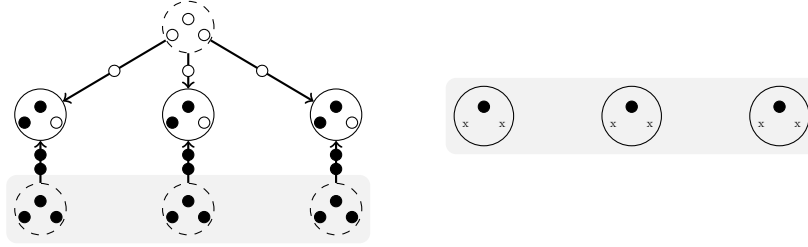
Figure 7: Left: A 3-elements gadget. The only dissolution where A wins all districts requires to dissolve the top district and move exactly one B-supporter from the top district to each neighbor. Right: Gadget symbol in the construction.
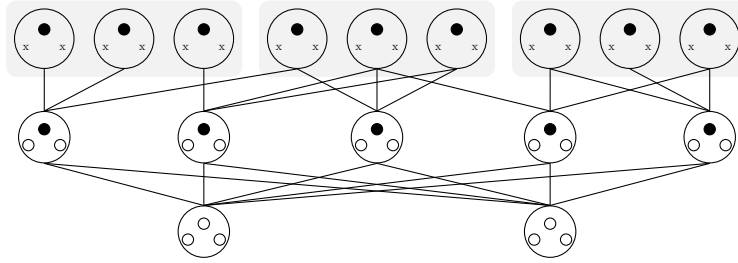

Figure 8: Illustration of the construction for $t = 3$, $m = 4$ and $q = 3$.

Then, we connect each set district v_Y with each element district $v_x, x \in Y$ and connect each dummy district with each set district. We set the number r_α of winning districts to $(t + 1) \cdot q$.

We now describe how many A-supporters each district contains (i.e. the function α).

- The dummy district contains no A-supporters.

- Each set district contains exactly one A-supporter.

- For each t-elements gadget, the center district contains no A-supporters, each inner district contains exactly two A-supporters, and each element district contains t A-supporters.

This construction is illustrated for $t = 3$ in Figure 8.

Now, we show that $(X, \mathcal{C})$ is a yes-instance of EXACT COVER BY t-SETS if and only if the constructed BIASED DISSOLUTION instance $(G, t, t, (t + 1)q, \alpha)$ is a yes-instance.

"⇒": Let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcollection such that each element of X is contained in exactly one subset in $\mathcal{C}'$. A $(t+1)q$-biased (t, t)-dissolution can be constructed as follows. Dissolve each center district and move one B-supporter to each of its adjacent inner districts. Dissolve each element district and move $(t - 1)$ A-supporters to its uniquely determined adjacent inner district. For each element

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district $v_x, x \in X$ move the remaining one A-supporter to the set district $v_Y, Y \in \mathcal{C}'$ with $x \in Y$. Since $\mathcal{C}'$ partitions X, v_Y is uniquely determined. The set R_α of winning districts consists of all inner districts and the set districts which correspond to the sets in $\mathcal{C}'$. For each dummy district v_{dummy}, uniquely choose one of the set districts $v_Y, Y \notin \mathcal{C}'$, and move all voters from v_{dummy} to v_Y. This is possible because there are $m - q$ dummy districts and $m - q$ set districts $v_Y, Y \notin \mathcal{C}'$, and each dummy district is adjacent to each set district.

To show that this indeed gives a $(t+1)q$-biased (t, t)-dissolution observe that we move all t voters from each dissolved district to the adjacent non-dissolved districts. Each inner district receives $\Delta_s = t$ voters: $t - 1$ A-supporters and one B-supporter. Since each inner district initially contained two A-supporters, party A wins a total of $t \cdot q$ inner districts. Each set district $v_Y, Y \in \mathcal{C}'$ receives t A-supporters and initially contains one A-supporter. Furthermore, $|\mathcal{C}'| = q$, and hence, party A wins q set districts in total and loses the remaining $m - q$ set districts. Thus, we indeed constructed a $(t+1)q$-biased (t, t)-dissolution.

"$\Leftarrow$": Assume that there is some $(t+1)q$-biased (t, t)-dissolution for the constructed instance. Since $s = \Delta_s$ and G has $2t \cdot q + 2m$ districts, after the dissolution, a total of $t \cdot q + m$ districts is dissolved and party A wins at least $(t + 1)q$ districts and loses at most $m - q$ districts. Observe that the only neighbors of the dummy districts are the set districts and hence, by the construction of function α, party A cannot win any non-dissolved district that receives/contains at least one voter from a dummy district. Furthermore, since the set of the $(m - q)$ dummy districts and the set of their neighboring districts build a bipartite induced subgraph, there are $(m - q)$ non-dissolved districts which may receive/contain any voters from the dummy districts. Thus, party A loses at least $m - q$ non-dissolved districts. Since $r_\alpha = (t + 1)q$, party A loses *exactly* $m - q$ districts. In particular, each of the losing districts contains at least one voter (originally) from a dummy district. This implies that party A has to win each non-dissolved set district, element district, inner district, or center district. However, the construction of α forbids A to win a center district or to win an inner district if one moves two B-supporters to it. Thus, we dissolve each center district and move exactly one B-supporter from this center districts to each of its adjacent inner district. As a direct consequence, all element districts are to be dissolved and $t - 1$ voters are moved from each element district to its adjacent inner district such that A wins all $t \cdot q$ inner districts. There are $t \cdot q$ A-supporters left, one A-supporter from each element district. These voters are to be moved to a set of exactly q winning set districts each. Since each of these districts needs at least t A-supporters to win and have exactly t adjacent element districts, $C' := \{S \in \mathcal{C} \mid v_S \in R_\alpha\}$ partitions X. $\square$

5 Special graph classes

In this section, we discuss the complexity of BIASED DISSOLUTION on graphs from special graph classes. In a companion paper [4], we have shown that computing star partitions – and hence by Lemma 2 also problem DISSOLUTION –

remains NP-hard even on subcubic grid graphs and split graphs.

First, in Section 5.1, we consider BIASED DISSOLUTION on planar graphs. This problem restriction is interesting especially in the political districting context since the neighborhood relation between voting districts on a map is planar (except, possibly, if districts may have enclaves or exclaves). Unfortunately, we will see that DISSOLUTION and, thus, BIASED DISSOLUTION, remains NP-hard for many choices of s and Δ_s. Second, in Section 5.2, we consider the case where the neighborhood graph is a clique, that is, voters may be moved unrestrictedly between districts. Finally, in Section 5.3, we consider BIASED DISSOLUTION on graphs of bounded treewidth. This problem restriction is interesting in the context of distributed systems since computers are often interconnected using a tree, star, or bus topology. We will see that BIASED DISSOLUTION is solvable in linear time on graphs of bounded treewidth when s and Δ_s are constant.

5.1 Planar graphs

By giving a polynomial-time reduction from the following NP-hard problem, it is easy to derive NP-hardness results for DISSOLUTION.

> PERFECT PLANAR H-MATCHING
> *Input:* Planar graph $G = (V, E)$.
> *Question:* Does G contain an H-*factor* $V_1, V_2, \ldots, V_{\lceil |V|/|V(H)| \rceil}$ that partitions the vertex set V such that $G[V_i]$ is isomorphic to H for all i?

PERFECT PLANAR H-MATCHING is NP-complete for any connected outerplanar graph H with three or more vertices [3]. In particular, PERFECT PLANAR H-MATCHING is NP-complete for any H being a star of size at least three. This makes it easy to prove the following theorem:

Theorem 4. DISSOLUTION *on planar graphs is NP-complete for all $s \neq \Delta_s$ such that Δ_s divides s or s divides Δ_s. It is polynomial-time solvable for $s = \Delta_s$.*

Proof. We have already shown in Theorem 2 how to solve DISSOLUTION in polynomial time for $s = \Delta_s$. Hence, now assume that $\Delta_s \neq s$ divides s. Let $x := s/\Delta_s \geq 2$. Due to Lemma 2 and the fact that PERFECT PLANAR $K_{1,x}$-MATCHING is NP-complete [3] we can conclude that DISSOLUTION is NP-complete even on planar graphs. □

It seems to be a challenging task to transfer the dichotomy result for DISSOLUTION on general graphs (Theorem 2) to the case of planar graphs. The main problem is that the proof of Theorem 2 exploits EXACT COVER BY t-SETS to be NP-hard for all $t \geq 3$. The reduction from EXACT COVER BY t-SETS to DISSOLUTION produces a graph that, as a subgraph, contains the incidence graph of the EXACT COVER BY t-SETS instance. To obtain a reduction to DISSOLUTION on planar graphs, it is necessary to have planar incidence graphs of EXACT COVER BY t-SETS. It is, however, unknown whether this problem variant, called PLANAR EXACT COVER BY t-SETS, is NP-hard for $t \geq 4$. One might be misled to think that EXACT COVER BY t-SETS is NP-hard for $t \geq 4$

since it already is NP-hard for $t = 3$. However, the closely related problem PLANAR 3-SAT, that is, 3-SAT with planar clause-literal incidence graphs, is NP-complete, whereas PLANAR 4-SAT is polynomial-time solvable: one can show that the clause-literal incidence graph of a PLANAR 4-SAT instance allows for a matching such that each clause is matched to some literal. These literals can then be simply set to true in order to satisfy all clauses. We consider the question whether PLANAR EXACT COVER BY 4-SETS is NP-hard of independent interest.

5.2 Cliques

In case that the neighborhood graph is a clique, that is, the districts are fully connected such that voters can move from any district to any other district, the existence of an (s, Δ_s)-dissolution depends only on the number $|V|$ of districts, the district size s and the size increase Δ_s. Clearly, a DISSOLUTION instance is a yes-instance if and only if $d := |V| \cdot \Delta_s/(s + \Delta_s)$ is an integer. We now show that BIASED DISSOLUTION can likewise be solved in polynomial time if the neighborhood graph is a clique. The basic idea is to dissolve districts with a large number of A-supporters while minimizing the number of losing districts by letting the districts with the smallest number of A-supporters lose.

Theorem 5. BIASED DISSOLUTION *is solvable in* $O(|V|^2)$ *time on cliques* $(V, \binom{V}{2})$.

Proof. In fact, we show how to solve the optimization version of BIASED DISSO-LUTION, where we maximize the number r_α of winning districts. Intuitively, it appears to be a reasonable approach to dissolve districts pursuing the following two objectives: Any losing district should contain as few A-supporters as possible and any winning district should contain exactly the amount that is required to have a majority. Dissolving districts this way minimizes the number of "wasted" A-supporters. We now show that this greedy strategy is indeed optimal.

To this end, let $G = (V, \binom{V}{2})$ be a clique, let α be an A-supporter distribution over V, and let s and Δ_s be the district size and the district size increase. With G being complete, we are free to move voters from any dissolved district to any non-dissolved district. Let $\mu := \lfloor (s + \Delta_s)/2 \rfloor + 1$ be the minimum number of A-supporters required to win a district. Thus, a district with less than $(\mu - \Delta_s)$ A-supporters can never win. Denote by $\mathcal{L} := \{v \in V \mid \alpha(v) < \mu - \Delta_s\}$ the set of *non-winnable* districts.

Our first claim corresponds to the first objective above, that is, the losing districts should contain a minimal number of A-supporters.

Claim 1. *Let* $v, w \in V$ *be two districts with* $\alpha(v) \leq \alpha(w)$. *If there exists an* r_α-*biased dissolution where* v *is winning and* w *is losing, then there also exists an* r_α-*biased dissolution where* v *is losing and* w *is winning.*

To verify this, let $(D, z, z_\alpha, R_\alpha)$ be an r_α-biased dissolution. Let $v \in R_\alpha$ and $w \in V \setminus D \setminus R_\alpha$ be two districts such that $\alpha(v) \leq \alpha(w)$. Now, we simply

exchange v and w, that is, we set $R'_\alpha := R_\alpha \setminus \{v\} \cup \{w\}$ and define for all $(x, y) \in Z(D, G)$:

$$z'(x,y) := \begin{cases} z(x,w), & \text{if } y = v \\ z(x,v), & \text{if } y = w \\ z(x,y), & \text{else,} \end{cases} \quad z'_\alpha(x,y) := \begin{cases} z_\alpha(x,w), & \text{if } y = v \\ z_\alpha(x,v), & \text{if } y = w \\ z_\alpha(x,y), & \text{else.} \end{cases}$$

Since $\alpha(v) \leq \alpha(w)$, it is clear that $(D, z', z'_\alpha, R'_\alpha)$ is also a well-defined r_α-biased dissolution.

The next claim basically corresponds to the second objective, in the sense that districts with a large number of A-supporters (possibly too large, that is, more than the required μ) should be dissolved in order to distribute the voters more efficiently.

Claim 2. *Let $v, w \in V$ be two districts with $\alpha(v) \leq \alpha(w)$. Assume that there exists an r_α-biased dissolution where r_α is optimal. If v is dissolved, then the following holds:*

    (i) If w is losing, then there also exists an r_α-biased dissolution where w is dissolved and v is losing.

    (ii) If w is winning and v is winnable, that is, $v \notin \mathcal{L}$, then there exists an r_α-biased dissolution where w is dissolved and v is winning.

This claim also holds by an exchange argument similar to the one above: Let $(D, z, z_\alpha, R_\alpha)$ be an r_α-biased dissolution and let $v \in D$, $w \in V \setminus D$ be two districts such that $\alpha(v) \leq \alpha(w)$. Again, we exchange v and w by setting $D' := D \setminus \{v\} \cup \{w\}$. Since $\sum_{x \in D'} \alpha(x) \geq \sum_{x \in D} \alpha(x)$ and since we are free to move voters arbitrarily between districts, it is clear that it is always possible to find an r_α-biased dissolution such that D' is the set of dissolved districts. In particular, if v is a winnable district, then it is always possible to make v a winning district.

Using the two claims above, we now show how to compute an optimal biased dissolution. In order to find a biased dissolution with the maximum number of winning districts, we seek a dissolution which loses a minimum number of remaining districts. Thus, for each $\ell \in \{0, \ldots, r\}$, we check whether it is possible to dissolve d districts such that at most ℓ of the remaining r districts lose. To this end, assume that the districts $v_1, \ldots, v_n$ are ordered by increasing number of A-supporters, that is, $\alpha(v_1) \leq \alpha(v_2) \leq \ldots \leq \alpha(v_n)$ and let $V_\ell := \{v_1, \ldots, v_\ell\}$. Now, if there exists an $(r-\ell)$-biased dissolution, then there also exists an $(r-\ell)$-biased dissolution where the losing districts are exactly V_ℓ. This follows by repeated application of the exchange arguments of Claim 1 and Claim 2(i). Hence, given ℓ, we have to check whether there is a set $D \subseteq V \setminus V_\ell$ of d districts that can be dissolved in such a way that all non-dissolved districts in $V \setminus (V_\ell \cup D)$ win and the districts in V_ℓ lose.

First, note that in order to achieve this, all districts in $\mathcal{L} \setminus V_\ell$ have to be dissolved because they cannot win in any way. Clearly, if $|\mathcal{L} \setminus V_\ell| > d$, then it is simply not possible to lose only ℓ districts and we can immediately go

to the next iteration with $\ell := \ell + 1$. Therefore, we assume that $|\mathcal{L} \setminus V_\ell| \le d$ and let $d' := d - |\mathcal{L} \setminus V_\ell|$ be the number of additional districts to dissolve in $V \setminus (\mathcal{L} \cup V_\ell)$. By Claim 2(ii), it follows that we can assume that the d' districts with the maximum number of A-supporters are dissolved, that is, $V^{d'} := \{v_{n-d'+1}, \ldots, v_n\}$. Thus, we set $D := \mathcal{L} \setminus V_\ell \cup V^{d'}$ and check whether there are enough A-supporters in D to let all $r - \ell$ remaining districts in $V \setminus (V_\ell \cup D)$ win.

Sorting the districts by the number of A-supporters (as preprocessing) requires $O(n \log n)$ arithmetic operations. Then, for up to n values of ℓ, to check whether the remaining districts in $V \setminus (V_\ell \cup D)$ can win requires $O(n)$ arithmetic operations each. Thus, assuming constant-time arithmetics, we end up with a total running time in $O(n^2)$. □

5.3 Graphs of bounded treewidth

Yuster [22, Theorem 2.3] showed that the H-FACTOR problem is solvable in linear time on graphs of bounded treewidth when the size of H is constant. This includes the case of finding x-star partitions, that is, $(x, 1)$-dissolutions resp. $(1, x)$-dissolutions, when x is constant. We can show that the more general problem BIASED DISSOLUTION is solvable in linear time on graphs of bounded treewidth when s and Δ_s are constants. In terms of parameterized complexity theory [7, 11, 18], this shows that BIASED DISSOLUTION is fixed-parameter tractable with respect to the combined parameter (t, s, Δ_s), where t is the treewidth of the neighborhood graph.

Theorem 6. BIASED DISSOLUTION *is solvable in linear time on graphs of bounded treewidth when s and Δ_s are constant.*

To prove Theorem 6, we exploit a general result that a maximum-cardinality set satisfying a constant-size formula in monadic second-order logic for graphs can be computed in linear time on graphs of bounded treewidth [2]. The set whose size we want to maximize is the set R_α of winning districts.

Definition 3 (Monadic second-order logic for graphs)**.** A formula ϕ of the monadic second-order logic for graphs may consist of the logic operators $\vee, \wedge, \neg$, vertex variables, edge variables, set variables, quantifiers $\exists$ and $\forall$ over vertices, edges, and sets, and the predicates

i) $x \in X$ for a vertex or edge variable x and a set X,

ii) $\mathrm{inc}(e, v)$, being true if e is an edge incident to the vertex v,

iii) $\mathrm{adj}(v, w)$, being true if v and w are adjacent vertices,

iv) equality of vertex variables, edge variables, and set variables.

We point out that a constant-size formula in monadic second-order logic for a problem does not only prove the mere existence of a linear-time algorithm on graphs of bounded treewidth; the formula itself can be converted into a linear-time algorithm [6, Chapter 6].
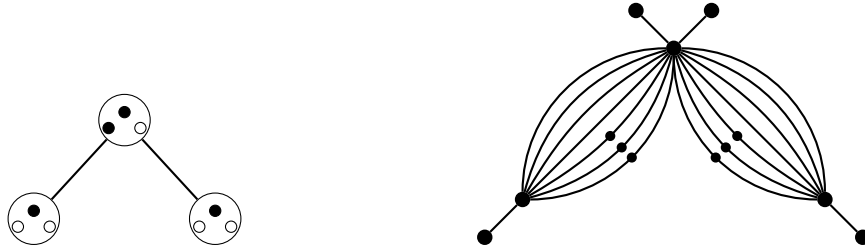
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Figure 9: Illustration of transforming a BIASED DISSOLUTION instance (left) into an instance of the auxiliary graph problem (right).

Proof of Theorem 6. We only have to model BIASED DISSOLUTION as a formula in monadic second-order logic. Since monadic second-order logic does not allow us to count the number of voters moved from one district to another or to count how many A-supporters a district contains, we first model BIASED DISSOLUTION as a problem on an auxiliary graph. For constant s and Δ_s, the transformation of a BIASED DISSOLUTION instance to this auxiliary graph can be done in linear time and works as follows:

1. For each input district of BIASED DISSOLUTION, introduce a vertex and attach to it as many degree-one vertices as the district has A-supporters.

2. Between two neighboring districts, add $s + 1$ parallel edges between their representing vertices. The $s + 1$ parallel edges represent potential moves of voters from one district to another.

3. Finally, connect each pair of vertices representing a pair of neighboring districts by s parallel subdivided edges. These represents potential moves of A-supporters.

Note that, in the graph resulting from this construction, a vertex has degree one if and only if it represents a party A-supporter; it belongs to the district represented by its neighbor. Moreover, a vertex has degree two if and only if it represents a possible movement of an A-supporter of one district to another.

A dissolution now does not contain a function z moving voters from one district to another (see Definition 1), but a set Z of selected edges representing such movements. Similarly, the A-supporter movement is no longer modelled as a function z_α (see Definition 2), but as a set of vertices Z_α representing such movements. Hence, we search for a maximum set R_α that satisfies the following formula in monadic second-order logic of graphs:

$$\max R_\alpha \text{ s.t. } \exists D \exists Z \exists Z_\alpha [\text{movements}(Z) \wedge \text{A-movements}(Z_\alpha) \\ \wedge \text{districts}(D) \wedge \text{districts}(R_\alpha) \\ \wedge a \wedge b \wedge c \wedge d \wedge e],$$

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where a, b, c, d, e will be predicates ensuring that the properties a–e of Definitions 1 and 2 are satisfied, D will be the set of dissolved districts, Z the set of voter movements and Z_α the set of A-supporter movements. To ensure this, we define

$$\text{districts}(X) := \forall x[x \in X \implies \text{degree-greater-two}(x)]$$

so that it is true if and only if all objects in X are districts, that is, vertices with degree more than two, where

$$\text{degree-greater-two}(x) := \exists v_1 \exists v_2 \exists v_3[\, v_1 \neq v_2 \wedge v_1 \neq v_3 \wedge v_2 \neq v_3$$
$$\wedge \text{adj}(v_1, x) \wedge \text{adj}(v_2, x) \wedge \text{adj}(v_3, x)]$$

is true if and only if x has at least three neighbors. Moreover, we define

$$\text{movements}(X) := \forall x[\, x \in X \implies \exists v_1 \exists v_2[\text{inc}(x, v_1) \wedge \text{inc}(x, v_2)$$
$$\wedge \text{degree-greater-two}(v_1) \wedge v_1 \in D$$
$$\wedge \text{degree-greater-two}(v_2) \wedge v_2 \notin D]]$$

so that it is true if and only if each object in the set X is a movement and

$$\text{A-movements}(X) := \forall x[\, x \in X \implies \exists v_1 \exists v_2[\text{adj}(x, v_1) \wedge \text{adj}(x, v_2)$$
$$\wedge v_1 \in D \wedge v_2 \notin D \wedge \neg\text{degree-greater-two}(x)]]$$

so that it is true if and only if each object in the set X is a movement of a A-supporter. It remains to give proper definitions of the predicates a, b, c, d, and e. We define

$$a := \forall x[x \in D \implies \exists Z'[\text{card}_s(Z') \wedge (\forall y[y \in Z' \iff \text{move-from}(x, y)])]]$$

so that it is true if and only if there is a set with cardinality s of movements out of each dissolved district x, where

$$\text{move-from}(x, y) := x \in D \wedge y \in Z \wedge \text{inc}(y, x)$$

is true if and only if y is a move out of x and

$$\text{card}_i(X) := \exists x_1 \exists x_2 \ldots \exists x_i \left[\left(\bigwedge_{j=1}^{i} x_i \in X \right) \wedge \left(\bigwedge_{j=1}^{i} \bigwedge_{k=j+1}^{i} (x_i \neq x_j) \right) \right.$$
$$\left. \wedge \forall x \left[x \in X \implies \bigvee_{j=1}^{i} x_i = x \right] \right]$$

for $1 \leq i \leq s$ is a constant size formula that is true if and only if the set X has cardinality i. Next, we define

$$b := \forall x[\, (\text{degree-greater-two}(x) \wedge x \notin D) \implies$$
$$\exists Z'[\text{card}_{\Delta_s}(Z') \wedge (\forall y[y \in Z' \iff \text{move-to}(x, y)])]]$$

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so that it is true if and only if there is a set Z' of moves to each non-dissolved district x with cardinality Δ_s, where

$$\text{move-to}(x, y) := x \notin D \wedge y \in Z \wedge \text{inc}(y, x)$$

is true if and only if y is a move to x. Next, we define

$$
\begin{aligned}
c := \forall x \forall y [x \in D \implies \exists Z' \exists Z'_\alpha [&\text{smaller}(Z'_\alpha, Z') \\
&\wedge (\forall z[z \in Z' \iff \text{move-from}(x, z) \\
&\qquad\qquad\qquad\quad \wedge \text{move-to}(y, z)]) \\
&\wedge (\forall z[z \in Z'_\alpha \iff \text{A-move-from}(x, z) \\
&\qquad\qquad\qquad\quad \wedge \text{A-move-to}(y, z)])]]
\end{aligned}
$$

so that it is true if and only if the A-supporters moved from x to y are at most the number of total moves from x to y, where

$$\text{smaller}(X, Y) := \bigvee_{i=1}^{s} \bigvee_{j=i+1}^{s} (\text{card}_i(X) \wedge \text{card}_j(Y))$$

is a constant-size formula that is true if and only if $|X| \leq |Y|$ and

$$
\begin{aligned}
\text{A-move-from}(x, y) &:= x \in D \wedge y \in Z_\alpha \wedge \text{adj}(x, y) \\
\text{A-move-to}(x, y) &:= x \notin D \wedge y \in Z_\alpha \wedge \text{adj}(x, y)
\end{aligned}
$$

are true if and only if y is an A-supporter move from or to x, respectively. Next, we define

$$
\begin{aligned}
d := \forall x [x \in D \implies \exists Z'_\alpha \exists A [&\text{equal-card}(Z'_\alpha, A) \\
&\wedge \forall v[v \in A \iff \text{A-supporter-of}(x, v)] \\
&\wedge \forall v[v \in Z'_\alpha \iff \text{A-move-from}(x, v)]]]
\end{aligned}
$$

so that it is true if and only if the number of A-supporter moves out of a district v equals the number of its A-supporters, where

$$\text{equal-card}(X, Y) := \bigvee_{i=1}^{s} (\text{card}_i(X) \wedge \text{card}_i(Y))$$

is a constant-size formula that is true if and only if $|X| = |Y|$ and

$$\text{A-supporter-of}(x, y) := \text{adj}(x, y) \wedge \forall v[\text{adj}(v, x) \implies v = y]$$

is true if and only if x is a A-supporter in district v. Finally, we define

$$
\begin{aligned}
e := \forall x [x \in R_\alpha \implies \exists A [&\text{card}_{>(s+\Delta_s)/2}(A) \\
&\wedge \forall v[v \in A \iff \text{A-supporter-of}(x, v) \\
&\qquad\qquad\qquad\quad \vee \text{A-move-to}(x, v)]]]
\end{aligned}
$$

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so that it is true if and only if each district in R_α has more than $(s + \Delta_s)/2$ A-supporters, where

$$\operatorname{card}_{>i}(X) := \bigvee_{j=\lfloor i \rfloor+1}^{s} \operatorname{card}_j(X).$$

$\square$

We claim that one can also prove Theorem 6 using an explicit dynamic programming algorithm that works on a so-called tree decomposition of a graph. The algorithm runs in $(\Delta_s + s)^{O(t^2)}$ time, but it is very technical and its correctness proof is very tedious, while its practical applicability still seems very limited.

6 Conclusion

We initiated a graph-theoretic combinatorial approach to concrete redistribution problems occurring in various application domains. Obviously, the two basic problems DISSOLUTION and BIASED DISSOLUTION concern highly simplified situations and will not be able to model all interesting aspects of redistribution scenarios. For instance, our constraint that before and after the dissolution all vertex loads are perfectly balanced may be too restrictive for many applications. We stress that even these simplistic problems are computationally intractable, and that more general models will lead to even harder problems. All in all, we consider our simple (and yet fairly realistic) models as a first step into a fruitful research direction, that might yield a stronger linking of graph-theoretic concepts with districting methods and other application scenarios.

We end with a few specific challenges for future research. We have left open whether the P vs NP dichotomy for general graphs fully carries over to the planar case: it might be possible that planar graphs allow for some further tractable cases with respect to the relation between old and new district sizes. Moreover, with redistricting applications in mind it might be of interest to study special cases of planar graphs (such as grid-like structures) in quest of finding polynomial-time solvable special cases of network-based dissolution problems. Having identified several NP-hard special cases of DISSOLUTION and BIASED DISSOLUTION, it is a natural endeavor to investigate their polynomial-time approximability and their parameterized complexity; in the latter case one also needs to identify new, application-driven parameterizations.

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