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Citation for published version (APA): Custic, A., Klinz, B., & Woeginger, G. J. (2014). *Geometric versions of the 3-dimensional assignment problem* under general norms. (arXiv; Vol. 1409.0845 [math.CO]). s.n.

Document status and date: Published: 01/01/2014

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.

• The final author version and the galley proof are versions of the publication after peer review.

• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

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Geometric versions of the 3-dimensional assignment problem under general norms

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Abstract

We discuss the computational complexity of special cases of the 3-dimensional (axial) assignment problem where the elements are points in a Cartesian space and where the cost coefficients are the perimeters of the corresponding triangles measured according to a certain norm. (All our results also carry over to the corresponding special cases of the 3-dimensional matching problem.)

The minimization version is NP-hard for every norm, even if the underlying Cartesian space is 2-dimensional. The maximization version is polynomially solvable, if the dimension of the Cartesian space is fixed and if the considered norm has a polyhedral unit ball. If the dimension of the Cartesian space is part of the input, the maximization version is NP-hard for every L_p norm; in particular the problem is NP-hard for the Manhattan norm L_1 and the Maximum norm L_{∞} which both have polyhedral unit balls.

Keywords: combinatorial optimization, computational complexity, 3-dimensional assignment problem, 3-dimensional matching problem, polyhedral norm.

1 Introduction

The 3-dimensional (axial) assignment problem (3AP) is an important and well-studied problem in combinatorial optimization. An instance of the 3AP consists of three sets X, Y, Z with |X| = |Y| = |Z| = n, and a cost function $c : X \times Y \times Z \to \mathbb{R}$. The goal is to find a set of n triples in $X \times Y \times Z$ that cover every element in $X \cup Y \cup Z$ exactly once, such that the sum of the costs of these triples is minimized. In the closely related maximization version max-3AP of the 3AP, this sum is to be maximized. The book [3] by Burkard, Dell'Amico & Martello contains a wealth of information on the 3AP and other assignment problems.

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A prominent special case of the 3AP is centered around some metric space (S, d)where S is a set and where d is a distance function on $S \times S$ (that hence is symmetric, non-negative, and satisfies the triangle inequality). The elements in $X \cup Y \cup Z$ are points in S, and the cost c(x, y, z) of a triple $(x, y, z) \in X \times Y \times Z$ is given by

$$c(x, y, z) = d(x, y) + d(y, z) + d(z, x).$$
(1)

Costs of this type are called *perimeter costs*; intuitively speaking, they measure the perimeter of the triangle determined by three points x, y, z in the metric space.

The 3AP is well-known to be NP-hard; see for instance Karp [8] or Garey & Johnson [7]. Spieksma & Woeginger [13] establish NP-hardness of the special case of perimeter costs (1) where the underlying metric space is the two-dimensional Euclidean plane with standard Euclidean distances. Polyakovskiy, Spieksma & Woeginger [11] show that 3AP and max-3AP with perimeter costs are polynomially solvable, if the underlying metric space satisfies the so-called Kalmanson conditions; their results cover convex Euclidean point sets and tree metric spaces as special cases. Crama & Spieksma [5] design a polynomial time approximation algorithm with worst case guarantee 4/3 for the 3AP with perimeter costs; their approach works for arbitrary metric spaces without imposing any additional structural constraints. Burkard, Rudolf & Woeginger [4] exhibit a polynomially solvable special case of the max-3AP where the costs are decomposable and products of certain parameters.

Results of this paper. We study 3AP and max-3AP with perimeter costs in Cartesian spaces under arbitrary distance functions. On the negative side, we derive NP-hardness results that contain and generalize the known results from the literature for the standard Euclidean distances. On the positive side, we derive polynomial time algorithms for certain special cases of max-3AP where the distances are defined via norms with polyhedral unit balls. Our main results are the following:

- (A) Problem max-3AP is polynomially solvable, if the dimension of the underlying Cartesian space is a fixed constant and if the underlying norm has a polyhedral unit ball.
- (B) Problem max-3AP is NP-hard, if the dimension of the underlying Cartesian space is part of the input and if the underlying norm is any fixed L_p norm. This hardness result in particular holds for the Manhattan norm L_1 and the Maximum norm L_{∞} which both have polyhedral unit balls.
- (C) Finally, the minimization problem 3AP is NP-hard for any fixed norm, even if the underlying Cartesian space is 2-dimensional.

Result (A) heavily builds on the machinery developed by Barvinok, Fekete, Johnson, Tamir, Woeginger & Woodroofe [2] for the Travelling Salesman Problem (TSP). Also the TSP is polynomially solvable, if the cities are points in some Cartesian space of fixed dimension and if the distances are defined via norms with polyhedral unit balls. While the framework for our result (A) is taken from [2], the technical details and the combinatorial features are very different and require a number of new ideas. Result (B) is done by a routine NP-hardness reduction from a closely related NP-hard graph problem. Result (C) builds on the NP-hardness reductions of Spieksma & Woeginger [13] and Pferschy, Rudolf & Woeginger [10] for Euclidean distances. In the Euclidean case, one may use Pythagorean triangles as simple building blocks to control the distances between points and to ensure rational coordinates that can be processed by a Turing machine. In the general case (C), it is much more tedious to prove the existence of the corresponding building blocks.

Organization of this paper. Section 2 summarizes some standard geometric definitions around distances, norms and unit balls. Result (A) for the max-3AP is derived in two steps. First Section 3 derives an auxiliary result on the max-3AP under so-called tunneling distances, and then Section 4 establishes that max-3AP under polyhedral norms is a special case of the tunneling case. Section 5 contains the proof of result (B). Section 6 constructs certain lattices with certain useful properties; these lattices are then used in Section 7 to prove the NP-hardness result (C). Section 8 translates our results (A), (B) and (C) into corresponding results for the maximization version and the minimization version of the 3-dimensional matching problem. Finally, Section 9 concludes the paper with a short discussion and some open questions.

2 Technical preliminaries

Let R denote a compact and convex subset of the s-dimensional Cartesian space \mathbb{R}^s that has non-empty interior and that is centrally symmetric with respect to the origin. The corresponding norm L_R with unit ball R determines for any two points $x, y \in \mathbb{R}^s$ a distance $d_R(x, y)$ in the following way. First translate the space so that one of the two points (say point x) lies in the origin. Then determine the unique scaling factor λ by which one must rescale the unit ball R (shrinking for $\lambda < 1$, expanding for $\lambda > 1$), such that the other point (point y, in our case) lies on its boundary. The distance is then given by $d_R(x, y) = \lambda$. Note that since R is centrally symmetric, it does not matter whether we choose point x or point y for the origin. See Figure 1 for an illustration.

The most popular norms for \mathbb{R}^s are the Manhattan norm, the Euclidean norm, and the Maximum norm. These three norms are special cases of the well-known L_p norm, respectively for p = 1, for p = 2, and for $p = \infty$. We recall that for $1 \leq p < \infty$, the L_p distance between two points $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$ in s-dimensional space is given by

$$d(x,y) = \left(\sum_{i=1}^{s} |x_i - y_i|^p\right)^{1/p}.$$
(2)



Figure 1: Three examples of unit balls for an L_R norm in \mathbb{R}^2

For $p = \infty$, the corresponding distance under the Maximum norm L_{∞} is given by

$$d(x,y) = \max_{i=1}^{s} |x_i - y_i|.$$
(3)

3 The maximization problem under tunneling distances

In this section we consider a variant of the max-3AP with perimeter costs that will be useful in Section 4 of the paper. The distances between the elements of $X \cup Y \cup Z$ are specified with the help of a system of $k \ge 2$ so-called *tunnels* t_1, \ldots, t_k ; we stress that throughout this section the number k of tunnels is a constant that does not depend on the input. Each tunnel acts as a bidirectional passage with a front entry and a back entry. For every element $x \in X \cup Y \cup Z$ and every tunnel t, we denote by F(x,t) the distance between x and the front entry of t and by B(x,t) the distance between x and the back entry of t. Intuitively speaking, the only way of moving from x to y is to first move from x to some tunnel, then to traverse the tunnel in either direction (either from front entry to back entry, or from back entry to front entry), and finally to move from the other end of the tunnel to y. The *tunneling distance* between two elements x and y in $X \cup Y \cup Z$ is then given by

$$d(x,y) = \max \{F(x,t_i) + B(y,t_i), \ B(x,t_i) + F(y,t_i): \ 1 \le i \le k\}.$$
(4)

(We note in passing that the lengths of the tunnels do not play any role in this formula, as these lengths can easily be encoded in the values F(x,t) and B(x,t).)

We construct an undirected, edge-labeled, bipartite multigraph G whose vertex set are the elements of $X \cup Y \cup Z$ together with the tunnels t_1, \ldots, t_k . Between any element x of $X \cup Y \cup Z$ and any tunnel t there are four edges, two of which are labeled B and have cost B(x,t), whereas the other two are labeled F and have cost F(x,t).

A six-cycle in G is a closed walk $x - t_i - y - t_j - z - t_\ell - x$ with $(x, y, z) \in X \times Y \times Z$ and three (not necessarily distinct) tunnels t_i, t_j, t_ℓ . The six-cycle is *legal*, if the labels of the two edges incident to t_i are distinct, if the labels of the two edges incident to t_j are distinct, and if the labels of the two edges incident to t_{ℓ} are distinct. We stress that we do not require tunnel t_i to be the maximizer of the expression for x and y in the right hand side of (4), nor that t_j and t_{ℓ} are the maximizers for the corresponding expressions for y and z, respectively for z and x.

A legal set C of six-cycles consists of n legal six-cycles in G that cover every vertex of $X \cup Y \cup Z$ exactly once. We define G[C] as the subgraph of G that is induced by the 6n edges in C. Then we coarsen the subgraph G[C] by anonymizing the identities of the vertices in $X \cup Y \cup Z$: every vertex in X is simply labeled X, every vertex in Y is labeled Y, and every vertex in Z is labeled Z. The resulting anonymized graph $G^*[C]$ is called an *outline* of C and G[C].

Lemma 3.1 The optimal objective value of the considered max-3AP instance coincides with the largest cost taken over all subgraphs G[C] of G with a legal set C of six-cycles.

Proof. Let C be an arbitrary legal set of six-cycles. Every six-cycle $x-t_i-y-t_j-z-t_\ell-x$ in C yields a corresponding triple (x, y, z) in $X \times Y \times Z$. The cost c(x, y, z) of triple (x, y, z) may be computed according to (4), by replacing the three tunnels t_i, t_j, t_ℓ by three other tunnels that maximize the value. Hence, the cost of the triple is an upper bound on the cost of the six-cycle, and the cost of all n corresponding triples is an upper bound on the cost of G[C]. This shows that the optimal objective max-3AP value is an upper bound on the cost of every subgraph G[C].

Next, consider a set T of n triples in $X \times Y \times Z$ that constitutes an optimal solution for the max-3AP instance. We translate every triple $(x, y, z) \in T$ into a legal six-cycle: we let t_i (respectively, t_j and t_ℓ) denote the tunnel that maximizes the expression (4) for x and y (respectively, for y and z and for z and x), and we choose the labels B and F appropriately in the obvious way. For the resulting legal set C_T , the cost of $G[C_T]$ coincides with the optimal objective max-3AP value. \Box

Lemma 3.2 Let G^* be a given outline. Then one can compute in polynomial time $O(n^3)$ the largest cost of all the induced subgraphs G[C] (with a legal set C of six-cycles), whose outline $G^*[C]$ coincides with G^* .

Proof. The problem boils down to assigning the elements of X (respectively, of Y and Z) to the n vertices in G^* that are labeled X (respectively, labeled Y and Z). The cost of assigning an element $x \in X$ to some vertex v only depends on x and on the two edges incident to v in G^* . Hence, we are dealing with a classical two-dimensional assignment problem which can be solved in polynomial time $O(n^3)$; see for instance Burkard, Dell'Amico & Martello [3]. \Box

Lemma 3.3 There exist only $O(n^{8k^3})$ distinct outlines G^* for graph G, and they can all be enumerated in polynomial time.

Proof. After anonymizing the identities of the vertices in $X \cup Y \cup Z$, a legal six-cycle is determined by the three tunnels t_i, t_j, t_ℓ and the labels of its first, third, and fifth edge.

Hence there remain only $8k^3$ combinatorially different legal six-cycles, and each of them may be used at most n times in any outline. \Box

The three lemmas suggest the following approach to max-3AP under tunneling distances: enumerate all possible outlines in polynomial time according to Lemma 3.3, and for each such outline compute the maximum possible cost of a corresponding induced subgraph according to Lemma 3.2. Return the largest cost over all outlines, which by Lemma 3.1 coincides with the optimal objective value of the max-3AP instance.

Theorem 3.4 Problem max-3AP with perimeter costs under tunneling distances can be solved within a time complexity that depends polynomially on the instance size n (and exponentially on the number k of tunnels).

Proof. The above approach computes the optimal objective value and the corresponding graphs $G^*[C]$ and G[C], but does neither yield the corresponding optimal solution $T \subset X \times Y \times Z$ for the max-3AP instance nor the underlying legal set C_T of six-cycles. We briefly sketch how these objects can also be determined in polynomial time. The set C_T can be determined in polynomial time by invoking Lenstra's algorithm [9] for integer programming in constant dimension. For each of the $8k^3$ combinatorially different legal six-cycles, we introduce a corresponding integer variable that counts the number of occurrences of this cycle in C_T . The constraints in the integer program enforce that $G[C_T]$ coincides with G[C]. And once we have found C_T through the integer program, it is straightforward to identify the optimal solution T (as outlined in the proof of Lemma 3.1). \Box

4 The maximization problem under polyhedral norms

Throughout this section, we consider the s-dimensional Cartesian space \mathbb{R}^s endowed with some fixed norm with polyhedral unit ball R. We investigate the special case of max-3AP with perimeter costs where the elements in $X \cup Y \cup Z$ are points in \mathbb{R}^s and where the distances are measured according to d_R . We stress that both the dimension s of the underlying space and the number of faces of the unit ball R are constants that do not depend on the input.

The unit ball R is a polytope with 2k faces that is centrally symmetric with respect to the origin. Then for certain vectors $h_1, \ldots, h_k \in \mathbb{R}^s$, this polytope R can be written as the intersection of a collection of half-spaces:

$$R = \left(\bigcap_{i=1}^{k} \{x : h_i \cdot x \le 1\}\right) \cap \left(\bigcap_{i=1}^{k} \{x : h_i \cdot x \ge -1\}\right)$$
(5)

As an example, for the Manhattan norm in \mathbb{R}^2 the corresponding vectors are $h_1 = (1, 1)$ and $h_2 = (-1, 1)$, and for the Maximum norm in \mathbb{R}^2 the corresponding vectors are $h_1 = (1,0)$ and $h_2 = (0,1)$. The distance $d_R(x,y)$ between two points $x, y \in \mathbb{R}^s$ may then be written as

$$d_{R}(x,y) = \max \{ |h_{i} \cdot (x-y)| : 1 \le i \le s \}$$

=
$$\max \{ h_{i} \cdot (x-y), h_{i} \cdot (y-x) : 1 \le i \le s \}$$

=
$$\max \{ h_{i} \cdot x - h_{i} \cdot y, -h_{i} \cdot x + h_{i} \cdot y : 1 \le i \le s \}$$
 (6)

We model a max-3AP instance under a polyhedral norm as a special instance of max-3AP under tunneling distances as discussed in Section 3. The k vectors h_1, \ldots, h_k serve as tunnels, and we set $F(x, h_i) = x \cdot h_i$ and $B(x, h_i) = -x \cdot h_i$. With this choice, the polyhedral distance $d_R(x, y)$ between two points x and y in $X \cup Y \cup Z$ in (6) coincides with the tunneling distance given in (4). Hence Theorem 3.4 yields the following.

Theorem 4.1 For any polyhedral norm L_R with unit ball R in s-dimensional space \mathbb{R}^s , problem max-3AP with perimeter costs measured according to L_R can be solved within a time complexity that depends polynomially on the instance size n (and exponentially on the number k of facets of the polyhedral unit ball). \Box

Theorem 4.1 also implies the existence of a polynomial time approximation scheme (PTAS) for max-3AP under any arbitrary norm with a not necessarily polyhedral unit ball R. One simply approximates the unit ball R by a polyhedral unit ball. Since the dimension s of the underlying space and the ball R are fixed, one may choose a fixed polyhedral approximation of the ball that approximates the distances between any two points within a factor $1 \pm \varepsilon$. (This trick of approximating the unit ball by a polyhedral unit ball is essentially due to Barvinok [1] who applied it to the maximum Travelling Salesman Problem.)

Theorem 4.2 For any fixed (not necessarily polyhedral) norm L_R with unit ball R in s-dimensional space \mathbb{R}^s , problem max-3AP with perimeter costs measured according to L_R possesses a PTAS. \Box

5 The maximization problem in non-fixed dimension

The polynomial time results for max-3AP in the preceding section assumed that the dimension s of the underlying Cartesian space \mathbb{R}^s as well as the number of faces of the underlying unit ball are constants that do not depend on the input. In this section we discuss problem max-3AP with perimeter costs measured according to a standard L_p norm (with $1 \leq p \leq \infty$) when the dimension s is not fixed, but part of the input. Our reductions are from the following variant of Partition into Triangles.

Problem: Partition into Triangles (PIT)

Instance: A 6-regular, tripartite graph G = (V, E) with tripartition $V = V_1 \cup V_2 \cup V_3$, where $|V_1| = |V_2| = |V_3| = q$.

Question: Does there exist a set T of q triples in $V_1 \times V_2 \times V_3$ such that every vertex in V occurs in exactly one triple and such that every triple induces a triangle in G?

We have not been able to locate an NP-hardness proof of PIT on 6-regular tripartite graphs in the literature (though we strongly expect that this result has been observed before). For instance Van Rooij, Van Kooten Niekerk & Bodlaender [14] establish NPhardness for 4-regular graphs, but their graphs are not tripartite.

Proposition 5.1 Problem PIT on 6-regular tripartite graphs is NP-complete.

Proof. The argument is routine, and we only sketch the main ideas. The NP-hardness proof on pages 68 and 69 of Garey & Johnson [7] for Partition into Triangles is a reduction from the Exact Cover By 3-Sets problem. We perform essentially the same reduction, but start it from another NP-hard feasibility version of the 3-dimensional assignment problem with bounded occurrence of elements (Instance: three sets X, Y, Z with |X| = |Y| = |Z| = q, and a set $T \subseteq X \times Y \times Z$ of triples such that every element of $X \cup Y \cup Z$ occurs in at most three triples of T. Question: Does there exist a subset T^* of q triples in T such that each element of $X \cup Y \cup Z$ is contained in precisely one triple of T^* ?). Then the resulting graph G is tripartite and all vertex degrees lie in $\{3, 4, 5, 6\}$.

Hence it remains to make the graph 6-regular. This can be reached by various gadget constructions. We sketch a particularly simple approach that increases the minimum degree of G by 1, while keeping the maximum degree unchanged. Take the graph G = (V, E), and construct a copy G' = (V', E') of it (so that for every $v \in V$ there is a corresponding copy $v' \in V'$, and there is an edge $[u, v] \in E$ if and only if there is an edge $[u', v'] \in E'$). Define a new graph on the vertex set $V \cup V'$, and all edges in $E \cup E'$, and furthermore an additional edge between v and v' whenever vertex v has degree in $\{3, 4, 5\}$. The new graph is still tripartite, and it has a partition into triangles if and only if the old graph allows a partition into triangles (note that the additional edges [v, v'] do not occur in any triangle, and hence are irrelevant for partitions into triangles). If we repeat this construction two more times, the resulting graph will be 6-regular and tripartite. \Box

The following two lemmas establish NP-hardness of max-3AP with perimeter costs for all values p with $1 \le p \le \infty$.

Lemma 5.2 For any fixed p with $1 \le p < \infty$, problem max-3AP with perimeter costs measured according to the L_p norm is NP-hard.

Proof. We consider an arbitrary instance G = (V, E) of PIT with |V| = 3q, and we construct the following instance of max-3AP with perimeter costs from it. For every

vertex v in part V_1 (respectively, part V_2 and part V_3), we create a corresponding point P(v) that belongs to the set X (respectively, set Y and set Z). We choose the dimension $s = \binom{3q}{2}$, and we make every coordinate correspond to one 2-element set of vertices in V. The coordinate of point P(v) corresponding to some set $\{u, w\}$ with $u, w \in V$ is chosen as follows: If $v \in \{u, w\}$ and [u, w] is not an edge in E, then the coordinate has value 1; in all other cases the coordinate has value 0.

Since G is 6-regular, every vertex v has exactly 3q - 7 non-neighbors and hence every point P(v) has exactly 3q - 7 coordinates with value 1 (and all other coordinates at 0). Furthermore, if $[u, v] \in E$ then the L_p distance between P(u) and P(v) equals $\ell^* := \sqrt[p]{6q - 14}$, and if $[u, v] \notin E$ then their L_p distance equals $\sqrt[p]{6q - 16}$. In other words, non-edges correspond to short distances and edges correspond to long distances. It can be seen that the PIT instance has answer YES, if and only if the constructed max-3AP instance has a feasible solution with objective value at least $3q \cdot \ell^*$. \Box

Lemma 5.3 Problem max-3AP with perimeter costs measured according to the Maximum norm L_{∞} is NP-hard.

Proof. The argument is very similar to the argument in Lemma 5.2. Again we start from an arbitrary instance G = (V, E) of PIT, and we create for every vertex v in $V_1 \cup V_2 \cup V_3$ a corresponding point P(v). We choose the dimension s = |E|, and we make every coordinate correspond to one edge in E. For an edge $e = [u, v] \in E$, the coordinates corresponding to e are 0 for all points with the exception of points P(u) and P(v); one of P(u) and P(v) receives coordinate +1 and the other one receives coordinate -1.

Then non-edges correspond to short distances 1 and edges correspond to long distances $\ell^* := 2$. It can be seen that the PIT instance has answer YES, if and only if the constructed max-3AP instance has a feasible solution with objective value at least 6q.

6 A useful lattice

In this section, we derive a technical result that will be central in our NP-hardness reduction in Section 7; since this reduction should be implementable on a standard Turing machine, we want to have all involved numbers to be rational or integer (so that they can be represented by simple finite strings). Throughout this section we consider a fixed norm L_R with a fixed unit ball R in the Cartesian plane \mathbb{R}^2 .

Theorem 6.1 For any norm L_R with unit ball R in the Cartesian plane \mathbb{R}^2 , there exist two integer vectors v_1 and v_2 , such that the lattice generated by v_1 and v_2 has the following properties.

(i) The fundamental triangle of the lattice with vertices in 0, in v_1 and in v_2 has a certain perimeter Δ (measured in the L_R norm).

(ii) Any three (distinct) points q_1, q_2, q_3 in the lattice either form a fundamental triangle, or otherwise form a triangle with perimeter at least $\Delta + 1$ (measured in the L_R norm).

The rest of this section is entirely devoted to the proof of Theorem 6.1. We start by introducing five points p_0, p_1, p_2, p_3, p_4 whose definition is based on a positive integer α ; the value of α will be fixed after the proof of Lemma 6.2. The Cartesian coordinates of the first four points are given by $p_0 = (0,0), p_1 = (\alpha,0), p_2 = (2\alpha,0), \text{ and } p_3 = (3\alpha,0).$ These points lie on the x-axis, and we assume without loss of generality that the L_R distance between them is given by $d_R(p_i, p_j) = (i - j) \alpha$ for all i, j with $0 \le i \le j \le 3$. The final point p_4 is chosen in the region above the x-axis so that its distances from p_0, p_1, p_2, p_3 satisfy the following inequalities:

$$\alpha < d_R(p_1, p_4), \ d_R(p_2, p_4) \le \frac{4}{3}\alpha < d_R(p_0, p_4), \ d_R(p_3, p_4)$$
 (7)

See Figure 2 for an illustration.



Figure 2: The five points p_0, p_1, p_2, p_3, p_4 with the fundamental triangle $p_1 p_2 p_4$.

Lemma 6.2 For any integer $\alpha > 0$, there exists a point p_4 that satisfies the inequalities in (7) and that furthermore has rational coordinates.

Proof. We let S denote the set of all points s in the upper halfplane that satisfy $d_R(p_1, s) = d_R(p_2, s) = 4\alpha/3$. Then set S is the intersection of the boundary of two copies of the unit ball R that are scaled by the factor 4/3 and that are centered in points p_1 and p_2 , respectively. Since R is convex and compact, set S either consists of a single point or otherwise is a horizontal line segment. We claim that S contains some point s^* that simultaneously satisfies

$$d_R(p_0, s^*) > 4\alpha/3$$
 and $d_R(p_3, s^*) > 4\alpha/3$. (8)

First consider the case where S consists of a single point $s = (\beta, \gamma)$. Then the horizontal line ℓ through this point s contains two points that are at L_R distance $4\alpha/3$ from point p_1 : the point s and some other point that is farther to the left of s. (In a degenerate case, the line ℓ contains an entire interval of points whose L_R distance to p_1 equals $4\alpha/3$; in this case point s forms the right endpoint of the interval.) If we traverse the points on line ℓ from left to right, their distances to point p_1 will follow a convex function; in particular for every point strictly to the right of s the L_R distance to p_1 will be strictly larger than $d_R(p_1, s)$. Since the auxiliary point $s' = (\beta + \alpha, \gamma)$ lies strictly to the right of s on line ℓ , we conclude

$$d_R(p_1, s') > d_R(p_1, s) = 4\alpha/3.$$

Since the line segment p_0s results by shifting line segment p_1s' a distance α to the left, we derive the desired inequality $d_R(p_0, s) > 4\alpha/3$. A symmetric argument yields $d_R(p_3, s) > 4\alpha/3$. Summarizing, the point $s^* = s$ satisfies the inequalities in (8).

Next consider the case where S is a horizontal line segment between a left endpoint s_1 and a right endpoint s_2 . Then the horizontal line ℓ through S contains an interval of points whose L_R distance to p_1 equals $4\alpha/3$, and another interval of points whose L_R distance to p_2 equals $4\alpha/3$; the line segment S is the intersection of these two intervals. The arguments in the preceding paragraph yield the two inequalities

$$d_R(p_0, s_2) > 4\alpha/3$$
 and $d_R(p_3, s_1) > 4\alpha/3.$ (9)

Now let S_0 denote the set of all points $s \in S$ with $d_R(p_0, s) \leq 4\alpha/3$, and let S_3 denote the set of all points $s \in S$ with $d_R(p_3, s) \leq 4\alpha/3$. The convexity and the compactness of the unit ball R imply that S_0 and S_3 are closed intervals. Furthermore (9) implies $S_0 \neq S$ and $S_3 \neq S$. Now suppose for the sake of contradiction that $S = S_0 \cup S_3$. Then the intersection $S_0 \cap S_3$ is non-empty and contains a point t. But then the triangle p_0p_3t has one side p_0p_3 of length 3α and two sides of length at most $4\alpha/3$. This is the desired contradiction to the triangle inequality. We conclude that S contains a point s^* that is neither in S_0 nor in S_3 , and this point s^* by definition satisfies the desired inequalities (8).

To summarize, we have found a point $s^* \in S$ that satisfies $d_R(p_1, s) = d_R(p_2, s) = 4\alpha/3$ and (8). If s^* has rational coordinates, we are done. Otherwise, we consider a sufficiently small open neighborhood $N(s^*)$ of s^* whose points satisfy (8). Then the intersection of $N(s^*)$ with the halfplane below S has non-empty interior, and we can find the desired point with rational coordinates in it. \Box

Our lattice will have the triangle $p_1p_2p_4$ as fundamental triangle. Without loss of generality we assume from now on that the sides of this triangle satisfy

$$\alpha = d_R(p_1, p_2) < d_R(p_1, p_4) \le d_R(p_2, p_4).$$
(10)

Indeed, the first inequality follows from $d_R(p_1, p_2) = \alpha$ and (7), while the second inequality may be assumed by symmetry. We now fix the value of α so that p_4 has integer coordinates, and so that

$$d_R(p_2, p_4) + 1 \leq \min \{ d_R(p_0, p_4), \ d_R(p_3, p_4), \ 2\alpha \}$$
(11)

and that

$$\alpha + 1 \leq d_R(p_2, p_4). \tag{12}$$

The first one of these conditions can be reached by making α a multiple of the denominators of the *x*-coordinate and *y*-coordinate of point p_4 (which are rational by Lemma 6.3). The other conditions can be reached by choosing α sufficiently large so that (11) and (12) are implied by (7). In particular, we will assume from now on that $\alpha \geq 3$.



Figure 3: An illustration for the five cases in the proof of Lemma 6.3.

Lemma 6.3 Let p and q be two points in the lattice with fundamental triangle $p_1p_2p_4$. If p and q do not both belong to the same fundamental triangle, then $d_R(p,q) \ge d_R(p_2,p_4) + 1$.

Proof. Without loss of generality we assume that the y-coordinate of point q is at least as large as the y-coordinate of point p. We consider the horizontal line through point p together with the four lines through point p that are, respectively, parallel to the four line segments p_0p_4 , p_1p_4 , p_2p_4 , and p_3p_4 . These five lines partition the region above point p into five wedges; see Figure 3 for an illustration.

Let us first deal with the easy cases where point q lies on one of the five lines. If q lies on the horizontal line then $d_R(p,q) \ge 2\alpha$; if q lies on the line parallel to p_1p_4 then $d_R(p,q) \ge 2d_R(p_1,p_4)$; and if q lies on the line parallel to p_2p_4 then $d_R(p,q) \ge 2d_R(p_2,p_4)$. In each of these three cases, the desired inequality follows from (10) and (11). Similarly, if q lies on the line parallel to p_0p_4 then $d_R(p,q) \ge d_R(p_0,p_4)$, and if q lies on the line parallel to p_3p_4 then $d_R(p,q) \ge d_R(p_3,p_4)$. In these two cases the desired inequality follows directly from (11).

In the main part of the proof we distinguish five cases where point q lies in the interior of one of the five wedges. In the first case, assume that point q lies in the

leftmost wedge (like point q_1 in Figure 3). Draw a line through $q = q_1$ that is parallel to p_2p_4 , and consider its intersection point r with the upper bounding line of the wedge. In the triangle pqr, the side length $d_R(q,r)$ equals λ times $d_R(p_2, p_4)$ and the side length $d_R(p,r)$ equals μ times $d_R(p_3, p_4)$ where λ and μ are positive integers with $\lambda < \mu$. Now the triangle inequality together with (11) yields

$$d_R(p,q) \geq d_R(p,r) - d_R(q,r) = \mu \cdot d_R(p_3,p_4) - \lambda \cdot d_R(p_2,p_4)$$

= $\lambda (d_R(p_3,p_4) - d_R(p_2,p_4)) + (\mu - \lambda) d_R(p_3,p_4)$
 $\geq d_R(p_3,p_4) \geq d_R(p_2,p_4) + 1.$

This completes the discussion of the first case. The second, fourth, and fifth case can be handled analogously, and we only list the crucial inequalities for them. In the second case (where point q lies in the same wedge as point q_2 in Figure 3), we have

$$d_R(p,q) \geq \mu \cdot d_R(p_4,p_3) - \lambda \cdot d_R(p_2,p_3) \geq d_R(p_4,p_3).$$

In the fourth case (where point q lies in the same wedge as point q_4 in Figure 3),

$$d_R(p,q) \geq \mu \cdot d_R(p_4,p_0) - \lambda \cdot d_R(p_1,p_0) \geq d_R(p_4,p_0).$$

In the fifth case (where point q lies in the same wedge as point q_5 in Figure 3),

$$d_R(p,q) \geq \mu \cdot d_R(p_0,p_4) - \lambda \cdot d_R(p_1,p_4) \geq d_R(p_0,p_4).$$

In each of the above three cases, (11) leads to the desired inequality. It remains to consider the third case (where point q lies in the same wedge as point q_3 in Figure 3). In this case we derive

$$d_R(p,q) \geq \mu \cdot d_R(p_4,p_2) - \lambda \cdot d_R(p_3,p_2) = \lambda \left(d_R(p_4,p_2) - d_R(p_3,p_2) \right) + (\mu - \lambda) d_R(p_4,p_2).$$

Now $\lambda \geq 1$ and $\mu - \lambda \geq 1$ together with $d_R(p_4, p_2) - d_R(p_3, p_2) \geq 1$ in (12) yield the desired inequality. As all five cases have been settled, the proof of the lemma is complete. \Box

Now let us wrap things up. Let Δ denote the L_R perimeter of the fundamental triangle $p_1p_2p_4$. Note that (7) and (10) imply the bounds $3\alpha < \Delta \leq 11\alpha/3$. By Lemma 6.3 and by (7), the three shortest distances between (distinct) lattice points are the three side lengths $d_R(p_1, p_2)$ and $d_R(p_1, p_4)$ and $d_R(p_2, p_4)$ of the fundamental triangle. All other distances are at least $d_R(p_2, p_4) + 1$, that is, the longest side of the fundamental triangle plus 1.

Suppose for the sake of contradiction that for some non-fundamental triangle $r_1r_2r_3$ the L_R perimeter would be strictly smaller than $\Delta + 1$. Since this triangle is nonfundamental, by Lemma 6.3 one of its side lengths is at least $d_R(p_2, p_4) + 1$. Hence by (10) and by the above discussion, its two other side lengths must both be equal to $d_R(p_1, p_2) = \alpha$. But then the triangle $r_1 r_2 r_3$ is necessarily degenerate, with all three points on a line and with L_R perimeter $\alpha + \alpha + 2\alpha = 4\alpha$. Now $\alpha \ge 3$ implies the desired contradiction $4\alpha \ge 11\alpha/3 + 1 \ge \Delta + 1$. This finally completes the proof of Theorem 6.1.

7 The minimization problem

Throughout this section, we investigate versions of 3AP with perimeter costs where the elements of $X \cup Y \cup Z$ are points in the 2-dimensional Cartesian plane \mathbb{R}^2 . The distances between points are measured according to some fixed norm L_R with unit ball R.

We will show that for every (compact, convex, centrally symmetric) unit ball R, the resulting version of 3AP with perimeter costs is NP-hard. Our reduction is built around the fundamental triangle and the lattice introduced in Theorem 6.1. We recall that in this lattice only fundamental triangles have a cheap perimeter of Δ , whereas all non-fundamental triangles have an expensive perimeter of at least $\Delta + 1$. A *diamond* is a set of four lattice points obtained by gluing together two fundamental triangles along one side; see Figure 4. We partition the lattice points into three classes, so that every fundamental triangle contains exactly one point from each class. In the figures the three classes are depicted by circles (\bigcirc), squares (\square) and filled circles (\bullet); see Figure 5. We refer to this structure as *three-colored lattice*.



Figure 4: A diamond (to the left) and all possible six directions of a diamond incident to point p (to the right)

Our reduction uses ideas that are similar to those used by Spieksma & Woeginger [13] and Pferschy, Rudolf & Woeginger [10]. The reduction is from the following special case of 3AP whose NP-hardness has been established by Dyer & Frieze [6]. To avoid notational collisions between the variables in 3AP and the variables in planar-3AP, we will consistently denote objects in planar-3AP instances by primed variables.

Problem: Planar 3-dimensional assignment problem (planar-3AP)



Figure 5: The three-colored lattice

Instance: Three pairwise disjoint sets X', Y' and Z' with |X'| = |Y'| = |Z'| = q' and a set $T' \subseteq X' \times Y' \times Z'$ such that (i) every element of $X' \cup Y' \cup Z'$ occurs in two or three triples from T', and (ii) the corresponding graph G' is planar. (This graph G' contains a vertex for every element of $X' \cup Y' \cup Z'$ and a vertex for every triple in T'. There is an edge connecting a triple vertex to an element vertex if and only if the corresponding element is a member of the corresponding triple.)

Question: Does there exist a subset T^* of q' triples in T' such that each element of $X' \cup Y' \cup Z'$ is contained in precisely one triple from T^* ?

Hence let us consider an arbitrary instance of planar-3AP. In the first step, we compute a planar layout of the planar graph G' that maps the vertices of G' into integer points in \mathbb{Z}^2 and that maps its edges into straight line segments. This can be done in polynomial time, for instance by using the algorithm of Schnyder [12].

In the second step, we map the planar layout into the three-colored lattice. Every point (α, β) in the planar layout maps into a point that is in the close neighborhood of the point $100\alpha v_1 + 100\beta v_2$ in the three-colored lattice; here v_1 and v_2 are the integer vectors from Theorem 6.1 that generate the lattice. Every element of $X' \cup Y' \cup Z'$ is mapped into a corresponding *element point*; every element of X' goes into a circle (\bigcirc) , every element of Y' goes into a square (\Box) , and every element of Z' goes into a filled circle (\bullet) . Every triple in T' is mapped into a fundamental triangle called *triple triangle*. These element points and triple triangles roughly imitate the planar layout constructed above; there is plenty of leeway for doing this, since the main restriction is that the various objects should be embedded far away from each other.

In the third step, we introduce several chains of diamonds that connect certain element points to certain triple triangles; see Figure 6 for an illustration. Every such chain connects an element point (for some element x' of $X' \cup Y' \cup Z'$) to a triple triangle (whose corresponding triple t' in T' contains that element x'). These chains roughly



Figure 6: A chain of diamonds between two points p and q

follow the straight line segment that corresponds to the edge between x' and t' in the planar layout in the first step. Figure 7 shows how such a chain is attached to a triple triangle, and Figure 8 shows how such a chain is attached to an element point.



Figure 7: How chains of diamonds attach to a triple triangle

Three comments are in place. First, if an element x' of $X' \cup Y' \cup Z'$ occurs in only two triples in T', then the corresponding element point is attached to only two chains of diamonds. Secondly, for every chain of diamonds the attachment point in the triple triangle belongs to the same class $(\bigcirc, \square, \bullet)$ as the element point at the other end of the chain. Thirdly, we note that there are two combinatorially different ways of choosing a triple triangle in the lattice; one way has the vertices in the classes $\bigcirc, \square, \bullet$ clockwise, and the other way has the vertices in the classes $\bigcirc, \square, \bullet$ counter-clockwise. We always pick the way that allows a crossing-free attachment of the three chains of diamonds to the triple triangle; see Figure 9 for an illustration.



Figure 8: How chains of diamonds attach to an element point

The element points, triple triangles and chains of diamonds altogether contain 3n points from the three-colored lattice, and each of the three classes contains exactly n points. These three sets with n points form the three sets X, Y, Z in a 3AP instance with perimeter costs. We complete the reduction by defining the integer bound $B = \lceil n \Delta \rceil$. The following two lemmas establish the connections between the considered instance of planar-3AP and the newly constructed instance of 3AP.

Lemma 7.1 If the constructed instance of 3AP has a solution with objective value at most B, then the considered instance of planar-3AP has answer YES.

Proof. Assume that the 3AP instance has a solution with objective value at most B. Then by Theorem 6.1 all n triples in this solution have perimeter cost Δ and induce fundamental triangles in the lattice. Moreover, it is straightforward to verify that from any chain of diamonds the solution does either pick all the dashed triangles or does pick all the solid triangles; see Figures 7 and 8.

We define a subset T^* of the triples in T' by picking all the triples for which the corresponding triple triangle occurs in the solution for the 3AP instance. Consider some element $x' \in X' \cup Y' \cup Z'$. The corresponding element point is contained in exactly one solid triangle in the 3AP solution, and this triangle must belong to some chain; see Figure 8. Consequently, this is a chain of solid triangles which propagates to some triple triangle. Figure 7 shows that the corresponding triple triangle is in T^* . To summarize, every element $x' \in X' \cup Y' \cup Z'$ is contained in exactly one triple in T^* . Hence T^* yields the desired certificate that the planar-3AP instance has answer YES. \Box

Lemma 7.2 If the considered instance of planar-3AP has answer YES, then the constructed instance of 3AP has a solution with objective value at most B.



Figure 9: Connecting chains to triple triangles: the upper picture shows an infeasible clockwise choice, the picture at the bottom shows the feasible counter-clockwise choice

Proof. Assume that the planar-3AP instance has answer YES, so that there is a set T^* of q' triples in T' that covers every element of $X' \cup Y' \cup Z'$ exactly once. Then we construct the following solution for the 3AP instance. For every triple in T^* , we pick the corresponding triple triangle for the 3AP solution. For every element $x' \in X' \cup Y' \cup Z'$, we pick the solid triangles in the chain of diamonds that connects the element point for x' to the triple triangle whose triple covers x' in T^* ; in the other chains incident to this element point, we pick the dashed triangles. As all points in $X \cup Y \cup Z$ are covered by the n picked (fundamental!) triangles, their overall length equals $n \Delta$. \Box

Note that the bound B in our construction is integer, and note that all the points in $X \cup Y \cup Z$ have integer coordinates; hence the reduction can easily be implemented in polynomial time (and without worrying about computations with irrational numbers). Together with Lemmas 7.1 and 7.2 this yields the following theorem.

Theorem 7.3 For any fixed norm L_R with unit ball R in two-dimensional space \mathbb{R}^2 , problem 3AP with perimeter costs measured according to L_R is NP-hard. \Box

8 Implications for the 3-dimensional matching problem

Up to this point we have been solely concerned with the 3-dimensional assignment problem, where the underlying elements belonged to three classes X, Y and Z, and where every triple contained exactly one element from every class. In the closely related 3-dimensional matching problem (3DM) all the elements belong to the same class: An instance of 3DM consists of a ground set U with |U| = 3n and a cost function $c: U \times U \times U \to \mathbb{R}$. The goal is to find a set of n triples in $U \times U \times U$ that cover every element in U exactly once, such that the sum of the costs of these triples is minimized. In the maximization version max-3DM of 3DM, this sum is to be maximized.

The algorithmic behavior of 3DM is very similar to that of 3AP. Both problems are NP-hard in general, and (as a rule of thumb) algorithms for one problem usually translate into similar algorithms for the other problem. Pferschy, Rudolf & Woeginger [10] proved that 3DM with perimeter costs under Euclidean distances in \mathbb{R}^2 is NP-hard. Our hardness arguments in Section 7 can easily be adapted to 3DM by setting $U := X \cup Y \cup Z$, thus extending and generalizing the result of [10] to arbitrary norms.

Corollary 8.1 For any fixed norm L_R with unit ball R in two-dimensional space \mathbb{R}^2 , problem 3DM with perimeter costs measured according to L_R is NP-hard. \Box

Also the NP-hardness proofs in Section 5 for max-3AP (when the dimension is part of the input) can easily be carried over to the matching problem.

Corollary 8.2 For any fixed p with $1 \le p \le \infty$, problem max-3DM with perimeter costs measured according to the L_p norm is NP-hard. \Box

In a similar fashion, the positive results in Sections 3 and 4 for the maximization version carry over to the 3-dimensional matching problem. We leave the (fairly easy) technical details to the reader.

Corollary 8.3 Problem max-3DM with perimeter costs under tunneling distances can be solved within a time complexity that depends polynomially on the instance size n (and exponentially on the number of tunnels). \Box

Corollary 8.4 For any polyhedral norm L_R with unit ball R in s-dimensional space \mathbb{R}^s , problem max-3DM with perimeter costs measured according to L_R can be solved within a time complexity that depends polynomially on the instance size n (and exponentially on the number of facets of the polyhedral unit ball). \Box

Corollary 8.5 For any fixed (not necessarily polyhedral) norm L_R with unit ball R in s-dimensional space \mathbb{R}^s , problem max-3DM with perimeter costs measured according to L_R possesses a PTAS. \Box

9 Conclusions

We have derived a variety of results on the complexity of 3AP and max-3AP with perimeter costs, when distances are measured according to certain norms.

Problem 3AP turned out to be hard for all norms, even if the dimension of the underlying Cartesian space \mathbb{R}^s equals s = 2. This of course (trivially) implies NP-hardness also for all dimensions $s \geq 3$. Problem max-3AP with perimeter costs shows a more versatile behavior. If the dimension s is fixed then max-3AP is easy for polyhedral norms. If the dimension s is part of the input then max-3AP is NP-hard for any L_p norm. The following question does not seem to be within the reach of our methods.

Problem 9.1 Decide whether max-3AP with perimeter costs is NP-hard, if the elements are points in 2-dimensional space \mathbb{R}^2 and if the distances are measured according to the Euclidean norm L_2 .

The literature contains only a handful of results on the approximability of 3AP and max-3AP with perimeter costs. Our Theorem 4.2 yields the existence of a PTAS for max-3AP if the dimension s is fixed. Furthermore, there is a polynomial time approximation algorithm with worst case guarantee 4/3 for the 3AP with perimeter costs by Crama & Spieksma [5], which works for arbitrary metric spaces. The following open problem seems to be very challenging.

Problem 9.2 Establish APX-hardness of the minimization problem 3AP with perimeter costs, if the elements are points in 2-dimensional space \mathbb{R}^2 and if the distances are measured according to the Euclidean norm L_2 .

Acknowledgements. This research was supported by the Austrian Science Fund: W1230, Doctoral Program "Discrete Mathematics". Gerhard Woeginger acknowledges support by DIAMANT (a mathematics cluster of the Netherlands Organization for Scientific Research NWO), and by the Alexander von Humboldt Foundation, Bonn, Germany.

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