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HOMOGENIZATION OF A THERMO-DIFFUSION SYSTEM WITH SMOLUCHOWSKI INTERACTIONS

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Abstract. We study the solvability and homogenization of a thermal-diffusion reaction problem posed in a periodically perforated domain. The system describes the motion of populations of hot colloidal particles interacting together via Smoluchowski production terms. The upscaled system, obtained via two-scale convergence techniques, allows the investigation of deposition effects in porous materials in the presence of thermal gradients.

1. **Introduction.** We aim at understanding processes driven by coupled fluxes through media with microstructures. In this paper, we study a particular type of coupling: we look at the interplay between diffusion fluxes of a fixed number of colloidal populations and a heat flux, the effects included here are incorporating an approximation of the Dufour ad Soret effects (cf. Section 2.3, see also [6]. The type of system of evolution equations that we encounter in Section 2.4 resembles very much cross-diffusion and chemotaxis-like systems; see e.g. [29, 10]. The structure of the chosen equations is useful in investigating transport, interaction, and deposition of a large numbers of hot multiple-sized particles in porous media.

Practical applications of our approach would include predicting the response of refractory concrete to high-temperatures exposure in steel furnaces, propagation of combustion waves due to explosions in tunnels, drug delivery in biological tissues, etc.; see for instance [3, 4, 25, 28, 12, 11]. In the paper [15] we study quantitatively some of these effects, focusing on colloids deposition under thermal gradients. Within this framework, our focus lies exclusively on two distinct theoretical aspects:

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(i) the mathematical understanding of the microscopic problem (i.e. the well-posedness of the starting system);
(ii) the averaging of the thermo-diffusion system over arrays of periodically-distributed microstructures (the so-called, *homogenization asymptotics limit*; see, for instance, [5, 19] and references cited therein).

The complexity of the microscopic system makes numerical simulations on the macro scale very expensive. That is the reason that the aspect (ii) is of concern here. Obviously, the study does not close with these questions. Many other issues like derivation of corrector estimates, design of efficient convergent numerical multiscale schemes, multiscale parameter identification etc. need also to be treated. Possible generalizations could point out to coupling heat transfer with Nernst-Planck-Stokes systems (extending [24]) or with semiconductor equations [18]. The paper is structured in the following manner. We present the basic notation and explain the multiscale geometry as well as some of the relevant physical processes in Section 2. Section 3 contains the proof of the solvability of the microstructure model. Finally, the homogenization procedure is performed in Section 4. The strong formulation of the upscaled thermo-diffusion model with Smoluchowski interactions is emphasized in Section 4.3.

2. **Notations and assumptions.**

2.1. **Model description and geometry.**

The geometry of the problem is depicted in Figure 1. The standard cell is shown in Figure 2.

$$
\begin{array}{ll}
(0,T) & = \text{time interval of interest} \\
\Omega & = (0,L)\times\cdots\times(0,L) \text{ bounded domain in } \mathbb{R}^n \text{ for } L>0 \\
\varepsilon & = \frac{L}{\ell} \text{ for any integer } \ell \\
\partial\Omega & = \text{piecewise smooth boundary of } \Omega \\
\vec{e}_i & = i\text{th unit vector in } \mathbb{R}^n \\
Y & = \{\sum_{i=1}^n \lambda_i \vec{e}_i :\ 0<\lambda_i<1\} \text{ unit cell in } \mathbb{R}^n \\
Y_0 & = \text{open subset of } Y \text{ that represents the solid grain} \\
Y_1 & = Y\setminus \overline{Y}_0 \\
\Gamma & = \partial Y_0 \text{ piecewise smooth boundary of } Y_0 \\
X^k & = X+\sum_{i=1}^n k_i\vec{e}_i, \text{ where } k=(k_1,\dots,k_n)\in\mathbb{Z}^n \text{ and } X\subset Y \\
\Omega_0^\varepsilon & = \cup\{(\varepsilon Y_0)^k :\ (Y_0)^k\subset\Omega^\varepsilon, k\in\mathbb{Z}^n\} \text{ pore skeleton} \\
\Omega^\varepsilon & = \Omega\setminus\overline{\Omega}_0^\varepsilon \text{ pore space} \\
\Gamma^\varepsilon & = \partial\Omega_0^\varepsilon \text{ boundary of the pore skeleton}
\end{array}
$$

The cells regions without the grain εY_1^k are filled with water and we denote their union by Ω^ε. Colloidal species are dissolved in the pore water. They react between themselves and participate in diffusion and convective transport. The colloidal matter cannot penetrate the grain boundary Γ^ε, but it deposits there reducing the amount of mass floating inside Ω^ε. Here $\partial\Omega^\varepsilon=\partial\Omega\cup\Gamma^\varepsilon$, where $\Gamma^\varepsilon=\Gamma_N^\varepsilon\cup\Gamma_R^\varepsilon$ and $\Gamma_N^\varepsilon\cap\Gamma_R^\varepsilon=\emptyset$. The boundary Γ_N^ε is insulated to the heat flow, while Γ_R^ε admits flux.

The unknowns are:

- θ^ε – the temperature in Ω^ε.
- u_i^ε – the concentration of the species that contains i monomers in Ω^ε.
- v_i^ε – the mass of the deposited species on Γ^ε.

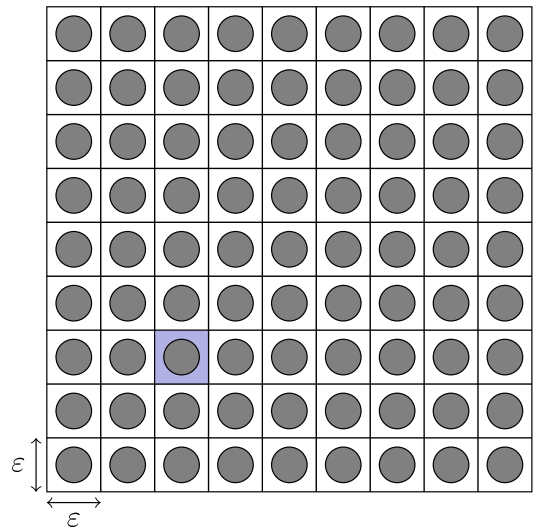

FIGURE 1. Porous medium geometry $\Omega^\varepsilon = \Omega \backslash \Omega_0^\varepsilon$, where the pore skeleton Ω_0^ε is marked with gray color and the pore space Ω^ε is white.

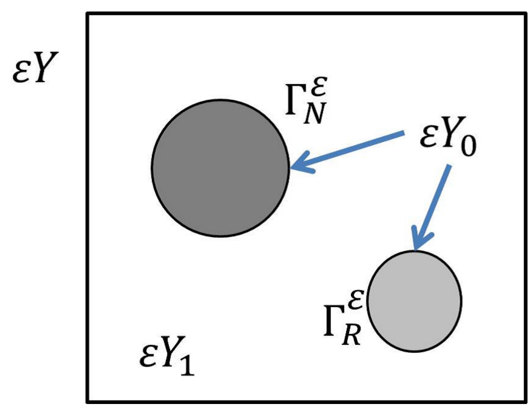

FIGURE 2. The unit cell geometry. The colloidal species u_i^ε and temperature θ^ε are defined in Ω^ε , while the deposited species v_i^ε are defined on $\Gamma^\varepsilon = \Gamma_R^\varepsilon \cup \Gamma_N^\varepsilon$. The boundary conditions for θ^ε differ on Γ_R and Γ_N, while the boundary conditions for u_i^ε are uniform on Γ^ε.

Furthermore, for a given $\delta > 0$ we introduce the mollifier:

$$J_\delta(s) := \begin{cases} Ce^{1/(|s|^2-\delta^2)} & \text{if } |s| < \delta, \\ 0 & \text{if } |s| \geq \delta, \end{cases} \tag{1}$$

where the constant $C > 0$ is selected such that

$$\int_{\mathbb{R}^d} J_\delta = 1,$$

see [8] for details.

Using J_δ from (1), define the mollified gradient:

$$\nabla^\delta f := \nabla \left[\int_{B(x,\delta)} J_\delta(x-y) f(y) dy \right]. \tag{2}$$

The following statement holds for all $1 \le p \le \infty$:

$$\|\nabla^\delta f \cdot g\|_{L^p(\Omega^\varepsilon)} \le c^\delta \|f\|_{L^\infty(\Omega^\varepsilon)} \|g\|_{L^p(\Omega^\varepsilon)^d} \text{ for } f \in L^\infty(\Omega^\varepsilon), g \in L^p(\Omega^\varepsilon)^d, \tag{3}$$

$$\|\nabla^\delta f\|_{L^p(\Omega^\varepsilon)} \le c^\delta \|f\|_{L^2(\Omega^\varepsilon)} \text{ for } f \in L^2(\Omega^\varepsilon). \tag{4}$$

In the equations below all norms are $L^2(\Omega^\varepsilon)$ unless specified otherwise, with c^δ independent of the choice of ε.

2.2. **Smoluchowski population balance equations.** We want to model the transport of aggregating colloidal particles under the influence of thermal gradients. We use the Smoluchowski population balance equation, originally proposed in [27], to account for colloidal aggregation:

$$R_i(s) := \frac{1}{2} \sum_{k+j=i} \beta_{kj} s_k s_j - \sum_{j=1}^{N} \beta_{ij} s_i s_j, \qquad i \in \{1, \ldots, N\};\ N > 2. \tag{5}$$

Here s_i is the concentration of the colloidal species that consists of i monomers, N is the number of species, i.e. the maximal aggregate size that we consider, $R_i(s)$ is the rate of change of s_i, and $\beta_{ij} > 0$ are the coagulation coefficients, which tell us the rate aggregation between particles of size i and j [7]. Colloidal aggregation rates are described in more detail in [14].

2.3. **Soret and Dufour effects.** The system we have in mind is inspired by the model proposed by Shigesada, Kawasaki and Teramoto [26] in 1979 when they have studied the segregation of competing species. For the case of two interacting species u and v, the diffusion term looks like:

$$\partial_t u = \Delta(d_1 u + \alpha u v), \tag{6}$$

where the second term in the flux is due to cross-diffusion. The second term can be expressed as:

$$\Delta(uv) = u\Delta v + v\Delta u + 2\nabla u \cdot \nabla v. \tag{7}$$

As a first step in our approach, we consider only the last term of (7), i.e. $\nabla u \cdot \nabla v$, as the driving force of cross-diffusion and we postpone the study of terms $u\Delta v$ and $v\Delta u$ until later.

From mathematical point of view, still it is not easy to treat the term $\nabla u \cdot \nabla v$. Hence, in the paper we approximate this term by $\nabla^\delta u \cdot \nabla v$ for $\delta > 0$.

2.4. **Setting of the model equations.** We consider the following balance equations for the temperature and colloid concentrations:

(P^ε):

$$\partial_t \theta^\varepsilon + \nabla \cdot (-\kappa^\varepsilon \nabla \theta^\varepsilon) - \tau^\varepsilon \sum_{i=1}^{N} \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon = 0, \qquad \text{in } (0,T) \times \Omega^\varepsilon, \tag{8}$$

$$\partial_t u_i^\varepsilon + \nabla \cdot (-d_i^\varepsilon \nabla u_i^\varepsilon) - \delta_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon = R_i(u^\varepsilon), \qquad \text{in } (0,T) \times \Omega^\varepsilon, \tag{9}$$

with boundary conditions:

$$-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = 0, \qquad \text{on } (0,T)\times\Gamma_N^\varepsilon, \tag{10}$$

$$-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = \varepsilon g_0 \theta^\varepsilon, \qquad \text{on } (0,T)\times\Gamma_R^\varepsilon, \tag{11}$$

$$-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = 0, \qquad \text{on } \partial\Omega, \tag{12}$$

$$-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu = 0, \qquad \text{on } \partial\Omega, \tag{13}$$

where ν is the outward normal vector on the boundary and a boundary condition for colloidal deposition:

$$-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu = \varepsilon(a_i u_i^\varepsilon - b_i v_i^\varepsilon), \qquad \text{on } (0,T)\times\Gamma^\varepsilon, \tag{14}$$

$$\partial_t v_i^\varepsilon = a_i u_i^\varepsilon - b_i v_i^\varepsilon \qquad \text{on } (0,T)\times\Gamma^\varepsilon. \tag{15}$$

As initial conditions, we take for $i \in \{1,\ldots,N\}$:

$$\theta^\varepsilon(0,x) = \theta^{\varepsilon,0}(x), \qquad \text{in } \Omega^\varepsilon, \tag{16}$$

$$u_i^\varepsilon(0,x) = u_i^{\varepsilon,0}(x), \qquad \text{in } \Omega^\varepsilon, \tag{17}$$

$$v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x), \qquad \text{on } \Gamma^\varepsilon. \tag{18}$$

Table 1. Physical parameters of (P^ε).

κ^ε	heat conduction coefficient
d_i^ε	diffusion coefficient
τ^ε	Soret coefficient
δ^ε	Dufour coefficient
g_i	Robin boundary coefficient, $i \in \{0,\ldots,N\}$
a_i	Deposition coefficient 1, $i \in \{1,\ldots,N\}$
b_i	Deposition coefficient 2, $i \in \{1,\ldots,N\}$

We refer to (8)- (18) as (P^ε) – our reference microscopic model. Note that the Soret and Dufour coefficients determine the structure of the particular cross-diffusion system (see [6], [26] [2], [3], [22], [29]). The coefficients a_i and b_i describe the deposition interaction between u_i^ε and v_i^ε. Consequently, each u_i^ε has a different affinity to sediment as well as a different mass.

All functions defined in Ω^ε are taken to be ε-periodic, i.e. $\kappa^\varepsilon(x) = \kappa(x/\varepsilon)$ and so on.

Note the use of the mollified gradient in the cross diffusion terms in (8) and (9). This is a choice that we have to make at this point in order to obtain the necessary estimates for our equations. From a physical point of view, smoothed gradients causing advection can be interpreted as there being no turbulence.

2.5. **Assumptions on data.**

(A_1): κ, τ, d_i, $\delta_i \in L^\infty(Y)$ for each $i \in \{1,\ldots,N\}$. Moreover, $\kappa_0 \le \kappa \le \kappa_*$, $\tau \le \tau_*$, $d_0 \le d_i \le d_*$, $\delta_i \le \delta_*$ on Y for $i \in \{1,\ldots,N\}$, where $\kappa_0, \kappa_*, d_0, d_*$ and δ_* are positive constants. Also, a_i and b_i are positive constants for $i \in \{1,\ldots,N\}$, and we put $a_0 = \min(a_1, a_2, \ldots, a_N)$, $a_* = \max(a_1, a_2, \ldots, a_N)$, and $b_* = \max(b_1, b_2, \ldots, b_N)$.

(A_2): $\theta^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $u_i^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $v_i^{\varepsilon,0} \in L_+^\infty(\Gamma^\varepsilon)$ for $i \in \{1,\ldots,N\}$ and $\varepsilon > 0$. Moreover, $||\theta^{\varepsilon,0}||_{H^1(\Omega^\varepsilon)} \le C_0$, $||u_i^{\varepsilon,0}||_{H^1(\Omega^\varepsilon)} \le C_0$, and $||v_i^{\varepsilon,0}||_{L^\infty(\Gamma^\varepsilon)} \le C_0$ for $i \in \{1,\ldots,N\}$ and $\varepsilon > 0$. Here C_0 is a positive constant independent of ε. Also, $L^\infty_+(\Omega^\varepsilon) = \{z \in L^\infty(\Omega^\varepsilon) : z \geq 0 \text{ a.e. on } \Omega^\varepsilon\}$ and $L^\infty_+(\Gamma^\varepsilon) = \{z \in L^\infty(\Gamma^\varepsilon) : z \geq 0 \text{ a.e. on } \Gamma^\varepsilon\}$.

Remark 2.1. By the definitions of κ^ε, d_i^ε, τ^ε, δ_i^ε and (A_1), it holds that $\kappa_0 \leq \kappa^\varepsilon \leq \kappa_*$, $\tau^\varepsilon \leq \tau_*$, $d_0 \leq d_i^\varepsilon \leq d_*$, $\delta_i^\varepsilon \leq \delta_*$ on Ω^ε for $i \in \{1, \ldots, N\}$ and each $\varepsilon > 0$.

3. Global solvability of problem (P^ε).

Definition 1. The triplet $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ is a solution to problem (P^ε)if the following holds:

$$\begin{aligned}
&\theta^\varepsilon, u_i^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap L^\infty(0,T;H^1(\Omega^\varepsilon)) \cap L^\infty((0,T)\times\Omega^\varepsilon),\\
&v_i^\varepsilon \in H^1(0,T;L^2(\Gamma^\varepsilon)) \cap L^\infty((0,T)\times\Gamma^\varepsilon),
\end{aligned} \tag{19}$$

for all $\phi \in H^1(\Omega^\varepsilon)$:

$$\int_{\Omega^\varepsilon} \partial_t\theta^\varepsilon\phi + \int_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla\phi + \varepsilon g_0\int_{\Gamma_R^\varepsilon}\theta^\varepsilon\phi = \sum_{i=1}^N\int_{\Omega^\varepsilon}\tau^\varepsilon\nabla^\delta u_i^\varepsilon\cdot\nabla\theta^\varepsilon\phi, \tag{20}$$

for all $\psi_i \in H^1(\Omega^\varepsilon)$:

$$\begin{aligned}
&\int_{\Omega^\varepsilon}\partial_t u_i^\varepsilon\psi_i + \int_{\Omega^\varepsilon} d_i^\varepsilon\nabla u_i^\varepsilon\cdot\nabla\psi_i + \varepsilon\int_{\Gamma^\varepsilon}(a_iu_i^\varepsilon - b_iv_i^\varepsilon)\psi_i\\
&\qquad = \int_{\Omega^\varepsilon}\delta_i^\varepsilon\nabla^\delta\theta^\varepsilon\cdot\nabla u_i^\varepsilon\psi_i + \int_{\Omega^\varepsilon}R_i(u^\varepsilon)\psi_i,
\end{aligned} \tag{21}$$

for all $\varphi_i \in L^2(\Gamma^\varepsilon)$:

$$\int_{\Gamma^\varepsilon}\partial_t v_i^\varepsilon\varphi_i = \int_{\Gamma^\varepsilon}(a_iu_i^\varepsilon - b_iv_i^\varepsilon)\varphi_i, \tag{22}$$

together with (16), (17) and (18) for a fixed value of $\varepsilon > 0$.

Remark 3.1. We note that each term appearing in Definition 1 is finite, since $\nabla^\delta u_i^\varepsilon$ and $\nabla^\delta\theta^\varepsilon$ are bounded in Ω^ε due to (3).

To prove the existence of solutions to problem (P^ε), we introduce the following auxiliary problems as iterations steps of the coupled system:

(P_1):

$$\begin{aligned}
&\partial_t\theta^\varepsilon + \nabla\cdot(-\kappa^\varepsilon\nabla\theta^\varepsilon) - \tau^\varepsilon\sum_{i=1}^N\nabla^\delta\bar{u}_i\cdot\nabla\theta^\varepsilon = 0, && \text{in } (0,T)\times\Omega^\varepsilon,\\
&-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\Gamma_N^\varepsilon,\\
&-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = \varepsilon g_0\theta^\varepsilon, && \text{on } (0,T)\times\Gamma_R^\varepsilon,\\
&-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\partial\Omega,\\
&\theta^\varepsilon(0,x) = \theta^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon,
\end{aligned}$$

and

(P_2):

$$
\begin{aligned}
&\partial_t u_i^\varepsilon + \nabla\cdot(-d_i^\varepsilon \nabla u_i^\varepsilon) - \delta_i^\varepsilon \nabla^\delta \bar{\theta}\cdot\nabla u_i^\varepsilon = R_i^M(u^\varepsilon), && \text{in } (0,T)\times\Omega^\varepsilon,\\
&- d_i^\varepsilon \nabla u_i^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\partial\Omega,\\
&- d_i^\varepsilon \nabla u_i^\varepsilon\cdot\nu = \varepsilon(a_i u_i^\varepsilon - b_i v_i^\varepsilon), && \text{on } (0,T)\times\Gamma^\varepsilon,\\
&u_i^\varepsilon(0,x) = u_i^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon,\\
&\partial_t v_i^\varepsilon = a_i u_i^\varepsilon - b_i v_i^\varepsilon, && \text{on } (0,T)\times\Gamma^\varepsilon,\\
&v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x), && \text{on } \Gamma^\varepsilon.
\end{aligned}
$$

Here

$$R_i^M(s) := R_i(\sigma_M(s_1),\sigma_M(s_2),\dots,\sigma_M(s_N)),\ \text{for } s\in\mathbb{R}^N \tag{23}$$

denotes our choice of truncation of R_i, where

$$\sigma_M(r) := \begin{cases} 0, & r<0,\\ r, & r\in[0,M],\\ M, & r>M,\end{cases} \tag{24}$$

where $M>0$ is a fixed threshold. Note that if M is large enough, the essential bounds obtained later in this paper will remain below M. This means that the existence result is obtained also for the uncut rates.

In the following, assuming (A_1)-(A_2), we show the existence, positivity and boundedness of solutions to (P_1) and (P_2).

When we denote the solutions of $P_1(\bar{u})$ by θ^ε and of $P_2(\bar{\theta})$ by $(u_i^\varepsilon, v_i^\varepsilon)$, respectively, we can define the solution operators $(\theta^\varepsilon, u_i^\varepsilon) = \mathbf{T}(\bar{\theta},\bar{u}_i)$ and $v_i^\varepsilon = \mathbf{T}_2(\bar{\theta},\bar{u}_i)$. We will show that the operator $\mathbf{T}$ is a contraction in the appropriate functional spaces and use the Banach fixed point theorem to prove the existence and uniqueness of solutions to (P^ε).

Notation 1. Let $K(T,M) := \{z\in L^2(0,T;L^2(\Omega^\varepsilon)) : |z|\le M \text{ a.e. on } (0,T)\times\Omega^\varepsilon\}$.

Lemma 3.2. Existence of solutions to (P_1). *Let $\bar{u}_i\in K(T,M)$, and assume that (A_1)-(A_2) hold. Then there exists $\theta^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon))\cap L^\infty(0,T;H^1(\Omega^\varepsilon))$ that solves (P_1) in the sense:*

for all $\phi\in H^1(\Omega^\varepsilon)$ and a.e. in $[0,T]$:

$$\int_{\Omega^\varepsilon}\partial_t\theta^\varepsilon\phi + \int_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla\phi + \varepsilon g_0\int_{\Gamma_R^\varepsilon}\theta^\varepsilon\phi = \sum_{i=1}^N\int_{\Omega^\varepsilon}\tau^\varepsilon\nabla^\delta\bar{u}_i\cdot\nabla\theta^\varepsilon\phi, \tag{25}$$

and

$$\theta^\varepsilon(0,x) = \theta^{\varepsilon,0}(x) \qquad \text{a.e. in } \Omega^\varepsilon. \tag{26}$$

Proof. Let $\{\xi_i\}$ be a Schauder basis of $H^1(\Omega^\varepsilon)$. Then for each $n\in\mathbb{N}$ there exists

$$\theta_n^{\varepsilon,0}(x) := \sum_{j=1}^n \alpha_j^{0,n}\xi_j(x) \text{ such that } \theta_n^{\varepsilon,0}\to\theta^{\varepsilon,0} \text{ in } H^1(\Omega^\varepsilon) \text{ as } n\to\infty. \tag{27}$$

We denote by θ_n^ε the Galerkin approximation of θ^ε, that is:

$$\theta_n^\varepsilon(t,x) := \sum_{j=1}^n \alpha_j^n(t)\xi_j(x) \qquad \text{for all } (t,x)\in(0,T)\times\Omega^\varepsilon. \tag{28}$$

By definition, θ_n^ε must satisfy (25) for all $\phi \in \mathrm{span}\{\xi_j\}_{j=1}^n$, i.e.:

$$\int\limits_{\Omega^\varepsilon} \partial_t \theta_n^\varepsilon \phi + \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \theta_n^\varepsilon \cdot \nabla \phi + \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \theta_n^\varepsilon \phi = \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \tau^\varepsilon \nabla^\delta \bar{u}_i \cdot \nabla \theta_n^\varepsilon \phi. \tag{29}$$

The coefficients $\alpha_i^n(t)$ can be found by testing (29) with $\phi := \xi_i$ and using (27) to solve the resulting ODE system:

$$\partial_t \alpha_i^n(t) + \sum_{j=1}^n (A_{ij} + B_{ij} - C_{ij}) \alpha_j^n(t) = 0, \qquad i \in \{1, \dots, n\}, \tag{30}$$

$$\alpha_i^n(0) = \alpha_i^{0,n}. \tag{31}$$

The coefficients in (30) and (31) are defined by the following expressions

$$A_{ij} := \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \xi_i \cdot \nabla \xi_j, \qquad i, j \in \{1, \dots, n\},$$

$$B_{ij} := \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \xi_i \xi_j, \qquad i, j \in \{1, \dots, n\},$$

$$C_{ij} := \sum_{k=1}^N \int\limits_{\Omega^\varepsilon} \tau^\varepsilon \nabla^\delta \bar{u}_k \cdot \nabla \xi_j \xi_i \qquad i, j \in \{1, \dots, n\}.$$

Since the system (30) is linear, there exists for each fixed $n \in \mathbb{N}$ a unique solution $\alpha_i^n \in C^1([0, T])$.

To prove uniform estimates for θ_n^ε with respect to n, we take in (29) $\phi = \theta_n^\varepsilon$. We obtain:

$$\frac{1}{2} \partial_t \|\theta_n^\varepsilon\|^2 + \kappa_0 \|\nabla \theta_n^\varepsilon\|^2 + \varepsilon g_0 \|\theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \leq \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \tau^\varepsilon |\nabla^\delta \bar{u}_i \cdot \nabla \theta_n^\varepsilon \theta_n^\varepsilon| := \tau_* \sum_{i=1}^N A_i.$$

Using the Cauchy-Schwarz inequality and Young's inequality in the form $ab \leq \eta a^2 + b^2/4\eta$, where $\eta > 0$, we get:

$$A_i \leq \eta \|\nabla \theta_n^\varepsilon\|^2 + \frac{1}{4\eta} \|\nabla^\delta \bar{u}_i \theta_n^\varepsilon\|^2 \leq \eta \|\nabla \theta_n^\varepsilon\|^2 + \frac{1}{4\eta} \|\nabla^\delta \bar{u}_i\|^2_{L^4(\Omega^\varepsilon)} \|\theta_n^\varepsilon\|^2_{L^4(\Omega^\varepsilon)}.$$

The mollifier property (3) yields $\|\nabla^\delta \bar{u}_i\|^2_{L^4(\Omega^\varepsilon)} \leq c^\delta \|\bar{u}_i\|^2_\infty$. Using Gagliardo-Nirenberg inequality (see [23] e.g.), we get:

$$\|\theta_n^\varepsilon\|^2_{L^4(\Omega^\varepsilon)} \leq c \|\theta_n^\varepsilon\|^{1/2} \|\nabla \theta_n^\varepsilon\|^{3/2}. \tag{32}$$

Applying Young's inequality, we obtain:

$$c \|\theta_n^\varepsilon\|^{1/2} \|\nabla \theta_n^\varepsilon\|^{3/2} \leq \eta \|\nabla \theta_n^\varepsilon\|^2 + c_\eta \|\theta_n^\varepsilon\|^2. \tag{33}$$

Finally, we obtain the structure:

$$\frac{1}{2} \partial_t \|\theta_n^\varepsilon\|^2 + (\kappa_0 - 2N\eta) \|\nabla \theta_n^\varepsilon\|^2 + \varepsilon g_0 \|\theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \leq c_\eta^\delta \sum_{i=1}^N \|\bar{u}_i\|^2 \|\theta_n^\varepsilon\|^2.$$

For a small $\eta > 0$ Gronwall's lemma gives:

$$\|\theta_n^\varepsilon(t)\|^2 + \kappa_0 \int_0^t \|\nabla \theta_n^\varepsilon(t)\|^2 < C \qquad \text{for } t \in (0, T),$$

where $C > 0$ is independent of n and ε, since $\bar{u}_i$ are uniformly bounded. This ensures that

$$\{\theta_n^\varepsilon\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega^\varepsilon)) \cap L^2(0,T;H^1(\Omega^\varepsilon)). \tag{34}$$

To show uniform estimates for $\partial_t\theta_n^\varepsilon$ with respect to n, we can take $\phi = \partial_t\theta_n^\varepsilon$ in (29). Indeed, by the formula (28) of θ_n^ε, $\partial_t\theta_n^\varepsilon = \sum_{j=1}^n (\partial_t\alpha_j^n)\xi_j$ so that $\partial_t\theta_n^\varepsilon \in \operatorname{span}\{\xi_j\}_{j=1}^n$. Then by using the Cauchy-Schwarz and Young's inequalities, as well as the mollifier property (3) we get:

$$\begin{aligned} \|\partial_t\theta_n^\varepsilon\|^2 + \frac{1}{2}\partial_t\|\sqrt{\kappa^\varepsilon}\nabla\theta_n^\varepsilon\|^2 + \varepsilon\frac{g_0}{2}\partial_t\|\theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} &\le \tau_*\sum_{i=1}^N\int\limits_{\Omega^\varepsilon}|\nabla^\delta\bar{u}_i\cdot\nabla\theta_n^\varepsilon\partial_t\theta_n^\varepsilon| \\ \le \left(c^\delta\tau_*\sum_{i=1}^N\|\bar{u}_i\|_{L^\infty(\Omega^\varepsilon)}\right)&(\eta\|\partial_t\theta_n^\varepsilon\|^2 + C_\eta\|\nabla\theta_n^\varepsilon\|^2) \text{ for } \eta > 0. \end{aligned} \tag{35}$$

By taking a small $\eta > 0$ and using (34), it holds that:

$$\kappa_0\|\nabla\theta_n^\varepsilon\|^2 + \int_0^t\|\partial_t\theta_n^\varepsilon\|^2 < C \qquad\qquad \text{for all } t\in(0,T),$$

where $C > 0$ depends on δ, but is independent of n and ε. Together with (34) this ensures that:

$$\{\theta_n^\varepsilon\} \text{ is bounded in } H^1(0,T;L^2(\Omega^\varepsilon)) \cap L^\infty(0,T;H^1(\Omega^\varepsilon)). \tag{36}$$

Hence, we can choose a subsequence $\theta_{n_k}^\varepsilon \rightharpoonup \theta^\varepsilon$ in $H^1(0,T;L^2(\Omega^\varepsilon))$ and $\theta_{n_k}^\varepsilon \overset{*}{\rightharpoonup} \theta^\varepsilon$ in $L^\infty(0,T;H^1(\Omega^\varepsilon))$ as $k\to\infty$.

Now, using

$$v_m(t,x) := \sum_{j=1}^m \beta_j^m(t)\xi_j(x) \tag{37}$$

as a test function in (29) and integrating with respect to time we get:

$$\begin{aligned} \int\limits_0^T\int\limits_{\Omega^\varepsilon}\partial_t\theta_{n_k}^\varepsilon v_m + \int\limits_0^T\int\limits_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta_{n_k}^\varepsilon\cdot\nabla v_m + \varepsilon g_0\int\limits_0^T\int\limits_{\Gamma_R^\varepsilon}\theta_{n_k}^\varepsilon v_m \\ = \sum_{i=1}^N\int\limits_0^T\int\limits_{\Omega^\varepsilon}\tau^\varepsilon\nabla^\delta\bar{u}_i\cdot\nabla\theta_{n_k}^\varepsilon v_m. \end{aligned} \tag{38}$$

Using (36), we pass to the limit as $k\to\infty$ to obtain: For each m

$$\int\limits_0^T\int\limits_{\Omega^\varepsilon}\partial_t\theta^\varepsilon v_m + \int\limits_0^T\int\limits_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla v_m + \varepsilon g_0\int\limits_0^T\int\limits_{\Gamma_R^\varepsilon}\theta^\varepsilon v = \sum_{i=1}^N\int\limits_0^T\int\limits_{\Omega^\varepsilon}\tau^\varepsilon\nabla^\delta\bar{u}_i\cdot\nabla\theta^\varepsilon v_m. \tag{39}$$

Note that (39) holds for all $v\in L^2(0,T;H^1(\Omega^\varepsilon))$ since we can approximate v with v_m in $L^2(0,T;H^1(\Omega^\varepsilon))$, hence

$$\int\limits_0^T\int\limits_{\Omega^\varepsilon}\partial_t\theta^\varepsilon v + \int\limits_0^T\int\limits_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla v + \varepsilon g_0\int\limits_0^T\int\limits_{\Gamma_R^\varepsilon}\theta^\varepsilon v = \sum_{i=1}^N\int\limits_0^T\int\limits_{\Omega^\varepsilon}\tau^\varepsilon\nabla^\delta\bar{u}_i\cdot\nabla\theta^\varepsilon v,$$

holds for all $v\in L^2(0,T;H^1(\Omega^\varepsilon))$.

Finally, we show the initial condition holds. Indeed, the Aubin-Lions lemma guarantees that $\theta^\varepsilon_{n_i} \to \theta^\varepsilon$ in $C([0,T];L^2(\Omega^\varepsilon))$. Then on account of $\theta^\varepsilon_{n_k}(0) \to \theta^{\varepsilon,0}$ in $L^2(\Omega^\varepsilon)$ as $k \to \infty$, we get $\theta^\varepsilon(0) = \theta^{\varepsilon,0}$. □

Lemma 3.3. Positivity and boundedness of solutions to (P_1). *Let $\bar{u}_i \in K(T,M)$, $M > 0$, and assume (A_1)-(A_2). Then $0 \le \theta^\varepsilon \le \|\theta^{\varepsilon,0}\|_{L^\infty(\Omega^\varepsilon)}$ a.e. in $(0,T)\times\Omega^\varepsilon$.*

Proof. Let $\theta^\varepsilon := \theta^{\varepsilon,+} - \theta^{\varepsilon,-}$, where $z^+ := \max(z,0)$ and $z^- := \max(-z,0)$. Testing (25) with $\phi := -\theta^{\varepsilon,-}$, and using (3) gives:

$$\frac{1}{2}\partial_t\|\theta^{\varepsilon,-}\|^2 + \kappa_0\|\nabla\theta^{\varepsilon,-}\|^2 + \varepsilon g_0\|\theta^{\varepsilon,-}\|^2_{L^2(\Gamma^\varepsilon_R)} \le c^\delta\tau^\varepsilon\sum_{i=1}^N\|\bar{u}_i\|_\infty\|\nabla\theta^{\varepsilon,-}\theta^{\varepsilon,-}\|_{L^1(\Omega^\varepsilon)}$$
$$\le \left(C^\delta_\eta\tau^\varepsilon\sum_{i=1}^N\|\bar{u}_i\|_\infty\right)\|\theta^{\varepsilon,-}\|^2 + \eta\|\nabla\theta^{\varepsilon,-}\|^2 \text{ for } \eta > 0.$$

Choosing $\eta < \kappa_0$ and taking into account that $\theta^{\varepsilon,-}(0) = 0$, Gronwall's lemma gives $\|\theta^{\varepsilon,-}\|^2 \le 0$. This means $\theta^\varepsilon \ge 0$ a.e. in Ω for all $t \in (0,T)$.

Let $\phi = (\theta^\varepsilon - M_0)^+$ in (25) with $M_0 \ge \|\theta^\varepsilon(0)\|_{L^\infty(\Omega^\varepsilon)}$: For $\eta > 0$

$$\frac{1}{2}\partial_t\|(\theta^\varepsilon - M_0)^+\|^2 + \kappa_0\|\nabla(\theta^\varepsilon - M_0)^+\|^2 + \varepsilon g_0\|(\theta^\varepsilon - M_0)^+\|^2_{L^2(\Gamma^\varepsilon_R)}$$
$$+ g_0\int_{\Gamma^\varepsilon_R} M_0(\theta^\varepsilon - M_0)^+ \le \tau_*\sum_{i=1}^N\int_{\Omega^\varepsilon}\nabla^\delta\bar{u}_i\cdot\nabla(\theta^\varepsilon - M_0)^+(\theta^\varepsilon - M_0)^+$$
$$\le \left(\tau_* c^\delta\sum_{i=1}^N\|\bar{u}_i\|_\infty\right)\left(c_\eta\|(\theta^\varepsilon - M_0)^+\|^2 + \eta\|\nabla(\theta^\varepsilon - M_0)^+\|^2\right).$$

Discarding the positive terms on the left side and then applying Gronwall's lemma leads to:

$$\|(\theta^\varepsilon - M_0)^+(t)\|^2 \le \|(\theta^\varepsilon - M_0)^+(0)\|^2\exp\left(\tau_* c^\delta c_\eta\sum_{i=1}^N\|\bar{u}_i\|_\infty t\right).$$

Since $\|(\theta^\varepsilon - M_0)^+(0)\| = 0$, we obtain $(\theta^\varepsilon - M_0)^+(t) = 0$. Thus the proof of the lemma is completed. □

Lemma 3.4. Existence of solutions to (P_2). *Let $\bar\theta \in K(T,M), M > 0$ and (A_1)-(A_2) hold. Then (P_2) has solutions $u^\varepsilon_i \in H^1(0,T;L^2(\Omega^\varepsilon))\cap L^\infty(0,T;H^1(\Omega))$ and $v^\varepsilon_i \in H^1(0,T;L^2(\Gamma^\varepsilon))$ in the following sense:*

For all $\psi_i \in H^1(\Omega^\varepsilon)$, it holds:

$$\begin{aligned}&\int_{\Omega^\varepsilon}\partial_t u^\varepsilon_i\psi_i + \int_{\Omega^\varepsilon} d^\varepsilon_i\nabla u^\varepsilon_i\cdot\nabla\psi_i + \varepsilon\int_{\Gamma^\varepsilon}(a_i u^\varepsilon_i - b_i v^\varepsilon_i)\psi_i \\ &\quad= \int_{\Omega^\varepsilon}\delta^\varepsilon_i\nabla^\delta\bar\theta\cdot\nabla u^\varepsilon_i\psi_i + \int_{\Omega^\varepsilon} R^M_i(u^\varepsilon)\psi_i\end{aligned} \tag{40}$$

$$u^\varepsilon_i(0,x) = u^{\varepsilon,0}_i(x) \quad a.e.\ in\ \Omega^\varepsilon, \tag{41}$$

and for all $\varphi_i \in L^2(\Gamma^\varepsilon)$*:*

$$\int_{\Gamma^\varepsilon} \partial_t v_i^\varepsilon \varphi_i = \int_{\Gamma^\varepsilon} (a_i u_i^\varepsilon - b_i v_i^\varepsilon)\varphi_i, \tag{42}$$

$$v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x) \quad a.e.\ on\ \Gamma^\varepsilon. \tag{43}$$

Proof. Let $\{\xi_j\}$ – Schauder basis of $H^1(\Omega^\varepsilon)$. Then, for each $n \in \mathbb{N}$, there exists

$$u_{i,n}^{\varepsilon,0}(x) := \sum_{j=1}^{n} \alpha_{i,j}^{0,n}\xi_j(x) \text{ such that } u_{i,n}^{\varepsilon,0} \to u_i^{\varepsilon,0} \text{ in } H^1(\Omega^\varepsilon) \text{ as } n \to \infty. \tag{44}$$

We denote by $u_{i,n}^\varepsilon$ the Galerkin approximation of u_i^ε, that is:

$$u_{i,n}^\varepsilon(t,x) := \sum_{j=1}^{n} \alpha_{i,j}^{n}(t)\xi_j(x) \qquad \text{for all } (t,x) \in (0,T) \times \Omega^\varepsilon. \tag{45}$$

$u_{i,n}^\varepsilon$ must satisfy (40), and hence,

$$\begin{aligned} &\int_{\Omega^\varepsilon} \partial_t u_{i,n}^\varepsilon \psi_i + \int_{\Omega^\varepsilon} d_i^\varepsilon \nabla u_{i,n}^\varepsilon \cdot \nabla\psi_i + \varepsilon\int_{\Gamma^\varepsilon} (a_i u_{i,n}^\varepsilon - b_i v_i^\varepsilon)\psi_i \\ &\quad = \int_{\Omega^\varepsilon} \delta_i^\varepsilon \nabla^\delta \bar{\theta} \cdot \nabla u_{i,n}^\varepsilon \psi_i + \int_{\Omega^\varepsilon} R_i^M(u_n^\varepsilon)\psi_i, \qquad \text{for all } \psi_i \in \operatorname{span}\{\xi_j\}_{j=1}^n. \end{aligned} \tag{46}$$

Accordingly, let $\{\eta_j\}$ – an orthonormal basis of $L^2(\Gamma^\varepsilon)$. Then for each $n \in \mathbb{N}$ there exists

$$v_{i,n}^{\varepsilon,0}(x) := \sum_{j=1}^{n} \beta_{i,j}^{0,n}\eta_j(x) \text{ such that } v_{i,n}^{\varepsilon,0} \to v_i^{\varepsilon,0} \text{ in } L^2(\Gamma^\varepsilon) \text{ as } n \to \infty. \tag{47}$$

We denote by $v_{i,n}^\varepsilon$ the Galerkin approximation of v_i^ε, that is:

$$v_{i,n}^\varepsilon(t,x) := \sum_{j=1}^{n} \beta_{i,j}^{n}(t)\eta_j(x), \qquad \text{for all } (t,x) \in (0,T) \times \Gamma^\varepsilon. \tag{48}$$

$v_{i,n}^\varepsilon$ must satisfy (42), and hence,

$$\int_{\Gamma^\varepsilon} \partial_t v_{i,n}^\varepsilon \varphi_i = \int_{\Gamma^\varepsilon} (a_i u_{i,n}^\varepsilon - b_i v_{i,n}^\varepsilon)\varphi_i, \qquad \text{for all } \varphi_i \in \operatorname{span}\{\eta_j\}_{j=1}^n. \tag{49}$$

$\alpha^n_{i,j}(t)$ and $\beta^n_{i,j}(t)$ can be found by substituting $u^\varepsilon_{i,n}$ and $v^\varepsilon_{i,n}$ into (40) – (43) and using ξ_k and η_k for $k \in \{1,\dots,n\}$ as test functions:

$$\begin{aligned}
&\partial_t \alpha^n_{i,k}(t) + \sum_{j=1}^n (A_{ijk} + C_{ijk} - D_{ijk})\alpha^n_{i,j}(t) - \sum_{j=1}^n E_{ijk}\beta^n_{i,j}(t) \\
&\quad = \int_{\Omega^\varepsilon} \xi_k \sum_{a=1}^{i-1} \beta_{a,i-a}\sigma_M\left(\sum_{b=1}^n \alpha^n_{a,b}(t)\xi_b\right)\sigma_M\left(\sum_{c=1}^n \alpha^n_{i-a,c}(t)\xi_c\right) \\
&\quad - \int_{\Omega^\varepsilon} \xi_k \sum_{a=1}^{N} \beta_{a,i}\sigma_M\left(\sum_{b=1}^n \alpha^n_{i,b}(t)\xi_b\right)\sigma_M\left(\sum_{c=1}^n \alpha^n_{a,c}(t)\xi_c\right),
\end{aligned} \tag{50}$$

$$\alpha^n_{i,j}(0) = \alpha^{0,n}_{i,j}, \tag{51}$$

$$\partial_t \beta^n_{i,k}(t) = \sum_{j=1}^n G_{ijk}\alpha^n_{i,j}(t) - H_{ijk}\beta^n_{i,j}(t), \tag{52}$$

$$\beta^n_{i,j}(0) = \beta^{0,n}_{i,j}. \tag{53}$$

The coefficients arising in (50) are defined by:

$$\begin{aligned}
&A_{ijk} := \int_{\Omega^\varepsilon} d^\varepsilon_i \nabla\xi_j \cdot \nabla\xi_k, \\
&C_{ijk} := \varepsilon a_i \int_{\Gamma^\varepsilon} \xi_j\xi_k, && D_{ijk} := \int_{\Omega^\varepsilon} \delta^\varepsilon_i \nabla^\delta\bar\theta \cdot \nabla\xi_j\xi_k, \\
&E_{ijk} := \varepsilon b_i \int_{\Gamma^\varepsilon} \xi_k\eta_j, && G_{ijk} := a_i \int_{\Gamma^\varepsilon} \xi_j\eta_k, \\
&H_{ijk} := b_i \int_{\Gamma^\varepsilon} \eta_j\eta_k.
\end{aligned}$$

The left-hand side of this system of ODEs is linear, while the right-hand side is globally Lipschitz. Thus there exists a unique solution $\alpha^n_{i,j}(t), \beta^n_{i,j}(t) \in H^1(0,T)$ to (50) - (53) for $t \in (0,T)$.

To show uniform estimates in n for $u^\varepsilon_{i,n}$ and $v^\varepsilon_{i,n}$, we take $\psi_i = u^\varepsilon_{i,n}$ and $\varphi_i = v^\varepsilon_{i,n}$ in (46) and (49) respectively. We get the inequality:

$$\begin{aligned}
&\frac{1}{2}\partial_t\|u^\varepsilon_{i,n}\|^2 + d_0\|\nabla u^\varepsilon_{i,n}\|^2 + \varepsilon a_0\|u^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} \\
&\quad \le \varepsilon b_* \int_{\Gamma^\varepsilon} |v^\varepsilon_{i,n}u^\varepsilon_{i,n}| + \delta_* c^\delta\|\bar\theta\|_\infty\|\nabla u^\varepsilon_{i,n}\|\|u^\varepsilon_{i,n}\| + \int_{\Omega^\varepsilon} R^M_i(u^\varepsilon_n)u^\varepsilon_{i,n} \\
&\quad \le \eta\|u^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} + C^\eta\|v^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} + \eta\|\nabla u^\varepsilon_{i,n}\|^2 \\
&\qquad + C^{\delta\eta}\|\bar\theta\|_\infty\|u_{i,n}\|^2 + C^M\|u_{i,n}\|, \\
&\frac{1}{2}\partial_t\|v^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} + b^\varepsilon_i\|v^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} \\
&\le \eta\|u^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} + C^\eta\|v^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} \\
&\le C\eta(\|\nabla u^\varepsilon_{i,n}\|^2 + \|u^\varepsilon_{i,n}\|^2) + C^\eta\|v^\varepsilon_{i,n}\|^2_{L^2(\Gamma^\varepsilon)} \text{ for } \eta > 0.
\end{aligned}$$

After taking a small η and adding the two inequalities, Gronwall's lemma gives:

$$\|u_{i,n}^\varepsilon\|^2 + d_0 \int_0^t \|\nabla u_{i,n}^\varepsilon\|^2 + \|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} < C \qquad \text{for all } t \in (0,T), \tag{54}$$

where $C > 0$ depends on δ, M and T, but is independent of n and ε, which ensures:

$$\{u_{i,n}^\varepsilon\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega^\varepsilon)) \cap L^2(0,T;H^1(\Omega^\varepsilon)), \tag{55}$$

$$\{v_{i,n}^\varepsilon\} \text{ is bounded in } L^\infty(0,T;L^2(\Gamma^\varepsilon)). \tag{56}$$

To show uniform estimates for $\partial_t u_{i,n}^\varepsilon$ and $\partial_t v_{i,n}^\varepsilon$ with respect to n, we take $\psi_i = \partial_t u_{i,n}^\varepsilon$ and $\varphi_i = \partial_t v_{i,n}^\varepsilon$ in (46) and (49) respectively, noticing that they are in $\operatorname{span}\{\xi_j\}_{j=1}^n$. We obtain:

$$\begin{aligned}
&\|\partial_t u_{i,n}^\varepsilon\|^2 + \int\limits_{\Omega^\varepsilon} \frac{d_i^\varepsilon}{2}\partial_t(\nabla u_{i,n}^\varepsilon)^2 + \frac{\varepsilon a_i}{2}\partial_t\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)}\\
=&\varepsilon \int\limits_{\Gamma^\varepsilon} b_i \partial_t u_{i,n}^\varepsilon v_{i,n}^\varepsilon + \int\limits_{\Omega^\varepsilon} \delta_i^\varepsilon \nabla^\delta \bar\theta \cdot \nabla u_{i,n}^\varepsilon \partial_t u_{i,n}^\varepsilon + \int\limits_{\Omega^\varepsilon} R_i^M(u_n^\varepsilon)\partial_t u_{i,n}^\varepsilon\\
=&\varepsilon\partial_t \int\limits_{\Gamma^\varepsilon} b_i u_{i,n}^\varepsilon v_{i,n}^\varepsilon - \varepsilon\int\limits_{\Gamma^\varepsilon} b_i u_{i,n}^\varepsilon \partial_t v_{i,n}^\varepsilon + \int\limits_{\Omega^\varepsilon} \delta_i^\varepsilon \nabla^\delta \bar\theta \cdot \nabla u_{i,n}^\varepsilon \partial_t u_{i,n}^\varepsilon + \int\limits_{\Omega^\varepsilon} R_i^M(u_n^\varepsilon)\partial_t u_{i,n}^\varepsilon,
\end{aligned} \tag{57}$$

$$\|\partial_t v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \frac{b_i}{2}\partial_t\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} = a_i \int\limits_{\Gamma^\varepsilon} u_{i,n}^\varepsilon \partial_t v_{i,n}^\varepsilon. \tag{58}$$

Adding them, and finally integrating the result over $(0,t)$, we get:

$$\begin{aligned}
&\int_0^t \|\partial_t u_{i,n}^\varepsilon\|^2 + \int_0^t \|\partial_t v_{i,n}^\varepsilon\|^2\\
&+ \frac{d_0}{2}\|\nabla u_{i,n}^\varepsilon(t)\|^2 + \frac{\varepsilon a_0}{2}\|u_{i,n}^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)} + \frac{b_i}{2}\|v_{i,n}^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)}\\
&\le b_*\|u_{i,n}^\varepsilon(t)\|_{L^2(\Gamma^\varepsilon)}\|v_{i,n}^\varepsilon(t)\|_{L^2(\Gamma^\varepsilon)} + b_*\|u_{i,n}^\varepsilon(0)\|_{L^2(\Gamma^\varepsilon)}\|v_{i,n}^\varepsilon(0)\|_{L^2(\Gamma^\varepsilon)}\\
&+ \eta\int_0^t \|\partial_t v_{i,n}^\varepsilon\|^2 + \varepsilon^2 c^\eta b_*^2 \int_0^t \|u_{i,n}^\varepsilon\|^2 + \frac{d_*}{2}\|\nabla u_{i,n}^\varepsilon(0)\|^2\\
&+ \frac{\varepsilon a_*}{2}\|u_{i,n}^\varepsilon(0)\|^2_{L^2(\Gamma^\varepsilon)} + \frac{b_*}{2}\|v_{i,n}^\varepsilon(0)\|^2_{L^2(\Gamma^\varepsilon)}\\
&+ \eta\int_0^t \|\partial_t u_{i,n}^\varepsilon\|^2 + \delta_*^2 c^\delta c^\eta \|\bar\theta\|_\infty \int_0^t \|\nabla u_{i,n}^\varepsilon\|^2\\
&+ C^M C^\eta + \eta a_* \int_0^t \|\partial_t u_{i,n}^\varepsilon\|^2 \text{ for } t \in (0,T] \text{ and } \eta > 0.
\end{aligned}$$

Denoting the initial condition terms on the right as C_0 and using (55) and (56), we get:

$$\begin{aligned}
&\int_0^t (1-2\eta)\|\partial_t u_{i,n}^\varepsilon\|^2 + \int_0^t (1-\eta)\|\partial_t v_{i,n}^\varepsilon\|^2 + \frac{d_0}{2}\|\nabla u_{i,n}^\varepsilon(t)\|^2\\
&\quad \le C_0 + a_*\delta_* c^\delta c^\varepsilon \|\bar\theta\|_\infty \int\limits_0^T \|\nabla u_{i,n}^\varepsilon\|^2 + C^M C^\varepsilon \qquad \text{for } t \in (0,T].
\end{aligned} \tag{59}$$

Then by using (55), again, we have:

$$\|\nabla u_{i,n}^{\varepsilon}(t)\|^2 + \int_0^T \|\partial_t u_{i,n}^{\varepsilon}\|^2 + \int_0^T \|\partial_t v_{i,n}^{\varepsilon}\|^2 \leq C \quad \text{for } t \in (0,T],$$

where $C > 0$ depends on δ, M and T, but is independent of n and ε. Namely, this gives:

$$\{u_{i,n}^{\varepsilon}\} \text{ is bounded in } H^1(0,T;L^2(\Omega^{\varepsilon})) \cap L^{\infty}(0,T;H^1(\Omega^{\varepsilon})), \tag{60}$$

$$\{v_{i,n}^{\varepsilon}\} \text{ is bounded in } H^1(0,T;L^2(\Gamma^{\varepsilon})). \tag{61}$$

Hence, we can choose subsequences $u_{i,n_j}^{\varepsilon} \rightharpoonup u_i^{\varepsilon}$ in $H^1(0,T;L^2(\Omega^{\varepsilon}))$ and $u_{i,n_j}^{\varepsilon} \to u_i^{\varepsilon}$ in $C([0,T],L^2(\Omega^{\varepsilon}))$ and weakly* in $L^{\infty}(0,T;H^1(\Omega^{\varepsilon}))$ and $v_{i,n_j}^{\varepsilon} \rightharpoonup v_i^{\varepsilon}$ in $H^1(0,T;L^2(\Gamma^{\varepsilon}))$ as $j \to \infty$. Since R_i^M is Lipschitz continuous, the rest of the proof follows the same line of arguments as in Lemma 3.2. $\square$

Lemma 3.5. Positivity and boundedness of solutions to (P_2). *Let $\bar{\theta} \in K(T,M)$, $M > 0$ and assume (A_1)-(A_2). Then $0 \leq u_i^{\varepsilon} \leq M_i(T+1)$ a.e. in $(0,T) \times \Omega^{\varepsilon}$, $0 \leq v_i^{\varepsilon} \leq \bar{M}_i(T+1)$ a.e. on $(0,T) \times \Gamma^{\varepsilon}$, where $M_i > 0$ and $\bar{M}_i > 0$ are independent of M.*

Proof. Testing (40) with $\psi_i = -u_i^{\varepsilon,-}$ and the definition of R_i^M give:

$$\begin{aligned}
&\frac{1}{2}\partial_t \|u_i^{\varepsilon,-}\|^2 + d_0\|\nabla u_i^{\varepsilon,-}\|^2 + g_i\|u_i^{\varepsilon,-}\|^2_{L^2(\Gamma_R^{\varepsilon})} + \varepsilon a_0 \|u_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} + \varepsilon \int_{\Gamma^{\varepsilon}} b_i v_i^{\varepsilon} u_i^{\varepsilon,-} \\
&\quad \leq \delta_* c^{\delta} \|\bar{\theta}\|_{\infty} \int_{\Omega} |\nabla u_i^{\varepsilon,-} u_i^{\varepsilon,-}| - \int_{\Omega^{\varepsilon}} \sum_{j=1}^{i-1} \beta_{j,i-j} u_j^{\varepsilon,+} u_{i-j}^{\varepsilon,+} u_i^{\varepsilon,-} \\
&\qquad + \int_{\Omega^{\varepsilon}} \sum_{j=1}^{N} \beta_{ij} u_i^{\varepsilon,+} u_j^{\varepsilon,+} u_i^{\varepsilon,-}.
\end{aligned}$$

The second term on the right is always negative, while the third is always zero. We can discard them and apply Cauchy-Schwarz and Young's inequalities to the first term on the right, as well as discard the positive terms on the left to obtain:

$$\frac{1}{2}\partial_t \|u_i^{\varepsilon,-}\|^2 + (d_0^{\varepsilon} - \eta)\|\nabla u_i^{\varepsilon,-}\|^2 \leq \delta_* c^{\delta} c^{\eta} \|\bar{\theta}\|_{\infty} \|u_i^{\varepsilon,-}\|^2 + b_* \int_{\Gamma^{\varepsilon}} v_i^{\varepsilon,-} u_i^{\varepsilon,-} \text{ for } \eta > 0. \tag{62}$$

Testing (42) with $\varphi_i = -v_i^{\varepsilon,-}$ gives:

$$\frac{1}{2}\partial_t \|v_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} \leq b_* \|v_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} + a_* \int_{\Gamma^{\varepsilon}} v_i^{\varepsilon,-} u_i^{\varepsilon,-}. \tag{63}$$

We rely on Cauchy-Schwarz, Young's and trace inequalities to estimate the last term. We obtain:

$$\begin{aligned}
\int_{\Gamma^{\varepsilon}} v_i^{\varepsilon,-} u_i^{\varepsilon,-} &\leq \|v_i^{\varepsilon,-}\|_{L^2(\Gamma^{\varepsilon})} \|u_i^{\varepsilon,-}\|_{L^2(\Gamma^{\varepsilon})} \leq c^{\eta} \|v_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} + \eta \|u_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} \\
&\leq c^{\eta} \|v_i^{\varepsilon,-}\|^2_{L^2(\Gamma^{\varepsilon})} + \eta C(\|u_i^{\varepsilon,-}\|^2 + \|\nabla u_i^{\varepsilon,-}\|^2) \text{ for } \eta > 0.
\end{aligned}$$

Adding (62) and (63) and choosing $\eta + \eta C < d_0$ and taking into account that $u_i^{\varepsilon,-}(0) \equiv 0$ and $v_i^{\varepsilon,-}(0) \equiv 0$, Gronwall's lemma gives $\|u_i^{\varepsilon,-}\|^2 + \|v_i^{\varepsilon,-}\|^2 \le 0$, that is $u_i^\varepsilon \ge 0$ a.e. in Ω^ε and $v_i^\varepsilon \ge 0$ a.e. in Γ^ε for all $t \in (0,T]$.

Next, let $i = 1$ and $\psi_1 := (u_1^\varepsilon - M_1)^+$ in (40) and $\varphi_1 := (v_1^\varepsilon - \bar{M}_1)^+$ in (42). Apply (3) for the cross-diffusion term to get:

$$\begin{aligned}
&\frac{1}{2}\partial_t\|(u_1^\varepsilon - M_1)^+\|^2 + d_0\|\nabla(u_1^\varepsilon - M_1)^+\|^2 + \varepsilon a_0\|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)}\\
&+\varepsilon\int\limits_{\Gamma^\varepsilon}(a_1M_1 - b_1\bar{M}_1)(u_1^\varepsilon - M_1)^+ + \varepsilon\int\limits_{\Gamma^\varepsilon} b_1(v_1^\varepsilon - \bar{M}_1)^-(u_1^\varepsilon - M_1)^+\\
&\le \varepsilon\int\limits_{\Gamma^\varepsilon} b_1(v_1^\varepsilon - \bar{M}_1)^+(u_1^\varepsilon - M_1)^+ + \delta_* c^\delta\|\bar{\theta}\|_\infty\|\nabla(u_1^\varepsilon - M_1)^+(u_1^\varepsilon - M_1)^+\|_{L^1(\Omega^\varepsilon)}\\
&+\int_{\Omega^\varepsilon} R_1^M(u^\varepsilon)(u_1^\varepsilon - M_1)^+,\\
&\frac{1}{2}\partial_t\|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + b_1\|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + \int\limits_{\Gamma^\varepsilon} a_1(u_1^\varepsilon - M_1)^-(v_1^\varepsilon - \bar{M}_1)^+\\
&\le \int\limits_{\Gamma^\varepsilon} a_1(v_1^\varepsilon - \bar{M}_1)^+(u_1^\varepsilon - M_1)^+ + \int\limits_{\Gamma^\varepsilon}(a_1M_1 - b_1\bar{M}_1)(v_1^\varepsilon - \bar{M}_1)^+.
\end{aligned}$$

Here, by the definition we note that $R_1^M(u^\varepsilon) \le 0$. Also, we choose M_1 and $\bar{M}_1$ such that $a_1M_1 - b_1\bar{M}_1 = 0$ and add the two inequalities, while dropping the positive terms on the left and using Cauchy-Schwarz and Young's inequalities on the right to obtain:

$$\begin{aligned}
&\frac{1}{2}\partial_t\|(u_1 - M_1)^+\|^2 + (d_0 - \eta)\|\nabla(u_1 - M_1)^+\|^2 + \varepsilon a_0\|(u_1 - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)}\\
&+\frac{1}{2}\partial_t\|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)}\\
\le\ &(a_* + \varepsilon b_*)(\eta\|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + c^\eta\|(v_1^\varepsilon - \bar{M}_1)^+\|_{L^2(\Gamma^\varepsilon)})\\
&+c^\eta(\delta_* c^\delta)^2\|\bar{\theta}\|^2_\infty\|(u_1 - M_1)^+\|^2 \text{ for } \eta > 0.
\end{aligned}$$

Then by taking a small $\eta > 0$ Gronwall's lemma gives:

$$\begin{aligned}
&\|(u_1^\varepsilon - M_1)^+(t)\|^2 + \|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)}\\
&\le (\|(u_1^\varepsilon - M_1)^+(0)\|^2 + \|(v_1^\varepsilon - \bar{M}_1)^+(0)\|^2_{L^2(\Gamma^\varepsilon)})\exp\left(C(\delta_i^\varepsilon, \bar{\theta}, \delta, M)t\right).
\end{aligned}$$

Since we choose $M_1 > 0$ to satisfy $\|(u_1^\varepsilon - M_1)^+(0)\| = 0$, and $\bar{M}_1 > 0$ to satisfy $\|(v_1^\varepsilon - \bar{M}_1)^+(0)\|_{L^2(\Gamma^\varepsilon)} = 0$, we get $0 \le u_1^\varepsilon \le M_1$ and $0 \le v_1^\varepsilon \le \bar{M}_1$.

Let $i = 2$ and $\psi_2 := (u_2^\varepsilon - M_2(t+1))^+$ in (40) and $\varphi_2 := (v_2^\varepsilon - \bar{M}_2(t+1))^+$ in (42) with $a_2M_2 = b_2\bar{M}_2$:

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)})\\
&\quad + \frac{d_0}{2}\|\nabla(u_2^\varepsilon - M_2(t+1))^+\|^2\\
&\quad + \varepsilon a_2\|(u_2^\varepsilon - M_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)} + b_2\|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)}
\end{aligned}$$

$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \int\limits_{\Omega^\varepsilon} R_2^M(u^\varepsilon)(u_2^\varepsilon - M_2(t+1))^+$$

$$- M_2 \int\limits_{\Omega^\varepsilon} (u_2^\varepsilon - M_2(t+1))^+ - \bar{M}_2 \int_{\Gamma^\varepsilon} (v_2^\varepsilon - \bar{M}_2(t+1))^+.$$

Here, we note that

$$R_2^M(u^\varepsilon) \leq \frac{1}{2}\beta_{11}\sigma_M(u_1^\varepsilon)^2 \leq \frac{1}{2}\beta_{11}u_1^{\varepsilon,2} \leq \frac{1}{2}\beta_{11}M_1^2.$$

Similarly, we have:

$$\frac{1}{2}\partial_t(\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)})$$

$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + (\frac{1}{2}\beta_{11}M_1^2 - M_2)\int\limits_{\Omega^\varepsilon}(u_2^\varepsilon - M_2(t+1))^+$$

$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2.$$

By applying Gronwall's lemma with $\frac{1}{2}\beta_{11}M_1^2 \leq M_2$, we see that $u_2^\varepsilon \leq M_2(T+1)$ in $(0,T)\times\Omega^\varepsilon$ and $v_2^\varepsilon \leq \bar{M}_2(T+1)$ on $(0,T)\times\Gamma^\varepsilon$. Recursively, we can obtain the same estimates for u_i^ε and v_i^ε for $i \geq 3$. □

Lemma 3.6. The boundedness of the concentration gradient for (P_2). *Let $\bar{\theta} \in K(T, M_0)$ and assume (A_1)-(A_2) to hold. Then there exists a positive constant $C(M_0)$ such that $\|\nabla u_i^\varepsilon(t)\| \leq C(M_0)$ and $\int_0^T ||\partial_t u_i^\varepsilon(t)||^2 dt \leq C(M_0)$ for $t \in (0,T)$.*

Proof. Let $u_{i,n}^\varepsilon$ be an approximate solution defined in the proof of Lemma 3.4 for each n. Then from (59) there exists a positive constant $C(M_0)$ depending on M_0 such that

$$\int\limits_0^T \|\partial_t u_{i,n}^\varepsilon\|^2 \leq C(M_0), \qquad \text{for each } n. \tag{64}$$

By letting $n\to\infty$ we have proved this Lemma. □

Lemma 3.7. The boundedness of the temperature gradient for (P_1).*Let $\bar{u}_i \in K(T, M_0)$ and assume (A_1)-(A_2) to hold. Then there exists a positive constant $C(M_0)$ such that $\|\nabla\theta^\varepsilon(t)\| \leq C(M_0)$ and $\int_0^T \partial_t\|\theta^\varepsilon(t)||^2 dt \leq C(M_0)$ for $t \in (0,T)$.*

Proof. From (35) we can prove this lemma in the similar way to that of Lemma 3.6. □

Theorem 3.8. Existence and uniqueness of weak solutions (P^ε)*Let (A_1)-(A_2) hold. Then there exists a unique solution to (P^ε).*

Proof. For any $M > 0$, $X_M := K(M,T) \times K(M,T)^N$ is a closed set of $X := L^2(0,T;L^2(\Omega^\varepsilon))^{N+1}$. Let $\bar{\theta}_1, \bar{\theta}_2, \bar{u}_{i,1}, \bar{u}_{i,2} \in K(M,T)$, for $i \in \{1,\dots,N\}$, and put $\bar{\theta} := \bar{\theta}_1 - \bar{\theta}_2$, $\bar{u}_i := \bar{u}_{i,1} - \bar{u}_{i,2}$, $(\theta_1^\varepsilon, u_{i,1}^\varepsilon) = \mathbf{T}(\bar{\theta}_1, \bar{u}_1)$ and $(\theta_2^\varepsilon, u_{i,2}^\varepsilon) = \mathbf{T}(\bar{\theta}_2, \bar{u}_2)$, $v_{i,1}^\varepsilon = \mathbf{T}_2((\bar{\theta}_1, \bar{u}_1)$ and $v_{i,2}^\varepsilon = \mathbf{T}_2((\bar{\theta}_2, \bar{u}_2)$. Moreover, we define $\theta^\varepsilon = \theta_1^\varepsilon - \theta_2^\varepsilon$ and $u_i^\varepsilon = u_{i,1}^\varepsilon - u_{i,2}^\varepsilon$ and $v_i^\varepsilon = v_{i,1}^\varepsilon - v_{i,2}^\varepsilon$.

By Lemma 3.3 and Lemma 3.5, $\mathbf{T} : X_M \to X_M$ for $M > \max(\|\theta^{\varepsilon,0}\|_{L^\infty(\Omega^\varepsilon)}, M_1, M_2(T+1), \dots, M_N(T+1))$. Hence, we want to prove the existence of a positive constant $C < 1$ such that

$$\|\mathbf{T}(\bar{\theta}_1, \bar{u}_{i,1}) - \mathbf{T}(\bar{\theta}_2, \bar{u}_{i,2})\|_X \leq C\|(\bar{\theta}_1, \bar{u}_{i,1}) - (\bar{\theta}_2, \bar{u}_{i,2})\|_X$$

for small $T > 0$. Substituting $\theta_1^\varepsilon, \theta_2^\varepsilon, u_{i,1}^\varepsilon, u_{i,2}^\varepsilon, v_1^\varepsilon, v_2^\varepsilon$ into the formulation:

$$\int\limits_{\Omega^\varepsilon} \partial_t\theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon) + \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta_1^\varepsilon\nabla(\theta_1^\varepsilon - \theta_2^\varepsilon) + \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon)$$
$$= \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \tau^\varepsilon\nabla^\delta\bar{u}_{i,1}\cdot\nabla\theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon),$$
$$\int\limits_{\Omega^\varepsilon} \partial_t\theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon) + \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta_2^\varepsilon\nabla(\theta_2^\varepsilon - \theta_1^\varepsilon) + \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon)$$
$$= \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \tau^\varepsilon\nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon).$$

Adding the last two equations we obtain:

$$\frac{1}{2}\partial_t\|\theta^\varepsilon\|^2 + \kappa^{\varepsilon,0}\|\nabla\theta^\varepsilon\|^2 + g_0\|\theta^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)}$$
$$\le \tau_*\sum_{i=1}^N \Big|\underbrace{\int\limits_{\Omega^\varepsilon}(\nabla^\delta\bar{u}_{i,1}\cdot\nabla\theta_1^\varepsilon - \nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta_2^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon)}_{A}\Big|.$$

The term A can be expressed as:

$$A = \int\limits_{\Omega^\varepsilon}(\nabla^\delta\bar{u}_{i,1}\cdot\nabla\theta_1^\varepsilon - \nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta_1^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon)$$
$$+ \int\limits_{\Omega^\varepsilon}(\nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta_1^\varepsilon - \nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta_2^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon)$$
$$= \underbrace{\int\limits_{\Omega^\varepsilon}\nabla^\delta\bar{u}_i\cdot\nabla\theta_1^\varepsilon\theta^\varepsilon}_{A_1} + \underbrace{\int\limits_{\Omega^\varepsilon}\nabla^\delta\bar{u}_{i,2}\cdot\nabla\theta^\varepsilon\theta^\varepsilon}_{A_2}.$$

With the help of Lemma 3.7, the terms B and C can be estimated as follows:

$$A_1 \le c^\delta M\|\bar{u}_i\|^2 + C(M)^2\|\theta^\varepsilon\|^2,$$
$$A_2 \le c^\delta\|\bar{u}_{i,2}\|_\infty(\eta\|\nabla\theta^\varepsilon\|^2 + \frac{1}{4\eta}\|\theta^\varepsilon\|^2) \text{ for } \eta > 0.$$

Looking at the formulation for the concentrations, we have:

$$\int\limits_{\Omega^\varepsilon}\partial_t u_{i,1}^\varepsilon(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon) + \int\limits_{\Omega^\varepsilon} d_i^\varepsilon\nabla u_{i,1}^\varepsilon\cdot\nabla(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon)$$
$$+ \varepsilon a_i\int\limits_{\Gamma^\varepsilon} u_{i,1}^\varepsilon(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon) - \varepsilon b_i\int\limits_{\Gamma^\varepsilon} v_{i,1}^\varepsilon(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon)$$
$$= \int\limits_{\Omega^\varepsilon}\delta_i\delta_i\nabla^\delta\bar{\theta}_1\cdot u_{i,1}^\varepsilon(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon) + \int\limits_{\Omega^\varepsilon} R_i(u_1^\varepsilon)(u_{i,1}^\varepsilon - u_{i,2}^\varepsilon),$$

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \partial_t u^\varepsilon_{i,2}(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}) + \int_{\Omega^\varepsilon} d^\varepsilon_i \nabla u^\varepsilon_{i,2} \cdot \nabla(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}) \\
&\quad + \varepsilon a_i \int_{\Gamma^\varepsilon} u^\varepsilon_{i,2}(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}) - \varepsilon b_i \int_{\Gamma^\varepsilon} v^\varepsilon_{i,2}(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}) \\
&\quad = \int_{\Omega^\varepsilon} \delta_i \nabla^\delta \bar\theta_2 \cdot u^\varepsilon_{i,2}(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}) + \int_{\Omega^\varepsilon} R_i(u^\varepsilon_2)(u^\varepsilon_{i,2} - u^\varepsilon_{i,1}).
\end{aligned}$$

We also test the deposition equation with v^ε_i to obtain:

$$\frac{1}{2}\partial_t \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} = \int_{\Gamma^\varepsilon} a_i v^\varepsilon_i u^\varepsilon_i - b_i \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)}.$$

After adding the three above equations, we obtain:

$$\begin{aligned}
&\frac{1}{2}\partial_t \|u^\varepsilon_i\|^2 + \frac{1}{2}\partial_t \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + d_0 \|\nabla u^\varepsilon_i\|^2 + \varepsilon a_0 \|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} \\
&\quad \le (a_* + \varepsilon b_*) \int_{\Gamma^\varepsilon} |v^\varepsilon_i u^\varepsilon_i| + \int_{\Omega^\varepsilon} |(\nabla^\delta \bar\theta_1 \cdot \nabla u^\varepsilon_{i,1} - \nabla^\delta \bar\theta_2 \cdot \nabla u^\varepsilon_{i,2}) u^\varepsilon_i| \\
&\quad + \int_{\Omega^\varepsilon} |(R_i(u_1) - R_i(u_2)) u_i|,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\partial_t \|u^\varepsilon_i\|^2 + \frac{1}{2}\partial_t \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + d_0 \|\nabla u^\varepsilon_i\|^2 + (a_0 - \eta) \|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} \\
\le \quad & \frac{(a_* + \varepsilon b_*)^2}{4\eta} \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + \underbrace{\delta_* \int_{\Omega^\varepsilon} |\nabla^\delta \bar\theta_1 \cdot \nabla u^\varepsilon_i u^\varepsilon_i|}_{B_1} \\
&+ \underbrace{\delta_* \int_{\Omega^\varepsilon} |\nabla u^\varepsilon_{i,2} \cdot \nabla^\delta \bar\theta u^\varepsilon_i|}_{B_2} + \underbrace{\int_{\Omega^\varepsilon} |(R_i(u^\varepsilon_1) - R_i(u^\varepsilon_2)) u^\varepsilon_i|}_{B_3},
\end{aligned}$$

where the sub-expressions can be estimated as:

$$\begin{aligned}
&B_1 \le \eta \|\nabla u^\varepsilon_i\|^2 + \frac{1}{4\eta} c^\delta \|\bar\theta_1\|^2_\infty \|u^\varepsilon_i\|^2 \text{ for } \eta > 0, \\
&B_2 \le c^\delta C(M) \|\bar\theta\|^2 + C(M) \|u^\varepsilon_i\|^2.
\end{aligned}$$

Note that with the boundedness of u^ε_i we can treat R^M_i as a Lipschitz continuous function with the Lipschitz constant C_L:

$$B_3 \le C_L \|u^\varepsilon_i\|^2.$$

Adding up the estimates for the temperature and concentrations:

$$\begin{aligned}
&\frac{d}{dt}(\|u^\varepsilon_i\|^2 + \|v^\varepsilon_i\|^2 + \|\theta^\varepsilon\|^2) + d_0 \|\nabla u^\varepsilon_i\|^2 + \kappa_0 \|\nabla \theta^\varepsilon\|^2 \\
&\quad \le c_1 \|u^\varepsilon_i\|^2 + c_2 \|v^\varepsilon_i\|^2 + c_3 \|\theta^\varepsilon\|^2 + c^\delta M(\|\bar u_i\|^2 + \|\bar\theta\|^2).
\end{aligned}$$

Gronwall's lemma gives the estimate:

$$\|\theta^\varepsilon(t)\|^2 + \|u^\varepsilon_i(t)\|^2 \le C \left(\|\bar\theta\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} + \|\bar u_i\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} \right).$$

Integrating over $(0,T)$, we have:

$$\int_0^T \|\theta^\varepsilon(t)\|^2 + \|u_i^\varepsilon(t)\|^2 \le CT\left(\|\bar{\theta}\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} + \|\bar{u}_i\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))}\right).$$

Accordingly, $\mathbf{T}$ is a contraction mapping for T' such that $CT' < 1$. Then the Banach fixed point theorem shows that (P^ε)admits a unique solution in the sense of Definition 1 on $[0,T']$. Next, we consider (P^ε)on $[T',T]$. Then we can solve uniquely this problem on $[T',2T']$. Recursively, we can construct a solution of (P^ε)on the whole interval $[0,T]$. □

4. Passing to $\varepsilon \to 0$ (the homogenization limit).

4.1. Preliminaries on periodic homogenization. Now that the well-posedness of our microscopic system is available, we can investigate what happens as the parameter ε vanishes. Recall that ε defines both the microscopic geometry and the periodicity in the model parameters.

Definition 2. (Two-scale convergence [21],[1]). Let (u^ε) be a sequence of functions in $L^2(0,T;L^2(\Omega))$, where Ω is an open set in $\mathbb{R}^n$ and $\varepsilon > 0$ tends to 0. (u^ε) two-scale converges to a unique function $u_0(t,x,y) \in L^2((0,T)\times\Omega\times Y)$ if and only if for all $\phi \in C_0^\infty((0,T)\times\Omega, C_\#^\infty(Y))$ we have:

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u^\varepsilon\phi(t,x,\frac{x}{\varepsilon})dxdt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t,x,y)dydxdt. \tag{65}$$

We denote (65) by $u^\varepsilon \overset{2}{\rightharpoonup} u_0$.

The space $C_\#^\infty(Y)$ refers to the space of all Y-periodic C^∞-functions. The spaces $H_\#^1(Y)$ and $C_\#^\infty(\Gamma)$ have a similar meaning; the index # is always indicating that is about Y-periodic functions.

Theorem 4.1. *(Two-scale compactness on domains)*

(i) From each bounded sequence (u^ε) in $L^2(0,T;L^2(\Omega))$, a subsequence may be extracted which two-scale converges to $u_0(t,x,y) \in L^2((0,T)\times\Omega\times Y)$.

(ii) Let (u^ε) be a bounded sequence in $L^2(0,T;H^1(\Omega))$, then there exists $\tilde{u} \in L^2((0,T)\times\Omega;H_\#^1(Y))$ such that up to a subsequence (u^ε) two-scale converges to $u_0 \in L^2(0,T;L^2(\Omega))$ and $\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_0 + \nabla_y\tilde{u}$.

Proof. See e.g. [21],[1]. □

Definition 3. (Two-scale convergence for ε-periodic hypersurfaces [20]). A sequence of functions $(u^\varepsilon) \in L^2((0,T)\times\Gamma_\varepsilon)$ is said to two-scale converge to a limit $u_0 \in L^2((0,T)\times\Omega^\varepsilon\times\Gamma)$ if and only if for all $\phi \in C_0^\infty((0,T)\times\Omega^\varepsilon;C_\#^\infty(\Gamma))$ we have

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma_\varepsilon} u^\varepsilon\phi(t,x,\frac{x}{\varepsilon}) = \frac{1}{|Y|}\int_0^T\int_\Omega\int_\Gamma u_0(t,x,y)\phi(t,x,y)d\gamma_y dxdt. \tag{66}$$

Theorem 4.2. *(Two-scale compactness on surfaces)*

(i) From each bounded sequence $(u^\varepsilon) \in L^2((0,T)\times\Gamma_\varepsilon)$ one can extract a subsequence u^ε which two-scale converges to $u_0 \in L^2((0,T)\times\Omega\times\Gamma)$.

(ii) If a sequence (u^ε) is bounded in $L^\infty((0,T)\times\Gamma_\varepsilon)$, then u^ε two-scale converges to a $u_0 \in L^\infty((0,T)\times\Omega\times\Gamma)$

Proof. See [20] for proof of (i), and [17] for proof of (ii). □

Lemma 4.3. *Let (A_1)-(A_2)hold. Denote by u_i^ε and θ^ε the Bochner extensions*[1] *in the space $L^2(0,T;H^1(\Omega))$ of the corresponding functions originally belonging to $L^2(0,T;H^1(\Omega^\varepsilon))$. Then the following statement holds:*

(i) $u_i^\varepsilon \rightharpoonup u_i$ and $\theta^\varepsilon \rightharpoonup \theta$ in $L^2(0,T;H^1(\Omega))$,
(ii) $u_i^\varepsilon \overset{*}{\rightharpoonup} u_i$ and $\theta^\varepsilon \overset{*}{\rightharpoonup} \theta$ in $L^\infty((0,T)\times\Omega)$,
(iii) $\partial_t u_i^\varepsilon \rightharpoonup \partial_t u_i$ and $\partial_t\theta^\varepsilon \rightharpoonup \partial_t\theta$ in $L^2(0,T;L^2(\Omega))$,
(iv) $u_i^\varepsilon \to u_i$ and $\theta^\varepsilon \to \theta$ strongly in $L^2(0,T;H^\beta(\Omega))$ for $\frac{1}{2} < \beta < 1$ and $\sqrt{\varepsilon}\|u_i^\varepsilon - u_i\|_{L^2((0,T)\times\Gamma_\varepsilon)} \to 0$ as $\varepsilon \to 0$,
(v) $u_i^\varepsilon \overset{2}{\rightharpoonup} u_i$, $\nabla u_i^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_i + \nabla_y u_i^1$ where $u_i^1 \in L^2((0,T)\times\Omega;H^1_\#(Y))$,
(vi) $\theta^\varepsilon \overset{2}{\rightharpoonup} \theta$, $\nabla\theta^\varepsilon \overset{2}{\rightharpoonup} \nabla_x\theta + \nabla_y\theta^1$ where $\theta^1 \in L^2((0,T)\times\Omega;H^1_\#(Y))$,
(vii) $v_i^\varepsilon \overset{2}{\rightharpoonup} v_i \in L^\infty((0,T)\times\Omega\times\Gamma)$ and $\partial_t v_i^\varepsilon \overset{2}{\rightharpoonup} \partial_t v_i \in L^2((0,T)\times\Omega\times\Gamma)$.

Proof. We obtain (i) and (ii) as a direct consequence of the fact that u_i^ε and θ^ε are uniformly bounded in $L^\infty(0,T;H^1(\Omega))\cap L^\infty((0,T)\times\Omega)$. A similar argument gives (iii). We get (iv) using the compact embedding $H^\alpha(\Omega) \hookrightarrow H^\beta(\Omega)$ for $\beta \in (\frac{1}{2},1)$ and $0<\beta<\alpha\le 1$, since Ω has Lipschitz boundary. Note that (iv) implies the strong convergence of u_i^ε up to the boundary.

Denote $W := \{w \in L^2(0,T;H^1(\Omega))$ and $\partial_t w \in L^2(0,T;L^2(\Omega))\}$. We have $u_i^\varepsilon, \theta^\varepsilon \in W$. Using Lions-Aubin lemma [16] we see that W is compactly embedded in $L^2(0,T;H^\beta(\Omega))$ for $\beta \in [0.5,1]$. We then use the trace inequality for perforated medium from [13], namely for all $\phi \in H^1(\Omega^\varepsilon)$ there exists a constant C independent of ε such that:

$$\varepsilon\|\phi\|_{L^2(\Gamma^\varepsilon)} \le C(\|\phi\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon^2\|\nabla\phi\|^2_{L^2(\Omega^\varepsilon)}). \tag{67}$$

Applying (67) to $u_i^\varepsilon - u_i$, we get:

$$\begin{aligned}\sqrt{\varepsilon}\|u_i^\varepsilon - u_i\|^2_{L^2((0,T)\times\Gamma^\varepsilon)} &\le C\|u_i^\varepsilon - u_i\|^2_{L^2(0,T;H^\beta(\Omega^\varepsilon))}\\ &\le C\|u_i^\varepsilon - u_i\|^2_{L^2(0,T;H^\beta(\Omega))},\end{aligned} \tag{68}$$

where $\|u_i^\varepsilon - u_i\|^2_{L^2(0,T;H^\beta(\Omega))} \to 0$ as $\varepsilon \to 0$. As for the rest of the statements (v)-(vii), since u_i^ε are bounded in $L^\infty(0,T;H^1(\Omega))$, up to a subsequence we have that $u_i^\varepsilon \overset{2}{\rightharpoonup} u_i$ in $L^2(0,T;L^2(\Omega))$, and $\nabla u_i^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_i + \nabla_y u_i^1$, where $u_i^1 \in L^2((0,T)\times\Omega;H^1_\#(Y))$. By Theorem 4.2, $v_i^\varepsilon \overset{2}{\rightharpoonup} v_i \in L^\infty((0,T)\times\Omega\times\Gamma^\varepsilon)$ and $\partial_t v_i^\varepsilon \overset{2}{\rightharpoonup} \partial_t v_i \in L^2((0,T)\times\Omega\times\Gamma^\varepsilon)$. □

4.2. **Two-scale homogenization procedure.**

Theorem 4.4. *Let (A_1)-(A_2)hold. The limit functions θ, u_i, v_i, θ^1 and u_i^1 satisfy (72), (73) and (74) for any $\alpha \in C^\infty((0,T)\times\Omega)$ and $\beta \in C^\infty((0,T)\times\Omega;C^\infty_\#(Y))$.*

[1] For our choice of microstructure, the interior extension from $H^1(\Omega^\varepsilon)$ into $H^1(\Omega)$ exists and the corresponding extension constant is independent of the choice of ε; see the standard extension result reported in Lemma 5 from [13].

Proof. Testing $(P^\varepsilon)$with oscillating functions $\phi(t,x)=\alpha(t,x)+\varepsilon\beta(t,x,\frac{x}{\varepsilon})$, where $\alpha\in C^\infty((0,T)\times\Omega)$ and $\beta\in C^\infty((0,T)\times\Omega;C^\infty_\#(Y))$, we obtain:

$$\int\limits_{\Omega^\varepsilon}\partial_t\theta^\varepsilon(\alpha+\varepsilon\beta)+\int\limits_{\Omega^\varepsilon}\kappa^\varepsilon(\frac{x}{\varepsilon})\nabla\theta^\varepsilon(\nabla_x\alpha+\varepsilon\nabla_x\beta+\nabla_y\beta)$$

$$+g_0\varepsilon\int\limits_{\Gamma^\varepsilon}\theta^\varepsilon(\alpha+\varepsilon\beta)=\sum_{i=1}^N\int\limits_{\Omega^\varepsilon}\tau^\varepsilon(\frac{x}{\varepsilon})\nabla^\delta u_i^\varepsilon\cdot\nabla\theta^\varepsilon(\alpha+\varepsilon\beta),\tag{69}$$

$$\int\limits_{\Omega^\varepsilon}\partial_t u_i^\varepsilon(\alpha+\varepsilon\beta)+\int\limits_{\Omega^\varepsilon}d_i^\varepsilon(\frac{x}{\varepsilon})\nabla u_i^\varepsilon(\nabla_x\alpha+\varepsilon\nabla_x\beta+\nabla_y\beta)$$

$$+\varepsilon\int\limits_{\Gamma^\varepsilon}(a_iu_i^\varepsilon-b_iv_i^\varepsilon)(\alpha+\varepsilon\beta)$$

$$=\int\limits_{\Omega^\varepsilon}\delta_i^\varepsilon(\frac{x}{\varepsilon})\nabla^\delta\theta^\varepsilon\cdot\nabla u_i^\varepsilon(\alpha+\varepsilon\beta)+\int\limits_{\Omega^\varepsilon}R_i(u^\varepsilon)(\alpha+\varepsilon\beta),\tag{70}$$

$$\varepsilon\int\limits_{\Gamma^\varepsilon}\partial_t v_i^\varepsilon(\alpha+\varepsilon\beta)=\varepsilon\int\limits_{\Gamma^\varepsilon}(a_iu_i^\varepsilon-b_iv_i^\varepsilon)(\alpha+\varepsilon\beta).\tag{71}$$

Using the concept of two-scale convergence for $\varepsilon\to0$ in (69), (70) and (71) yields:

$$\int_\Omega\partial_t\theta\alpha+\frac{1}{|Y_1|}\int_\Omega\int_{Y_1}\kappa(y)(\nabla\theta+\nabla_y\theta^1)(\nabla_x\alpha(x)+\nabla_y\beta(x,y))$$

$$+g_0\frac{|\Gamma_R|}{|Y_1|}\int_\Omega\theta\alpha=\sum_{i=1}^N\frac{1}{|Y_1|}\int_\Omega\int_{Y_1}\tau(y)\nabla^\delta u_i\cdot(\nabla\theta+\nabla_y\theta^1)\alpha,\tag{72}$$

$$\int_\Omega\partial_t u_i\alpha+\frac{1}{|Y_1|}\int_\Omega\int_{Y_1}d_i(y)(\nabla u_i+\nabla_y u_i^1)(\nabla_x\alpha+\nabla_y\beta)$$

$$+\frac{1}{|Y_1|}\int_\Omega\int\limits_\Gamma(a_iu_i-b_iv_i)\alpha$$

$$=\frac{1}{|Y_1|}\int_\Omega\int_{Y_1}\delta_i(y)\nabla^\delta\theta\cdot(\nabla u_i+\nabla_y u_i^1)\alpha+\int_\Omega R_i(u)\alpha,\tag{73}$$

$$\int_\Omega\int_\Gamma\partial_t v_i\alpha=\frac{1}{|Y_1|}\int_\Omega\int\limits_\Gamma(a_iu_i-b_iv_i)\alpha.\tag{74}$$

Note that we have used strong convergence for passing to the limit in the aggregation term in (73). ∎

Now we just need to find θ^1 and u_i^1.

Lemma 4.5. *The limit functions θ_1 and u_i^1 depend linearly on θ and u_i as follows:*

$$\theta^1:=\sum_{j=1}^3\partial_{x_j}\theta\bar\theta^j,\tag{75}$$

$$u_i^1:=\sum_{j=1}^3\partial_{x_j}u_i\bar u_i^j.\tag{76}$$

Moreover, $\bar{\theta}^j$ *and* $\bar{u}_i^j$ *solve the elliptic problems on the cell: (77) and (78), respectively:*

$$\begin{cases} -\nabla_y \cdot (\kappa(y)\nabla_y \bar{\theta}^j) = \dfrac{\partial \kappa}{\partial y_j} & \text{in } Y_1, \\ \kappa \nabla_y \bar{\theta}^j \cdot \nu = -\kappa \nu_j & \text{on } \Gamma, \\ \bar{\theta}^j \text{ is periodic in } Y, \end{cases} \tag{77}$$

$$\begin{cases} -\nabla_y \cdot (d_i(y)\nabla_y \bar{u}_i^j) = \dfrac{\partial d_i}{\partial y_j} & \text{in } Y_1, \\ d_i \nabla_y \bar{u}_i^j \cdot \nu = -d_i \nu_j & \text{on } \Gamma, \\ \bar{\theta}^j \text{ is periodic in } Y, \end{cases} \tag{78}$$

Proof. To do this we choose $\alpha = 0$ in (72) and (73). This gives for all $\beta \in C^\infty((0,T)\times\Omega; C_\#^\infty(Y))$ a system of decoupled equations:

$$\int_\Omega \int_{Y_1} \kappa(y)(\nabla\theta + \nabla_y \theta^1)\nabla_y \beta(x,y) = 0, \tag{79}$$

$$\int_\Omega \int_{Y_1} d_i(y)(\nabla u_i + \nabla_y u_i^1)\nabla_y \beta(x,y) = 0. \tag{80}$$

From these equations we can easily get the assertion of this lemma. □

4.3. **Strong formulation for the limit functions.** Here, we give the strong formulation (P^0) for limit functions θ, u_i and v_i obtained by Lemma 4.3.

Lemma 4.6. (Strong formulation). *Assume (A_1)-(A_2) to hold. Then the triplet (θ, u_i, v_i) of limit functions of weak solutions to the microscopic model is a the weak solution of the following macroscopic problem:*

$$\partial_t\theta - \nabla\cdot(\mathbb{K}\nabla\theta) + g_0\frac{|\Gamma_R|}{|Y_1|}\theta = \sum_{i=1}^N (\mathbb{T}\nabla^\delta u_i)\cdot\nabla\theta \ \text{ in } (0,T)\times\Omega,$$
$$-(\mathbb{K}\nabla\theta)\cdot\nu = 0 \ \text{ on } (0,T)\times\partial\Omega,$$

where $\mathbb{K}$ and $\mathbb{T}^i$ are matrices given by $\mathbb{K} = K_0\mathbb{I} + (K_{ij})_{ij}$ and $\mathbb{T} = T_0\mathbb{I} + (T^i_{jk})_{jk}$, respectively, $\mathbb{I}$ is the identity matrix,

$$K_0 = \frac{1}{|Y_1|}\int_{Y_1}\kappa dy, \quad K_{ij} = \frac{1}{|Y_1|}\int_{Y_1}\kappa\frac{\partial\bar{\theta}^j}{\partial y_i}dy,$$
$$T_0^i = \frac{1}{|Y_1|}\int_{Y_1}\tau_i dy, \quad T_{jk}^i = \frac{1}{|Y_1|}\int_{Y_1}\tau_i\frac{\partial\bar{\theta}^j}{\partial y_k}dy,$$

and

$$\partial_t u_i - \nabla\cdot(\mathbb{D}^i\nabla u_i) + A_i u_i - B_i v_i = (\mathbb{F}^i\nabla u_i)\cdot\nabla^\delta\theta + R_i(u) \ \text{ in } (0,T)\times\Omega,$$
$$-(\mathbb{D}^i\nabla u_i)\cdot\nu = 0 \ \text{ on } (0,T)\times\partial\Omega,$$

where $\mathbb{D}^i$ and $\mathbb{F}^i$ are matrices defined by $\mathbb{D}^i = D_i\mathbb{I} + \mathbb{D}_0^i$ and $\mathbb{F}^i = F_i\mathbb{I} + \mathbb{F}_0^i$,

$$D_i = \frac{1}{|Y_1|}\int_{Y_1} d_i dy, \quad \mathbb{D}_0^i = (\frac{1}{|Y_1|}\int_{Y_1} d_i\partial_{y_k}\bar{u}_i^j dy)_{jk},$$
$$F_i = \frac{1}{|Y_1|}\int_{Y_1}\delta_i dy, \quad \mathbb{F}^i = (\frac{1}{|Y_1|}\int_{Y_1}\delta_i\partial_{y_k}\bar{u}_i^j dy)_{jk},$$
$$A_i = \frac{1}{|Y_1|}\int_\Gamma a_i, \quad B_i = \frac{1}{|Y_1|}\int_\Gamma b_i,$$

and initial conditions:

$$\theta(0,x) = \theta^0(x) \qquad \text{in } \Omega, \tag{81}$$

$$u_i(0,x) = u_i^0(x) \qquad \text{in } \Omega, \tag{82}$$

$$v_i(0,x) = v_i^0(x) \qquad \text{on } \Gamma. \tag{83}$$

Proof. First, choose $\alpha \in C^\infty((0,T)\times\Omega)$ and $\beta = 0$ in (72) to obtain:

$$\int_\Omega \partial_t \theta\alpha + \frac{1}{|Y_1|}\int_\Omega\int_{Y_1} \kappa(y)(\nabla\theta + \nabla_y(\sum_{j=1}^3 \partial_{x_j}\theta\bar{\theta}^j)\nabla_x\alpha(x))$$

$$+ g_0\frac{|\Gamma_R|}{|Y_1|}\int_\Omega \theta\alpha = \sum_{i=1}^N \frac{1}{|Y_1|}\int_\Omega\int_{Y_1} \tau(y)\nabla^\delta u_i \cdot (\nabla\theta + \nabla_y(\sum_{j=1}^3 \partial_{x_j}\theta\bar{\theta}^j)\alpha. \tag{84}$$

Integrating (84) w.r.t. y leads to:

$$\int_\Omega \partial_t\theta\alpha + \int_\Omega \mathbb{K}\nabla\theta\nabla_x\alpha + g_0\frac{1}{|Y_1|}\int_\Omega\int_{Y_1}\theta\alpha = \sum_{i=1}^N \int_\Omega \mathbb{T}\nabla^\delta u_i \cdot \nabla\theta\alpha. \tag{85}$$

We can similarly derive from (73) that:

$$\int_\Omega \partial_t u_i\alpha + \int_\Omega \mathbb{D}^i\nabla u_i\nabla_x\alpha + \int_\Omega (A_i u_i - B_i v_i)\alpha = \int_\Omega \mathbb{F}^i\nabla^\delta\theta\cdot\nabla u_i\alpha + \int_\Omega R_i(u)\alpha, \tag{86}$$

$$\int_\Omega \partial_t v_i\alpha = \int_\Omega (A_i u_i - B_i v_i)\alpha. \tag{87}$$

See also [17] and [9] for a similar application of the two-scale convergence method. □

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