A thermo-diffusion system with Smoluchowski interactions : well-posedness and homogenization

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 14-09
April 2014

A thermo-diffusion system with Smoluchowski interations: well-posedness and homogenization

by

O. Krehel, A. Muntean, T. Aiki

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

A thermo-diffusion system with Smoluchowski interactions: well-posedness and homogenization

Oleh Krehel and Adrian Muntean and Toyohiko Aiki

April 4, 2014

Oleh Krehel

Department of Mathematics and Computer Science
CASA - Center for Analysis, Scientific computing and Engineering
Eindhoven University of Technology
5600 MB, PO Box 513, Eindhoven, The Netherlands

Toyohiko Aiki

Department of Mathematical and Physical Sciences, Faculty of Science
Japan Women University, Tokyo, Japan

Adrian Muntean

Department of Mathematics and Computer Science
CASA - Center for Analysis, Scientific computing and Engineering
ICMS - Institute for Complex Molecular Systems
Eindhoven University of Technology
5600 MB, PO Box 513, Eindhoven The Netherlands

Abstract

We study the solvability and homogenization of a thermal-diffusion reaction problem posed in a periodically perforated domain. The system describes the motion of populations of hot colloidal particles interacting together via Smoluchowski production terms. The upscaled system, obtained via two-scale convergence techniques, allows the investigation of deposition effects in porous materials in the presence of thermal gradients.

1 Introduction

We aim at understanding processes driven by coupled fluxes through media with microstructures. In this paper, we study a particular type of coupling: we look at the interplay between diffusion fluxes of a fixed number of colloidal populations and a heat flux, the effects included here incorporating an approximation of the Dufour and Sorret effects (cf. Section 2.3, see also [10]. The type of system of evolution equations that we encounter in Section 2.4 resembles very much cross-diffusion and chemotaxis-like systems; see [28, 8], e.g. The structure of the chosen equations is useful in investigating transport, interaction, and deposition of a large numbers of hot multiple-sized particles in porous media.

Practical applications of our approach would include predicting the response of refractory concrete to high-temperatures exposure in steel furnances, heat pollution from open geothermal wells, propagation of combustion waves due to explosions in tunnels, drug delivery in soils and in biological tissues, etc.; see for instance [3, 4, 24, 27, 9]. In a follow-up paper [13] we will study quantitatively some of these effects, focussing on colloids deposition under thermal gradients. Within this framework, our focus lies exclusively on two distinct theoretical aspects:

(i) the mathematical understanding of the microscopic problem (i.e. the well-posedness of the starting system);

(ii) the averaging of the thermal-diffusion system over arrays of periodically-distributed microstructures (the so-called, *homogenization asymptotics limit*; see, for instance, [5, 18] and references cited therein).

The complexity of the microscopic system makes numerical simulations on the macro scale very expensive. That is the reason why the aspect (ii) is of concern here. Obviously, the study does not close with these questions. Many other issues like derivation of corrector estimates, design of convergent numerical multiscale schemes, multiscale parameter identification etc. need also to be treated. Possible generalizations could point out to coupling heat transfer with Nernst-Planck-Stokes systems (extending [23]) or with semiconductor equations [17].

The paper is structured in the following manner. We present the basic notation and explain the multiscale geometry as well as some of the relevant physical processes in Section 2.

Section 3 contains the proof of the solvability of the microstructure model. Finally, the homogenization procedure is performed in Section 4. This is also the place where we list our macroscopic equations together with their effective coefficients.

2 Notations and Assumptions

2.1 Model description and geometry

The geometry of the problem is depicted in Figure 1, given a scale factor $\varepsilon > 0$.

$(0,T)$	= time interval of interest
Ω	= bounded domain in $\mathbb{R}^n$
$\partial\Omega$	$= \Gamma_R^u \cup \Gamma_N^u = \Gamma_R^\theta \cup \Gamma_D^\theta$ piecewise smooth boundary of Ω, $\Gamma_R^u \cap \Gamma_N^u = \Gamma_R^\theta \cap \Gamma_D^\theta = \emptyset$
$\vec{e}_i$	= ith unit vector in $\mathbb{R}^n$
Y	$= \{\sum_{i=1}^n \lambda_i \vec{e}_i : 0 < \lambda_i < 1\}$ unit cell in $\mathbb{R}^n$
Y_0	= open subset of Y that represents the solid grain
Y_1	$= Y \setminus \overline{Y}_0$
Γ	$= \partial Y_0$ piecewise smooth boundary of Y_0
X^k	$= X + \sum_{i=1}^n k_i \vec{e}_i$, where $k \in \mathbb{Z}^n$ and $X \subset Y$.

For simplicity, assume that Ω is a parallelipiped in $\mathbb{R}^n$. Then we define:

Ω_0^ε	$= \cup\{\varepsilon Y_0^k : Y_0^k \subset \Omega^\varepsilon, k \in \mathbb{Z}^n\}$ array of pores
Ω^ε	$= \Omega \setminus \overline{\Omega}_0^\varepsilon$ matrix skeleton
Γ^ε	$= \partial\Omega_0^\varepsilon$ pore boundaries.

We are dealing with a periodic system of cells, where each cell is the reference (standard) cell scaled by a small factor ε, which relates the the pore length scale to the domain length scale. The standard cell is a square region with a circular grain inclusion.

The cells regions without the grain $\varepsilon Y_1$are filled with water and we denote their union by Ω^ε. Colloidal species are dissolved in the pore water. They react between themselves and participate in diffusion and convective transport. The colloidal matter cannot penetrate the grain boundary Γ^ε, but it deposits there reducing the amount of mass floating inside Ω^ε. Here $\partial\Omega^\varepsilon = \Gamma_N^\varepsilon \cup \Gamma_R^\varepsilon \cup \Gamma^\varepsilon$, where $\Gamma_N^\varepsilon \cap \Gamma_R^\varepsilon = \emptyset$. Γ_N^ε is impenetrable, while Γ_R^ε admits flux. Here Γ_N^ε and Γ_R^ε are portions of the macroscopic part of $\partial\Omega^\varepsilon$.

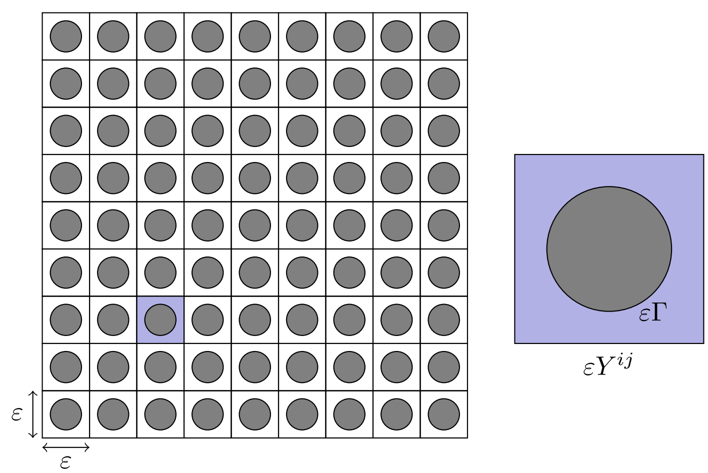

Figure 1: Porous medium geometry. Ω_0^ε is marked with gray color and Ω^ε is white.

The unknowns are:

- θ^ε – the temperature in Ω^ε.
- u_i^ε – the concentration of the species that contains i monomers in Ω^ε.
- v_i^ε – the mass of the deposited species on Γ^ε.

Furthermore, for a given $\delta > 0$, we introduce the mollifier

$$J_\delta(s) := \begin{cases} Ce^{1/(|s|^2-\delta^2)} & \text{if } |s| < \delta, \\ 0 & \text{if } |s| \geq \delta, \end{cases} \tag{1}$$

where the constant $C > 0$ is selected such that

$$\int_{\mathbb{R}^d} J_\delta = 1,$$

see [7, pp. 629-630] for details.

Using J_δ from (1), define the mollified gradient:

$$\nabla^\delta f := \nabla \left[\int_{B(x,\delta)} J_\delta(x-y) f(y) dy \right]. \tag{2}$$

The following statement holds for all $f \in L^\infty(\Omega^\varepsilon)$, $g \in L^p(\Omega^\varepsilon)^d$ and $1 \le p \le \infty$:

$$\|\nabla^\delta f \cdot g\|_{L^p(\Omega^\varepsilon)} \le c^\delta \|f\|_{L^\infty(\Omega^\varepsilon)} \|g\|_{L^p(\Omega^\varepsilon)^d}, \tag{3}$$

$$\|\nabla^\delta f\|_{L^p(\Omega^\varepsilon)} \le c^\delta \|f\|_{L^2(\Omega^\varepsilon)}. \tag{4}$$

In the equations below all norms are $L^2(\Omega^\varepsilon)$ unless specified otherwise, with c^δ independent of the choice of ε.

2.2 Smoluchowski population balance equations

We want to model the transport of aggregating colloidal particles under the influence of thermal gradients. For this purpose, we use the Smoluchowski population balance equation, originally proposed in [26], to account for colloidal aggregation:

$$R_i(s) := \frac{1}{2} \sum_{k+j=i} \beta_{kj} s_k s_j - \sum_{j=1}^{N} \beta_{ij} s_i s_j, \qquad i \in \{1, \dots, N\};\ N > 2. \tag{5}$$

Here s_i is the concentration of the colloidal species that consists of i monomers, N is the number of species, i.e. the maximal aggregate size that we consider, $R_i(s)$ is the rate of change of s_i, and $\beta_{ij} > 0$ are the coagulation coefficients, which tell us the rate aggregation between particles of size i and j [6]. Colloidal aggregation rates are described in more detail in [14].

2.3 Soret and Dufour effects

The structure of our target system is inspired by the model proposed by Shigesada, Kawasaki and Teramoto [25] in 1979 when they've studied the segregation of competing species. For the case of two interacting species u and v, the diffusion term looks like:

$$\partial_t u = \Delta(d_1 u + \alpha uv), \tag{6}$$

where the second term in the flux is due to cross-diffusion. The second term can be expressed as:

$$\Delta(uv) = u\Delta v + v\Delta u + 2\nabla u \cdot \nabla v \tag{7}$$

As a first step in our approach, we consider only the last term of (7) as the driving force of cross-diffusion and we postpone the study of terms $u\Delta v$ and $v\Delta u$ until later.

2.4 Setting of the model equations

We consider the following balance equations for the temperature and colloid concentrations:

(P^ε)

$$\partial_t \theta^\varepsilon + \nabla \cdot (-\kappa^\varepsilon \nabla \theta^\varepsilon) - \tau^\varepsilon \sum_{i=1}^{N} \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon = 0, \qquad \text{in } (0,T) \times \Omega^\varepsilon, \tag{8}$$

$$\partial_t u_i^\varepsilon + \nabla \cdot (-d_i^\varepsilon \nabla u_i^\varepsilon) - \delta_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon = R_i(u^\varepsilon), \qquad \text{in } (0,T) \times \Omega^\varepsilon, \tag{9}$$

with boundary conditions:

$$
\begin{aligned}
&-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = 0, && \text{on } (0,T)\times\Gamma_N^\varepsilon, && (10)\\
&-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = \varepsilon g_0 \theta^\varepsilon, && \text{on } (0,T)\times\Gamma_R^\varepsilon, && (11)\\
&-\kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nu = 0, && \text{on } (0,T)\times\Gamma^\varepsilon, && (12)\\
&-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu = 0, && \text{on } (0,T)\times\Gamma_N^\varepsilon, && (13)\\
&-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu = \varepsilon g_i u_i^\varepsilon, && \text{on } (0,T)\times\Gamma_R^\varepsilon, && (14)
\end{aligned}
$$

and a boundary condition for colloidal deposition:

$$
\begin{aligned}
&-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu = \varepsilon(a_i u_i^\varepsilon - b_i v_i^\varepsilon), && \text{on } (0,T)\times\Gamma^\varepsilon, && (15)\\
&\partial_t v_i^\varepsilon = a_i u_i^\varepsilon - b_i v_i^\varepsilon, && \text{on } (0,T)\times\Gamma^\varepsilon. && (16)
\end{aligned}
$$

As initial conditions, we take for $i \in \{1, \dots, N\}$:

$$
\begin{aligned}
&\theta^\varepsilon(0,x) = \theta^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon, && (17)\\
&u_i^\varepsilon(0,x) = u_i^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon, && (18)\\
&v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x), && \text{on } \Gamma^\varepsilon. && (19)
\end{aligned}
$$

κ^ε	heat conduction coefficient
d_i^ε	diffusion coefficient
τ^ε	Soret coefficient
δ^ε	Dufour coefficient
g_i	Robin boundary coefficient, $i \in \{0, \dots, N\}$
a_i	Deposition coefficient 1, $i \in \{1, \dots, N\}$
b_i	Deposition coefficient 2, $i \in \{1, \dots, N\}$.

We will refer to (8)- (19) as (P^ε) – our reference microscopic model. Note that the Soret and Dufour coefficients determine the structure of the particular cross-diffusion system (see [10], [25] [2], [3], [21], [28]). a_i and b_i describe the deposition interaction between u_i^ε and v_i^ε. Each u_i^ε has a different affinity to sediment as well as a different mass.

All functions defined in Ω^ε and on Γ^ε are taken to be ε-periodic, i.e. $\kappa^\varepsilon(x) = \kappa(x/\varepsilon)$ and so on.

2.5 Assumptions on data

(A_1) κ^ε, τ^ε, d_i^ε and δ_i^ε are functions of the variable x for $i \in \{1, \dots, N\}$ and ε, and g_i, a_i and b_i are positive constants. The meaning of the notation κ^ε is as follows: $\kappa^\varepsilon(x) = \kappa(\frac{x}{\varepsilon})$ (and similarly for all other coefficients with upper index ε), where κ is a bounded measurable function on Y. Moreover, $\kappa_0 \le \kappa \le \kappa_*$, $\tau \le \tau_*$, $d_0 \le d_i \le d_*$, $\delta_i \le \delta_*$ for $i \in \{1, \dots, N\}$ and $\epsilon > 0$, where $\kappa_0, \kappa_*, d_0, d_*, \delta_*$ are positive constants.

(A_2) $\theta^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $u_i^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $v_i^{\varepsilon,0} \in L_+^\infty(\Gamma^\varepsilon)$ for $i \in \{1, \dots, N\}$ and $\varepsilon > 0$. Moreover, $||\theta^{\varepsilon,0}||_{H^1(\Omega^\varepsilon)} \le C_0$, $||u_i^{\varepsilon,0}||_{H^1(\Omega^\varepsilon)} \le C_0$, and $||v_i^{\varepsilon,0}||_{L^\infty(\Gamma^\varepsilon)} \le C_0$ for $i \in \{1, \dots, N\}$ and $\varepsilon > 0$. where C_0 is a positive constant.

3 Global solvability of problem (P^ε)

Definition 1. *The triplet* $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ *is a solution to problem* (P^ε) *if the following holds:*

$$\begin{aligned}&\theta^\varepsilon, u_i^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap L^\infty(0,T;H^1(\Omega^\varepsilon)) \cap L^\infty((0,T)\times\Omega^\varepsilon),\\ &v_i^\varepsilon \in H^1(0,T;L^2(\Gamma^\varepsilon)) \cap L^\infty((0,T)\times\Gamma^\varepsilon),\end{aligned} \tag{20}$$

for all $\phi \in H^1(\Omega^\varepsilon)$:

$$\int_{\Omega^\varepsilon} \partial_t\theta^\varepsilon\phi + \int_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla\phi + \varepsilon g_0 \int_{\Gamma_R^\varepsilon} \theta^\varepsilon\phi = \tau^\varepsilon \sum_{i=1}^N \int_{\Omega^\varepsilon} \nabla^\delta u_i^\varepsilon \cdot \nabla\theta^\varepsilon\phi, \tag{21}$$

for all $\psi_i \in H^1(\Omega^\varepsilon)$:

$$\begin{aligned}&\int_{\Omega^\varepsilon} \partial_t u_i^\varepsilon\psi_i + \int_{\Omega^\varepsilon} d_i^\varepsilon\nabla u_i^\varepsilon\cdot\nabla\psi_i + \varepsilon g_i\int_{\Gamma_R^\varepsilon} u_i^\varepsilon\psi_i + \varepsilon\int_{\Gamma^\varepsilon}(a_iu_i^\varepsilon - b_iv_i^\varepsilon)\psi_i\\ &\qquad = \delta_i^\varepsilon\int_{\Omega^\varepsilon}\nabla^\delta\theta^\varepsilon\cdot\nabla u_i^\varepsilon\psi_i + \int_{\Omega^\varepsilon} R_i(u^\varepsilon)\psi_i,\end{aligned} \tag{22}$$

for all $\varphi_i \in L^2(\Gamma^\varepsilon)$:

$$\int_{\Gamma^\varepsilon} \partial_t v_i^\varepsilon\varphi_i = \int_{\Gamma^\varepsilon}(a_iu_i^\varepsilon - b_iv_i^\varepsilon)\varphi_i. \tag{23}$$

together with (17), (18) and (19) for a fixed value of $\varepsilon > 0$.

To prove the existence of solutions to problem (P^ε), we introduce the following auxiliary problems:

(P_1)

$$\begin{aligned}&\partial_t\theta^\varepsilon + \nabla\cdot(-\kappa^\varepsilon\nabla\theta^\varepsilon) - \tau^\varepsilon\sum_{i=1}^N \nabla^\delta \bar{u^\varepsilon}_i\cdot\nabla\theta^\varepsilon = 0, && \text{in } (0,T)\times\Omega^\varepsilon,\\ &-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\Gamma_N^\varepsilon,\\ &-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = \varepsilon g_0\theta^\varepsilon, && \text{on } (0,T)\times\Gamma_R^\varepsilon,\\ &-\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\Gamma^\varepsilon,\\ &\theta^\varepsilon(0,x) = \theta^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon,\end{aligned}$$

and

(P_2)

$$\begin{aligned}&\partial_t u_i^\varepsilon + \nabla\cdot(-d_i^\varepsilon\nabla u_i^\varepsilon) - \delta_i^\varepsilon\nabla^\delta\bar{\theta^\varepsilon}\cdot\nabla u_i^\varepsilon = R_i^M(u^\varepsilon), && \text{in } (0,T)\times\Omega^\varepsilon,\\ &-d_i^\varepsilon\nabla u_i^\varepsilon\cdot\nu = 0, && \text{on } (0,T)\times\Gamma_N^\varepsilon,\\ &-d_i^\varepsilon\nabla u_i^\varepsilon\cdot\nu = \varepsilon g_iu_i^\varepsilon, && \text{on } (0,T)\times\Gamma_R^\varepsilon,\\ &-d_i^\varepsilon\nabla u_i^\varepsilon\cdot\nu = \varepsilon(a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon), && \text{on } (0,T)\times\Gamma^\varepsilon,\\ &u_i^\varepsilon(0,x) = u_i^{\varepsilon,0}(x), && \text{in } \Omega^\varepsilon,\\ &\partial_t v_i^\varepsilon = a_iu_i^\varepsilon - b_iv_i^\varepsilon, && \text{on } (0,T)\times\Gamma^\varepsilon,\\ &v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x), && \text{on } \Gamma^\varepsilon.\end{aligned}$$

Here

$$R_i^M(s) := R_i(\sigma_M(s_1), \sigma_M(s_2), \ldots, \sigma_M(s_N)), \text{ for } s \in \mathbb{R}^N \tag{24}$$

denotes our choice of truncation of R_i, where

$$\sigma_M(r) := \begin{cases} 0 & \text{for } r < 0 \\ r & \text{for } r \in [0, M] \\ M & \text{for } r > M, \end{cases} \tag{25}$$

where $M > 0$ is a fixed threshold.

In the following, assuming (A_1)-(A_2), we show the existence, positivity and boundedness of solutions to (P_1) and (P_2).

When we denote the solutions as $\theta^\varepsilon = P_1(\bar{u^\varepsilon})$ and $(u_i^\varepsilon, v_i^\varepsilon) = P_2(\bar{\theta^\varepsilon})$, we can define the solution operator $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon) = \mathbf{T}(\bar{\theta^\varepsilon}, \bar{u^\varepsilon}_i)$. We will show that the operator $\mathbf{T}$ is a contraction in the appropriate functional spaces and use the Banach fixed-point theorem to prove the existence and uniqueness of solutions to (P^ε).

Let $K(T, M) := \{z \in L^2(0, T; L^2(\Omega^\varepsilon)) : |z| \le M \text{ a.e. on } (0, T) \times \Omega^\varepsilon\}$.

Lemma 3.1. Existence of solutions to (P_1).
Let $\bar{u^\varepsilon}_i \in K(T, M)$, and assume that (A_1)-(A_2) hold.
Then there exists $\theta^\varepsilon \in H^1(0, T; L^2(\Omega^\varepsilon)) \cap L^\infty(0, T; H^1(\Omega^\varepsilon))$
that solves (P_1) in the sense:

for all $\phi \in H^1(\Omega^\varepsilon)$ and a.e. in $[0, T]$:

$$\int_{\Omega^\varepsilon} \partial_t \theta^\varepsilon \phi + \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \theta^\varepsilon \cdot \nabla \phi + \varepsilon g_0 \int_{\Gamma_R^\varepsilon} \theta^\varepsilon \phi = \tau^\varepsilon \sum_{i=1}^N \int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla \theta^\varepsilon \phi, \tag{26}$$

and

$$\theta^\varepsilon(0, x) = \theta^{\varepsilon,0}(x) \qquad a.e. \text{ in } \Omega^\varepsilon. \tag{27}$$

Proof. Let $\{\xi_i\}$ be a Schauder basis of $H^1(\Omega^\varepsilon)$. Then for each $n \in \mathbb{N}$ there exists

$$\theta_n^{\varepsilon,0}(x) := \sum_{j=1}^n \alpha_j^{0,n} \xi_j(x) \text{ such that } \theta_n^{\varepsilon,0} \to \theta^{\varepsilon,0} \text{ in } H^1(\Omega^\varepsilon) \text{ as } n \to \infty. \tag{28}$$

We denote by θ_n^ε the Galerkin approximation of θ^ε, that is:

$$\theta_n^\varepsilon(t, x) := \sum_{j=1}^n \alpha_j^n(t) \xi_j(x) \qquad \text{for all } (t, x) \in (0, T) \times \Omega^\varepsilon. \tag{29}$$

By definition, θ_n^ε must satisfy (26) for all $\phi \in \operatorname{span}\{\xi_i\}_{i=1}^n$, i.e.:

$$\int_{\Omega^\varepsilon} \partial_t \theta_n^\varepsilon \phi + \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \theta_n^\varepsilon \cdot \nabla \phi + \varepsilon g_0 \int_{\Gamma_R^\varepsilon} \theta_n^\varepsilon \phi = \tau^\varepsilon \sum_{i=1}^N \int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla \theta_n^\varepsilon \phi. \tag{30}$$

The coefficients $\alpha_i^n(t)$ can be found by testing (30) with $\phi := \xi_i$ and using (28) to solve the resulting ODE system:

$$\begin{cases} \partial_t \alpha_i^n(t) + \sum_{j=1}^{n} (A_{ij} + B_{ij} - C_{ij}) \alpha_j^n(t) = 0, & i \in \{1, \dots, n\}, \qquad (31) \\ \alpha_i^n(0) = \alpha_i^{0,n}. & \qquad (32) \end{cases}$$

The coefficients in (31) and (32) are defined by the following expressions

$$A_{ij} := \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \xi_i \cdot \nabla \xi_j, \qquad i, j \in \{1, \dots, n\},$$

$$B_{ij} := \varepsilon g_0 \int_{\Gamma_R^\varepsilon} \xi_i \xi_j, \qquad i, j \in \{1, \dots, n\},$$

$$C_{ij} := \tau^\varepsilon \sum_{k=1}^{N} \int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_k \cdot \nabla \xi_j \xi_i \qquad i, j \in \{1, \dots, n\}.$$

Since the system (31) is linear, there exists for each fixed $n \in \mathbb{N}$ a unique solution $\alpha_i^n \in C^1([0,T])$.

To show uniform estimates for θ_n^ε with respect to n, we take in (30) $\phi = \theta_n^\varepsilon$. We obtain:

$$\frac{1}{2} \partial_t \|\theta_n^\varepsilon\|^2 + \kappa_0 \|\nabla \theta_n^\varepsilon\|^2 + \varepsilon g_0 \|\theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \le \tau^\varepsilon \sum_{i=1}^{N} \int_{\Omega^\varepsilon} |\nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla \theta_n^\varepsilon \theta_n^\varepsilon| := \tau^\varepsilon \sum_{i=1}^{N} A_i.$$

Using the Cauchy-Schwarz inequality and Young's inequality in the form $ab \le \eta a^2 + b^2/4\eta$, where $\eta > 0$, we get:

$$A_i \le \eta \|\nabla \theta_n^\varepsilon\|^2 + \frac{1}{4\eta} \|\nabla^\delta \bar{u^\varepsilon}_i \theta_n^\varepsilon\|^2 \le \eta \|\nabla \theta_n^\varepsilon\|^2 + \frac{1}{4\eta} \|\nabla^\delta \bar{u^\varepsilon}_i\|^2_{L^4(\Omega^\varepsilon)} \|\theta_n^\varepsilon\|^2_{L^4(\Omega^\varepsilon)}.$$

The mollifier property (3) yields $\|\nabla^\delta \bar{u^\varepsilon}_i\|^2_{L^4(\Omega^\varepsilon)} \le c^\delta \|\bar{u^\varepsilon}_i\|^2_\infty$. Using Gagliardo-Nirenberg inequality(see [22]) we get:

$$\|\theta_n^\varepsilon\|^2_{L^4(\Omega^\varepsilon)} \le c \|\theta_n^\varepsilon\|^{1/2} \|\nabla \theta_n^\varepsilon\|^{3/2}. \qquad (33)$$

Applying Young's inequality, we obtain:

$$c \|\theta_n^\varepsilon\|^{1/2} \|\nabla \theta_n^\varepsilon\|^{3/2} \le \eta \|\nabla \theta_n^\varepsilon\|^2 + c_\eta \|\theta_n^\varepsilon\|^2. \qquad (34)$$

Finally, we obtain the structure:

$$\frac{1}{2} \partial_t \|\theta_n^\varepsilon\|^2 + (\kappa_0 - 2N\eta) \|\nabla \theta_n^\varepsilon\|^2 + g_0 \|\nabla \theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \le c_\eta^\delta \sum_{i=1}^{N} \|\bar{u^\varepsilon}_i\|^2 \|\theta_n^\varepsilon\|^2.$$

Gronwall's lemma gives:

$$\|\theta_n^\varepsilon(t)\|^2 + \kappa_0 \int_0^t \|\nabla \theta_n^\varepsilon(t)\|^2 < C \qquad \text{for } t \in (0,T),$$

where $C > 0$ is independent of n. This ensues that

$$\theta_n^\varepsilon \in L^\infty(0,T;L^2(\Omega^\varepsilon)) \cap L^2(0,T;H^1(\Omega^\varepsilon)). \tag{35}$$

To show uniform estimates for $\partial_t\theta_n^\varepsilon$ with respect to n, we take $\phi = \partial_t\theta_n^\varepsilon$ in (30) and use the Cauchy-Schwarz and Young's inequalities, as well as the mollifier property (3) to get: For $\eta > 0$

$$\begin{aligned}\|\partial_t\theta_n^\varepsilon\|^2 + \frac{1}{2}\partial_t\|\kappa^\varepsilon\nabla\theta_n^\varepsilon\|^2 + \varepsilon\frac{g_0}{2}\partial_t\|\theta_n^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} &\le \tau^\varepsilon\sum_{i=1}^N\int_{\Omega^\varepsilon}|\nabla^\delta\bar{u^\varepsilon}_i\cdot\nabla\theta_n^\varepsilon\partial_t\theta_n^\varepsilon| \\ \le \left(c^\delta\tau^\varepsilon\sum_{i=1}^N\|\bar{u^\varepsilon}_i\|_{L^\infty(\Omega^\varepsilon)}\right)&(\eta\|\partial_t\theta_n^\varepsilon\|^2 + \frac{C_\eta}{\kappa_0}\|\kappa^\varepsilon\nabla\theta_n^\varepsilon\|^2).\end{aligned}$$

Gronwall's lemma gives:

$$\|\kappa^\varepsilon\nabla\theta_n^\varepsilon\|^2 + \int_0^t\|\partial_t\theta_n^\varepsilon\|^2 < C \qquad\qquad \text{for all } t\in(0,T),$$

where $C > 0$ depends on δ, but is independent of n. Together with (35) this ensues that:

$$\theta_n^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap L^\infty(0,T;H^1(\Omega^\varepsilon)). \tag{36}$$

Hence, we can choose a subsequence $\theta_{n_i}^\varepsilon \rightharpoonup \theta^\varepsilon$ in $H^1(0,T;L^2(\Omega^\varepsilon))$ and $\theta_{n_i}^\varepsilon \overset{*}{\rightharpoonup} \theta^\varepsilon$ in $L^\infty(0,T;H^1(\Omega^\varepsilon))$ as $i\to\infty$.

Now, using

$$v_m(t,x) := \sum_{j=1}^m \beta_j^m(t)\xi_j(x) \tag{37}$$

as a test function in (30) and integrating with respect to time we get:

$$\begin{aligned}\int_0^T\int_{\Omega^\varepsilon}\partial_t\theta_{n_i}^\varepsilon v_m + \int_0^T\int_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta_{n_i}^\varepsilon\cdot\nabla v_m + \varepsilon g_0\int_0^T\int_{\Gamma_R^\varepsilon}\theta_{n_i}^\varepsilon v_m \\ = \tau^\varepsilon\sum_{i=1}^N\int_0^T\int_{\Omega^\varepsilon}\nabla^\delta\bar{u^\varepsilon}_i\cdot\nabla\theta_{n_i}^\varepsilon v_m.\end{aligned} \tag{38}$$

Using (36), we pass to the limit as $i\to\infty$ to obtain:

$$\int_0^T\int_{\Omega^\varepsilon}\partial_t\theta^\varepsilon v_m + \int_0^T\int_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla v_m + \varepsilon g_0\int_0^T\int_{\Gamma_R^\varepsilon}\theta^\varepsilon v = \tau^\varepsilon\sum_{i=1}^N\int_0^T\int_{\Omega^\varepsilon}\nabla^\delta\bar{u^\varepsilon}_i\cdot\nabla\theta^\varepsilon v_m. \tag{39}$$

Note that (39) holds for all $v\in L^2(0,T;H^1(\Omega^\varepsilon))$ since we can approximate v with v_m in $L^2(0,T;H^1(\Omega^\varepsilon))$, hence

$$\int_{\Omega^\varepsilon}\partial_t\theta^\varepsilon v + \int_{\Omega^\varepsilon}\kappa^\varepsilon\nabla\theta^\varepsilon\cdot\nabla v + \varepsilon g_0\int_{\Gamma_R^\varepsilon}\theta^\varepsilon v = \tau^\varepsilon\sum_{i=1}^N\int_{\Omega^\varepsilon}\nabla^\delta\bar{u^\varepsilon}_i\cdot\nabla\theta^\varepsilon v,$$

holds for all $v \in H^1(\Omega)$ and a.e. $t \in [0,T]$.

To prove $\theta^\varepsilon(0) = \theta^{\varepsilon,0}$, note first from (39) that:

$$
\begin{aligned}
&-\int_0^T\int_{\Omega^\varepsilon} \theta^\varepsilon \partial_t v + \int_0^T\int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nabla v + \varepsilon g_0 \int_0^T\int_{\Gamma_R^\varepsilon} \theta^\varepsilon v \\
&\qquad = \tau^\varepsilon \sum_{i=1}^N \int_0^T\int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla\theta^\varepsilon v + \int_{\Omega^\varepsilon} \theta^\varepsilon(0) v(0).
\end{aligned}
\tag{40}
$$

We get a similar term from (38):

$$
\begin{aligned}
&-\int_0^T\int_{\Omega^\varepsilon} \theta^\varepsilon_{n_i} \partial_t v_m + \int_0^T\int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla\theta^\varepsilon_{n_i} \cdot \nabla v_m + \varepsilon g_0 \int_0^T\int_{\Gamma_R^\varepsilon} \theta^\varepsilon_{n_i} v_m \\
&\qquad = \tau^\varepsilon \sum_{i=1}^N \int_0^T\int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla\theta^\varepsilon_{n_i} v_m + \int_{\Omega^\varepsilon} \theta^\varepsilon_{n_i}(0) v_m(0).
\end{aligned}
\tag{41}
$$

Using now (36) and $\theta^\varepsilon_{n_i}(0) \to \theta^{\varepsilon,0}$ as $i \to \infty$ gives:

$$
\begin{aligned}
&-\int_0^T\int_{\Omega^\varepsilon} \theta^\varepsilon \partial_t v + \int_0^T\int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla\theta^\varepsilon \cdot \nabla v + \varepsilon g_0 \int_0^T\int_{\Gamma_R^\varepsilon} \theta^\varepsilon v \\
&\qquad = \tau^\varepsilon \sum_{i=1}^N \int_0^T\int_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla\theta^\varepsilon v + \int_{\Omega^\varepsilon} \theta^{\varepsilon,0} v(0).
\end{aligned}
\tag{42}
$$

Comparing (40) and (42), since $v(0)$ is chosen arbitrarily, we get $\theta^\varepsilon(0) = \theta^{\varepsilon,0}$. □

Lemma 3.2. Positivity and boundedness of solutions to (P_1).
Let $\bar{u^\varepsilon}_i \in K(T,M)$, $M > 0$, and assume (A_1)-(A_2).
Then $0 \le \theta^\varepsilon \le \|\theta^{\varepsilon,0}\|_{L^\infty(\Omega^\varepsilon)}$ a.e. in $(0,T)\times\Omega^\varepsilon$.

Proof. Let $\theta^\varepsilon := \theta^{\varepsilon,+} - \theta^{\varepsilon,-}$, where $z^+ := \max(z,0)$ and $z^- := \max(-z,0)$. Testing (26) with $\phi := -\theta^{\varepsilon,-}$ gives for $\eta > 0$ that

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\theta^{\varepsilon,-}\|^2 + \kappa_0\|\nabla\theta^{\varepsilon,-}\|^2 + \varepsilon g_0\|\theta^{\varepsilon,-}\|^2_{L^2(\Gamma_R^\varepsilon)} \le c^\delta\tau^\varepsilon \sum_{i=1}^N \|\bar{u^\varepsilon}_i\|_\infty \|\nabla\theta^{\varepsilon,-}\theta^{\varepsilon,-}\|_{L^1(\Omega^\varepsilon)} \\
&\le \left(C^\delta_\eta \tau^\varepsilon \sum_{i=1}^N \|\bar{u^\varepsilon}_i\|_\infty\right)\|\theta^{\varepsilon,-}\|^2 + \varepsilon\|\nabla\theta^{\varepsilon,-}\|^2.
\end{aligned}
$$

Choosing $\eta < \kappa^0$ and taking into account that $\theta^{\varepsilon,-}(0) = 0$, Gronwall's lemma gives $\|\theta^{\varepsilon,-}\|^2 \le 0$. This means $\theta^\varepsilon \ge 0$ a.e. in Ω for all $t \in (0,T)$.

Let $\phi = (\theta^\varepsilon - M_0)^+$ in (26) with $M_0 \geq \|\theta^\varepsilon(0)\|_{L^\infty(\Omega^\varepsilon)}$:

$$\frac{1}{2}\partial_t\|(\theta^\varepsilon - M_0)^+\|^2 + \kappa_0\|\nabla(\theta^\varepsilon - M_0)^+\|^2 + \varepsilon g_0\|(\theta^\varepsilon - M_0)^+\|^2_{L^2(\Gamma_R^\varepsilon)}$$
$$+ \int\limits_{\Gamma_R^\varepsilon} M_0(\theta^\varepsilon - M_0)^+ \leq \tau^\varepsilon \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla(\theta^\varepsilon - M_0)^+(\theta^\varepsilon - M_0)^+$$
$$\leq \left(\tau^\varepsilon c^\delta \sum_{i=1}^N \|\bar{u^\varepsilon}_i\|_\infty\right)\left(c_\eta\|(\theta^\varepsilon - M_0)^+\|^2 + \eta\|\nabla(\theta^\varepsilon - M_0)^+\|^2\right).$$

Discarding the positive terms on the left side and then applying Gronwall's lemma leads to:

$$\|(\theta^\varepsilon - M_0)^+(t)\|^2 \leq \|(\theta^\varepsilon - M_0)^+(0)\|^2 \exp\left(\tau^\varepsilon c^\delta c_\eta \sum_{i=1}^N \|\bar{u^\varepsilon}_i\|_\infty t\right).$$

Since $\|(\theta^\varepsilon - M_0)^+(0)\| = 0$, we obtain $(\theta^\varepsilon - M_0)^+(t) = 0$. Thus the proof of the lemma is completed. ◻

Lemma 3.3. Existence of solutions to (P_2).
Let $\bar{\theta^\varepsilon} \in K(T, M), M > 0$ and (A_1)-(A_2) hold.
Then (P_2) has solutions $u_i^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap L^\infty(0,T;H^1(\Omega))$ and $v_i^\varepsilon \in H^1(0,T;L^2(\Gamma^\varepsilon))$ in the following sense:

For all $\psi_i \in H^1(\Omega^\varepsilon)$, it holds:

$$\int\limits_{\Omega^\varepsilon} \partial_t u_i^\varepsilon \psi_i + \int\limits_{\Omega^\varepsilon} d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla\psi_i + \varepsilon g_i \int\limits_{\Gamma_R^\varepsilon} u_i^\varepsilon \psi_i + \varepsilon \int\limits_{\Gamma^\varepsilon} (a_i u_i^\varepsilon - b_i v_i^\varepsilon)\psi_i$$
$$= \delta_i^\varepsilon \int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{\theta^\varepsilon} \cdot \nabla u_i^\varepsilon \psi_i + \int\limits_{\Omega^\varepsilon} R_i^M(u^\varepsilon)\psi_i \tag{43}$$

$$u_i^\varepsilon(0,x) = u_i^{\varepsilon,0}(x) \quad a.e.\ in\ \Omega^\varepsilon, \tag{44}$$

and for all $\varphi_i \in L^2(\Gamma^\varepsilon)$:

$$\int\limits_{\Gamma^\varepsilon} \partial_t v_i^\varepsilon \varphi_i = \int\limits_{\Gamma^\varepsilon} (a_i u_i^\varepsilon - b_i v_i^\varepsilon)\varphi_i, \tag{45}$$

$$v_i^\varepsilon(0,x) = v_i^{\varepsilon,0}(x) \quad a.e.\ on\ \Gamma^\varepsilon \tag{46}$$

Proof. Let $\{\xi_j\}$ – Schauder basis of $H^1(\Omega^\varepsilon)$. Then, for each $n \in \mathbb{N}$, there exists

$$u_{i,n}^{\varepsilon,0}(x) := \sum_{j=1}^n \alpha_{i,j}^{0,n}\xi_j(x) \text{ such that } u_{i,n}^{\varepsilon,0} \to u_i^{\varepsilon,0} \text{ in } H^1(\Omega^\varepsilon) \text{ as } n \to \infty. \tag{47}$$

We denote by $u_{i,n}^\varepsilon$ the Galerkin approximation of u_i^ε, that is:

$$u_{i,n}^\varepsilon(t,x) := \sum_{j=1}^n \alpha_{i,j}^n(t)\xi_j(x), \qquad \text{for all } (t,x) \in (0,T) \times \Omega^\varepsilon. \tag{48}$$

$u_{i,n}^{\varepsilon}$ must satisfy (43):

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \partial_t u_{i,n}^{\varepsilon} \psi_i + \int_{\Omega^\varepsilon} d_i^{\varepsilon} \nabla u_{i,n}^{\varepsilon} \cdot \nabla \psi_i + \varepsilon g_i \int_{\Gamma_R^\varepsilon} u_{i,n}^{\varepsilon} \psi_i + \varepsilon \int_{\Gamma^\varepsilon} (a_i u_{i,n}^{\varepsilon} - b_i v_i^{\varepsilon}) \psi_i \\
&\qquad = \delta_i^{\varepsilon} \int_{\Omega^\varepsilon} \nabla^{\delta} \bar{\theta}^{\varepsilon} \cdot \nabla u_{i,n}^{\varepsilon} \psi_i + \int_{\Omega^\varepsilon} R_i^M(u_n^{\varepsilon}) \psi_i, \qquad \text{for all } \psi_i \in \operatorname{span}\{\xi_j\}_{j=1}^n.
\end{aligned} \tag{49}$$

Accordingly, let $\{\eta_j\}$ – an orthonormal basis of $L^2(\Gamma^\varepsilon)$. Then for each $n \in \mathbb{N}$ there exists

$$v_{i,n}^{\varepsilon,0}(x) := \sum_{j=1}^{n} \beta_{i,j}^{0,n} \eta_j(x) \text{ such that } v_{i,n}^{\varepsilon,0} \to v_i^{\varepsilon,0} \text{ in } L^2(\Gamma^\varepsilon) \text{ as } n \to \infty. \tag{50}$$

We denote by $v_{i,n}^{\varepsilon}$ the Galerkin approximation of v_i^{ε}, that is:

$$v_{i,n}^{\varepsilon}(t,x) := \sum_{j=1}^{n} \beta_{i,j}^{n}(t) \eta_j(x), \qquad \text{for all } (t,x) \in (0,T) \times \Gamma^\varepsilon. \tag{51}$$

It must satisfy (45):

$$\int_{\Gamma^\varepsilon} \partial_t v_{i,n}^{\varepsilon} \varphi_i = \int_{\Gamma^\varepsilon} (a_i u_{i,n}^{\varepsilon} - b_i v_{i,n}^{\varepsilon}) \varphi_i, \qquad \text{for all } \varphi_i \in \operatorname{span}\{\eta_j\}_{j=1}^n. \tag{52}$$

$\alpha_{i,j}^n(t)$ and $\beta_{i,j}^n(t)$ can be found by substituting $u_{i,n}^{\varepsilon}$ and $v_{i,n}^{\varepsilon}$ into (43) – (46) and using ξ_k and η_k for $k \in \{1, \ldots, n\}$ as test functions:

$$\begin{cases}
\partial_t \alpha_{i,k}^n(t) + \sum_{j=1}^{n} (A_{ijk} + B_{ijk} + C_{ijk} - D_{ijk}) \alpha_{i,j}^n(t) - \sum_{j=1}^{n} E_{ijk} \beta_{i,j}^n(t) \\
\qquad = \int_{\Omega^\varepsilon} \xi_k \sum_{a=1}^{i-1} \beta_{a,i-a} \sigma_M \left(\sum_{b=1}^{n} \alpha_{a,b}^n(t) \xi_b \right) \sigma_M \left(\sum_{c=1}^{n} \alpha_{i-a,c}^n(t) \xi_c \right) \\
\qquad - \int_{\Omega^\varepsilon} \xi_k \sum_{a=1}^{N} \beta_{a,i} \sigma_M \left(\sum_{b=1}^{n} \alpha_{i,b}^n(t) \xi_b \right) \sigma_M \left(\sum_{c=1}^{n} \alpha_{a,c}^n(t) \xi_c \right), & (53) \\
\alpha_{i,j}^n(0) = \alpha_{i,j}^{0,n}, & (54) \\
\partial_t \beta_{i,k}^n(t) = \sum_{j=1}^{n} G_{ijk} \alpha_{i,j}^n(t) - H_{ijk} \beta_{i,j}^n(t), & (55) \\
\beta_{i,j}^n(0) = \beta_{i,j}^{0,n}. & (56)
\end{cases}$$

The coefficients in (53) are defined by:

$$A_{ijk} := \int\limits_{\Omega^\varepsilon} d_i^\varepsilon \nabla\xi_j \cdot \nabla\xi_k, \qquad B_{ijk} := \varepsilon g_i \int\limits_{\Gamma_R^\varepsilon} \xi_j\xi_k,$$

$$C_{ijk} := \varepsilon a_i \int\limits_{\Gamma^\varepsilon} \xi_j\xi_k, \qquad D_{ijk} := \delta_i^\varepsilon \int\limits_{\Omega^\varepsilon} \nabla^\delta\bar{\theta}^\varepsilon \cdot \nabla\xi_j\xi_k,$$

$$E_{ijk} := \varepsilon b_i \int\limits_{\Gamma^\varepsilon} \xi_k\eta_j, \qquad G_{ijk} := \varepsilon a_i \int\limits_{\Gamma^\varepsilon} \xi_j\eta_k,$$

$$H_{ijk} := \varepsilon b_i \int\limits_{\Gamma^\varepsilon} \eta_j\eta_k.$$

The left-hand side of this system of ODEs is linear, while the right-hand side is globally Lipschitz. Thus there exists a unique solution $\alpha_{i,j}^n(t), \beta_{i,j}^n(t) \in H^1(0,T)$ to (53)–(56) for $t \in (0,T)$.

To show uniform in n estimates for $u_{i,n}^\varepsilon$ and $v_{i,n}^\varepsilon$, we take $\psi_i = u_{i,n}^\varepsilon$ and $\varphi_i = v_{i,n}^\varepsilon$ in (49) and (52), respectively. We get the inequalities: For $\eta > 0$,

$$\begin{aligned}
&\frac{1}{2}\partial_t\|u_{i,n}^\varepsilon\|^2 + d_0\|\nabla u_{i,n}^\varepsilon\|^2 + \varepsilon g_i\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} + \varepsilon a_1\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \\
\leq\ & \varepsilon b_1 \int_{\Gamma^\varepsilon} |v_{i,n}^\varepsilon u_{i,n}^\varepsilon| + \delta_* c^\delta\|\bar{\theta}^\varepsilon\|_\infty\|\nabla u_{i,n}^\varepsilon\|\|u_{i,n}^\varepsilon\| + \int_{\Omega^\varepsilon} R_i^M(u_n^\varepsilon)u_{i,n}^\varepsilon \\
\leq\ & \varepsilon\eta\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \varepsilon C_\eta\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \eta\|\nabla u_{i,n}^\varepsilon\|^2 + C_\eta\|u_{i,n}^\varepsilon\|^2 + C^M + \|u_{i,n}^\varepsilon\|^2,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\partial_t\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + b_1\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} &\leq \eta\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + C_\eta\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \\
&\leq \eta C\|u_{i,n}^\varepsilon\|^2_{H^1(\Omega^\varepsilon)} + C_\eta\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \\
&\leq \eta C\|\nabla u_{i,n}^\varepsilon\|^2 + C\|u_{i,n}^\varepsilon\|^2 + C_\eta\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)}.
\end{aligned}$$

After adding two inequalities, Gronwall's lemma gives:

$$\|u_{i,n}^\varepsilon\|^2 + d_0\int_0^t \|\nabla u_{i,n}^\varepsilon\|^2 + \|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} < C \qquad \text{for all } t \in (0,T),$$

where $C > 0$ depends on δ, M and T, but is independent of n and ε, which ensures:

$$(u_{i,n}^\varepsilon) \text{ is bounded in } L^\infty(0,T;L^2(\Omega^\varepsilon) \text{ and } L^2(0,T;H^1(\Omega^\varepsilon)),$$

$$(v_{i,n}^\varepsilon) \text{ is bounded in } L^\infty(0,T;L^2(\Gamma^\varepsilon)).$$

To show uniform estimates for $\partial_t u_{i,n}^\varepsilon$ and $\partial_t v_{i,n}^\varepsilon$ with respect to n, we take $\psi_i = \partial_t u_{i,n}^\varepsilon$ and $\varphi_i = \partial_t v_{i,n}^\varepsilon$ in (49) and (52), respectively, and obtain for $\eta > 0$ that

$$\begin{aligned}
&\|\partial_t u_{i,n}^\varepsilon\|^2 + \partial_t\|\sqrt{d_i^\varepsilon}\nabla u_{i,n}^\varepsilon\|^2 + \frac{\varepsilon g_i}{2}\partial_t\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} + \frac{\varepsilon a_i}{2}\partial_t\|u_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \\
\leq\ & \varepsilon\int_{\Gamma^\varepsilon} b_i v_{i,n}^\varepsilon\partial_t u_{i,n}^\varepsilon + \delta_* c^\delta\|\bar{\theta}^\varepsilon\|_\infty\|\nabla u_{i,n}^\varepsilon\|\|\partial_t u_{i,n}^\varepsilon\| + \int_{\Omega^\varepsilon} R_i^M(u_n^\varepsilon)\partial_t u_{i,n}^\varepsilon \\
\leq\ & \varepsilon\partial_t\int_{\Gamma^\varepsilon} b_i v_{i,n}^\varepsilon u_{i,n}^\varepsilon - \int_{\Gamma^\varepsilon} b_i\partial_t v_{i,n}^\varepsilon u_{i,n}^\varepsilon + 2\eta\|\partial_t u_{i,n}^\varepsilon\|^2 + C_\eta\|\nabla u_{i,n}^\varepsilon\|^2 + C_M \\
\leq\ & \varepsilon\partial_t\int_{\Gamma^\varepsilon} b_i v_{i,n}^\varepsilon u_{i,n}^\varepsilon + \eta\|v_{i,n}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + C_\eta\|u_{i,n}^\varepsilon\|^2 + 2\eta\|\partial_t u_{i,n}^\varepsilon\|^2 + C_\eta\|\nabla u_{i,n}^\varepsilon\|^2 + C_M,
\end{aligned}$$

$$\begin{aligned}\|\partial_t v_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 + \frac{b_i}{2}\|v_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 &\le \eta\|v_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 + C_\eta\|u_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \\ &\le \eta\|v_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 + C_\eta\|u_{i,n}^\varepsilon\|_{H^1(\Omega^\varepsilon)}^2.\end{aligned}$$

By adding two inequalities and integrating it, we can get

$$\int_0^t \|\partial_t u_{i,n}^\varepsilon\|^2 + \|\nabla u_{i,n}^\varepsilon\|^2 + \int_0^t \|\partial_t v_{i,n}^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \le C \quad \text{for all } t \in (0,T),$$

where $C > 0$ depends on δ, M and T, but is independent of n and ε. Together with (58) this gives:

$$(u_{i,n}^\varepsilon) \text{ is bounded in } H^1(0,T;L^2(\Omega^\varepsilon) \text{ and } L^\infty(0,T;H^1(\Omega^\varepsilon)),$$

$$(v_{i,n}^\varepsilon) \text{ is bounded in } H^1(0,T;L^2(\Gamma^\varepsilon)).$$

Hence, we can choose subsequences $u_{i,n_j}^\varepsilon \rightharpoonup u_i^\varepsilon$ in $H^1(0,T;L^2(\Omega^\varepsilon))$ and $u_{i,n_j}^\varepsilon \to u_i^\varepsilon$ in $C([0,T],L^2(\Omega^\varepsilon))$ and weakly* in $L^\infty(0,T;H^1(\Omega^\varepsilon))$ as $i \to \infty$ and $v_{i,n_j}^\varepsilon \rightharpoonup v_i^\varepsilon$ in $H^1(0,T;L^2(\Gamma^\varepsilon))$ as $j \to \infty$. Since R_i^M is Lipschitz continuous, the rest of the proof follows the same line of arguments as in Lemma 3.1. □

Lemma 3.4. Positivity and boundedness of solutions to (P_2).
Let $\bar\theta^\varepsilon \in K(T,M)$, $M > 0$ and assume (A_1)-(A_2). Then $0 \le u_i^\varepsilon \le M_i(T+1)$ a.e. in $(0,T)\times\Omega^\varepsilon$, $0 \le v_i^\varepsilon \le \bar M_i(T+1)$ a.e. on $(0,T)\times\Gamma^\varepsilon$, where $M_i > 0$ and $\bar M_i > 0$ are independent of M.

Proof. Testing (43) with $\psi_i = -u_i^{\varepsilon,-}$ gives:

$$\begin{aligned}&\frac{1}{2}\partial_t\|u_i^{\varepsilon,-}\|^2 + d_0\|\nabla u_i^{\varepsilon,-}\|^2 + \varepsilon g_i\|u_i^{\varepsilon,-}\|_{L^2(\Gamma_R^\varepsilon)}^2 + \varepsilon a_i\|u_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 + \varepsilon b_i\int_{\Gamma^\varepsilon} v_i^\varepsilon u_i^{\varepsilon,-} \\ &\quad\le \delta_i^\varepsilon c^\delta\|\bar\theta^\varepsilon\|_\infty\int_\Omega \nabla u_i^{\varepsilon,-}u_i^{\varepsilon,-} - \int_{\Omega^\varepsilon}\sum_{j=1}^{i-1}\beta_{j,i-j}u_j^{\varepsilon,+}u_{i-j}^{\varepsilon,+}u_i^{\varepsilon,-} + \int_{\Omega^\varepsilon}\sum_{j=1}^{N}\beta_{ij}u_i^{\varepsilon,+}u_j^{\varepsilon,+}u_i^{\varepsilon,-}.\end{aligned}$$

The second term on the right is always negative, while the third is always zero. We can discard them and apply Cauchy-Schwarz and Young's inequalities to the first term on the right, as well as discard the positive terms on the left to obtain for $\eta > 0$ that

$$\frac{1}{2}\partial_t\|u_i^{\varepsilon,-}\|^2 + (d_0-\eta)\|\nabla u_i^{\varepsilon,-}\|^2 \le \delta_i^\varepsilon c^\delta\|\bar\theta^\varepsilon\|_\infty c^\eta\|u_i^{\varepsilon,-}\|^2 + \varepsilon b_i\int_{\Gamma^\varepsilon} v_i^{\varepsilon,-}u_i^{\varepsilon,-}. \quad (57)$$

Testing (45) with $\varphi_i = -v_i^{\varepsilon,-}$ gives:

$$\frac{1}{2}\partial_t\|v_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 \le b_i\|v_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 + a_i\int_{\Gamma^\varepsilon} v_i^{\varepsilon,-}u_i^{\varepsilon,-} \quad (58)$$

We rely on Cauchy-Schwarz, Young's and trace inequalities to estimate the last term. For $\eta > 0$, we obtain

$$\begin{aligned}\int_{\Gamma^\varepsilon} v_i^{\varepsilon,-}u_i^{\varepsilon,-} &\le \|v_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}\|u_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)} \le c^\eta\|v_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 + \eta\|u_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 \\ &\le c^\eta\|v_i^{\varepsilon,-}\|_{L^2(\Gamma^\varepsilon)}^2 + \eta C(\|u_i^{\varepsilon,-}\|^2 + \|\nabla u_i^{\varepsilon,-}\|^2).\end{aligned}$$

Adding (57) and (58) and choosing $\eta + \eta C < d_0$ and taking into account that $u_i^{\varepsilon,-}(0) \equiv 0$, Gronwall's lemma gives $\|u_i^{\varepsilon,-}\|^2 + \|v_i^{\varepsilon,-}\|^2 \leq 0$, that is $u_i^\varepsilon \geq 0$ a.e. in Ω^ε and $v_i^\varepsilon \geq 0$ a.e. in Γ^ε for all $t \in (0, T)$.

Let $i = 1$ and $\psi = (u_1^\varepsilon - M_1)^+$ in (43) and $\varphi = (v_1^\varepsilon - \bar{M}_1)^+$ in (45). Apply (3) for the cross-diffusion term to get:

$$
\begin{aligned}
& \frac{1}{2}\partial_t \|(u_1^\varepsilon - M_1)^+\|^2 + d_0 \|\nabla (u_1^\varepsilon - M_1)^+\|^2 + \varepsilon g_1 \|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma_R^\varepsilon)} \\
& + \varepsilon g_1 \int_{\Gamma_R^\varepsilon} M_1 (u_1^\varepsilon - M_1)^+ \\
& + \varepsilon a_1 \|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + \varepsilon \int_{\Gamma^\varepsilon} (a_1 M_1 - b_1 \bar{M}_1)(u_1^\varepsilon - M_1)^+ \\
& - \varepsilon \int_{\Gamma^\varepsilon} b_1 (v_1^\varepsilon - \bar{M}_1)(u_1^\varepsilon - M_1)^+ \\
\leq \; & \delta_* c^\delta \|\bar{\theta}\|_\infty \|\nabla (u_1^\varepsilon - M_1)^+ (u_1^\varepsilon - M_1)^+\|_{L^1(\Omega^\varepsilon)} + \int_{\Omega^\varepsilon} R_1(u)(u_1^\varepsilon - M_1)^+,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2}\partial_t \|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + b_1 \|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)} \\
\leq \; & \int_{\Gamma_R^\varepsilon} a_1 (u_1^\varepsilon - M_1)(v_1^\varepsilon - \bar{M}_1)^+ + \int_{\Gamma^\varepsilon} (a_1 M_1 - b_1 \bar{M}_1)(v_1^\varepsilon - \bar{M}_1)^+.
\end{aligned}
$$

Here, we note that

$$
R_1(u)(u_1^\varepsilon - M_1)^+ = -\sum_{j=1}^{N} \beta_{1j} u_1^\varepsilon u_j^\varepsilon (u_1^\varepsilon - M_1)^+ \leq 0.
$$

Now, we add the two inequalities, while dropping the positive terms on the left, putting $a_1 M_1 - b_1 \bar{M}_1 = 0$ and using Cauchy-Schwarz and Young's inequalities on the right to obtain: For $\eta > 0$

$$
\begin{aligned}
& \frac{1}{2}\partial_t \|(u_1^\varepsilon - M_1)^+\|^2 + (d_0 - \eta) \|\nabla (u_1^\varepsilon - M_1)^+\|^2 + \varepsilon a_1 \|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)} \\
& + \frac{1}{2}\partial_t \|(v_1^\varepsilon - \bar{M}_1)^+\|^2_{L^2(\Gamma^\varepsilon)} \\
\leq \; & C^\eta \|(u_1^\varepsilon - M_1)^+\|^2 + \varepsilon \eta \|(u_1^\varepsilon - M_1)^+\|^2_{L^2(\Gamma^\varepsilon)} + C^\eta \|(v_1^\varepsilon - \bar{M})^+\|^2_{L^2(\Gamma^\varepsilon)}.
\end{aligned}
$$

Then Gronwall's lemma gives:

$$
\begin{aligned}
& \frac{1}{2}\|(u_1^\varepsilon - M_1)^+(t)\|^2 + \|(v_1^\varepsilon - \bar{M}_1)^+(t)\|^2_{L^2(\Gamma^\varepsilon)} \\
\leq \; & \left(\frac{1}{2}\|(u_1^\varepsilon - M_1)^+(0)\|^2 + \|(v_1^\varepsilon - \bar{M}_1)^+(0)\|^2_{L^2(\Gamma^\varepsilon)}\right) \exp(Ct).
\end{aligned}
$$

Since we can choose M_1 and $\bar{M}_1$ to satisfy $\|(u_1^\varepsilon - M_1)(0)^+\| = 0$, $\|(v_1^\varepsilon - \bar{M}_1)(0)^+\|$ and $a_1 M_1 - b_1 \bar{M}_1 = 0$, we get $u_1^\varepsilon \in L_+^\infty((0, T) \times \Omega_\varepsilon)$ and $v_1^\varepsilon \in L_+^\infty((0, T) \times \Gamma_\varepsilon)$.

Let $i = 2$ and $\psi_2 := (u_2^\varepsilon - M_2(t+1))^+$ in (43) and $\varphi_2 := (v_2^\varepsilon - \bar{M}_2(t+1))^+$ in

(45):

$$\frac{1}{2}\partial_t(\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)})$$
$$+ \frac{d_0}{2}\|\nabla(u_2^\varepsilon - M_2(t+1))^+\|^2$$
$$+ \varepsilon a_2\|(u_2^\varepsilon - M_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)} + \varepsilon b_2\|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)}$$
$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \int\limits_{\Omega^\varepsilon} R_2^M(u^\varepsilon)(u_2^\varepsilon - M_2(t+1))^+$$
$$- M_2\int\limits_{\Omega^\varepsilon}(u_2^\varepsilon - M_2(t+1))^+ - \bar{M}_2(v_2^\varepsilon - \bar{M}_2(t+1))^+.$$

Here, we note that

$$R_2^M(u^\varepsilon) \leq \frac{1}{2}\beta_{11}\sigma_M(u_1^\varepsilon)^2 \leq \frac{1}{2}\beta_{11}u_1^{\varepsilon,2} \leq \frac{1}{2}\beta_{11}M_1^2.$$

Similarly, we have:

$$\frac{1}{2}\partial_t(\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + \|(v_2^\varepsilon - \bar{M}_2(t+1))^+\|^2_{L^2(\Gamma^\varepsilon)})$$
$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2 + (\frac{1}{2}\beta_{11}M_1^2 - M_2)\int\limits_{\Omega^\varepsilon}(u_2^\varepsilon - M_2(t+1))^+$$
$$\leq C\|(u_2^\varepsilon - M_2(t+1))^+\|^2.$$

By applying Gronwall's lemma with $\frac{1}{2}\beta_{11}M_1^2 \leq M_2$, we see that $u_2^\varepsilon \leq M_2(T+1)$ in $(0,T)\times\Omega^\varepsilon$ and $v_2^\varepsilon \leq \bar{M}_2(T+1)$ on $(0,T)\times\Gamma^\varepsilon$. Recursively, we can obtain the same estimates for u_i^ε and v_i^ε for $i \geq 3$. □

Lemma 3.5. The boundedness of the concentration gradient for (P_2). *Let $\bar{\theta}^\varepsilon \in K(T, M_0)$ and assume (A_1)-(A_2) to hold. Then there exists a positive constant $C(M_0)$ such that $\|\nabla u_i^\varepsilon(t)\| \leq C(M_0)$ and $\int_0^T ||\partial_t u_i^\varepsilon(t)||^2 dt \leq C(M_0)$ for $t \in (0,T)$.*

Proof. Let $\psi_i = \partial_t u_i^\varepsilon$ in (43):

$$\|\partial_t u_i^\varepsilon\|^2 + \frac{d_0}{2}\partial_t\|\nabla u_i^\varepsilon\|^2 + \frac{g_i}{2}\partial_t\|u_i^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} + \frac{\varepsilon a_i}{2}\partial_t\|u_i^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \leq$$
$$\underbrace{\varepsilon b_i\int\limits_{\Gamma^\varepsilon} v_i^\varepsilon\partial_t u_i^\varepsilon}_{A} + \underbrace{|\delta_i^\varepsilon\int\limits_{\Omega^\varepsilon}\nabla^\delta\bar{\theta}^\varepsilon\cdot\nabla u_i^\varepsilon\partial_t u_i^\varepsilon|}_{B} + \underbrace{|\int\limits_{\Omega^\varepsilon} R_i(u^\varepsilon)\partial_t u_i^\varepsilon|}_{C}$$

We shall now estimate one by one the terms A, B, and C. Note first that

$$A = \varepsilon b_i\partial_t\int\limits_{\Gamma^\varepsilon} v_i^\varepsilon u_i^\varepsilon - \varepsilon b_i\int\limits_{\Gamma^\varepsilon} u_i^\varepsilon\partial_t v_i^\varepsilon \leq \varepsilon b_i\partial_t\int\limits_{\Gamma^\varepsilon} v_i^\varepsilon u_i^\varepsilon + \frac{1}{2}\|\partial_t v_i^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \frac{\varepsilon^2}{2}b_i^2\|u_i^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)}.$$

Then we have

$$B \leq \frac{1}{2}\|\partial_t u_i^\varepsilon\|^2 + \frac{\delta^{\varepsilon,2}}{2}\int\limits_{\Omega^\varepsilon}(\nabla^\delta\bar{\theta}^\varepsilon)^2(\nabla u_i^\varepsilon)^2 \leq \frac{1}{2}\|\partial_t u_i^\varepsilon\|^2 + \frac{\delta^{\varepsilon,2}}{2}c^\delta\|\bar{\theta}^\varepsilon\|^2_\infty\|\nabla u_i^\varepsilon\|^2,$$

and

$$C \le C_{B,\varepsilon} + \varepsilon \|\partial_t u_i^\varepsilon\|^2.$$

After integration from 0 to T, all these lead to

$$\begin{aligned}
(\frac{1}{2}-\eta)\int_0^T \|\partial_t u_i^\varepsilon\|^2 &+ \frac{d_0}{2}\|\nabla u_i^\varepsilon(T)\|^2 + \frac{\varepsilon g_i}{2}\|u_i^\varepsilon(T)\|^2_{L^2(\Gamma_R^\varepsilon)} + \frac{\varepsilon a_i}{2}\|u_i^\varepsilon(T)\|^2_{L^2(\Gamma^\varepsilon)} \\
&+ \varepsilon b_i \int_{\Gamma^\varepsilon} \bar{v^\varepsilon}(0) u_i^\varepsilon(0) \le T C_{B,\varepsilon} + \frac{\delta^{\varepsilon,2}}{2} c^\delta \|\bar{\theta^\varepsilon}\|^2_\infty \int_0^T \|\nabla u_i^\varepsilon\|^2 \\
&+ \frac{d_0}{2}\|\nabla u_i^\varepsilon(0)\|^2 + \varepsilon\frac{g_i}{2}\|u_i^\varepsilon(0)\|^2_{L^2(\Gamma_R^\varepsilon)} + \varepsilon\frac{a_i}{2}\|u_i^\varepsilon(0)\|^2_{L^2(\Gamma^\varepsilon)} \\
&+ \underbrace{\varepsilon b_i \int_{\Gamma^\varepsilon} \bar{v^\varepsilon}(T) u_i^\varepsilon(T)}_{D} + \varepsilon b_i \int_0^T \int_{\Gamma^\varepsilon} u_i^\varepsilon \partial_t \bar{v^\varepsilon}.
\end{aligned}$$

Removing some positive terms on the left and using Cauchy-Schwarz and Young's inequalities to obtain an upper bound for D, we finally get for $\eta > 0$ that

$$\begin{aligned}
(\frac{1}{2}-\eta)\int_0^T \|\partial_t u_i^\varepsilon\|^2 &+ \frac{d_0}{2}\|\nabla u_i^\varepsilon(T)\|^2 + \varepsilon(\frac{a_i}{2}-\eta)\|u_i^\varepsilon(T)\|^2_{L^2(\Gamma^\varepsilon)} \\
&\le T C_{B,\varepsilon} + \frac{\delta^{\varepsilon,2}}{2} c^\delta \|\bar{\theta^\varepsilon}\|^2_\infty \int_0^T \|\nabla u_i^\varepsilon\|^2 + C_0 \\
&+ b_i^\varepsilon \|\bar{v^\varepsilon}(T)\|^2_{L^2(\Gamma^\varepsilon)} + b_i^\varepsilon \|u_i^\varepsilon\|_\infty \|\bar{v^\varepsilon}\|_\infty,
\end{aligned}$$

where C_0 depends on $\|u_i^{\varepsilon,0}\|$. Using Gronwall's lemma, we obtain the statement of the Lemma. □

Lemma 3.6. The boundedness of the temperature gradient for (P_1) .

Let $\bar{u^\varepsilon}_i \in K(T, M_0)$ and assume (A_1)-(A_2) to hold. Then there exists a positive constant $C(M_0)$ such that $\|\nabla\theta^\varepsilon(t)\| \le C(M_0)$ and $\int_0^T ||\partial_t\theta^\varepsilon(t)||^2 dt \le C(M_0)$ for $t \in (0, T)$.

Proof. Let $\phi_i = \partial_t \theta_i^\varepsilon$ in (26), then for $\eta > 0$ we have

$$\|\partial_t\theta^\varepsilon\|^2 + \frac{\kappa_0}{2}\partial_t\|\nabla\theta^\varepsilon\|^2 + \varepsilon\frac{g_0}{2}\|\theta^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \le c^\delta M N(\eta\|\partial_t\theta^\varepsilon\|^2 + \frac{1}{4\eta}\|\nabla\theta^\varepsilon\|^2).$$

Applying Gronwall's lemma gives us the desired statement. □

Theorem 3.7. Existence and uniqueness of weak solutions (P^ε)
Let (A_1)-(A_2) hold.
Then there exists a unique solution to (P^ε).

Proof. For any $M > 0$, $X_M := K(M, T) \times K(M, T)^N$ is a closed set of $X := L^2(0, T; L^1(\Omega^\varepsilon))^{N+1}$. Let $\bar{\theta^\varepsilon}_1, \bar{\theta^\varepsilon}_2, \bar{u^\varepsilon}_{i,1}, \bar{u^\varepsilon}_{i,2} \in K(M, T)$, for $i \in \{1, \dots, N\}$, and put $\bar{\theta^\varepsilon} := \bar{\theta^\varepsilon}_1 - \bar{\theta^\varepsilon}_2$, $\bar{u^\varepsilon}_i := \bar{u^\varepsilon}_{i,1} - \bar{u^\varepsilon}_{i,2}$, $(\theta_1^\varepsilon, u_{i,1}^\varepsilon, v_{i,1}^\varepsilon) = \mathbf{T}(\bar{\theta^\varepsilon}_1, \bar{u^\varepsilon}_1)$ and $(\theta_2^\varepsilon, u_{i,2}^\varepsilon, v_{i,2}^\varepsilon) = \mathbf{T}(\bar{\theta^\varepsilon}_2, \bar{u^\varepsilon}_2)$. Moreover, we define $\theta^\varepsilon = \theta_1^\varepsilon - \theta_2^\varepsilon$ and $u_i^\varepsilon = u_{i,1}^\varepsilon - u_{i,2}^\varepsilon$ and $v_i^\varepsilon = v_{i,1}^\varepsilon - v_{i,2}^\varepsilon$.

By Lemma 3.2 and Lemma 3.4, $\mathbf{T} : X_M \to X_M$ for $M > \max(\|\theta^{\varepsilon,0}\|_{L^\infty(\Omega^\varepsilon)}, M_1(T+1), M_2(T+1), \ldots, M_N(T+1),)$. Hence, we want to prove the existence of a positive constant $C < 1$ such that

$$\|\mathbf{T}(\bar{\theta^\varepsilon}_1, \bar{u^\varepsilon}_{i,1}) - \mathbf{T}(\bar{\theta^\varepsilon}_2, \bar{u^\varepsilon}_{i,2})\|_X \leq C\|(\bar{\theta^\varepsilon}_1, \bar{u^\varepsilon}_{i,1}) - (\bar{\theta^\varepsilon}_2, \bar{u^\varepsilon}_{i,2})\|_X$$

for small $T > 0$. We substitute $\theta_1^\varepsilon, \theta_2^\varepsilon, u_{i,1}^\varepsilon, u_{i,2}^\varepsilon, v_1^\varepsilon, v_2^\varepsilon$ in the corresponding formulations to get:

$$\begin{aligned}
&\int\limits_{\Omega^\varepsilon} \partial_t\theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon) + \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta_1^\varepsilon\nabla(\theta_1^\varepsilon - \theta_2^\varepsilon) + \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon) \\
&\qquad = \tau^\varepsilon \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_{i,1} \cdot \nabla\theta_1^\varepsilon(\theta_1^\varepsilon - \theta_2^\varepsilon), \\
&\int\limits_{\Omega^\varepsilon} \partial_t\theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon) + \int\limits_{\Omega^\varepsilon} \kappa^\varepsilon\nabla\theta_2^\varepsilon\nabla(\theta_2^\varepsilon - \theta_1^\varepsilon) + \varepsilon g_0 \int\limits_{\Gamma_R^\varepsilon} \theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon) \\
&\qquad = \tau^\varepsilon \sum_{i=1}^N \int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta_2^\varepsilon(\theta_2^\varepsilon - \theta_1^\varepsilon).
\end{aligned}$$

Adding the last two equations we obtain:

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\theta^\varepsilon\|^2 + \kappa_0\|\nabla\theta^\varepsilon\|^2 + \varepsilon g_0\|\theta^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \\
&\qquad \leq \tau^\varepsilon \sum_{i=1}^N \underbrace{\Big|\int\limits_{\Omega^\varepsilon} (\nabla^\delta \bar{u^\varepsilon}_{i,1} \cdot \nabla\theta_1^\varepsilon - \nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta_2^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon)\Big|}_{A}.
\end{aligned}$$

The term A can be expressed as:

$$\begin{aligned}
A &= \int\limits_{\Omega^\varepsilon} (\nabla^\delta \bar{u^\varepsilon}_{i,1} \cdot \nabla\theta_1^\varepsilon - \nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta_1^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon) \\
&\quad + \int\limits_{\Omega^\varepsilon} (\nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta_1^\varepsilon - \nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta_2^\varepsilon)(\theta_1^\varepsilon - \theta_2^\varepsilon) \\
&= \underbrace{\int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_i \cdot \nabla\theta_1^\varepsilon\theta^\varepsilon}_{B} + \underbrace{\int\limits_{\Omega^\varepsilon} \nabla^\delta \bar{u^\varepsilon}_{i,2} \cdot \nabla\theta^\varepsilon\theta^\varepsilon}_{C}.
\end{aligned}$$

With the help of Lemma 3.6, the terms B and C can be estimated as follows: For $\eta > 0$

$$\begin{aligned}
B &\leq c^\delta M\|\bar{u^\varepsilon}_i\|^2 + M\|\theta^\varepsilon\|^2, \\
C &\leq c^\delta\|\bar{u^\varepsilon}_{i,2}\|_\infty(\eta\|\nabla\theta^\varepsilon\|^2 + \frac{1}{4\eta}\|\theta^\varepsilon\|^2).
\end{aligned}$$

Looking at the formulation for the concentrations, we have:

$$\int_{\Omega^\varepsilon} \partial_t u^\varepsilon_{i,1}(u^\varepsilon_{i,1}-u^\varepsilon_{i,2}) + \int_{\Omega^\varepsilon} d^\varepsilon_i \nabla u^\varepsilon_{i,1}\cdot\nabla(u^\varepsilon_{i,1}-u^\varepsilon_{i,2}) + \varepsilon g_i \int_{\Gamma^\varepsilon_N} u^\varepsilon_{i,1}(u^\varepsilon_{i,1}-u^\varepsilon_{i,2})$$
$$+ \varepsilon a_i \int_{\Gamma^\varepsilon} u^\varepsilon_{i,1}(u^\varepsilon_{i,1}-u^\varepsilon_{i,2}) - \varepsilon b_i \int_{\Gamma^\varepsilon} v^\varepsilon_{i,1}(u^\varepsilon_{i,1}-u^\varepsilon_{i,2})$$
$$= \delta^\varepsilon \int_{\Omega^\varepsilon} \nabla^\delta \bar{\theta}^\varepsilon{}_1 \cdot u^\varepsilon_{i,1}(u^\varepsilon_{i,1}-u^\varepsilon_{i,2}) + \int_{\Omega^\varepsilon} R_i(u^\varepsilon_1)(u^\varepsilon_{i,1}-u^\varepsilon_{i,2}),$$
$$\int_{\Omega^\varepsilon} \partial_t u^\varepsilon_{i,2}(u^\varepsilon_{i,2}-u^\varepsilon_{i,1}) + \int_{\Omega^\varepsilon} d^\varepsilon_i \nabla u^\varepsilon_{i,2}\cdot\nabla(u^\varepsilon_{i,2}-u^\varepsilon_{i,1}) + \varepsilon g_i \int_{\Gamma^\varepsilon_N} u^\varepsilon_{i,2}(u^\varepsilon_{i,2}-u^\varepsilon_{i,1})$$
$$+ \varepsilon a_i \int_{\Gamma^\varepsilon} u^\varepsilon_{i,2}(u^\varepsilon_{i,2}-u^\varepsilon_{i,1}) - \varepsilon b_i \int_{\Gamma^\varepsilon} v^\varepsilon_{i,2}(u^\varepsilon_{i,2}-u^\varepsilon_{i,1})$$
$$= \delta^\varepsilon \int_{\Omega^\varepsilon} \nabla^\delta \bar{\theta}^\varepsilon{}_2 \cdot u^\varepsilon_{i,2}(u^\varepsilon_{i,2}-u^\varepsilon_{i,1}) + \int_{\Omega^\varepsilon} R_i(u^\varepsilon_2)(u^\varepsilon_{i,2}-u^\varepsilon_{i,1}).$$

We also test the deposition equation with v^ε_i to obtain:

$$\frac{1}{2}\partial_t \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} = a_i \int_{\Gamma^\varepsilon} v^\varepsilon_i u^\varepsilon_i - b_i \|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)}.$$

After adding the three above equations, we obtain for $\eta > 0$ that

$$\frac{1}{2}\partial_t\|u^\varepsilon_i\|^2 + \frac{1}{2}\partial_t\|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + d^{\varepsilon,0}_i\|\nabla u^\varepsilon_i\|^2 + g_i\|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon_R)} + a^\varepsilon_i\|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)}$$
$$\leq (a_i+b_i)\int_{\Gamma^\varepsilon}|v^\varepsilon_i u^\varepsilon_i| - b^\varepsilon_i\|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + \int_{\Omega^\varepsilon}|(\nabla^\delta\bar{\theta}^\varepsilon{}_1\cdot\nabla u^\varepsilon_{i,1} - \nabla^\delta\bar{\theta}^\varepsilon{}_2\cdot\nabla u^\varepsilon_{i,2})u^\varepsilon_i|$$
$$+ \int_{\Omega^\varepsilon}|(R_i(u_1)-R_i(u_2))u_i|,$$

$$\frac{1}{2}\partial_t\|u^\varepsilon_i\|^2 + \frac{1}{2}\partial_t\|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + d^{\varepsilon,0}_i\|\nabla u^\varepsilon_i\|^2 + g_i\|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon_R)} + (a_i-\eta)\|u^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} \leq$$
$$\left(\frac{(a_i+b_i)^2}{4\eta} - b_i\right)\|v^\varepsilon_i\|^2_{L^2(\Gamma^\varepsilon)} + \underbrace{\int_{\Omega^\varepsilon}|\nabla^\delta\bar{\theta}^\varepsilon{}_1\cdot\nabla u^\varepsilon_i u^\varepsilon_i|}_{A}$$
$$+ \underbrace{\int_{\Omega^\varepsilon}|\nabla u^\varepsilon_{i,2}\cdot\nabla^\delta\bar{\theta}^\varepsilon u^\varepsilon_i|}_{B} + \underbrace{\int_{\Omega^\varepsilon}|(R_i(u^\varepsilon_1)-R_i(u^\varepsilon_2))u^\varepsilon_i|}_{C},$$

where the sub-expressions can be estimated as:

$$A \leq \eta\|\nabla u^\varepsilon_i\|^2 + \frac{1}{4\eta}c^\delta\|\bar{\theta}^\varepsilon\|^2_\infty\|u^\varepsilon_i\|^2,$$
$$B \leq c^\delta M\|\bar{\theta}^\varepsilon\|^2 + M\|u^\varepsilon_i\|^2.$$

Note that with the boundedness of u_i^ε we can treat R_i as Lipschitz:

$$C \leq C_L \|u_i^\varepsilon\|^2.$$

Adding up the estimates for the temperature and concentrations:

$$\begin{aligned}&\frac{1}{2}\|u_i^\varepsilon\|^2 + \frac{1}{2}\|v_i^\varepsilon\|^2 + \frac{1}{2}\|\theta^\varepsilon\|^2 + \hat{d_i^\varepsilon}\|\nabla u_i^\varepsilon\|^2 + \hat{\kappa^\varepsilon}\|\nabla\theta^\varepsilon\|^2 + \hat{g}_i\|u_i^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)}\\ &\quad + \hat{a_i^\varepsilon}\|u_i^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \hat{g_0}\|\theta^\varepsilon\|^2_{L^2(\Gamma_R^\varepsilon)} \leq c_1\|u_i^\varepsilon\|^2 + c_2\|v_i^\varepsilon\|^2 + c_3\|\theta^\varepsilon\|^2\\ &\quad + c^\delta M(\|\bar{u^\varepsilon}_i\|^2 + \|\bar{\theta^\varepsilon}\|^2).\end{aligned}$$

Gronwall's lemma gives the estimate:

$$\|\theta^\varepsilon(t)\|^2 + \|u_i^\varepsilon(t)\|^2 \leq C\left(\|\bar{\theta^\varepsilon}\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} + \|\bar{u^\varepsilon}_i\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))}\right).$$

Integrating over $(0,T)$, we have:

$$\int_0^T \|\theta^\varepsilon(t)\|^2 + \|u_i^\varepsilon(t)\|^2 \leq CT\left(\|\bar{\theta^\varepsilon}\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} + \|\bar{u^\varepsilon}_i\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))}\right).$$

Accordingly, $\mathbf{T}$ is a contraction mapping for T' such that $CT' < 1$. Then the Banach fixed point theorem shows that (P^ε) admits a unique solution in the sense of Definition 1 on $[0,T']$. Next, we consider (P^ε) on $[T',T]$. Then we can solve uniquely this problem on $[T',2T']$. Recursively, we can construct a solution of (P^ε) on the whole interval $[0,T]$. □

4 Passing to $\varepsilon \to 0$ (the homogenization limit)

4.1 Preliminaries

Now that the well-posedness of our microscopic system is available, we can investigate what happens as the parameter ε vanishes. Recall that ε defines both the microscopic geometry and the periodicity in the model parameters.

Definition 2. *(Two-scale convergence [20],[1]). Let (u^ε) be a sequence of functions in $L^2(0,T;L^2(\Omega))$ and $\varepsilon > 0$ tends to 0. (u^ε) two-scale converges to a unique function $u_0(t,x,y) \in L^2((0,T)\times\Omega\times Y)$ if and only if for all $\phi \in C_0^\infty((0,T)\times\Omega, C_\#^\infty(Y))$ we have:*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u^\varepsilon\phi(t,x,\frac{x}{\varepsilon})dxdt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t,x,y)dydxdt. \tag{59}$$

We denote (59) by $u^\varepsilon \stackrel{2}{\rightharpoonup} u_0$.

The space $C_\#^\infty(Y)$ refers to the space of all Y-periodic C^∞-functions. The spaces $H_\#^1(Y)$ and $C_\#^\infty(\Gamma)$ have a similar meaning; the index # is always indicating that is about Y-periodic functions.

Theorem 4.1. *(Two-scale compactness on domains)*

(i) *From each bounded sequence (u^ε) in $L^2(0,T;L^2(\Omega))$, a subsequence may be extracted which two-scale converges to $u_0(t,x,y) \in L^2((0,T)\times\Omega\times Y)$. Moreover, for $\sigma \in L^2_\#(Y)$ with $\sigma^\varepsilon(x) = \sigma(\frac{x}{\varepsilon})$, we have $\sigma^\varepsilon u^\varepsilon \overset{2}{\rightharpoonup} \sigma u_0$.*

(ii) *Let (u^ε) be a bounded sequence in $L^2(0,T;H^1(\Omega))$ and $u^\varepsilon \overset{2}{\rightharpoonup} u$. Then there exists $\tilde{u} \in L^2((0,T)\times H^1_\#(Y))$ such that up to a subsequence (u^ε) two-scale converges to $u_0 \in L^2(0,T;L^2(\Omega))$ and $\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u + \nabla_y u^1$.*

Proof. See e.g. [20],[1], [11]. □

Definition 3. *(Two-scale convergence for ε-periodic hypersurfaces [19]). A sequence of functions $(u^\varepsilon) \in L^2((0,T)\times\Gamma_\varepsilon)$ is said to two-scale converge to a limit $u_0 \in L^2((0,T)\times\Omega^\varepsilon\times\Gamma)$ if and only if for all $\phi \in C_0^\infty((0,T)\times\Omega^\varepsilon; C^\infty_\#(\Gamma))$ we have*

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u^\varepsilon \phi(t,x,\frac{x}{\varepsilon}) = \frac{1}{|Y|}\int_0^T\int_\Omega\int_\Gamma u_0(t,x,y)\phi(t,x,y)d\gamma_y dxdt. \tag{60}$$

Theorem 4.2. *(Two-scale compactness on surfaces)*

(i) *From each bounded sequence $(u^\varepsilon) \in L^2((0,T)\times\Gamma_\varepsilon)$ one can extract a subsequence u^ε which two-scale converges to $u_0 \in L^2((0,T)\times\Omega^\varepsilon\times\Gamma)$.*

(ii) *If a sequence (u^ε) is bounded in $L^\infty((0,T)\times\Gamma_\varepsilon)$, then u^ε two-scale converges to a $u_0 \in L^\infty((0,T)\times\Omega^\varepsilon\times\Gamma)$*

Proof. See [19] for proof of (i), and [16] for proof of (ii). □

Lemma 4.3. *Let (A_1) and (A_2) hold and denote by u^ε and θ^ε the Bochner extensions in the space $L^2(0,T;H^1(\Omega))$ of the corresponding functions originally belonging to $L^2(0,T;H^1(\Omega^\varepsilon))$. Then the following statements hold for subsequence u^ε:*

(i) $u_i^\varepsilon \rightharpoonup u_i$ and $\theta^\varepsilon \rightharpoonup \theta$ in $L^2(0,T;H^1(\Omega))$.
(ii) $u_i^\varepsilon \overset{*}{\rightharpoonup} u_i$ and $\theta^\varepsilon \overset{*}{\rightharpoonup} \theta$ in $L^\infty((0,T)\times\Omega)$.
(iii) $\partial_t u_i^\varepsilon \rightharpoonup \partial_t u_i$ and $\partial_t\theta^\varepsilon \rightharpoonup \partial_t\theta$ in $L^2(0,T;L^2(\Omega))$.
(iv) $u_i^\varepsilon \to u_i$ and $\theta^\varepsilon \to \theta$ in $L^2(0,T;L^2(\Omega))$.
(v) $u_i^\varepsilon \overset{2}{\rightharpoonup} u_i$, $\nabla_x u_i^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_i + \nabla_y u_i^1$, where $u_i^1 \in L^2((0,T)\times\Omega; H^1_\#(Y))$.
(vi) $\theta^\varepsilon \overset{2}{\rightharpoonup} \theta$, $\nabla_x\theta^\varepsilon \overset{2}{\rightharpoonup} \nabla_x\theta + \nabla_y\theta^1$, where $\theta^1 \in L^2((0,T)\times\Omega; H^1_\#(Y))$.
(vii) $v_i^\varepsilon \overset{2}{\rightharpoonup} v_i$ and $\partial_t v_i^\varepsilon \overset{2}{\rightharpoonup} \partial_t v_i \in L^2((0,T)\times\Omega\times\Gamma)$.

Proof. We obtain (i) and (ii) as a direct consequence of the fact that (u_i^ε) and (θ^ε) are uniformly bounded in $L^\infty(0,T;H^1(\Omega))$ and $L^\infty((0,T)\times\Omega)$. A similar argument gives (iii). Since (u^ε) and (θ^ε) are bounded in $L^2(0,T;H^1(\Omega))$ and $H^1(0,T;L^2(\Omega))$, by using Lions-Aubin lemma [15] (iv) holds for subsequences. As for the rest of the statements (v) -(vii), since (u^ε) is bounded in $L^\infty(0,T;H^1(\Omega))$, by Theorem 4.2 up to a subsequence we have that $u_i^\varepsilon \overset{2}{\rightharpoonup} u_i$ in $L^2(0,T;L^2(\Omega))$, and $\nabla_x u_i^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_i + \nabla_y u_i^1$ for some $u_i^1 \in L^2((0,T)\times\Omega; H^1_\#(Y))$. By Theorem 4.1 it is easy to get (vii). See [12] for similar arguments. □

4.2 Two-scale homogenization procedure

Theorem 4.4. *Let (A_1) and (A_2) hold and $\theta, \theta^1, u_i, u_i^1, v_i$ be functions obtained in Lemma 4.3. Then θ, u_i and v_i satisfy (61), (62) and (63) for $t \in [0,T]$, respectively:*

$$\begin{aligned} &\int_\Omega (\partial_t \theta)\alpha + \frac{1}{|Y|}\int_\Omega \int_{Y_1} \kappa(\nabla_x \theta + \nabla_y \theta^1)(\nabla_x \alpha + \nabla_y \beta) + g_0 \frac{|\Gamma_R|}{|Y|}\int_\Omega \theta\alpha \\ = &\sum_i^N \frac{1}{|Y|}\int_\Omega \int_{Y_1} \tau \nabla_x^\delta u_i (\nabla_x \theta + \nabla_y \theta^1)\alpha, , \end{aligned} \tag{61}$$

$$\begin{aligned} &\int_\Omega (\partial_t u_i)\alpha + \frac{1}{|Y|}\int_\Omega \int_{Y_1} d_i(\nabla_x u_i + \nabla_y u_i^1)(\nabla_x \alpha + \nabla_y \beta) \\ &+ g_i \frac{|\Gamma_R|}{|Y|}\int_\Omega u_i \alpha + \frac{1}{|Y|}\int_\Omega \int_\Gamma (a_i u_i - b_i v_i)\alpha \\ = &\sum_i^N \frac{1}{|Y|}\int_\Omega \int_{Y_1} \delta_i \nabla_x^\delta u_i (\nabla_x u_i + \nabla_y u_i^1)\alpha + \int_\Omega R_i(u)\alpha, \end{aligned} \tag{62}$$

$$\int_\Omega \int_\Gamma (\partial_t v_i)\alpha = \int_\Omega \int_\Gamma (a_i u_i - b_i v_i)\alpha \text{ for } \alpha \in C^\infty(\Omega), \beta \in C^\infty(\Omega; C^\infty_\#(Y)). \tag{63}$$

Moreover,

$$\partial_t \theta - \frac{1}{|Y|}\nabla \cdot K \nabla \theta + g_0 \frac{|\Gamma_R|}{|Y|}\theta = \frac{1}{|Y|}\sum_i^N \mathcal{T} \nabla_x^\delta u_i \nabla_x \theta \text{ on } (0,T) \times \Omega, \tag{64}$$

$$-K\nabla_x \theta \cdot n = 0 \text{ on } (0,T) \times \partial\Omega, \tag{65}$$

$$\begin{aligned} &\partial_t u_i - \frac{1}{|Y|}\nabla \cdot D_i \nabla u_i + g_0 \frac{|\Gamma_R|}{|Y|} u_i + \frac{1}{|Y|}(A_i u_i - \int_\Gamma b_i v_i) \\ &\quad = \frac{1}{|Y|}\mathcal{F}_i \nabla_x u_i \cdot \nabla_x^\delta \theta + R_i(u) \text{ on } (0,T) \times \Omega, \end{aligned} \tag{66}$$

$$-D_i \nabla_x u_i \cdot n = 0 \text{ on } (0,T) \times \partial\Omega, \tag{67}$$

$$\partial_t v_i = a_i u_i - b_i v_i \text{ on } (0,T) \times \Omega \times \Gamma. \tag{68}$$

Here, K, τ, D_i and $\mathcal{F}_i$ are matrices given by $K = K_0\mathbb{I} + (K_{kj})$, $\mathcal{T} = T_0\mathbb{I} + (T_{kj})$, $D_i = D_0^i + (D_{kj}^i)$ and $\mathcal{F}_i = F_0^i\mathbb{I} + (F_{kj})$, where $\mathbb{I}$ is the identity matrix. Furthermore, $K_0 = \int_{Y_1} \kappa dy$, $K_{kj} = \int_{Y_1} \kappa \partial_{y_k} \bar{\theta}^j dy$, $T_{kj} = \int_{Y_1} \tau \partial_{y_k} \bar{\theta}^j$, $D_0^i = \int_{Y_1} d_i dy$, $D_{kj}^i = \int_{Y_1} d_i \partial_{y_k} \bar{u}_i^j dy$, $F_0^i = \int_{Y_1} \delta_i dy$, $D_{kj}^i = \int_{Y_1} d_i \partial_{y_k} \bar{u}_i^j dy$, and $A_i = \int_\Gamma a_i d\gamma_y$.

Here $\bar{\theta}^j$ and $\bar{u}_i^j$ are called cell functions. *They satisfy*

$$\begin{cases} -\nabla_y(\kappa \nabla_y \bar{\theta}^j) = \partial_{y_j}\kappa \text{ in } Y_1, \\ \kappa \nabla_y \bar{\theta}^j \cdot n = -\kappa n_j. \end{cases} \tag{69}$$

$$\begin{cases} -\nabla_y(d_i \nabla_y \bar{u}_i^j) = \partial_{y_j} d_i \text{ in } Y_1, \\ \kappa \nabla_y \bar{u}_i^j \cdot n = -d_i n_j. \end{cases} \tag{70}$$

Proof. Let $\alpha \in C^\infty((0,T) \times \Omega)$ and $\beta \in C^\infty((0,T) \times \Omega; C^\infty_\#(Y))$. By testing with $\phi(t,x) = \alpha(t,x) + \varepsilon\beta(t,x,\frac{x}{\varepsilon})$, we obtain

$$
\begin{aligned}
&\int_0^T \int_{\Omega^\varepsilon} (\partial_t \theta^\varepsilon)(\alpha + \varepsilon\beta) + \int_0^T \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla_x \theta^\varepsilon (\nabla_x \alpha + \varepsilon \nabla_x \beta + \nabla_y \beta) \\
&+\varepsilon g_0 \int_0^T \int_{\Gamma_R^\varepsilon} \theta^\varepsilon (\alpha + \varepsilon\beta) \\
= \quad & \sum_i^N \frac{1}{|Y|} \int_{\Omega^\varepsilon} \tau^\varepsilon \nabla_x^\delta u_i^\varepsilon \nabla_x \theta^\varepsilon (\alpha + \varepsilon\beta),
\end{aligned} \tag{71}
$$

Here, we have

$$
\int_0^T \int_{\Omega^\varepsilon} (\partial_t \theta^\varepsilon)(\alpha + \varepsilon\beta) = \int_0^T \int_{\Omega} \chi^\varepsilon (\partial_t \theta^\varepsilon)(\alpha + \varepsilon\beta),
$$

where χ^ε is the characteristic function of Ω^ε. Then it is easy to see that $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, where χ is the characteristic function of Y_1. By Lemma 4.3 and Theorem 4.1 (i) we get that

$$
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^\varepsilon} (\partial_t \theta^\varepsilon)(\alpha + \varepsilon\beta) = \frac{1}{|Y|} \int_0^T \int_\Omega \int_Y \chi (\partial_t \theta)\alpha = \frac{|Y_1|}{|Y|} \int_0^T \int_\Omega (\partial_t \theta)\alpha.
$$

Similarly, as $\varepsilon \to 0$, we have

$$
\begin{aligned}
\int_0^T \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla_x \theta^\varepsilon (\nabla_x \alpha + \varepsilon \nabla_x \beta + \nabla_y \beta) \quad &= \quad \int_0^T \int_\Omega \chi^\varepsilon \kappa^\varepsilon \nabla_x \theta^\varepsilon (\nabla_x \alpha + \varepsilon \nabla_x \beta + \nabla_y \beta) \\
&\overset{2}{\rightharpoonup} \quad \int_0^T \frac{1}{|Y|} \int_\Omega \int_{Y_1} \kappa (\nabla_x \theta + \nabla_y \theta^1)(\nabla_x \alpha + \nabla_y \beta).
\end{aligned}
$$

Next, Theorem 3 guarantees that

$$
\lim_{\varepsilon \to 0} \varepsilon g_0 \int_0^T \int_{\Gamma_R^\varepsilon} \theta^\varepsilon (\alpha + \varepsilon\beta) = \frac{g_0}{|Y|} \int_0^T \int_\Omega \int_{\Gamma_R} \theta\alpha = g_0 \frac{|\Gamma_R|}{|Y|} \int_0^T \int_\Omega \theta\alpha.
$$

Since $\nabla_x^\delta u_i^\varepsilon \to \nabla_x^\delta u_i$ in $L^2(0,T;L^2(\Omega))$, on account of Theorem 4.1 (i) it holds that $\chi^\varepsilon \tau^\varepsilon \nabla_x^\delta u_i^\varepsilon \nabla_x \theta^\varepsilon \overset{2}{\rightharpoonup} \chi \tau \nabla_x^\delta u_i \nabla_x \theta$. Hence, by letting $\varepsilon \to 0$ in (74) we have

$$
\begin{aligned}
&\int_0^T \int_\Omega (\partial_t \theta)\alpha + \int_0^T \frac{1}{|Y|} \int_\Omega \int_{Y_1} \kappa (\nabla_x \theta + \nabla_y \theta^1)(\nabla_x \alpha + \nabla_y \beta) + g_0 \frac{|\Gamma_R|}{|Y|} \int_0^T \int_\Omega \theta\alpha \\
= \quad & \sum_i^N \frac{1}{|Y|} \int_0^T \int_\Omega \int_{Y_1} \tau \nabla_x^\delta u_i (\nabla_x \theta + \nabla_y \theta^1)\alpha.
\end{aligned}
$$

Thus we get (61). In a similar way we can prove (62) and (63). □

Acknowledgements

AM and OK gratefully acknowledge financial support by the European Union through the Initial Training Network *Fronts and Interfaces in Science and Technology* of the Seventh Framework Programme (grant agreement number 238702).

References

[1] G. Allaire. “Homogenization and two-scale convergence”. In: *SIAM Journal on Mathematical Analysis* 23.6 (1992), pp. 1482–1518.

[2] B. Andreianov, M. Bendahmane, and R. Ruiz-Baier. “Analysis of a finite volume method for a cross-diffusion model in population dynamics”. In: *Mathematical Models and Methods in Applied Sciences* 21.02 (2011), pp. 307–344.

[3] M. Beneš and R. Štefan. “Global weak solutions for coupled transport processes in concrete walls at high temperatures”. In: *ZAMM* 93.4 (2013), pp. 233–251.

[4] M. Beneš, R. Štefan, and J. Zeman. “Analysis of coupled transport phenomena in concrete at elevated temperatures”. In: *Applied Mathematics and Computation* 219.13 (2013), pp. 7262–7274.

[5] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. Vol. 374. American Mathematical Soc., 2011.

[6] M. Elimelech et al. *Particle Deposition and Aggregation: Measurement, Modelling and Simulation*. Elsevier, 1995.

[7] L. Evans. *Partial Differential Equations*. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, 1998. ISBN: 0821807722.

[8] T. Funaki, H. Izuhara, and M. Mimura. “A link between microscopic and macroscopic models of self-organized aggregation”. In: *Networks and Heterogeneous Media* 7 (2012), pp. 705–740.

[9] Z.-X. Gong and A. S. Mujumdar. “Development of drying schedules for one-side-heating drying of refractory concrete slabs based on a finite element model”. In: *Journal of the American Ceramic Society* 79.6 (1996), pp. 1649–1658.

[10] S. de Groot and P. Mazur. *Non-equilibrium Thermodynamics*. Series in physics. North-Holland Publishing Company - Amsterdam, 1962.

[11] U. Hornung, W. Jäger, and A. Mikelić. “Reactive transport through an array of cells with semipermeable membranes”. In: *M2AN* 28.1 (1994), pp. 59–94.

[12] U. Hornung and W. Jäger. “Diffusion, convection, adsorption, and reaction of chemicals in porous media”. In: *Journal of differential equations* 92.2 (1991), pp. 199–225.

[13] O. Krehel, A. Muntean, and P. Knabner. “Multiscale modeling of colloidal dynamics in porous media including aggregation and deposition”. In: *in preparation* (2014).

[14] O. Krehel, A. Muntean, and P. Knabner. “On modeling and simulation of flocculation in porous media”. In: *Proceedings of XIX International Conference on Water Resources* (2012).

[15] J. Lions. *Quelques méthodes de résolution des problèmes aux limites non linèaires*. Dunod, Paris, 1969.

[16] A. Marciniak-Czochra and M. Ptashnyk. “Derivation of a macroscopic receptor-based model using homogenization techniques”. In: *SIAM Journal on Mathematical Analysis* 40.1 (2008), pp. 215–237.

[17] N. Masmoudi and M. L. Tayeb. “Diffusion limit of a semiconductor Boltzmann-Poisson system”. In: *SIAM Journal on Mathematical Analysis* 38.6 (2007), pp. 1788–1807.

[18] C. C. Mei and B. Vernescu. *Homogenization Methods for Multiscale Mechanics.* World Scientific, 2010.

[19] M. Neuss-Radu. “Some extensions of two-scale convergence”. In: *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* 322.9 (1996), pp. 899–904.

[20] G. Nguetseng. “A general convergence result for a functional related to the theory of homogenization”. In: *SIAM Journal on Mathematical Analysis* 20.3 (1989), pp. 608–623.

[21] W.-M. Ni. “Diffusion, cross-diffusion, and their spike-layer steady states”. In: *Notices of the AMS* 45.1 (1998), pp. 9–18.

[22] L. Nirenberg. “On elliptic partial differential equations”. In: *Ann. Scuola Norm. Sup. Pisa* 13 (1959), pp. 115–162.

[23] N. Ray, A. Muntean, and P. Knabner. “Rigorous homogenization of a Stokes–Nernst–Planck–Poisson system”. In: *Journal of Mathematical Analysis and Applications* 390.1 (2012), pp. 374–393.

[24] S. Rothstein, W. Federspiel, and S. Little. “A unified mathematical model for the prediction of controlled release from surface and bulk eroding polymer matrices”. In: *Biomaterials* 30.8 (2009), pp. 1657–1664.

[25] N. Shigesada, K. Kawasaki, and E. Teramoto. “Spatial segregation of interacting species”. In: *Journal of Theoretical Biology* 79.1 (1979), pp. 83–99.

[26] M. Smoluchowski. “Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen”. In: *Z. Phys. Chem* 92 (1917), pp. 129–168.

[27] J. Soares and P. Zunino. “A mixture model for water uptake, degradation, erosion and drug release from polydisperse polymeric networks”. In: *Biomaterials* 31.11 (2010), pp. 3032–3042.

[28] V. K. Vanag and I. R. Epstein. “Cross-diffusion and pattern formation in reaction–diffusion systems”. In: *Physical Chemistry Chemical Physics* 11.6 (2009), pp. 897–912.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
14-05	L.M.J. Florack	Canonical coordinates for Retino-Cortical magnification	March '14
14-06	M.J.H. Anthonissen J.H.M. ten Thije Boonkkamp	The complete flux scheme in cylindrical coordinates	March '14
14-07	N. Kumar J.H.M. ten Thije Boonkkamp B. Koren	A new discretization method for the convective terms in the incompressible Navier-Stokes equations	March '14
14-08	A. Corbetta A. Muntean F. Toschi K. Vafayi	Parameter estimation of social forces in crowd dynamics models via a probabilistic method	March '14
14-09	O. Krehel A. Muntean T. Aiki	A thermo-diffusion system with Smoluchowski interactions: well-posedness and homogenization	Apr. '14

Ontwerp: de Tantes,
Tobias Baanders, CWI