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Observer-based output-feedback control to eliminate torsional drill-string vibrations

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Abstract

Torsional stick-slip vibrations decrease the performance and reliability of drilling systems used for the exploration of energy and mineral resources. In this work, we present the design of a nonlinear observer-based output-feedback control strategy to eliminate these vibrations. We apply the controller to a drill-string model based on a real-life rig. Conditions, guaranteeing asymptotic stability of the desired equilibrium, corresponding to nominal drilling operation, are presented. The proposed control strategy has a significant advantage over existing vibration control systems in current drilling rigs as it only requires surface measurements instead of expensive down-hole measurements and can handle multiple modes of torsional vibration. Case study results using the proposed control strategy show that stick-slip oscillations can indeed be eliminated in realistic drilling scenarios.

I. INTRODUCTION

Drilling systems are used to drill deep wells for the exploration and production of oil and gas, mineral resources and geo-thermal energy. Fig. 1 shows a schematic drilling system. Surface and down-hole measurements [1], [2], [3] show that these systems experience different types of oscillations, which significantly decrease the drilling rate of penetration due to damage to the drill bit (e.g. bit tooth wear), drill pipes and bottom hole assembly (e.g. twisted pipe). Different modes of vibration, such as axial, lateral and torsional vibrations, lead to bit bouncing, whirling and torsional stick-slip, respectively. The focus of the current paper is on the aspect of mitigation of torsional stick-slip oscillations by means of control as these vibrations are known to be highly detrimental to drilling efficiency, reliability and safety.

For the design of controllers to eliminate torsional vibrations most studies rely on one- or two degree-of-freedom (DOF) models for the torsional drill-string dynamics only, see e.g. [4], [5], [6]. In these models, it is generally assumed that the resisting torque at the bit-rock interface can be modeled as a frictional contact with a velocity weakening effect as reported in [7], [8]. In fact, modelling of the coupled axial and torsional dynamics, as for example in [9], shows that the velocity weakening effect in the torque-on-bit (TOB) is a consequence of the drilling dynamics, rather than an intrinsic property of the bit-rock interface. The fact that such coupling effectively leads to a velocity weakening effect of the TOB (see e.g. [10], [11]) motivates to adopt a modelling-for-control approach for drill-string dynamics involving the torsional dynamics only, as we will pursue in this paper.
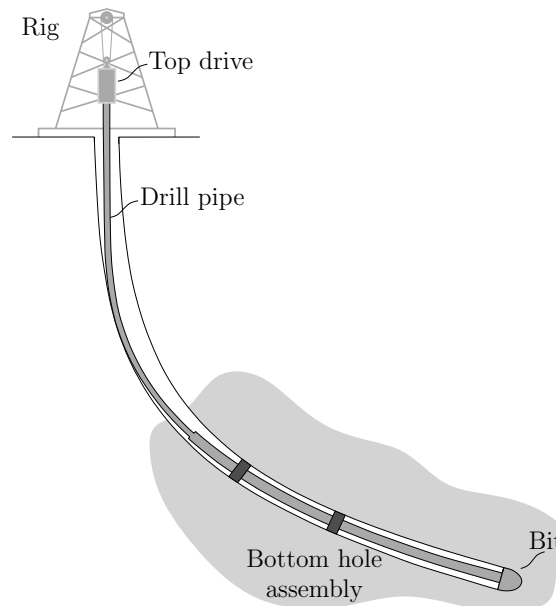
Fig. 1. Schematic drilling system (adapted from [12]).

Different control strategies to suppress torsional vibrations can be found in literature. In [13], the use of torque feedback in addition to a speed controller is investigated. The underlying idea is making the top rotary system behave in a "soft" manner, hence the name *Soft Torque Rotary System*, see also [5]. In these research contributions, it is assumed that the drilling system behaves like a 2-DOF torsional pendulum of which the first torsional mode can be damped using a PI-controller based on the surface angular velocity. In [8], [14], the above soft torque approach is compared with a control method based on torsional rectification, which outperforms the soft torque approach in simulation studies by using improved torque feedback based on the twist of the drill-string near the rotary. A linear $\mathcal{H}_\infty$ controller synthesis approach is presented in [6]. Herein, the bit-rock interaction, key in causing stick-slip, is not taken into account in the controller design and stability analysis of the closed-loop dynamics. A control design approach, where information of the nonlinear bit-rock interaction model is explicitly taken into account in the controller synthesis, is proposed in [4], [15], [16]. Drawbacks of the approaches in [4], [6] are, firstly, the necessity of down-hole measurements reflecting the twist of the drill-string between surface and bit, which can not be measured in practice, and, secondly, the fact that only one torsional mode of the drill-string is taken into account.

Increasing demands on the operating envelope and a tendency towards drilling deeper and inclined wells impose higher demands on the controllers used in drilling systems. Industrial controllers are not always able to eliminate stick-slip vibrations under the imposed operating conditions. Two main reasons for this deficiency are the influence of multiple dynamical modes of the drill-string on torsional vibrations and the uncertainty in the bit-rock interaction law. The main contribution of this paper is an output-feedback control strategy mitigating torsional stick-slip vibrations while 1) only using surface measurements, 2) taking into account a multi-modal drill-string model and 3) taking into account severe velocity weakening and uncertainty in the bit-rock interaction.

This paper is organized as follows. Section II introduces the drill-string model based on a finite element model of a real-life rig. Subsequently, in Section III we present the design of an output-feedback controller including a robust stability analysis of the resulting closed-loop system. Section IV will present simulation results illustrating the effectiveness of the proposed approach while comparing the results with those obtained using an industrial controller. Finally, we draw conclusions in Section V.

Preliminaries: In support of the controller design result in Section III-A we present the following definitions on input-to-state-stability (ISS) and the strict passivity property.

The concept of input-to-state stability has been introduced in [17]. Its local version has first appeared in [18].

Definition 1: The system $\dot{x}(t) \in F(x(t), e(t))$ is locally input-to-state stable (LISS) if there exist constants $c_1, c_2 > 0$, a function ρ of class $\mathcal{KL}$ and a function μ of class $\mathcal{K}$ such that for each initial condition $x(0) = x_0$, such that $\|x_0\| \leq c_1$, and each piecewise continuous bounded input function $e(t)$ defined on $[0, \infty)$ and satisfying $\sup_{\tau \in [0, \infty)} \|e(\tau)\| \leq c_2$, it holds that

- all solutions $x(t)$ exist on $[0, \infty)$ and,
- all solutions satisfy

$$\|x(t)\| \leq \rho(\|x_0\|, t) + \mu \left(\sup_{\tau \in [0, t]} \|e(\tau)\| \right), \ \forall t \geq 0. \tag{1}$$

Consider the linear time-invariant minimal realization

$$\begin{aligned} \dot{x} &= Ax + Gw \\ z &= Hx + Dw \end{aligned} \tag{2}$$

with the state $x \in \mathbb{R}^n$, input and output $w, z \in \mathbb{R}$.

Definition 2: The system (2) or the quadruple (A, G, H, D) is said to be strictly passive if there exist an $\varepsilon > 0$ and a matrix $P = P^\top > 0$ such that

$$\begin{bmatrix} A^\top P + PA + \varepsilon I & PG - H^\top \\ G^\top P - H & -D - D^\top \end{bmatrix} \leq 0. \tag{3}$$

II. DRILL-STRING MODEL

The system we will investigate is a realistic drill-string model of a jack-up drilling rig and the reservoir sections of the wells are drilled with a 6" bit to reach depths of >6000 m and with an inclination angle up to 60°, resulting in significant resistive torques along the drill-string. The rig is equipped with an AC top drive and fitted with a modern Soft Torque system [19]. However, for this depth and hole size stick-slip vibrations have been observed in the field for this rig. A finite element model of this drilling system has been developed and the simulation results of this model have been validated with field data under different conditions (such as weight-on-bit (WOB) and angular velocity).

The finite element method (FEM) representation of the drill-string is a model with 18 elements. The element at the top is a rotational inertia to model the top drive inertia, the following elements are equivalent pipe sections based on the dimensions and material properties of the drill-string. The resulting model can be written as

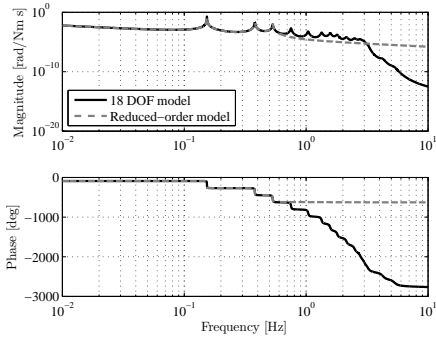$$M\ddot{q} + D\dot{q} + K_t q_d = S_w T_w(\dot{q}) + S_b T_{bit}(\dot{q}_1) + S_m T_m \tag{4}$$

Fig. 2. Frequency response function of the full-order and reduced-order model from input torque T_m to bit velocity ω_{bit}.
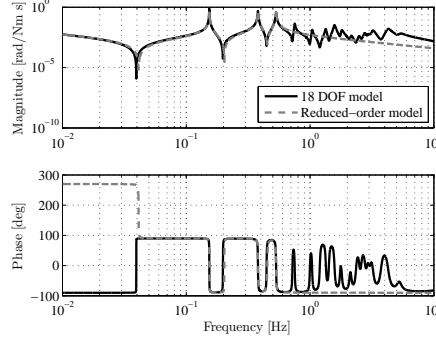
Fig. 3. Frequency response function of the full-order and reduced-order model from bit torque T_{bit} to bit velocity ω_{bit}, i.e. bit mobility.
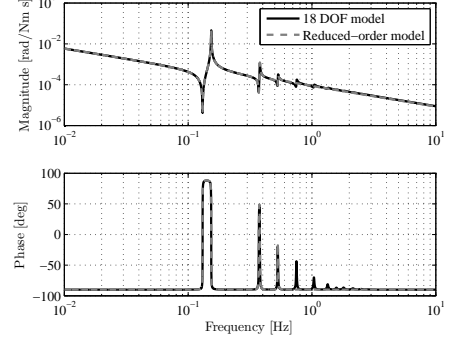
Fig. 4. Frequency response function of the full-order and reduced-order model from input torque T_m to top drive velocity ω_{td}.

with the coordinates $q \in \mathbb{R}^{\bar{n}}$ with $\bar{n} = 18$, the top drive motor torque input $T_m \in \mathbb{R}$ being the control input, the bit-rock inter-action torque $T_{bit} \in \mathbb{R}$ and the interaction torques $T_w \in \mathbb{R}^{\bar{n}-1}$ between the borehole and the drill-string acting on the nodes of the FEM model. The coordinates q represent the angular displacements of the nodes of the finite element representation. Next, we define the difference in angular position between adjacent nodes as follows; $q_d := \begin{bmatrix} q_1 - q_2 & q_2 - q_3 & \cdots & q_{17} - q_{18} \end{bmatrix}^\top$. In (4), the mass, damping and stiffness matrices are, respectively, given by $M \in \mathbb{R}^{\bar{n} \times \bar{n}}$, $D \in \mathbb{R}^{\bar{n} \times \bar{n}}$ and $K_t \in \mathbb{R}^{\bar{n} \times \bar{n}-1}$, the matrices $S_w \in \mathbb{R}^{\bar{n} \times \bar{n}-1}$, $S_b \in \mathbb{R}^{\bar{n} \times 1}$ and $S_m \in \mathbb{R}^{\bar{n} \times 1}$ represent the generalized force directions of the input torque, the bit torque and the interaction torques, respectively. The coordinates q are chosen such that the first element (q_1) describes the rotation of the bit and the last element (q_{18}) the rotation of the top drive at surface. The interaction between the borehole and the drill-string is modeled as Coulomb friction, that is

$$T_{w,i} \in T_i \operatorname{Sign}(\dot{q}_i), \quad \text{for } i = 2, \ldots, 18, \tag{5}$$

with T_i representing the amount of friction at each element and the set-valued sign function defined as

$$\operatorname{Sign}(y) \triangleq \begin{cases} -1, & y < 0 \\ [-1, 1], & y = 0 \\ 1, & y > 0. \end{cases} \tag{6}$$

The bit-rock interaction model is given by

$$T_{bit}(\dot{q}_1) = \operatorname{Sign}(\dot{q}_1) \left(T_d + (T_s - T_d) e^{-v_d |\dot{q}_1|} \right) \tag{7}$$

with T_s the static torque, T_d the dynamic torque and $v_d := \frac{30}{N_d \pi}$ indicating the decrease from static to dynamic torque. The model (4), (5) and (7) together forms a differential inclusion that we can write in state-space Lur'e-type form as:

$$\begin{aligned}
\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{G}\bar{w} + \bar{G}_2 \bar{w}_2 + \bar{B}u \\
\bar{z} &= \bar{H}\bar{x} \\
\bar{z}_2 &= \bar{H}_2 \bar{x} \\
\bar{y} &= \bar{C}\bar{x} \\
\bar{w} &\in -\varphi(\bar{z}) \\
\bar{w}_2 &\in -\phi(\bar{z}_2),
\end{aligned} \tag{8}$$

where $\bar{x} := \begin{bmatrix} q_d^\top & \dot{q}^\top \end{bmatrix}^\top \in \mathbb{R}^{2\bar{n}-1}$ is the state, $\bar{z} := \omega_{bit} \in \mathbb{R}$ and $\bar{z}_2 := \begin{bmatrix} \dot{q}_2, \ldots, \dot{q}_{18} \end{bmatrix}^\top \in \mathbb{R}^{\bar{n}-1}$ are the angular velocity arguments of the set-valued nonlinearities φ and ϕ, respectively. The bit-rock interaction torque is given by $\bar{w} \in \mathbb{R}$ and the drill-string-borehole interaction torques are given by $\bar{w}_2 \in \mathbb{R}^{\bar{n}-1}$, $u := T_m \in \mathbb{R}$ is the control input and $\bar{y} := \begin{bmatrix} \omega_{td} & T_{pipe} \end{bmatrix}^\top \in \mathbb{R}^2$ is the measured output. Note that the latter implies that only surface measurements will be employed in the output-feedback control strategy proposed in Section III. The angular velocities of the top drive and the bit are defined as $\omega_{td} := \dot{q}_{18}$ and $\omega_{bit} := \dot{q}_1$, respectively, and the pipe torque T_{pipe} is the torque in the drill-string directly below the top drive. The matrices $\bar{A}$, $\bar{B}$, $\bar{G}$, $\bar{G}_2$, $\bar{H}$ and $\bar{H}_2$ in (8), with appropriate dimensions are given by

$$\bar{A} = \begin{bmatrix} 0_{17 \times 17} & \bar{a} \\ -M^{-1}K_t & -M^{-1}D \end{bmatrix}, \quad \bar{a} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0_{17 \times 1} \\ M^{-1}S_m \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} 0_{17 \times 1} \\ M^{-1}S_b \end{bmatrix}, \quad \bar{G}_2 = \begin{bmatrix} 0_{17 \times 17} \\ M^{-1}S_w \end{bmatrix},$$

$$\bar{H} = \begin{bmatrix} 0_{1 \times 17} & 1 & 0_{1 \times 17} \end{bmatrix}, \quad \bar{H}_2 = \begin{bmatrix} 0_{17 \times 18} & I_{17} \end{bmatrix},$$

and $\bar{C} \in \mathbb{R}^{2 \times 2\bar{n}-1}$ indicates the measured output. Note that $\varphi(\bar{z}) := T_{bit}(\bar{z})$ and $\phi(\bar{z}_2) := [T_{w,2}(\dot{q}_2), \ldots, T_{w,18}(\dot{q}_{18})]^\top$.

A. Reduced-order model

To facilitate the design and to decrease the implementation burden of observer-based output-feedback controllers (see Section III), we apply model reduction to obtain a low-order approximation of the drilling system dynamics (8), that approximates the input-output behavior from inputs u and $\bar{w}$ to outputs $\bar{y}$ and $\bar{z}$. The inputs and outputs related to the drill-string-borehole interaction ($T_{w,i}$) are not taken into account in the reduction process, but can be approximated using the transformation matrix obtained from the reduction procedure. With this assumption, system (8) can be represented as a Lur'e type system $\bar{\Sigma} = (\bar{\Sigma}_{lin}, \varphi)$, consisting of high-order linear dynamics $\bar{\Sigma}_{lin}$ with a single static output-dependent nonlinearity φ, related to the bit-rock interaction, in the feedback loop. We will use the model reduction approach for Lur'e-type systems as proposed in [20], which employs a linear model reduction technique (such as balanced truncation) for the reduction of the linear part of the Lur'e-type system. In doing so, we combine the inputs u and $\bar{w}$ and the outputs $\bar{y}$ and $\bar{z}$, yielding the new input matrix $\begin{bmatrix} \bar{B} & \bar{G} \end{bmatrix}$ and the new output matrix $\begin{bmatrix} \bar{C}^\top & \bar{H}^\top \end{bmatrix}^\top$. By applying balanced truncation to the linear part of the Lur'e-type system we obtain the reduced-order linear system Σ_{lin}. Now, the reduced-order linear part is interconnected with the original nonlinearity yielding the reduced-order drill-string system $\Sigma = (\Sigma_{lin}, \varphi)$.

Using the approach outlined above, we obtain a reduced-order model with state $x \in \mathbb{R}^n$, with $n = 7$. The equations of motion for the reduced-order system are written as

$$
\begin{aligned}
\dot{x} &= Ax + Gw + G_2 w_2 + Bu \\
z &= Hx \\
z_2 &= H_2 x \\
y &= Cx \\
w &\in -\varphi(z) \\
w_2 &\in -\phi(z_2),
\end{aligned}
\tag{9}
$$

and an experimentally validated bit-rock interaction model is shown in Fig. 5. The relevant frequency response functions for the linear part of the dynamics in (9) are shown in Figs. 2, 3 and 4. Clearly, the first three resonance modes (and the rigid-body mode) are accurately captured in the reduced-order model. The so-called bit mobility, shown in Fig. 3, gives an indication of the important resonance modes in the onset of stick-slip vibrations. It can be seen that the first three resonance modes are dominant, which motivates the choice to reduce to a model order of $n = 7$.

III. DESIGN OF AN OUTPUT-FEEDBACK CONTROLLER

We employ an observer-based controller synthesis strategy for Lur'e-type systems with discontinuities as in [4], [21]. In these previous works the controller and observer were designed for a drill-string model with a single flexibility mode and with the assumption on the availability of down-hole measurements. The conditions for controller synthesis as in [4] achieving *global* asymptotic stability are infeasible for the realistic drill-string model presented here for three reasons: firstly, the incorporation of more realistic drill-string dynamics including multiple torsional flexibility modes of the drill-string, see Figs. 2, 3 and 4, secondly, the incorporation of a bit-rock interaction model based on field data, which shows a rather severe velocity weakening effect, see Fig. 5 and, thirdly, the restriction on the availability of only surface measurements. Therefore, we employ a controller synthesis strategy to design *locally* stabilizing controllers and we show that such local stability properties suffice in realistic drilling scenarios. In Section III-A, we will propose the state-feedback controller, in Section III-B, the observer design and in Section III-C, the resulting output-feedback control strategy, all including stability guarantees.

A. State-feedback controller

In this section, we discuss the design of a state-feedback controller for systems in the form

$$
\begin{aligned}
\dot{\xi} &= A\xi + Bu_{fb} + G\tilde{w} \\
\tilde{z} &= H\xi \\
\tilde{w} &\in -\tilde{\varphi}(\tilde{z})
\end{aligned}
\tag{10}
$$

that stabilizes the origin $\xi = 0$ of the system state $\xi \in \mathbb{R}^n$. Stabilization of the origin of (10) corresponds to the desired operation of constant angular velocity of the drilling system. The relation between systems (9) and (10) will be explained in more detail in Section IV, while this section focusses on the design of controllers for generic systems of the form (10). The control input is given by $u_{fb} \in \mathbb{R}^m$, the input and output of the set-valued nonlinearity $\tilde{\varphi}$ are given by $\tilde{z} \in \mathbb{R}$ and $\tilde{w} \in \mathbb{R}$, respectively, and the system matrices are $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times 1}$ and $H \in \mathbb{R}^{1 \times n}$. We introduce the linear static state-feedback law, where we take the "measurement" (or observer) error $e := \xi - \hat{\xi}$ into account:

$$
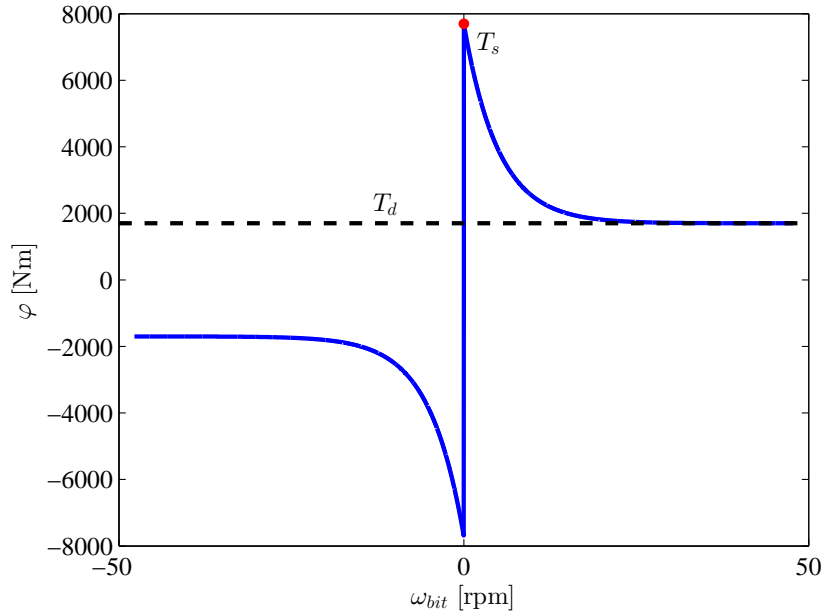u_{fb} = K\hat{\xi} = K(\xi - e),
\tag{11}
$$

Fig. 5.   Bit-rock interaction model.

where $K \in \mathbb{R}^{m \times n}$ is the control gain matrix and $\hat{\xi}$ the observer estimate of the state ξ; the observer will be treated in more detail in Section III-B. The resulting closed-loop system is described by the following differential inclusion:

$$
\begin{array}{rcl}
\dot{\xi} & = & (A + BK)\,\xi + G\tilde{w} - BKe \\
\tilde{z} & = & H\xi \\
\tilde{w} & \in & -\tilde{\varphi}(\tilde{z}).
\end{array}
\tag{12}
$$

The transfer function $G_{cl}(s)$ from the input $\tilde{w}$ to the output $\tilde{z}$ of system (12) is given by $G_{cl} = H(sI - (A + BK))^{-1}G$, $s \in \mathbb{C}$. Now, let us state the following assumptions on the properties of the set-valued nonlinearity $\tilde{\varphi}(\tilde{z})$. Hereto, we first define a set $\mathcal{S}_a$ for which a particular sector condition is satisfied $\mathcal{S}_a := \{\tilde{z} \in \mathbb{R} | \tilde{z}_{a1} < \tilde{z} < \tilde{z}_{a2}\}$ with $\tilde{z}_{a1} < 0 < \tilde{z}_{a2}$.

Assumption 1: The set-valued nonlinearity $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ satisfies

- $0 \in \tilde{\varphi}(0)$;
- $\tilde{\varphi}$ is continuously differentiable and bounded $\forall \ \tilde{z} \in \mathcal{S}_a$;
- $\tilde{\varphi}$ locally satisfies the $[0, k]$ sector condition, with $k > 0$, in the sense that

$$
\tilde{w}\,[\tilde{w} + k\tilde{z}] \leq 0 \quad \forall \ \tilde{w} \in \{\tilde{w} \in -\tilde{\varphi}(\tilde{z}) | \tilde{z} \in \mathcal{S}_a\} .
\tag{13}
$$

Remark 1: The case where $\tilde{\varphi}$ is non-smooth or even discontinuous on the domain $\mathcal{S}_a$ can also be treated using the approach presented in [4], however, this is not necessary for the application presented in this paper.

The intended control goal is to render the closed-loop system (12) *locally input-to-state stable* with respect to the input e, as formalized in Definition 1, by a proper design of the controller gain K. We use the concept of a dynamic multiplier to transform the original system into a feedback interconnection of two passive systems. In Fig. 6, a block diagram of the system including the dynamic multiplier with transfer function $M(s) = 1 + \gamma s$, $s \in \mathbb{C}$, is shown; furthermore the loop transformation gain $\frac{1}{k}$ is included given the fact that the nonlinearity $\tilde{\varphi}(\cdot)$ belongs to the sector $[0, k]$. The linear system Σ_1 in Fig. 6 can be written in state-space form as follows:

$$
\Sigma_1 : \left\{
\begin{array}{rcl}
\dot{\xi} & = & (A + BK)\,\xi + G\tilde{w} - BKe \\
\breve{z} & = & \breve{H}\xi + \breve{D}\tilde{w} + \breve{Z}e
\end{array}
\right.
\tag{14}
$$

with $\breve{H} := H + \gamma H\,(A + BK)$, $\breve{D} := \frac{1}{k} + \gamma HG$ and $\breve{Z} := -\gamma HBK$. For system Σ_2 in Fig. 6 we can write:

$$
\Sigma_2 : \left\{
\begin{array}{rcl}
\dot{\tilde{z}} & = & -\frac{1}{\gamma}\tilde{z} + \frac{1}{\gamma}\breve{z} - \frac{1}{\gamma k}\tilde{w} \\
\tilde{w} & \in & -\tilde{\varphi}(\tilde{z})
\end{array}
\right.
\tag{15}
$$

The following theorem states sufficient conditions under which system (12) is LISS with respect to input e.

Theorem 1: Consider system (12) and suppose there exists a constant $\gamma > 0$ such that $\left(A + BK, G, \breve{H}, \breve{D} \right)$ is strictly passive. Then system (12) is LISS, with respect to input e for any $\tilde{\varphi}(\cdot)$ satisfying Assumption 1.

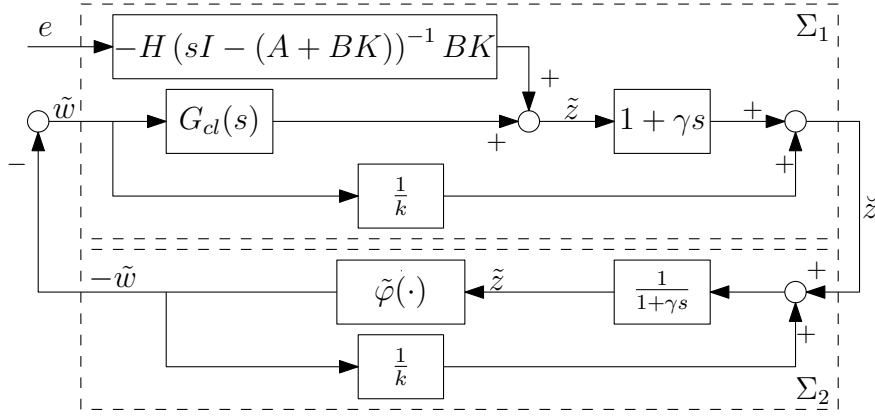*Proof:* The proof can be found in [22]. ■

Fig. 6.   Schematic representation of system (12) after transformation using a dynamic multiplier.

B. Observer design

The observer will be used to find an estimate of the states of system (10), this estimate is necessary since we only rely on surface measurements. The observer design proposed here is inspired by that in [21]. We propose the following observer, with measured output $\tilde{y} = C\xi$ ($\tilde{y} \in \mathbb{R}^k$ and $C \in \mathbb{R}^{n \times k}$):

$$
\begin{aligned}
\dot{\hat{\xi}} &= (A - LC)\hat{\xi} + Bu_{fb} + G\hat{w} + L\tilde{y} \\
\hat{z} &= (H - NC)\hat{\xi} + N\tilde{y} \\
\hat{w} &\in -\tilde{\varphi}(\hat{z}) \\
\hat{y} &= C\hat{\xi}
\end{aligned}
\tag{16}
$$

and observer gain matrices $L \in \mathbb{R}^{n \times k}$ and $N \in \mathbb{R}^{1 \times k}$. Next, we state an additional assumption on the nonlinearity $\tilde{\varphi}(\cdot)$. Hereto, we first define the set $\mathcal{S}_b$ as $\mathcal{S}_b := \{\tilde{z} \in \mathbb{R} | \tilde{z}_{b1} < \tilde{z} < \tilde{z}_{b2}\}$ with $\tilde{z}_{b1} < 0 < \tilde{z}_{b2}$, such that for all $\tilde{z} \in \mathcal{S}_b$ the monotonicity property holds.

Assumption 2: The set-valued nonlinearity $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ is such that $\tilde{\varphi}$ is monotone for all $\tilde{z} \in \mathcal{S}_b$, i.e. for all $z_1 \in \mathcal{S}_b$ and $z_2 \in \mathcal{S}_b$ with $w_1 \in \tilde{\varphi}(z_1)$ and $w_2 \in \tilde{\varphi}(z_2)$, it holds that $(w_1 - w_2)(z_1 - z_2) \geq 0$.

The observer error has been defined as $e = \xi - \hat{\xi}$ before. Consequently, the observer error dynamics can be written as

$$
\begin{aligned}
\dot{e} &= (A - LC)e + G(\tilde{w} - \hat{w}) \\
\tilde{w} &\in -\tilde{\varphi}(\tilde{z}) \\
\hat{w} &\in -\tilde{\varphi}\left(H\hat{\xi} + N\left(\tilde{y} - C\hat{\xi}\right)\right).
\end{aligned}
\tag{17}
$$

The following theorem provides sufficient conditions for the design of the observer gains L and N such that the origin $e = 0$ is a locally exponentially stable (LES) equilibrium point of the observer error dynamics (17).

Theorem 2: Consider system (10) and the observer (16) with $(A - LC, G, H - NC, 0)$ strictly passive and the matrix G being of full column rank. If it holds that

$$
\|\xi(t)\| \leq \epsilon \frac{\tilde{z}_{b,min}}{\|H\|}, \quad \forall t \geq 0,
$$

for some $\epsilon \in (0,1)$ and $\tilde{z}_{b,min} := \min(\tilde{z}_{b1}, \tilde{z}_{b2})$, then $e = 0$ is a locally exponentially stable equilibrium point of the observer error dynamics (17) for any $\tilde{\varphi}$ satisfying Assumptions 1 and 2 with the region of attraction containing the set

$$
\left\{ e \in \mathbb{R}^n | \|e_0\| \leq (1 - \epsilon) \frac{\tilde{z}_{b,min}}{\|H - NC\|} \left(\frac{\lambda_{max}(P_o)}{\lambda_{min}(P_o)}\right)^{-\frac{1}{2}} \right\}
$$

with the initial observer error $e(0) = e_0$. The matrix P_o results from the existence of $P_o = P_o^\top > 0$ and $Q_o = Q_o^\top > 0$ such that $P_o(A - LC) + (A - LC)P_o = -Q_o$ and $G^\top P_o = H - NC$, which is equivalent to the strict passivity of $(A - LC, G, H - NC, 0)$.

Proof: The proof can be found in [22]. ∎

C. Output-feedback control

The state-feedback controller and the observer from the previous sections together form an observer-based output-feedback controller. We use the estimated state $\hat{\xi}$ of the observer (16) in the feedback law (11) of system (12) and prove local asymptotic stability of the equilibrium $(\xi, e) = (0, 0)$ of the interconnected system (12), (17).
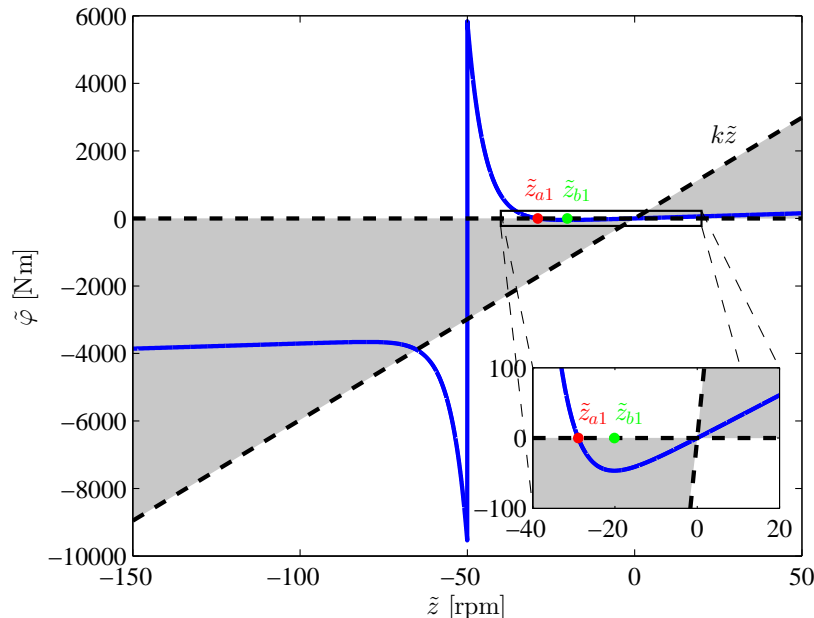
Fig. 7.   Transformed bit-rock interaction model $\tilde{\varphi}(\tilde{z})$.

Theorem 3: Consider system (12) and observer (16). Suppose the conditions in Theorem 1 are satisfied for system (12) and that the observer error dynamics in (17) satisfies the conditions in Theorem 2. Then, $(\xi, e) = (0, 0)$ is a locally asymptotically stable equilibrium point of the interconnected system (12), (17) for any $\tilde{\varphi}$ satisfying Assumptions 1 and 2.

Proof: The proof can be found in [22]. ∎

IV. SIMULATION RESULTS

In this section, we will show the application of the observer-based output-feedback controller (see Section III) to the reduced-order drill-string model presented in Section II. To stabilize the desired equilibrium x_{eq} of system (9) we have to design the controller gain K and the observer gains L and N to apply the control torque

$$u = u_{ff} + K\hat{\xi}, \tag{18}$$

with u_{ff} a constant feedforward torque to obtain the equilibrium (desired constant positive rotational velocity) and $K\hat{\xi}$ the feedback torque based on the observer estimate $\hat{\xi}$. If we assume that we can indeed operate the drill-string system at positive angular velocity, the Coulomb friction terms $T_{w,i}$ along the drill-string do not affect the dynamics of the system, at least locally near the desired operating condition, and can consequently be represented by constant resistive torques. These constant resistive torques, can then be compensated by the feedforward torque. The equilibrium x_{eq} and feedforward torque u_{ff} can be obtained from the equilibrium condition of system (9) that has to satisfy $Ax_{eq} - G\varphi(Hx_{eq}) - G_2\phi(H_2 x_{eq}) + Bu_{ff} = 0$ and we require that $y_1 = \omega_{td}$ matches the desired equilibrium velocity ω_{eq}.

Now, we have to write the system (9) in the form (10) and such that the set-valued nonlinearity satisfies the conditions in Assumptions 1 and 2. Therefore, we write the reduced-order drill-string system in perturbation states, i.e. $\xi := x - x_{eq}$. Furthermore, we apply a linear loop transformation to change the properties of the nonlinearity φ. This results in the following state-space representation

$$\begin{aligned}
\dot{\xi} &= A_t\xi + Bu_{fb} + G\tilde{w} \\
\tilde{z} &= H\xi \\
\tilde{w} &\in -\tilde{\varphi}(\tilde{z})
\end{aligned} \tag{19}$$

with $\delta > 0$ a constant to apply the linear loop transformation, $\tilde{\varphi}(\tilde{z}) := \varphi(\tilde{z} + Hx_{eq}) - \varphi(Hx_{eq}) + \delta\tilde{z}$ and $\tilde{w} = w - w_{eq} - \delta\tilde{z}$ and we define $A_t := A + \delta GH$. The transformed nonlinearity $\tilde{\varphi}(\tilde{z})$ is shown in Fig. 7. As can be seen in this figure, $\tilde{\varphi}(\tilde{z})$ belongs locally to the sector $[0, k]$ with $k = 570$ Nms/rad (note $\delta = 29.2$ Nms/rad in this case). The physical meaning of this condition is that the amount of velocity weakening in the bit-rock interaction is limited. A larger sector, including the total nonlinearity $\tilde{\varphi}(\tilde{z})$, would result in high control gains K. Such high gains result in high control torques u that can not be realized by the top drive and are therefore infeasible in practice. In Fig. 7, we have also indicated the point $\tilde{z}_{a1} = -28.9$ rpm for which holds that for $\tilde{z}_{a1} < \tilde{z} < \tilde{z}_{a2}$ the sector condition is satisfied (i.e. $\tilde{z}_{a2}$ can be chosen arbitrarily large in this case) and the point $\tilde{z}_{b1} = -20.1$ rpm such that for $\tilde{z}_{b1} < \tilde{z} < \tilde{z}_{b2}$ it holds that $\tilde{\varphi}$ is monotonically increasing (i.e. $\tilde{z}_{b2}$ can also be chosen arbitrarily large in this case), as stated in Assumption 1 and 2, respectively. The last condition in Assumption 1 states that the bit-rock interaction model is bounded by a linear function.
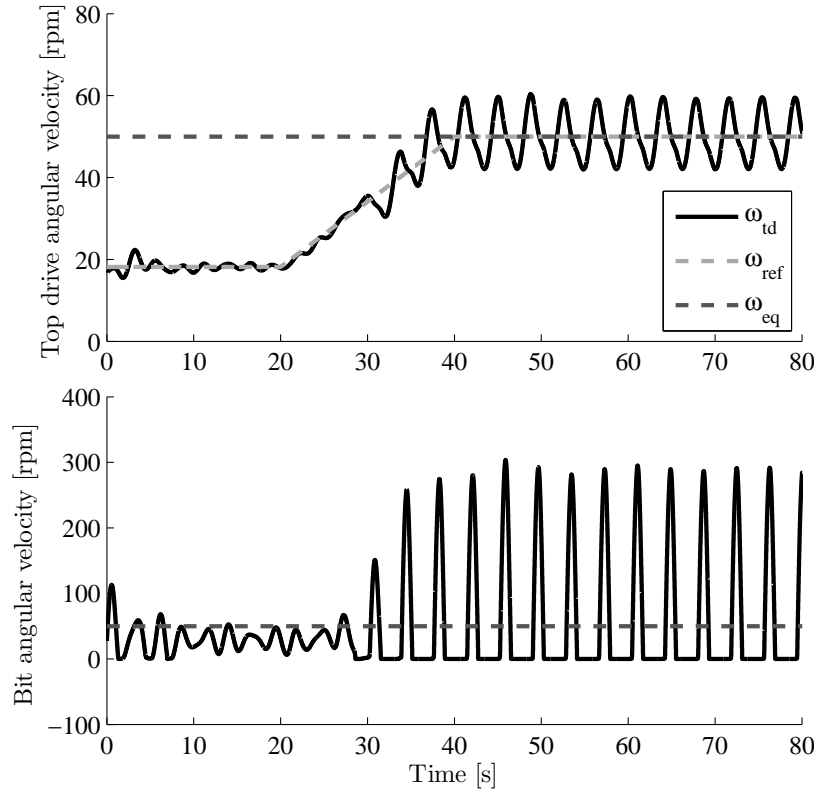
Fig. 8. Simulation result of the reduced-order model with an existing industrial controller.

The controller and observer gains are designed according to the conditions given in Theorem 1 and Theorem 2, respectively. The results are obtained by using SeDuMi 1.3 [23], a linear matrix inequality (LMI) solver and the YALMIP interface [24]. Hence, the controller gains K are determined by finding a solution such that $(A_t + BK, G, \check{H}, \check{D})$ is strictly passive, with $\check{H}$ and $\check{D}$ as defined in (14). To find the observer gains L and N we have to satisfy the strict passivity conditions for $(A_t - LC, G, H - NC, 0)$.

Before we show the simulation results of the designed output-feedback controller, we will show a simulation result of the reduced-order drill-string system in closed-loop with an existing industrial controller (based on [5]). For the simulations, we introduce a so-called startup scenario, which is based on practical startup procedures for drilling rigs. Herein, the drill-string is first accelerated to a low constant rotational velocity with the bit above the formation (off bottom) and, subsequently, the angular velocity and weight-on-bit (WOB) are gradually increased to the desired operating conditions. The startup scenario is built up as follows:

1) Start with WOB $= 0$, such that there is no velocity weakening effect in the bit-rock interaction model and use the industrial PI-controller to operate at relatively low velocity and build up torque in drill-string to overcome static torques due to drag in the time window $0 < t < 20$ s;

2) Turn on the controller (18) and slowly increase the reference angular velocity until the desired operating velocity (ω_{eq}) is reached (in the time window $20 \leq t < 40$ s). At the same time, slowly increase the WOB to let the bit bite the formation and finally obtain the nominal operating condition in the angular velocity and WOB.

A simulation result of the reduced-order model with the industrial controller is shown in Fig. 8. The controller is a properly tuned active damping system (i.e. PI-control of the angular velocity) which aims at damping the first torsional mode of the drill-string dynamics. In the upper plot the top drive velocity (ω_{td}) is shown along with the reference velocity ω_{ref} that starts at a velocity of approximately 20 rpm and is gradually increased to the desired equilibrium velocity, ω_{eq}, of 50 rpm. From the bit response, in the bottom plot, we can clearly recognize stick-slip oscillations. The increasing amplitude of the oscillations in the top drive velocity, demonstrates that these vibrations arise when the WOB is increased ($20 \leq t < 40$ s).

For the designed output-feedback controller, we immediately activate (at $t = 0$) the observer to obtain the state estimate $\hat{\xi}$; however, this estimate is not used by the industrial PI-controller in the first 20 seconds (since this controller only uses the top drive velocity as a measured output). When the state-feedback controller is switched on at $t = 20$, it uses the state estimate $\hat{\xi}$, based on the surface measurements ω_{td} and T_{pipe} only. Fig. 9 shows a simulation result of the closed-loop system with output-feedback controller, where we used the same initial conditions ξ_0 as for the previous simulation (Fig. 8). Furthermore, the initial states for the observer $\hat{\xi}_0$ have a 10% offset from the initial states ξ_0. It can be seen that after some transient behavior, the observer estimates converge to the actual states within approximately 5 seconds. After 20 seconds the
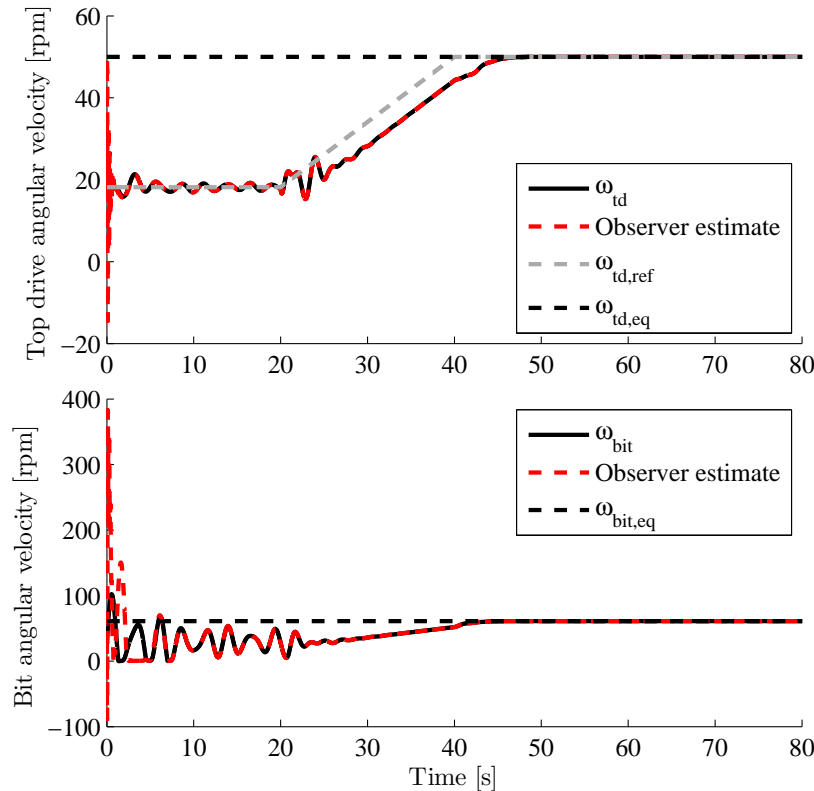
Fig. 9. Simulation result of the reduced-order model with the designed output-feedback controller.

output-feedback controller is turned on and the WOB and desired velocity are increased. The simulation results show that the top drive and bit velocity converge to their equilibrium value and stick-slip oscillations are avoided. The equilibrium velocity of the bit $\omega_{bit,eq} := Hx_{eq}$ is slightly higher than the equilibrium velocity of the top drive $\omega_{td,eq} = \omega_{eq} = 50$ rpm. This small mismatch is due to the reduction as the outputs z and y of the reduced-order system slightly differ from the original outputs $\bar{z}$ and $\bar{y}$ and the feedforward is designed such that in equilibrium the top drive velocity of the reduced-order model matches the desired velocity. The lag between the desired velocity and the top drive velocity between 25 and 45 s in Fig. 9 can be explained as follows. We have designed a low-gain controller to accommodate the practical limitations of the top drive actuator aiming at the stabilization of the desired equilibrium (not at achieving a high bandwidth). Most importantly, it can be concluded that the stick-slip vibrations are eliminated with the designed controller.

V. CONCLUSIONS

In this work, an observer-based output-feedback control strategy is proposed to eliminate torsional stick-slip vibrations in drilling systems. Particular benefits of the proposed approach with respect to existing ones are, firstly, the fact that a realistic multi-modal model of the drill-string dynamics is taken into account, secondly, that severe velocity weakening in the bit-rock interaction is taken into account, thirdly, that only surface measurements are employed and, finally, that a guarantee for (local) asymptotic stability of the closed-loop system is given for bit-rock interaction laws lying within a certain sector (which is beneficial as the bit-rock interaction is subject to uncertainty in practice). Simulation results of applying the proposed controller to a realistic drill-string model show that stick-slip oscillations can be eliminated, while under the same conditions the existing industrial controller is unable to do so.

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