

## Are there any nicely structured preference~profiles~nearby?

***Citation for published version (APA):***

Bredereck, R., Chen, J., & Woeginger, G. J. (2015). *Are there any nicely structured preference~profiles~nearby?* (arXiv; Vol. 1509.04595 [cs.GT]). arXiv.org.

***Document status and date:***

Published: 01/01/2015

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

# Are there any nicely structured preference profiles nearby?\*

Robert Brederbeck<sup>1</sup>, Jiehua Chen<sup>1</sup>, and Gerhard J. Woeginger<sup>2</sup>

<sup>1</sup> Institut für Softwaretechnik und Theoretische Informatik,  
Technische Universität Berlin,  
D-10587 Berlin, Germany  
{[robert.bredereck](mailto:robert.bredereck@tu-berlin.de), [jiehua.chen](mailto:jiehua.chen@tu-berlin.de)}@tu-berlin.de

<sup>2</sup> Department of Mathematics and Computer Science,  
TU Eindhoven,  
5600 MB Eindhoven, Netherlands  
[gwoegi@win.tue.nl](mailto:gwoegi@win.tue.nl)

## Abstract

We investigate the problem of deciding whether a given preference profile is close to having a certain nice structure, as for instance single-peaked, single-caved, single-crossing, value-restricted, best-restricted, worst-restricted, medium-restricted, or group-separable profiles. We measure this distance by the number of voters or alternatives that have to be deleted to make the profile a nicely structured one. Our results classify the problem variants with respect to their computational complexity, and draw a clear line between computationally tractable (polynomial-time solvable) and computationally intractable (NP-hard) questions.

## 1 Introduction

The area of Social Choice (and in particular the subarea of *Computational* Social Choice) is full of so-called *negative* results. On the one hand there are many axiomatic impossibility results, and on the other hand there are just as many computational intractability results. For instance, the famous impossibility result of Arrow [3] states that there is no perfectly fair way (satisfying certain desirable axioms) of aggregating the preferences of a society of voters into a single preference order. As another example, Bartholdi III et al. [8] establish that it is computationally intractable (NP-hard) to determine whether some particular alternative has won an election under a voting scheme designed by Lewis Carroll. Most of these negative results hold for general preference profiles where *any* combination of preference orders may occur.

One branch of Social Choice studies *restricted domains* of preference profiles, where only certain nicely structured combinations of preference orders are admissible. The standard example for this approach are *single-peaked* preference profiles as introduced by Black [10]: a preference profile is single-peaked if there exists a linear order of the alternatives such that every voter's preference along this order is either always strictly increasing, always strictly decreasing, or first strictly increasing and then strictly decreasing. Determining whether a profile is single-peaked is solvable in polynomial time [7, 22, 30].

---

\*A preliminary short version of this work has been presented at the 23rd International Joint Conference on Artificial Intelligence (IJCAI 2013), Beijing, August, 2013 [13].

Single-peakedness implies a number of nice properties, as for instance strategy-proofness of a family of voting rules [42] and transitivity of the majority relation [36]. Furthermore, Arrow’s impossibility result collapses for single-peaked profiles. In a similar spirit (but in the algorithmic branch), Walsh [51], Brandt et al. [11], and Faliszewski et al. [31] show that many electoral bribery, control and manipulation problems that are NP-hard in the general case become tractable under single-peaked profiles. Besides the single-peaked domain, the literature contains many other *restricted domains* of nicely structured preference profiles (see Section 2 for precise mathematical definitions).

- Sen [47] and Sen and Pattanaik [46] introduced the domain of *value-restricted* preference profiles which satisfy the following: for every triple of alternatives, one alternative is not preferred most by any individual (*best-restricted* profile), or one is not preferred least by any individual (*worst-restricted* profile), or one is not considered as the intermediate alternative by any individual (*medium-restricted* profile).
- Inada [35, 36] considered the domain of *group-separable* preference profiles which satisfy the following: the alternatives can be split into two groups such that every voter prefers every alternative in the first group to those in the second group, or prefers every alternative in the second group to those in the first group. Every group-separable profile is also medium-restricted.
- *Single-caved* preference profiles are derived from single-peaked profiles by reversing the preferences of every voter. Sometimes single-caved profiles are also called *single-dipped* [37].
- *Single-crossing* preference profiles go back to the seminal papers of Mirrlees [40] and Roberts [45] on income taxation. A preference profile is single-crossing if there exists a linear order of the voters such that each pair of alternatives separates this order into two sub-orders where in each sub-order, all voters agree on the relative order of this pair. Similar to the single-peaked property, single-crossing profiles can also be recognized in polynomial time [12, 22, 26].

Unfortunately, real-world elections are almost never single-peaked, value-restricted, group-separable, single-caved or single-crossing. Usually there are maverick voters whose preferences are determined for instance by race, religion, or gender, and whose misfit behaviors destroy all nice structures in the preference profile. In a very recent line of research, Faliszewski et al. [32] searched for a cure against such mavericks, and arrived at *nearly* single-peaked preference profiles: a profile is nearly single-peaked if it is very close to a single-peaked profile. Of course there are many mathematical ways of measuring the closeness of profiles. Natural ways to make a given profile single-peaked are (i) by deletion of voters and (ii) by deletion of alternatives. This leads to the two central problems of our work for a specific property  $\Pi$  and a given number  $k$ :

1. The  $\Pi$  MAVERICK DELETION problem asks whether it is possible to delete at most  $k$  voters to make a given profile satisfy the  $\Pi$ -property.
2. The  $\Pi$  ALTERNATIVE DELETION problem asks whether it is possible to delete at most  $k$  alternatives to make a given profile satisfy the  $\Pi$ -property.

In this paper, we have  $\Pi \in \{\text{worst-restricted, medium-restricted, best-restricted, value-restricted, single-peaked, single-caved, single-crossing, group-separable, } \beta\text{-restricted}\}$ . We provide the formal definitions of these properties in Section 2.

Restriction	Maverick deletion		Alternative deletion	
Single-peaked	NP-complete	(*, Corollary 1)	P	(*)
Single-caved	NP-complete	(*, Corollary 1)	P	(*)
Group-separable	NP-complete	(Corollary 1)	NP-complete	(Corollary 2)
Single-crossing	P	(Theorem 6)	NP-complete	(Theorem 5)
Value-restricted	NP-complete	(Theorem 1)	NP-complete	(Theorem 2)
Best-restricted	NP-complete	(Proposition 1)	NP-complete	(Proposition 2)
Worst-restricted	NP-complete	(Proposition 1)	NP-complete	(Proposition 2)
Medium-restricted	NP-complete	(Proposition 1)	NP-complete	(Proposition 2)
$\beta$ -restricted	NP-complete	(Theorem 3)	NP-complete	(Theorem 4)

Table 1: Summary of the results where P means polynomial-time solvable. Entries marked by “\*” are due to Erdélyi et al. [29]. The definition of the respective domain restrictions can be found in Section 2.

**Results of this paper.** We investigate the problem of deciding how close (by deletion of voters and by deletion of alternatives) a given preference profile is to having a nice structure (like being single-peaked, or single-crossing, or group-separable). We focus on the most fundamental definitions of closeness and on the most popular restricted domains. Our results draw a clear line between computationally easy (polynomial-time solvable) and computationally intractable (NP-hard) questions as they classify all considered problem variants with respect to their computational complexity. In particular, for most of the cases both of our central problems are computationally intractable (with the exceptions of maverick deletion when the specific property  $\Pi$  is single-crossing, and of alternative deletion when  $\Pi$  is either single-peaked or single-caved). Our results are summarized in Table 1.

**Related work.** As to different notions of closeness to restricted domains,

- Erdélyi et al. [29] study various concepts of nearly single-peakedness. Besides deletion of voters and deletion of alternatives, they also study closeness measures that are based on swapping alternatives in the preference orders of some voters, or on introducing additional political axes.
- Yang and Guo [52] study *k-peaked* domains, where every preference order can have at most  $k$  peaks (that is, at most  $k$  rising streaks that alternate with falling streaks).
- Cornaz et al. [19, 20] introduce a closeness measure, the *width*, for single-peaked, single-crossing, and group-separable profiles which is based on the notion of a clone set [50]. For instance, the *single-peaked width* of a preference profile is the smallest number  $k$  such that partitioning all alternatives into disjoint *intervals*, each with size at most  $k+1$ , and replacing each of these intervals with a single alternative results in a single-peaked profile. An interval of alternatives is a set of alternatives that appear consecutively (in any order) in the preference orders of all voters.

There are several generalizations of the single-peaked property. For instance, Barberà et al. [6] introduce the concept of multi-dimensional single-peaked domains. The 1-dimensional special case is equivalent to our single-peaked property. Sui et al. [49] study this concept empirically. They present approximation algorithms (for several optimization goals) of finding multi-dimensional single-peaked profiles and show that their two real-world data sets

are far from being single-peaked but are nearly 2-dimensional single-peaked. While Erdélyi et al. [29] and this paper show that deciding the distance to restricted domains is NP-hard in most cases, Elkind and Lackner [25] present efficient approximation and fixed-parameter tractable algorithms for deciding the distance to restricted domains such as single-peakedness and single-crossingness.

Finally, we remark that the closeness concept can also be used to characterize voting rules [4, 27, 39]. The basic idea is to first fix a specific property, for instance, the transitivity of the pairwise majority relation (also known as Condorcet consistency), and then to define a closeness measure from a given profile to the “nearest” profile with this specific property. For instance, the Young rule [53] takes the subprofile that is closest to being Condorcet consistent by deleting the fewest number of voters and selects the corresponding Condorcet winner as a winner; see Elkind et al. [27] for more information on this.

For restricted and nearly restricted domains, there are various studies on single-winner determination [11], on multi-winner determination [9, 48], on control, manipulation, and bribery [11, 15, 31, 32], and on possible/necessary winner problems [51]. Usually, the expectation is that domain restrictions help in lowering the computational complexity of many voting problems. Many publications, however, report that this is not always the case. For instance, Faliszewski et al. [32] show that the computational complexity of “controlling approval-based rules” for nearly single-peaked profiles is polynomial-time solvable if the distance to single-peaked is a constant, and thus, coincides with the one for single-peaked profiles, whereas the computational complexity of “manipulating the veto rule” for nearly single-peaked profiles is still NP-hard and thus, coincides with the one for unrestricted profiles.

**Article outline.** This paper is organized as follows. Section 2 summarizes all the basic definitions and notations. Our results are presented in Sections 3 to 5:

1. Section 3 presents results for the value-restricted, best-restricted, worst-restricted, and medium-restricted properties. All results are NP-hardness results and are obtained through reduction from the NP-complete VERTEX COVER problem (see the beginning of Section 3 for the definition).
2. Section 4 shows results for single-peakedness, single-cavedness, and group-separability. In addition, this section shows results for the  $\beta$ -restricted property, a necessary condition for group-separability. Again, all results are NP-hardness results and are obtained through reduction from VERTEX COVER.
3. Section 5 shows that achieving the single-crossing property by deleting as few alternatives as possible is NP-hard; the reduction is from the NP-complete MAXIMUM 2-SATISFIABILITY problem (see the beginning of Section 5 for the definition), and shows that finding a single-crossing profile with the largest voter set is polynomial-time solvable; this is done by reducing the problem to finding a longest path in a directed acyclic graph.

We conclude with some future research directions in the last section.

## 2 Preliminaries and basic notations

Let  $a_1, \dots, a_m$  be  $m$  alternatives and let  $v_1, \dots, v_n$  be  $n$  voters. A *preference profile* specifies the *preference orders* of the  $n$  voters, where voter  $v_i$  ranks the alternatives according to a linear order  $\succ_i$  over the  $m$  alternatives. For alternatives  $a$  and  $b$ , the relation  $a \succ_i b$  means

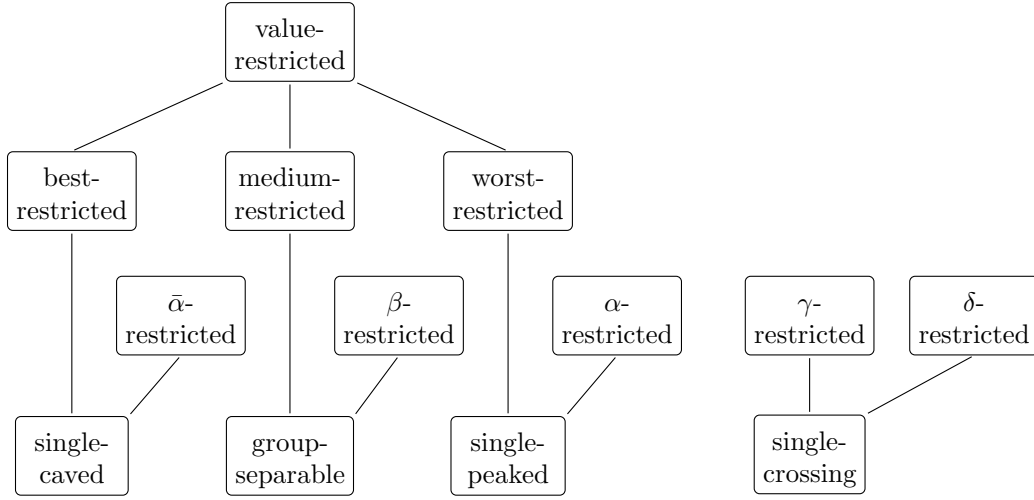


Figure 1: A Hasse diagram for the relation of the different properties. An edge between two properties means that a profile with the property in the lower tier implies the property in the upper tier. For instance, there is an edge between “value-restricted” and “best-restricted”, because a best-restricted profile is also value-restricted.

that voter  $v_i$  strictly prefers  $a$  to  $b$ . We omit the subscript  $i$  if it is clear from the context whose preference order we are referring to.

Given two disjoint sets  $A$  and  $B$  of alternatives, we write  $A \succ_i B$  to express that voter  $v_i$  prefers set  $A$  to set  $B$ , that is, for each alternative  $a \in A$  and each alternative  $b \in B$  it holds that  $a \succ_i b$ . We write  $A \succ_i b$  as shorthand for  $A \succ_i \{b\}$  and  $b \succ_i A$  for  $\{b\} \succ_i A$ . We sometimes fix a canonical order of the alternatives in  $A$  and denote this order by  $\langle A \rangle$ . The expression  $\langle A_1 \rangle \succ \langle A_2 \rangle$  denotes the preference order that is consistent with  $\langle A_1 \rangle$ ,  $\langle A_2 \rangle$ , and  $A_1 \succ A_2$ .

Next, we review some preference profiles with special properties studied in the literature [5, 12, 35, 36, 46, 47]. We call such profiles *configurations* and we use them to characterize some properties of the preference profiles. We illustrate the relation between the respective properties in Figure 1.

## 2.1 Value-restricted profiles

The first three configurations [5] describe profiles with three alternatives where each alternative is in the best, medium, or worst position in some voter’s preference order.

**Definition 1** (Best-diverse configuration). A profile with three voters  $v_1, v_2, v_3$ , and three distinct alternatives  $a, b, c$  is a *best-diverse configuration* if it satisfies the following:

- voter  $v_1$ :  $a \succ \{b, c\}$ ,
- voter  $v_2$ :  $b \succ \{a, c\}$ ,
- voter  $v_3$ :  $c \succ \{a, b\}$ .

**Definition 2** (Medium-diverse configuration). A profile with three voters  $v_1, v_2, v_3$ , and three distinct alternatives  $a, b, c$  is a *medium-diverse configuration* if it satisfies the following:

- voter  $v_1$ :  $b \succ a \succ c$  or  $c \succ a \succ b$ ,
- voter  $v_2$ :  $a \succ b \succ c$  or  $c \succ b \succ a$ ,
- voter  $v_3$ :  $a \succ c \succ b$  or  $b \succ c \succ a$ .

**Definition 3** (worst-diverse configuration). A profile with three voters  $v_1, v_2, v_3$  and three distinct alternatives  $a, b, c$  is a *worst-diverse configuration* if it satisfies the following:

$$\begin{aligned} \text{voter } v_1: & \{b, c\} \succ a, \\ \text{voter } v_2: & \{a, c\} \succ b, \\ \text{voter } v_3: & \{a, b\} \succ c. \end{aligned}$$

We use these three configurations to characterize several restricted domains: A profile is *best-restricted* (resp. *medium-restricted*, *worst-restricted*) *with respect to a triple of alternatives* if it contains no three voters that form a best-diverse configuration (resp. a medium-diverse configuration, a worst-diverse configuration) with respect to this triple. A profile is *best-restricted* (resp. *medium-restricted*, *worst-restricted*) if it is best-restricted (resp. medium-restricted, worst-restricted) with respect to every possible triple of alternatives.

A profile is *value-restricted* [47] if for every triple  $T$  of alternatives, it is best-restricted, medium-restricted, or worst-restricted with respect to  $T$ . In other words, a profile is *not* value-restricted if and only if it contains a triple of alternatives and three voters  $v_1, v_2, v_3$  that form a best-diverse configuration, a medium-diverse configuration, and a worst-diverse configuration with respect to this triple.

**Definition 4** (Cyclic configuration). A profile with three voters  $v_1, v_2, v_3$  and three distinct alternatives  $a, b, c$  is a *cyclic configuration* if it satisfies the following:

$$\begin{aligned} \text{voter } v_1: & a \succ b \succ c, \\ \text{voter } v_2: & b \succ c \succ a, \\ \text{voter } v_3: & c \succ a \succ b. \end{aligned}$$

The set of the preference orders of all voters in a value-restricted profile is also known as *acyclic domain of linear orders*. Many research groups [1, 2, 41, 44] investigate maximal acyclic domains for a given number  $m$  of alternatives, where an acyclic domain is *maximal* if adding any new linear order destroys the value-restricted property.

## 2.2 Single-peaked profiles and single-caved profiles

Given a set  $A$  of alternatives and a linear order  $L$  over  $A$ , we say that a voter  $v$  is *single-peaked* with respect to  $L$  if his preference along  $L$  is always strictly increasing, always strictly decreasing, or first strictly increasing and then strictly decreasing. Formally, a voter  $v$  is single-peaked with respect to  $L$  if for each three distinct alternatives  $a, b, c \in A$ , it holds that

$$(a \succ_L b \succ_L c \text{ or } c \succ_L b \succ_L a) \text{ implies that if } a \succ_v b, \text{ then } b \succ_v c.$$

A profile with the alternative set  $A$  is *single-peaked* if there is a linear order over  $A$  such that every voter is single-peaked with respect to this order. Single-peaked profiles are necessarily worst-restricted. To see this, we observe that in a profile with at least three alternatives, the alternative that is ranked last by at least one voter must not be placed between the other two along any single-peaked order. But then, none of the alternatives  $a, b$ , and  $c$  from a worst-diverse configuration can be placed between the other two in any single-peaked order. Thus, a profile with worst-diverse configurations cannot be single-peaked.

To fully characterize the single-peaked domain, we additionally need the following configuration.

**Definition 5** ( $\alpha$ -configuration). A profile with two voters  $v_1$  and  $v_2$ , and four distinct alternatives  $a, b, c, d$  is an  $\alpha$ -configuration if it satisfies the following:

$$\begin{aligned} \text{voter } v_1: & \{a, d\} \succ b \succ c, \\ \text{voter } v_2: & \{c, d\} \succ b \succ a. \end{aligned}$$

The  $\alpha$ -configuration describes a situation where two voters have opposite opinions on the order of three alternatives  $a, b$  and  $c$  but agree that a fourth alternative  $d$  is “better” than the one ranked in the middle. A profile with this configuration is not single-peaked as we must put alternatives  $b$  and  $d$  between alternatives  $a$  and  $c$ , but then voter  $v_1$  prevents us from putting  $b$  next to  $a$  and voter  $v_2$  prevents us from putting  $b$  next to  $c$ .

A profile is *single-peaked* if and only if it contains neither worst-diverse configurations nor  $\alpha$ -configurations [5]. Since reversing the preference orders of a single-peaked profile results in a single-caved one, an analogous characterization of single-caved profiles follows. A profile is *single-caved* if and only if it contains neither best-diverse configurations nor  $\bar{\alpha}$ -configurations where a  $\bar{\alpha}$ -configuration is a  $\alpha$ -configuration with both “preference orders” being inverted:

**Definition 6** ( $\bar{\alpha}$ -configuration). A profile with two voters  $v_1$  and  $v_2$ , and four distinct alternatives  $a, b, c, d$  is an  $\bar{\alpha}$ -configuration if it satisfies the following:

$$\begin{aligned} \text{voter } v_1: & a \succ b \succ \{c, d\}, \\ \text{voter } v_2: & c \succ b \succ \{a, d\}. \end{aligned}$$

## 2.3 Group-separable profiles

Given a profile with  $A$  being the set of alternatives, the *group-separable* property requires that every size-at-least-three subset  $A' \subseteq A$  can be partitioned into two disjoint non-empty subsets  $A'_1$  and  $A'_2$  such that for each voter  $v_i$ , either  $A'_1 \succ_i A'_2$  or  $A'_2 \succ_i A'_1$  holds. One can verify that group-separable profiles are necessarily medium-restricted. Ballester and Haeringer [5] characterized the group-separable property using the following configuration.

**Definition 7** ( $\beta$ -configuration).

A profile with two voters  $v_1$  and  $v_2$  and four distinct alternatives  $a, b, c, d$  is a  $\beta$ -configuration if it satisfies the following:

$$\begin{aligned} \text{voter } v_1: & a \succ b \succ c \succ d, \\ \text{voter } v_2: & b \succ d \succ a \succ c. \end{aligned}$$

The  $\beta$ -configuration describes a situation where the most preferred alternative and the least preferred alternative of voter  $v_1$  are  $a$  and  $d$  which are different from the ones of voter  $v_2$ :  $b$  and  $c$ . Both voters agree that  $b$  is better than  $c$ , but disagree on whether  $d$  is better than  $a$ . This profile is not group-separable: We can not partition  $\{a, b, c, d\}$  into one singleton and one three-alternatives set as each alternative is ranked in the middle once, but neither can we partition them into two sets each of size two since voter  $v_1$  prevents us from putting alternatives  $a$  and  $c$  or alternatives  $a$  and  $d$  together and voter  $v_2$  prevents us from putting alternatives  $a$  and  $b$  together. Profiles without  $\beta$ -configurations are called  *$\beta$ -restricted* [5].

A profile is *group-separable* if and only if it contains neither medium-diverse configurations nor  $\beta$ -configurations [5].

## 2.4 Single-crossing profiles

The single-crossing property describes the existence of a “natural” linear order of the voters. A preference profile is *single-crossing* if there exists a *single-crossing order* of the voters, that is, a linear order  $L$  of the voters, such that each pair of alternatives separates  $L$  into two sub-orders where in each sub-order, all voters agree on the relative order of this pair. Formally, this means that for each pair of alternatives  $a$  and  $b$  such that the first voter along the order  $L$  prefers  $a$  to  $b$  and for each two voters  $v, v'$  with  $v \succ_L v'$ ,

$$b \succ_v a \text{ implies } b \succ_{v'} a.$$

To characterize single-crossing profiles, we need the following two configurations.



**Definition 8** ( $\gamma$ -configuration).

A profile with three voters  $v_1, v_2, v_3$ , and six (not necessarily distinct) alternatives  $a, b, c, d, e, f$  is a  $\gamma$ -configuration, if it satisfies the following:

voter  $v_1$ :  $b \succ a$  and  $c \succ d$  and  $e \succ f$ ,  
voter  $v_2$ :  $a \succ b$  and  $d \succ c$  and  $e \succ f$ ,  
voter  $v_3$ :  $a \succ b$  and  $c \succ d$  and  $f \succ e$ .

The  $\gamma$ -configuration describes a situation where each voter disagrees with the other two voters on the order of exactly two distinct alternatives. The profile is not single-crossing as none of the three voters can be put between the other two: The pair  $\{a, b\}$  prevents us from putting  $v_1$  in the middle, the pair  $\{c, d\}$  forbids voter  $v_2$  in the middle, and the pair  $\{e, f\}$  forbids  $v_3$  in the middle.

**Definition 9** ( $\delta$ -configuration).

A profile with four voters  $v_1, v_2, v_3, v_4$ , and four (not necessarily distinct) alternatives  $a, b, c, d$  is a  $\delta$ -configuration, if it satisfies the following:

voter  $v_1$ :  $a \succ b$  and  $c \succ d$ ,  
voter  $v_2$ :  $a \succ b$  and  $d \succ c$ ,  
voter  $v_3$ :  $b \succ a$  and  $c \succ d$ ,  
voter  $v_4$ :  $b \succ a$  and  $d \succ c$ .

The  $\delta$ -configuration shows a different kind of voter behavior: Two voters disagree with the other two voters on the order of two alternatives, but also disagree between each other on the order of two further alternatives. This profile is not single-crossing as the pair  $\{a, b\}$  forces us to place  $v_1$  and  $v_2$  next to each other, and to put  $v_3$  and  $v_4$  next to each other; the pair  $\{c, d\}$  forces us to place  $v_1$  and  $v_3$  next to each other, and to put  $v_2$  and  $v_4$  next to each other. This means that no voter can be placed in the first position.

A profile is *single-crossing* if and only if it contains neither  $\gamma$ -configurations nor  $\delta$ -configurations [12].

## 2.5 Two central problems

As already discussed before, two natural ways of measuring the closeness of profiles to some restricted domains are to count the number of voters resp. alternatives which have to be deleted to make a profile single-crossing. Hence, for  $\Pi \in \{\text{worst-restricted, medium-restricted, best-restricted, value-restricted, single-peaked, single-caved, single-crossing, group-separable, } \beta\text{-restricted}\}$ , we study the following two decision problems:  $\Pi$  MAVERICK DELETION and  $\Pi$  ALTERNATIVE DELETION.

$\Pi$  MAVERICK DELETION

**Input:** A profile with  $n$  voters and a non-negative integer  $k \leq n$ .

**Question:** Can we delete at most  $k$  voters so that the resulting profile satisfies the  $\Pi$ -property?

$\Pi$  ALTERNATIVE DELETION

**Input:** A profile with  $m$  alternatives and a non-negative integer  $k \leq m$ .

**Question:** Can we delete at most  $k$  alternatives so that the resulting profile satisfies the  $\Pi$ -property?

An upper bound for the computational complexity of  $\Pi$  MAVERICK DELETION and  $\Pi$  ALTERNATIVE DELETION is easy to see. Both problems are contained in **NP** for each property  $\Pi$

we study: Given a preference profile one can check in polynomial time whether it has property  $\Pi$  since  $\Pi$  is characterized by a finite set of forbidden finite substructures. Thus, in order to show NP-completeness of  $\Pi$  MAVERICK DELETION and  $\Pi$  ALTERNATIVE DELETION, we only have to show their NP-hardness.

### 3 Value-restricted properties

In this section, we show NP-hardness for the value-restricted, best-restricted, worst-restricted, and medium-restricted domains, respectively. Notably, we show all these results by reducing from the NP-complete VERTEX COVER problem [34].

VERTEX COVER

**Input:** An undirected graph  $G = (U, E)$  and a non-negative integer  $k \leq |U|$ .

**Question:** Is there a vertex cover  $U' \subseteq U$  of at most  $k$  vertices, that is,  $|U'| \leq k$  and  $\forall e \in E: e \cap U' \neq \emptyset$ ?

In every reduction from VERTEX COVER we describe, the vertex cover size  $k$  coincides with the maximum number  $k$  of voters (resp. alternatives) to delete. Hence, we use the same variable name.

We first deal with the case of maverick voter deletion (Section 3.1) and then, with the case of deleting alternatives (Section 3.2). In both cases, the general idea is to transform every edge of a given graph into an appropriate forbidden configuration.

#### 3.1 Maverick Voter Deletion

**Theorem 1.** VALUE-RESTRICTED MAVERICK DELETION is NP-complete.

*Proof.* We provide a polynomial-time reduction from VERTEX COVER to show NP-hardness. We will present an example (see Figure 2) for the reduction right after this proof.

Let  $(G = (U, E), k)$  denote a VERTEX COVER instance with vertex set  $U = \{u_1, \dots, u_r\}$  and edge set  $E = \{e_1, \dots, e_s\}$ ; without loss of generality we assume that the input graph  $G$  is connected and that graph  $G$  has at least four vertices, that is,  $r \geq 4$ , and that  $k \leq r - 3$ .

The set of alternatives consists of three *edge alternatives*  $a_j, b_j$ , and  $c_j$  for each edge  $e_j \in E$ . For each vertex in  $U$ , we construct one voter. That is, we define  $A := \{a_j, b_j, c_j \mid e_j \in E\}$  and  $V := \{v_i \mid u_i \in U\}$ . In total, the number  $m$  of alternatives is  $3s$  and the number  $n$  of voters is  $r$ .

Now we specify the preference order of each voter. Every voter prefers  $\{a_j, b_j, c_j\}$  to  $\{a_{j'}, b_{j'}, c_{j'}\}$  whenever  $j < j'$ . For each edge  $e_j$  with two incident vertices  $u_i$  and  $u_{i'}$ ,  $i < i'$ , and for each non-incident vertex  $u_{i''} \notin e_j$ , the following holds:

$$\begin{aligned} \text{voter } v_i &: c_j \succ a_j \succ b_j, \\ \text{voter } v_{i'} &: b_j \succ c_j \succ a_j, \\ \text{voter } v_{i''} &: a_j \succ b_j \succ c_j. \end{aligned}$$

In this way, the two vertex voters that correspond to the vertices in  $e_j$  and any voter  $v_z$  not in  $e_j$  form a cyclic configuration with regard to the three edge alternatives  $a_j, b_j$ , and  $c_j$ . By the definition of cyclic configurations, this configuration is also best-diverse, medium-diverse, and worst-diverse.

The maximum number of voters to delete equals the maximum vertex cover size  $k$ . This completes the construction which can be done in polynomial time.

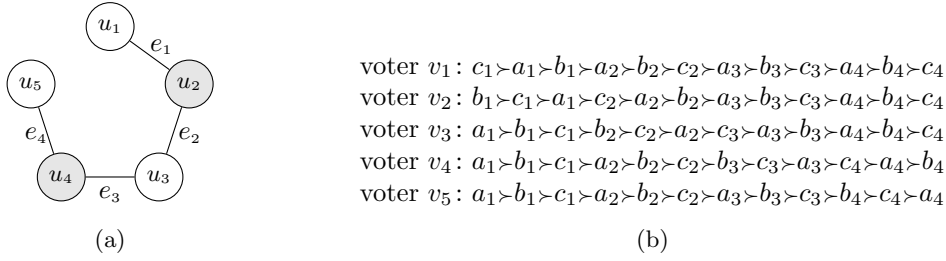


Figure 2: (a) An undirected graph with 5 vertices and 4 edges. The graph has a vertex cover of size 2 (filled in gray). (b) A reduced instance  $((A, V), k = 2)$  of VALUE-RESTRICTED MAVERICK DELETION, where  $A = \{a_i, b_i, c_i \mid 1 \leq i \leq 4\}$  and  $V = \{v_1, v_2, v_3, v_4, v_5\}$ . Deleting  $v_2$  and  $v_4$  results in a value-restricted profile. In fact, the resulting profile is also best-restricted, single-peaked (and hence worst-restricted), and group-separable (and hence medium-restricted).

It remains to show its correctness. In particular, we show that  $(G = (U, E), k)$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made value-restricted by deleting at most  $k$  voters.

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover. We show that, after deleting the voters corresponding to the vertices in  $U'$  the resulting profile is value-restricted. Suppose for the sake of contradiction that the resulting profile is *not* value-restricted. That is, it still contains a cyclic configuration  $\sigma$ . By the definition of cyclic configurations, it must hold that for each pair of alternatives  $x$  and  $y$  in  $\sigma$ , there are two voters, one preferring  $x$  to  $y$  and the other preferring  $y$  to  $x$ . Together with the fact that all voters agree on the relative order of two edge alternatives that correspond to different edges, this implies that the three alternatives  $a_j$ ,  $b_j$ , and  $c_j$  in  $\sigma$  correspond to the same edge  $e_j$ . Furthermore,  $\sigma$  involves two voters corresponding to the incident vertices from  $e_j$ , and one other voter, because all voters corresponding to vertices not in  $e_j$  have the same ranking  $a_j \succ b_j \succ c_j$ . Then, edge  $e_j$  is not covered by any vertex in  $U'$ —a contradiction.

For the “if” part, suppose that the profile becomes value-restricted after the removal of a subset  $V' \subseteq V$  of voters with  $|V'| \leq k$ . That is, no three remaining voters form a cyclic configuration. We show by contradiction that  $V'$  corresponds to a vertex cover of graph  $G$ . Assume towards a contradiction that an edge  $e_j$  is not covered by the vertices corresponding to the voters in  $V'$ . Then, the two voters corresponding to the vertices that are incident with edge  $e_j$  together with a third voter form a cyclic configuration with regard to the three alternatives  $a_j$ ,  $b_j$  and  $c_j$ —a contradiction. Thus,  $V'$  corresponds to a vertex cover of graph  $G$  and its size is at most  $k$ .  $\square$

We illustrate our reduction through an example. Figure 2(a) depicts an undirected graph with 5 vertices and 4 edges. Vertices  $u_2$  and  $u_4$  form a vertex cover of size two. Figure 2(b) shows the reduced instance with 5 voters and  $3 \cdot 4 = 12$  alternatives. Deleting voters  $v_2$  and  $v_4$  results in a value-restricted profile which is also best-restricted, single-peaked (and hence worst-restricted), and group-separable (and hence medium-restricted).

Taking a closer look at the reduction shown in the proof of Theorem 1, the constructed profile contains cyclic configurations which are simultaneously best-diverse, worst-diverse, and medium-diverse. It turns out that we can use the same construction to show the following three NP-hardness results with regard to the best-restricted, worst-restricted, and medium-restricted properties.

**Proposition 1.**  $\Pi$  MAVERICK DELETION is NP-complete for every property  $\Pi \in \{\text{best-restricted}, \text{worst-restricted}, \text{medium-restricted}\}$ .

*Proof.* Let  $(G = (U, E), k)$  be a VERTEX COVER instance and let  $A$  and  $V$  be the set of alternatives and the set of voters that are constructed in the same way as in the proof of Theorem 1. Let  $k$  be the number of voters to be deleted. As we already observed in that proof, for each edge  $e_j \in E$ , the two vertex voters that correspond to the vertices in  $e_j$  and any other voter  $v_z$  form a cyclic configuration, that is, a best-diverse, worst-diverse, and medium-diverse configuration. It remains to show that  $(G = (U, E), k)$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made best-restricted (or worst-restricted or medium-restricted) by deleting at most  $k$  voters.

For the “if” part, suppose that the profile becomes best-restricted (or worst-restricted or medium-restricted) by deleting a subset  $V' \subseteq V$  of voters with  $|V'| \leq k$ . Then, the resulting profile is also value-restricted. Thus, we can use the “if” part in the proof of Theorem 1 and obtain that the vertices corresponding to  $V'$  form a vertex cover of size at most  $k$ .

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover. As in the “only if” part proof of Theorem 1, we can show that deleting the voters corresponding to  $U'$  results in a best-restricted and a worst-restricted profile.

As for the medium-restricted property, suppose towards a contradiction that after deleting the voters corresponding to the vertices in  $U'$  there is still a medium-diverse configuration  $\sigma$ . By the definition of medium-diverse configurations, we know that in  $\sigma$ , each alternative is ranked between the other two by one voter. This implies that  $\sigma$  involves three alternatives that correspond to the same edge  $e_j$  and involves two voters that correspond to  $e_j$ ’s incident vertices. Thus,  $e_j$  is an uncovered edge—a contradiction.  $\square$

### 3.2 Alternative Deletion

Next, we consider the case of deleting alternatives. Just as for the voter deletion case, we first show NP-hardness of deciding the distance to value-restricted profiles. Then, we show how to adapt the reduction to also work for deciding the distance to best-restricted, worst-restricted, and medium-restricted profiles, respectively.

**Theorem 2.** VALUE-RESTRICTED ALTERNATIVE DELETION is NP-complete.

*Proof.* We reduce from VERTEX COVER to VALUE-RESTRICTED ALTERNATIVE DELETION. Let  $(G = (U, E), k)$  denote a VERTEX COVER instance with vertex set  $U = \{u_1, \dots, u_r\}$  and edge set  $E = \{e_1, \dots, e_s\}$ . The set of alternatives consists of one *vertex alternative*  $a_j$  for each vertex  $u_j$  in  $U$  and of  $k + 1$  additional *dummy alternatives*. Let  $A$  denote the set of all vertex alternatives and let  $D$  denote the set of all dummy alternatives. We arbitrarily fix a canonical order  $\langle D \rangle$  of  $D$  and we set  $\langle A \rangle := a_1 \succ a_2 \succ \dots \succ a_r$ . The number  $m$  of constructed alternatives is  $r + k + 1$ .

We introduce a voter  $v_0$  with the canonical preference order  $\langle D \rangle \succ \langle A \rangle$ . For each edge  $e_i = \{u_j, u_{j'}\}$  with  $j < j'$ , we introduce two *edge voters*  $v_{2i-1}$  and  $v_{2i}$  with preference orders

$$\begin{aligned} \text{voter } v_{2i-1}: & \quad a_j \succ a_{j'} \succ \langle D \rangle \succ \langle A \setminus \{a_j, a_{j'}\} \rangle, \\ \text{voter } v_{2i} : & \quad a_{j'} \succ \langle D \rangle \succ \langle A \setminus \{a_{j'}\} \rangle. \end{aligned}$$

Together with voter  $v_0$ , the two voters  $v_{2i-1}$  and  $v_{2i}$  form a cyclic configuration with respect to the two vertex alternatives  $a_j, a_{j'}$  and an arbitrary dummy alternative from  $D$ . Let  $V$  denote the set of all voters. In total, the number  $n$  of constructed voters is  $2s + 1$ . The maximum number of alternatives to delete equals the maximum vertex cover size  $k$ . This completes the construction.

Our reduction runs in polynomial time. It remains to show that graph  $G$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made value-restricted by deleting at most  $k$  alternatives.

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover. We show that after deleting the vertex alternatives corresponding to  $U'$ , the resulting profile is value-restricted. Suppose for the sake of contradiction that the resulting profile is not value-restricted, that is, it contains a cyclic configuration  $\sigma$ . By definition, it must hold that for each pair of alternatives  $x$  and  $y$  in  $\sigma$ , there are two voters, one preferring  $x$  to  $y$  and the other preferring  $y$  to  $x$ . Together with the fact that each voter agrees on the relative order of two distinct dummy alternatives, this implies that  $\sigma$  contains at most one dummy alternative. But if  $\sigma$  contains one dummy alternative  $d \in D$ , then there is a voter with preferences  $a_j \succ a_{j'} \succ d$  where  $a_j, a_{j'} \in A$ , which means that edge  $\{u_j, u_{j'}\}$  is not covered by  $U'$ . Hence,  $\sigma$  contains no dummy alternative. This means that  $\sigma$  contains three vertex alternatives  $a_j, a_{j'}$ , and  $a_{j''}$  with  $j < j' < j''$  and by the definition of cyclic configurations,  $\sigma$  involves three voters with preferences  $\{a_j, a_{j'}\} \succ a_{j''}$ ,  $\{a_j, a_{j''}\} \succ a_{j'}$ , and  $\{a_{j'}, a_{j''}\} \succ a_j$ , respectively. However, the last preference implies that  $\{u_{j'}, u_{j''}\}$  is an edge which is not covered by  $U'$ —a contradiction.

For the “if” part, suppose that the constructed profile is a yes-instance of VALUE-RESTRICTED ALTERNATIVE DELETION. Let  $A' \subseteq A \cup D$  be the set of deleted vertex alternatives with  $|A'| \leq k$ . We show that the vertex set  $U'$  corresponding to  $A'$  form a vertex cover of graph  $G$  and has size at most  $k$ . Clearly,  $|U'| \leq k$ . Assume towards a contradiction that  $e_i = \{u_j, u_{j'}\}$ ,  $j < j'$ , is not covered by  $U'$ . Since  $|D| > k$ , at least one dummy alternative  $d$  is not deleted. Then,  $v_0$  and the two edge voters  $v_{2i-1}, v_{2i}$  form a cyclic configuration with regard to  $a_j, a_{j'}, d$ —a contradiction.  $\square$

Using the same construction as in the last proof, we can show that achieving best-restriction, worst-restriction, or medium-restriction via deleting the fewest number of alternatives is intractable.

**Proposition 2.**  $\Pi$  ALTERNATIVE DELETION is NP-complete for every property  $\Pi \in \{\text{best-restricted}, \text{worst-restricted}, \text{medium-restricted}\}$ .

*Proof.* Let  $((U, E), k)$  denote a VERTEX COVER instance with vertex set  $U = \{u_1, \dots, u_r\}$  and edge set  $E = \{e_1, \dots, e_s\}$ . Let  $A, D$ , and  $V$  be the sets constructed in the same way as in the proof of Theorem 2.

It remains to show that  $((U, E), k)$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made best-restricted (or worst-restricted or medium-restricted) by deleting at most  $k$  alternatives.

For the “if” part, suppose that the profile becomes best-restricted (or worst-restricted or medium-restricted) after deleting a set  $A' \subseteq A \cup D$  of at most  $k$  alternatives. Then, the resulting profile is also value-restricted. Thus, we can use the “if” part in the proof of Theorem 2 and obtain that the vertices corresponding to  $A'$  form a vertex set of size at most  $k$ .

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover. Just as in the “only if” part proof of Theorem 2, we can show that deleting the alternatives corresponding to  $U'$  results in a best-restricted and a worst-restricted profile.

Now, we consider the medium-restricted property. Suppose for the sake of contradiction that the resulting profile is not medium-restricted, that is, it contains a medium-diverse configuration  $\sigma$ . Since all voters rank  $\langle D \rangle$  and since no voter rank  $d \succ a_j \succ d'$  with  $d, d' \in D$  and  $a_j \in A$ , configuration  $\sigma$  contains at most one dummy alternative. Now, if  $\sigma$  involves one dummy alternative  $d \in D$  and two vertex alternatives  $a_j, a_{j'} \in A$  with  $j < j'$ , then the voter ranking  $a_{j'}$  between  $a_j$  and  $d$  must rank  $a_j \succ a_{j'} \succ d$ . But this means that edge  $\{u_j, u_{j'}\}$

is uncovered—a contradiction. Hence,  $\sigma$  contains no dummy alternative. This means that  $\sigma$  involves three vertex alternatives  $a_j, a_{j'}, a_{j''}$  with  $j < j' < j''$ . By the definition of medium-diverse configurations,  $\sigma$  must contain a voter that ranks  $a_{j''}$  between  $a_j$  and  $a_{j'}$ , that is, a voter ranks either  $a_j \succ a_{j''} \succ a_{j'}$  or  $a_{j'} \succ a_{j''} \succ a_j$ . This, however, implies that either edge  $\{u_j, u_{j''}\}$  or edge  $\{u_{j'}, u_{j''}\}$  is uncovered—a contradiction.  $\square$

## 4 Single-peaked, single-caved, and group-separable properties

Since single-peaked, group-separable, and single-caved profiles are necessarily worst-restricted, medium-restricted, and best-restricted, respectively, it seems reasonable to expect that the intractability result (Proposition 1) transfers. Indeed, we can show that this immediately follows from the proofs of Proposition 1 (and hence of Theorem 1) because the profile constructed in the NP-hardness reduction contains neither  $\alpha$ -configurations, nor  $\beta$ -configurations, nor  $\bar{\alpha}$ -configurations. Note that NP-hardness of SINGLE-PEAKED MAVERICK DELETION is already known by a different proof of Erdélyi et al. [29]. However, their proof does not work for  $\Pi$  MAVERICK DELETION with  $\Pi \in \{\text{best-restricted, medium-restricted, worst-restricted, group-separable}\}$ .

**Corollary 1.**  $\Pi$  MAVERICK DELETION is NP-complete for every property  $\Pi \in \{\text{single-peaked, single-caved, group-separable}\}$ .

*Proof.* First, the profile constructed in the proof of Proposition 1 does not contain any three alternatives  $x, y$ , and  $z$  such that there is one voter with  $x \succ y \succ z$  and one voter with  $z \succ y \succ x$ . Thus, the profile does not contain any  $\alpha$ -configuration or any  $\bar{\alpha}$ -configuration.

Second, one can partition every set  $T$  of four alternatives into two non-empty subsets  $T_1$  and  $T_2$  such that  $T_1 \succ T_2$  holds for each voter because at most three alternatives can correspond to the same edge and all voters have the same ranking over the alternatives that correspond to different edges. However, this is not possible in a  $\beta$ -configuration. Thus, the profile does not contain any  $\alpha$ -configuration,  $\bar{\alpha}$ -configuration, or  $\beta$ -configuration.

As a consequence, the reduction in the proof of Proposition 1 also works for SINGLE-PEAKED MAVERICK DELETION, SINGLE-CROSSING MAVERICK DELETION, and GROUP-SEPARABLE MAVERICK DELETION.  $\square$

Just as the result for the maverick deletion, the NP-hardness result of MEDIUM-RESTRICTED ALTERNATIVE DELETION also transfers to the group-separable case. After deleting the alternatives corresponding to a vertex cover, the resulting profile from the proof of Proposition 2 does not contain any  $\beta$ -configurations. Thus, the following holds.

**Corollary 2.** GROUP-SEPARABLE ALTERNATIVE DELETION is NP-complete.

*Proof.* The profile constructed in the proof of Proposition 2 may contain  $\beta$ -configurations, but we show that destroying all medium-diverse configurations by deleting at most  $k$  alternatives also destroys all  $\beta$ -configurations. Consider the profile  $\mathcal{P}$  after the deletion of the alternatives. Assume towards a contradiction that profile  $\mathcal{P}$  contains a  $\beta$ -configuration which involves four alternatives  $w, x, y, z$  and two voters  $v, v'$  with preferences

$$\text{voter } v: w \succ x \succ y \succ z \quad \text{and} \quad \text{voter } v': x \succ z \succ w \succ y.$$

Observe that a  $\beta$ -configuration may contain at most one dummy alternative, because no two alternatives appear consecutively in both preference orders of a  $\beta$ -configuration, but all

dummy alternatives appear consecutively in all preference orders of the profile  $\mathcal{P}$ . Furthermore,  $w$ ,  $y$ , and  $z$  are vertex alternatives since in  $\mathcal{P}$ , no voter prefers more than one vertex alternative to a dummy alternative. Then by the definition of  $\beta$ -configurations, we have that voter  $v$  ranks  $a_j \succ x \succ a_{j'} \succ a_{j''}$  and voter  $v'$  ranks  $x \succ a_{j''} \succ a_j \succ a_{j'}$  with  $a_j, a_{j'}, a_{j''} \in A$ . However, the preference order of voter  $v$  implies that  $j' < j''$  and the preference order of voter  $v'$  implies that  $j'' < j'$ —a contradiction. As consequence, the reduction in the proof of Proposition 2 also works for GROUP-SEPARABLE MAVERICK DELETION.  $\square$

In order to be group-separable, a preference profile must be medium-restricted and  $\beta$ -restricted. As already shown in Corollary 1 and in Corollary 2, deleting as few maverick voters (or alternatives) as possible to obtain the group-separable property is NP-hard. Alternatively, we can also derive this intractability result from the following two theorems.

**Theorem 3.**  $\beta$ -RESTRICTED MAVERICK DELETION is NP-complete.

*Proof.* We reduce from VERTEX COVER to show NP-hardness. Let  $(G = (U, E), k)$  denote a VERTEX COVER instance with vertex set  $U = \{u_1, \dots, u_r\}$  and edge set  $E = \{e_1, \dots, e_s\}$ ; without loss of generality we assume that the input graph  $G$  is connected and that it has at least four vertices, that is,  $r \geq 4$ . The set of alternatives consists of four *edge alternatives*  $a_j, b_j, c_j, d_j$  for each edge  $e_j \in E$ . For each vertex in  $U$ , we construct one voter. That is, we define  $A := \{a_j, b_j, c_j, d_j \mid e_j \in E\}$  and  $V := \{v_i \mid u_i \in U\}$ . In total, the number  $m$  of alternatives is  $4s$  and the number  $n$  of voters is  $r$ .

Now we specify the preference order of each voter. Every voter prefers  $\{a_j, b_j, c_j, d_j\}$  to  $\{a_{j'}, b_{j'}, c_{j'}, d_{j'}\}$  whenever  $j < j'$ . For each edge  $e_j$  with two incident vertices  $u_i$  and  $u_{i'}$ ,  $i < i'$ , and for each non-incident vertex  $u_{i''} \notin e_j$ , the following holds:

$$\begin{aligned} \text{voter } v_i &: a_j \succ b_j \succ c_j \succ d_j, \\ \text{voter } v_{i'} &: b_j \succ d_j \succ a_j \succ c_j, \\ \text{voter } v_{i''} &: d_j \succ a_j \succ b_j \succ c_j, \end{aligned}$$

In this way, any  $\beta$ -configuration regarding alternatives  $a_j, b_j, c_j, d_j$  must involve voters  $v_i$  and  $v_{i'}$ . The maximum number of voters to delete equals the maximum vertex cover size  $k$ . This completes the construction.

Clearly, the whole construction runs in polynomial time. It remains to show that  $(G = (U, E), k)$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made  $\beta$ -restricted by deleting at most  $k$  voters.

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover. We show that after deleting the voters corresponding to the vertices in  $U'$  the resulting profile is  $\beta$ -restricted. Suppose for the sake of contradiction that the resulting profile is *not*  $\beta$ -restricted. That is, it still contains a  $\beta$ -configuration  $\sigma$ . Since all voters prefer  $\{a_j, b_j, c_j, d_j\}$  to  $\{a_{j'}, b_{j'}, c_{j'}, d_{j'}\}$  whenever  $j < j'$ , the profile restricted to every four alternatives that correspond to at least two edges, is group-separable. But since  $\sigma$  is not group-separable,  $\sigma$  involves four alternatives  $a_j, b_j, c_j$ , and  $d_j$  that correspond to a single edge  $e_j$ . As already observed, any  $\beta$ -configuration regarding alternatives  $a_j, b_j, c_j, d_j$  must involve voters  $v_i$  and  $v_{i'}$  that correspond to both endpoints of  $e_j$ . Then edge  $e_j$  is not covered by any vertex in  $U'$ —a contradiction.

For the “if” part, suppose that the profile becomes  $\beta$ -restricted by deleting a subset  $V' \subseteq V$  of voters with  $|V'| \leq k$ . That is, no two voters form a  $\beta$ -configuration. We show by contradiction that  $V'$  corresponds to a vertex cover of graph  $G$  and has size at most  $k$ . Clearly,  $|V'| \leq k$ . Assume towards a contradiction that an edge  $e_j$  is not covered by the vertices corresponding to the voters in  $V'$ . Then, the two voters corresponding to the vertices



that are incident with edge  $e_j$  form a  $\beta$ -configuration with regard to  $a_j$ ,  $b_j$ ,  $c_j$ , and  $d_j$ —a contradiction. Thus,  $V'$  corresponds to a vertex cover of graph  $G$  and its size is at most  $k$ .  $\square$

**Theorem 4.**  $\beta$ -RESTRICTED ALTERNATIVE DELETION is NP-complete.

*Proof.* We reduce from VERTEX COVER to  $\beta$ -RESTRICTED ALTERNATIVE DELETION. Let  $(G = (U, E), k)$  denote a VERTEX COVER instance with vertex set  $U = \{u_1, \dots, u_r\}$  and edge set  $E = \{e_1, \dots, e_s\}$ ; without loss of generality we assume that the input graph  $G$  is connected and that  $r \geq k + 2$ . The set of alternatives consists of one *vertex alternative*  $a_j$  and one *dummy alternative*  $d_j$  for each vertex  $u_j$  in  $U$ . Let  $A$  denote the set of all vertex alternatives and let  $D$  denote the set of all dummy alternatives. The number  $m$  of constructed alternatives is  $2r$ .

We fix the canonical order of  $A \cup D$  to be

$$\langle A \cup D \rangle := d_1 \succ a_1 \succ d_2 \succ a_2 \succ \dots \succ d_r \succ a_r.$$

We introduce a voter  $v_0$  with the preference order  $\langle A \cup D \rangle$ . For each edge  $e_i = \{u_j, u_{j'}\}$  with  $j < j'$ , we introduce one *edge voter*  $v_i$  with preference order

$$a_j \succ a_{j'} \succ \langle A \cup D \setminus \{a_j, a_{j'}\} \rangle.$$

Observe that voter  $v_0$  and  $v_i$  form a  $\beta$ -configuration with respect to the four alternatives corresponding to the vertices in edge  $\{u_j, u_{j'}\}$  with  $j < j'$ , that is, with respect to  $a_j$ ,  $a_{j'}$ ,  $d_j$ , and  $d_{j'}$ .

$$\begin{aligned} \text{voter } v_0: & \quad d_j \succ a_j \succ d_{j'} \succ a_{j'}, \\ \text{voter } v_i: & \quad a_j \succ a_{j'} \succ d_j \succ d_{j'}. \end{aligned}$$

Furthermore, for each pair of alternatives  $a$  and  $b$ , if there is a voter  $v$  preferring  $a$  to  $b$ , then the following holds:

- (i) If neither  $a$  nor  $b$  is in the first two positions of voter  $v$ 's preference order, then  $a$  and  $b$  correspond to two (not necessarily distinct) vertices  $u_j$  and  $u_{j'}$  with  $j \leq j'$ .
- (ii) If  $a$  and  $b$  correspond to two vertices  $u_j$  and  $u_{j'}$  with  $j > j'$  and if there is a third alternative  $c$  such that  $v$  prefers  $c$  to  $a$ , then  $c$  and  $a$  correspond to two adjacent vertices.

We will utilize these two facts several times to show some contradictions.

Let  $V$  denote the set of all voters. In total, the number  $n$  of constructed voters is  $s + 1$ . The maximum number of alternatives to delete equals the maximum vertex cover size  $k$ . This completes the construction.

Our construction runs in polynomial time. It remains to show that  $(G = (U, E), k)$  has a vertex cover of size at most  $k$  if and only if the constructed profile can be made  $\beta$ -restricted by deleting at most  $k$  alternatives.

For the “only if” part, suppose that  $U' \subseteq U$  with  $|U'| \leq k$  is a vertex cover of graph  $G$ . We show that after deleting the vertex alternatives corresponding to  $U'$ , denoted by  $A'$ , the resulting profile is  $\beta$ -restricted. Suppose for the sake of contradiction that the resulting profile still contains a  $\beta$ -configuration  $\sigma$  with regard to four alternatives  $w, x, y, z$  and two voters  $v, v'$  with preferences

$$\text{voter } v: w \succ x \succ y \succ z \quad \text{and} \quad \text{voter } v': x \succ z \succ w \succ y.$$



Let  $w, x, y, z$  correspond to four non-deleted vertices  $u_j, u_{j'}, u_{j''}, u_{j'''}$ , respectively. By Property (i), the preference order of voter  $v$  implies that  $j'' \leq j'''$  (note that voter  $v$  ranks neither  $y$  nor  $z$  in the first two positions) and the preference order of voter  $v'$  implies that  $j \leq j''$  (note that voter  $v'$  ranks neither  $w$  nor  $y$  in the first two positions). Since  $x, y$ , and  $z$  correspond to at least two distinct vertices, it follows that  $j < j'''$ . Since voter  $v'$  rank  $x \succ z \succ w \succ y$ , by Property (ii) the inequality  $j < j'''$  implies that  $u_{j'}$  and  $u_{j'''}$  are adjacent—a contradiction to  $U'$  being a vertex cover. Indeed, the resulting profile is group-separable. To see this, note that any size-at-least-three subset  $T \subseteq (A \cup D) \setminus A'$  of alternatives can be partitioned into two non-empty subsets  $\{a\}$  and  $T \setminus \{a\}$  with  $a$  being the last alternative in the canonical order restricted to the alternatives in set  $T$ .

For “if” part, suppose that the constructed profile is a yes-instance of  $\beta$ -RESTRICTED ALTERNATIVE DELETION. Let  $A' \subseteq A \cup D$  be the set of deleted alternatives with  $|A'| \leq k$ . Consider the vertex set  $U'$  containing all vertices corresponding to a vertex alternative or to a dummy alternative in  $A'$ , that is,  $U' := \{u_j \mid a_j \in A' \vee d_j \in A'\}$ . Obviously,  $|U'| \leq k$ . We show that set  $U'$  is a vertex cover. Suppose towards a contradiction that there is an uncovered edge  $e_i = \{u_j, u_{j'}\}$  with  $j < j'$ . By the definition of  $U'$ , we have that  $A' \cap \{d_j, a_j, d_{j'}, a_{j'}\} = \emptyset$ . Then, voters  $v_0$  and  $v_i$  form a  $\beta$ -configuration with respect to the alternatives  $d_j, a_j, d_{j'}, a_{j'}$ —a contradiction.  $\square$

## 5 Single-crossing properties

In this section, we show that for the single-crossing property, the alternative deletion problem is NP-hard while the maverick deletion problem is polynomial-time solvable. The NP-hardness proof is based on the following NP-complete MAXIMUM 2-SATISFIABILITY problem [34].

MAXIMUM 2-SATISFIABILITY (MAX2SAT)

**Input:** A set  $U$  of Boolean variables, a collection  $C$  of size-two clauses over  $U$  and a positive integer  $h$ .

**Question:** Is there a truth assignment for  $U$  such that at least  $h$  clauses in  $C$  are satisfied?

**Theorem 5.** SINGLE-CROSSING ALTERNATIVE DELETION is NP-complete.

*Proof.* For the NP-hardness result we reduce from MAX2SAT [34]. We will provide an example for the reduction (Table 2) right after this proof.

Let  $(U, C, h)$  be a MAX2SAT instance with variable set  $U = \{x_1, \dots, x_r\}$  and clause set  $C = \{c_1, \dots, c_s\}$ . We construct two sets  $O$  and  $\bar{O}$  of *dummy alternatives* with  $|O| = |\bar{O}| = 2(r \cdot s + r + s) + 1$ . For each variable  $x_i \in U$ , we construct two sets  $X_i$  and  $\bar{X}_i$  of *variable alternatives* with  $|X_i| = |\bar{X}_i| = s + 1$ . We say that  $X_i$  corresponds to  $x_i$  and that  $\bar{X}_i$  corresponds to  $\bar{x}_i$ . The canonical orders  $\langle O \rangle, \langle \bar{O} \rangle, \langle X_i \rangle$  and  $\langle \bar{X}_i \rangle, i \in \{1, \dots, r\}$ , are arbitrary but fixed. Let  $X$  be the union  $\bigcup_{i=1}^r X_i \cup \bar{X}_i$  of all variable alternatives. The canonical order  $\langle X \rangle$  is defined as

$$\langle X \rangle := \langle X_1 \rangle \succ \langle \bar{X}_1 \rangle \succ \langle X_2 \rangle \succ \langle \bar{X}_2 \rangle \succ \dots \succ \langle X_r \rangle \succ \langle \bar{X}_r \rangle.$$

For each clause  $c_j \in C$ , we construct two *clause alternatives*  $a_j$  and  $b_j$ . Let  $A$  denote the set of all clause alternatives. The canonical order  $\langle A \rangle$  is defined as

$$\langle A \rangle := a_1 \succ b_1 \succ a_2 \succ b_2 \succ \dots \succ a_s \succ b_s.$$

The total number  $m$  of alternatives is  $6(r \cdot s + r + s) + 2$ .

We will introduce voters and their preference orders such that

- (1) deleting all alternatives in  $X_i$  corresponds to setting variable  $x_i$  to true,
- (2) deleting all alternatives in  $\overline{X_i}$  corresponds to setting variable  $x_i$  to false, and
- (3) deleting  $b_j$  or  $a_j$  corresponds to clause  $c_j$  not being satisfied.

We construct two sets  $V$  and  $W$  of voters with  $|V| = 2r$  and  $|W| = 4s$ . Voter set  $V$  consists of two voters  $v_{2i-1}$  and  $v_{2i}$  for each variable  $x_i$ ,  $1 \leq i \leq r$ . Their preference orders are

$$\begin{aligned} \langle O \rangle &\succ \langle \overline{O} \rangle \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \dots \succ \langle X_{i-1} \rangle \succ \langle \overline{X_{i-1}} \rangle \succ \\ &\quad \langle \overline{X_i} \rangle \succ \langle X_i \rangle \succ \langle X_{i+1} \rangle \succ \langle \overline{X_{i+1}} \rangle \succ \dots \succ \langle X_r \rangle \succ \langle \overline{X_r} \rangle \succ \langle A \rangle, \\ \langle \overline{O} \rangle &\succ \langle O \rangle \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \dots \succ \langle X_{i-1} \rangle \succ \langle \overline{X_{i-1}} \rangle \succ \\ &\quad \langle \overline{X_i} \rangle \succ \langle X_i \rangle \succ \langle X_{i+1} \rangle \succ \langle \overline{X_{i+1}} \rangle \succ \dots \succ \langle X_r \rangle \succ \langle \overline{X_r} \rangle \succ \langle A \rangle, \end{aligned}$$

respectively. These two voters together with any other two voters  $v_\ell$  and  $v_{\ell'} \in V \setminus \{v_{2i-1}, v_{2i}\}$  with odd number  $\ell$  and even number  $\ell'$  form a  $\delta$ -configuration with regard to each four alternatives  $o, \overline{o}, x, \overline{x}$  such that  $o \in O, \overline{o} \in \overline{O}, x \in X_i, \overline{x} \in \overline{X_i}$ :

$$\begin{aligned} \text{voter } v_{2i-1}: & \quad o \succ \overline{o} \text{ and } \overline{x} \succ x, \\ \text{voter } v_{2i}: & \quad \overline{o} \succ o \text{ and } \overline{x} \succ x, \\ \text{voter } v_\ell: & \quad o \succ \overline{o} \text{ and } x \succ \overline{x}, \\ \text{voter } v_{\ell'}: & \quad \overline{o} \succ o \text{ and } x \succ \overline{x}. \end{aligned}$$

Voter set  $W$  consists of four voters  $w_{4j-3}, w_{4j-2}, w_{4j-1}, w_{4j}$  for each clause  $c_j$ ,  $1 \leq j \leq s$ . These four voters have the same preference order

$$\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle A_1 \rangle \succ \langle X \rangle \succ \langle A_2 \rangle$$

over the set  $O \cup \overline{O} \cup A_1 \cup A_2 \cup X$ , where  $A_1 = \{a_{j'}, b_{j'} \mid j' < j\}$  and  $A_2 = \{a_{j'}, b_{j'} \mid j' > j\}$ . Note that  $A_1 \cup A_2 = A \setminus \{a_j, b_j\}$ . Thus, it remains to specify the exact positions of  $a_j$  and  $b_j$  in the four voters' preference orders: Let  $\widehat{X}_1^j$  denote the set of variable alternatives corresponding to the literal in  $c_j$  with the lower index and  $\widehat{X}_2^j$  denote the set of variable alternatives corresponding to the literal in  $c_j$  with the higher index. For instance, if  $c_j = \overline{x_2} \vee x_4$ , then  $\widehat{X}_1^j$  equals  $\overline{X_2}$  and  $\widehat{X}_2^j$  equals  $\overline{X_4}$ .

Voters  $w_{4j-3}$  and  $w_{4j-2}$  rank the clause alternative  $a_j$  right below the last alternative in  $\langle \widehat{X}_1^j \rangle$  while voters  $w_{4j-1}$  and  $w_{4j}$  rank it right above the first alternative in  $\langle \widehat{X}_1^j \rangle$ . As for alternative  $b_j$ , voters  $w_{4j-3}$  and  $w_{4j-1}$  rank  $b_j$  right above the first variable alternative in  $\langle \widehat{X}_2^j \rangle$  while voters  $w_{4j-2}$  and  $w_{4j}$  rank it right below the last variable alternative in  $\langle \widehat{X}_2^j \rangle$ . Thus, these four voters form a  $\delta$ -configuration with regard to  $a_j, b_j, x \in \widehat{X}_1^j$ , and  $y \in \widehat{X}_2^j$ :

$$\begin{aligned} \text{voter } w_{4j-3}: & \quad x \succ a_j \text{ and } b_j \succ y, \\ \text{voter } w_{4j-2}: & \quad x \succ a_j \text{ and } y \succ b_j, \\ \text{voter } w_{4j-1}: & \quad a_j \succ x \text{ and } b_j \succ y, \\ \text{voter } w_{4j}: & \quad a_j \succ x \text{ and } y \succ b_j. \end{aligned}$$

We complete the construction by setting the number  $k$  of alternatives that may be deleted to  $k := r(s+1) + (s-h)$ .

The construction clearly runs in polynomial time. It remains to show that  $(U, C, h)$  is a yes-instance of MAX2SAT if and only if the constructed profile together with  $k$  is a yes-instance of SINGLE-CROSSING ALTERNATIVE DELETION.

For the “only if” part, suppose that there is a truth assignment  $U \rightarrow \{\text{true}, \text{false}\}^r$  of the variables such that at least  $h$  clauses are satisfied. We delete all variable alternatives in  $X_i$  if  $x_i$  is assigned to true. Otherwise, we delete all variable alternatives in  $\overline{X_i}$ . Furthermore, we delete the clause alternative  $b_j$  if  $c_j$  is not satisfied by the assignment. Let  $X_{\text{rem}}$  be the set of remaining variable alternatives, and let  $A_{\text{rem}}$  be the set of remaining clause alternatives. Then,  $|X_{\text{rem}}| = r(s+1)$  and  $|A'| \geq s+h$ , implying that the number of deleted alternatives is  $|X| + |A| - (|X_{\text{rem}}| + |A_{\text{rem}}|) \leq r(s+1) + (s-h) = k$ .

For each  $j \in \{1, \dots, s\}$ , we define  $\langle z_j \rangle = w_{4j-2} \succ w_{4j} \succ w_{4j-3} \succ w_{4j-1}$  if the literal in clause  $c_j$  with the lower index is satisfied; otherwise,  $\langle z_j \rangle = w_{4j-3} \succ w_{4j-2} \succ w_{4j-1} \succ w_{4j}$ . The resulting profile is single-crossing with respect to the voter order  $L$ :

$$v_1 \succ v_3 \succ \dots \succ v_{2r-1} \succ v_2 \succ v_4 \succ \dots \succ v_{2r} \succ \langle z_1 \rangle \succ \langle z_2 \rangle \succ \dots \succ \langle z_s \rangle.$$

Suppose for the sake of contradiction that  $L$  is not a single-crossing order, which means that there is a pair  $\{a, a'\} \subset O \cup \overline{O} \cup X_{\text{rem}} \cup A_{\text{rem}}$  of alternatives and three voters  $u, v, w$  with  $u \succ_L v \succ_L w$  such that voter  $v$  disagrees with voters  $u$  and  $w$  on the relative order of  $a$  and  $a'$ .

Note that all voters along  $L$  up to and including voter  $v_{2r-1}$  rank  $\langle O \rangle \succ \langle \overline{O} \rangle \succ \langle X \rangle$  while all voters from  $v_2$  onwards rank  $\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle X \rangle$ . Hence,  $a$  and  $a'$  can neither both be in  $O \cup \overline{O}$ , nor both be in  $X_{\text{rem}}$ . Furthermore,  $a$  and  $a'$  cannot both be in  $A_{\text{rem}}$  as all voters rank  $\langle A \rangle$ . Moreover, since all voters rank  $\langle O \cup \overline{O} \rangle \succ \langle X \cup A \rangle$ , neither  $a$  nor  $a'$  belongs to  $O \cup \overline{O}$ . This means, without loss of generality, that  $a \in X_{\text{rem}}$  and  $a' \in A_{\text{rem}}$ .

Assume that  $a'$  corresponds to clause  $c_j$  for some  $j$ , that is,  $a' \in \{a_j, b_j\}$ . Then, for each alternative  $a'' \in X_{\text{rem}} \setminus (\widehat{X}_1^j \cup \widehat{X}_2^j)$  that does not correspond to a literal in  $c_j$  (recall that  $\widehat{X}_1^j$  and  $\widehat{X}_2^j$  denote the two sets of variable alternatives corresponding to the literal in  $c_j$  with the lower index and the literal in  $c_j$  with the lower index, respectively), the following holds. If the first voter in  $\langle z_j \rangle$  prefer  $a''$  to  $a'$ , which means either that  $a''$  is ranked in front of  $\widehat{X}_1^j \cup \widehat{X}_2^j$  (by all voters) or that  $a' = a_j$  and  $a''$  is ranked in front of  $\widehat{X}_2^j$  (by all voters), then all voters along the order  $L$  up to and including the last voter in  $\langle z_j \rangle$  prefer  $a''$  to  $a'$  while all remaining voters prefer  $a'$  to  $a''$ ; otherwise all voters along the order  $L$  up to and including the last voter in  $\langle z_{j-1} \rangle$  prefer  $a''$  to  $a'$  while all remaining voters prefer  $a'$  to  $a''$ . Thus,  $a'$  cannot be in  $X_{\text{rem}} \setminus (\widehat{X}_1^j \cup \widehat{X}_2^j)$ . That is, we have  $a \in \widehat{X}_1^j \cup \widehat{X}_2^j$ . We distinguish four cases regarding  $a$  and  $a'$ .

- (i) If  $a \in \widehat{X}_1^j$  and if  $a' = a_j$ , then the literal corresponding to  $\widehat{X}_1^j$  is not satisfied because  $\widehat{X}_1^j$  is not deleted. Thus,  $\langle z_j \rangle$  is defined as  $w_{4j-3} \succ w_{4j-2} \succ w_{4j-1} \succ w_{4j}$ . All voters along  $L$  up to and including  $w_{4j-2}$  prefer  $a$  to  $a'$ , and all remaining voters prefer  $a'$  to  $a$ .
- (ii) If  $a \in \widehat{X}_1^j$  and if  $a' = b_j$ , then all voters along  $L$  up to and including the last voter in  $\langle z_j \rangle$  prefer  $a$  to  $a'$ , and all remaining voters prefer  $a'$  to  $a$ .
- (iii) If  $a \in \widehat{X}_2^j$  and if  $a' = a_j$ , then all voters along  $L$  up to and including the last voter in  $\langle z_{j-1} \rangle$  prefer  $a$  to  $a'$ , and all remaining voters prefer  $a'$  to  $a$ .
- (iv) If  $a \in \widehat{X}_2^j$  and if  $a' = b_j$ , then clause  $c_j$  is satisfied because  $b_j$  is not deleted. Furthermore, since  $\widehat{X}_2^j$  is not deleted,  $\widehat{X}_1^j$  must be deleted because clause  $c_j$  is satisfied. This implies that the literal in clause  $c_j$  with the lower index is satisfied. Thus,  $\langle z_j \rangle$  is defined as  $w_{4j-2} \succ w_{4j} \succ w_{4j-3} \succ w_{4j-1}$ . All voters along  $L$  up to and including  $w_{4j}$  prefer  $a$  to  $a'$ , and all remaining voters prefer  $a'$  to  $a$ .

voter $v_1$ :	$\langle O \rangle \succ \langle \overline{O} \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_1 \rangle \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $v_2$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_1 \rangle \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $v_3$ :	$\langle O \rangle \succ \langle \overline{O} \rangle \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \langle \overline{X_2} \rangle \succ \langle X_2 \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $v_4$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ \langle X_2 \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_1$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle X_1 \rangle \succ a_1 \succ \langle \overline{X_1} \rangle \succ b_1 \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_2$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ \langle X_1 \rangle \succ a_1 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_1 \succ \langle \overline{X_2} \rangle \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_3$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ b_1 \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_4$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_1 \succ \langle \overline{X_2} \rangle \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_5$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ \langle X_1 \rangle \succ a_2 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_2 \succ \langle \overline{X_2} \rangle \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_6$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ \langle X_1 \rangle \succ a_2 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_7$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_2 \succ \langle \overline{X_2} \rangle \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_8$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ b_2 \succ a_3 \succ b_3 \succ a_4 \succ b_4$
voter $w_9$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ a_3 \succ b_3 \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_4 \succ b_4$
voter $w_{10}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ a_3 \succ \langle X_2 \rangle \succ b_3 \succ \langle \overline{X_2} \rangle \succ a_4 \succ b_4$
voter $w_{11}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ \langle X_1 \rangle \succ a_3 \succ \langle \overline{X_1} \rangle \succ b_3 \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ a_4 \succ b_4$
voter $w_{12}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ \langle X_1 \rangle \succ a_3 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_3 \succ \langle \overline{X_2} \rangle \succ a_4 \succ b_4$
voter $w_{13}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ a_4 \succ \langle X_2 \rangle \succ b_4 \succ \langle \overline{X_2} \rangle$
voter $w_{14}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ \langle X_1 \rangle \succ \langle \overline{X_1} \rangle \succ a_4 \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ b_4$
voter $w_{15}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ \langle X_1 \rangle \succ a_4 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ b_4 \succ \langle \overline{X_2} \rangle$
voter $w_{16}$ :	$\langle \overline{O} \rangle \succ \langle O \rangle \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ a_3 \succ b_3 \succ \langle X_1 \rangle \succ a_4 \succ \langle \overline{X_1} \rangle \succ \langle X_2 \rangle \succ \langle \overline{X_2} \rangle \succ b_4$

Table 2: An instance  $((A, V), k = 11)$  with alternative set  $O \cup \overline{O} \cup X_1 \cup \overline{X_1} \cup X_2 \cup \overline{X_2} \cup \{a_i, b_i \mid 1 \leq i \leq 4\}$  and voter set  $\{v_i, w_{4i-3}, w_{4i-2}, w_{4i-1}, w_{4i} \mid 1 \leq i \leq 4\}$  reduced from the MAX2SAT instance with two variables  $x_1$  and  $x_2$ , and with four clauses  $c_1 = x_1 \wedge x_2$ ,  $c_2 = x_1 \wedge \overline{x_2}$ ,  $c_3 = \overline{x_1} \wedge x_2$ , and  $c_4 = \overline{x_1} \wedge \overline{x_2}$ . The maximum number  $h$  of satisfied clauses is three.

In summary, there is single a voter  $v$  along the order  $L$  such that all voters up to and including  $v$  have the same preference over  $\{a, a'\}$  and all remaining voters have the same preference over  $\{a, a'\}$ —a contradiction to the assumption that  $L$  is not a single-crossing order.

For the “if” part, suppose that deleting a set  $K$  of at most  $k$  alternatives makes the remaining profile single-crossing. Since  $|O| = |\overline{O}| \geq k$ , at least one pair  $\{o, \overline{o}\}$  of dummy alternatives is not deleted, where  $o \in O$  and  $\overline{o} \in \overline{O}$ . Let  $X_{\text{del}}$  denote the set of all deleted variable alternatives, and  $A_{\text{del}}$  denote the set of all deleted clause alternatives. Clearly,  $|X_{\text{del}}| + |A_{\text{del}}| \leq |K|$ . For each  $x_i \in U$ , at least one set of  $X_i$  and  $\overline{X_i}$  must be deleted to destroy all  $\delta$ -configurations involving alternatives in  $\{o, \overline{o}\} \cup X_i \cup \overline{X_i}$ . This means that  $|X_{\text{del}}| \geq r(s+1)$ . Thus,  $|A_{\text{del}}| \leq |K| - |X_{\text{del}}| \leq k - r(s+1) \leq s - h$ . Let  $C_{\text{both}}$  denote the set of clauses such that neither  $a_j$  nor  $b_j$  is deleted,  $1 \leq j \leq s$ , that is,  $C_{\text{both}} := \{c_j \mid \{a_j, b_j\} \cap A_{\text{del}} = \emptyset\}$ . Set  $C_{\text{both}}$  has cardinality at least  $h$  because  $|A_{\text{del}}| \leq s - h$ . We show that by setting variable  $x_i \in U$  to true if  $X_i \subseteq X_{\text{del}}$ , and false otherwise, all clauses  $c_j$  from  $C_{\text{both}}$  are satisfied. Suppose for the sake of contradiction that clause  $c_j \in C_{\text{both}}$  is not satisfied. This means that  $\{a_j, b_j\} \cap A_{\text{del}} = \emptyset$ , and that both  $\widehat{X}_1^j$  and  $\widehat{X}_2^j$  are not completely contained in  $X_{\text{del}}$ . But then, voters  $w_{4i-3}, w_{4i-2}, w_{4i-1}$ , and  $w_{4i}$  form a  $\delta$ -configuration with regard to  $a_j, b_j, x, x'$  with  $x \in \widehat{X}_1^j \setminus X_{\text{del}}$  and  $x' \in \widehat{X}_2^j \setminus X_{\text{del}}$ —a contradiction.  $\square$

We illustrate our NP-hardness reduction through an example. Consider a MAX2SAT instance with two variables  $x_1$  and  $x_2$  and four clauses  $c_1 = x_1 \wedge x_2$ ,  $c_2 = x_1 \wedge \overline{x_2}$ ,  $c_3 = \overline{x_1} \wedge x_2$ , and  $c_4 = \overline{x_1} \wedge \overline{x_2}$ . The maximum number  $h$  of satisfied clauses is three. For instance, the truth assignment  $x_1 \mapsto \text{true}$  and  $x_2 \mapsto \text{false}$  satisfies clauses  $c_1, c_2, c_4$ . Table 2 de-

picts the reduced instance of SINGLE-CROSSING MAVERICK DELETION with alternative set  $O \cup \overline{O} \cup X_1 \cup \overline{X_1} \cup X_2 \cup \overline{X_2} \cup \{a_i, b_i \mid 1 \leq i \leq 4\}$  and voter set  $\{v_i, w_{4i-3}, w_{4i-2}, w_{4i-1}, w_{4i} \mid 1 \leq i \leq 4\}$ . The number  $k$  of alternatives that may be deleted is set to  $2 \cdot (4+1) + (4-3) = 11$ . We can verify that deleting the alternatives from  $X_1 \cup \overline{X_2} \cup \{b_3\}$  results in a single-crossing profile where a single-crossing order is  $v_1 \succ v_3 \succ v_2 \succ v_4 \succ w_2 \succ w_4 \succ w_1 \succ w_3 \succ w_6 \succ w_8 \succ w_5 \succ w_7 \succ w_9 \succ w_{10} \succ \dots \succ w_{16}$ .

In contrast to the other NP-hard II MAVERICK DELETION problems, SINGLE-CROSSING MAVERICK DELETION is polynomial-time solvable. The algorithm, which is similar to the single-crossing detection algorithm by Elkind et al. [26], not only solves the decision problem, but also the optimization problem asking for the maximum-size subset of voters such that the profile restricted to this subset is single-crossing.

Before we proceed to describe the algorithm, we define some notions and make some observations about single-crossing profiles. We call a set  $S$  of preference orders *single-crossing*, if there is a single-crossing order of the elements in  $S$ . We introduce the notion  $\Delta(\succ, \succ')$  of the set of *conflict pairs* for two given preference orders  $\succ$  and  $\succ' \in \mathcal{S}$ . By  $\Delta(\succ, \succ')$ , we denote the set of pairs  $\{a, b\}$  of alternatives whose relative order differs between preference orders  $\succ$  and  $\succ'$ . Formally,  $\Delta(\succ, \succ') := \{\{a, b\} \mid a \succ b \wedge b \succ' a\}$ . For instance, given three alternatives  $a, b, c$ , if the preference orders  $\succ, \succ'$  are the same, then  $\Delta(v, v') = \emptyset$ ; if the two preference orders are specified as follows:

$$b \succ a \succ c \text{ and } c \succ' b \succ' a,$$

then  $\Delta(\succ, \succ') := \{\{a, c\}, \{b, c\}\}$ .

Based on this notion, we can redefine the single-crossing property of a set of preference orders using set inclusions. For the sake of readability, we will use the vector notation  $(\cdot, \dots, \cdot)$  to denote a linear order over a set of preference orders.

**Lemma 1.** *A linear order  $(\succ_1^*, \succ_2^*, \dots, \succ_n^*)$  over a set of  $n$  preference orders is single-crossing if and only if for each two preference orders  $\succ_i^*$  and  $\succ_j^*$  with  $1 \leq i < j \leq n$  it holds that  $\Delta(\succ_i^*, \succ_i^*) \subseteq \Delta(\succ_1^*, \succ_j^*)$ .*

*Proof.* The “only if” part follows directly from the definition of the single-crossing property and the set of conflict pairs. For the “if” part, suppose towards a contradiction that the order  $(\succ_1^*, \succ_2^*, \dots, \succ_n^*)$  is not single-crossing. This means that there are two alternatives  $a, b$ , and there are two preference orders  $\succ_i^*, \succ_j^*$  with  $1 < i < j$  such that  $a \succ_i^* b$  and  $a \succ_j^* b$ , but  $b \succ_i^* a$ . Then it follows that  $\{a, b\} \in \Delta(\succ_i^*, \succ_i^*)$  but  $\{a, b\} \notin \Delta(\succ_1^*, \succ_j^*)$ —a contradiction.  $\square$

The following observation states that the single-crossing property only depends on the preference orders, not on the voters.

**Observation 1.** *Let  $V$  be a set of voters and let  $w \notin V$  be an additional voter such that there is a voter in  $V$  who has the same preference order as voter  $w$ . Then, the profile with voter set  $V$  is single-crossing if and only if the profile with voter set  $V \cup \{w\}$  is single-crossing.*

*Proof.* By the definition of single-crossing orders, a profile is single-crossing if and only if the set of the preference orders of all voters in this profile is single-crossing. Since adding voter  $w$  to voter set  $V$  does not change the set of the preference orders of all voters in  $V$ , the statement follows.  $\square$

Based on the notions of conflicting pairs and single-crossing sets of preference orders, and Lemma 1 and Observation 1, we can solve the maximization variant of the SINGLE-CROSSING MAVERICK DELETION problem by reducing it to finding a longest path in an appropriately constructed directed acyclic graph. This implies the following theorem.

**Theorem 6.** SINGLE-CROSSING MAVERICK DELETION is solvable in  $O(n^3 \cdot m^2)$  time, where  $n$  denotes the number of voters and  $m$  denotes the number of alternatives.

*Proof.* Suppose that we are given a profile with  $A$  being the set of  $m$  alternatives and  $V$  being the set of  $n$  voters, each voter having a preference order over  $A$ . Now, the goal is to find a maximum-size subset of voters such that the profile restricted to this subset is single-crossing. To achieve this goal, we use two further notions: Let  $\mathcal{S}(V) := \{\succ_v \mid v \in V\}$  be the set of the preference orders of all voters from  $V$ . Without loss of generality, let  $\mathcal{S}(V) := \{\succ_1, \succ_2, \dots, \succ_{n'}\}$ . For each preference order  $\succ \in \mathcal{S}(V)$ , let  $\#(\succ, V)$  denote the number of voters in  $V$  with the same preference order  $\succ$ . By Observation 1, it follows that finding the maximum-size single-crossing voter subset is equivalent to finding a *single-crossing* subset  $\mathcal{S}' \subseteq \mathcal{S}(V)$  of preference orders that maximizes the sum  $\sum_{\succ \in \mathcal{S}'} \#(\succ, V)$ .

Now, observe that if  $\succ$  is the first preference order along the single-crossing order over set  $\mathcal{S}'$ , then for each two further preference orders  $\succ', \succ'' \in \mathcal{S}'$  with  $\succ'$  being the predecessor of  $\succ''$  along the single-crossing order, by Lemma 1, it holds that  $\Delta(\succ, \succ') \subseteq \Delta(\succ, \succ'')$ . This inspires us to build a directed graph based on the set inclusion relation and then, to find a maximum-weight path. Thus, the idea of our algorithm is to first construct a directed graph with weighted arcs and then to find a maximum-weight path on this graph. We will provide an example to illustrate this idea right after this proof.

The construction of the desired directed graph works as follows: For each two numbers  $z, i \in \{1, 2, \dots, n'\}$ , we construct one vertex  $u_i^z$ ; this vertex will represent the preference order  $\succ_i$  in a linear order starting with preference order  $\succ_z$ . Then, for each further number  $i' \in \{1, 2, \dots, n'\}$  with  $i \neq i'$ , we add an arc with weight  $\#(\succ_{i'}, V)$  from vertex  $v_i^z$  to vertex  $v_{i'}^z$  if  $\Delta(\succ_z, \succ_i) \subseteq \Delta(\succ_z, \succ_{i'})$ . Finally, we construct a root vertex  $u_r$ , and for each number  $z \in \{1, 2, \dots, n'\}$ , we add an arc with weight  $\#(\succ_z, V)$  from root  $u_r$  to  $u_z^z$ . This completes the construction. Observe that the constructed directed graph is acyclic:

1. For each three numbers  $z, z', i \in \{1, 2, \dots, n'\}$  with  $z \neq z'$ , there are no arcs between vertices  $u_i^z$  and  $u_i^{z'}$ .
2. For each three numbers  $z, i, i' \in \{1, 2, \dots, n'\}$  with  $i \neq i'$ , a path from  $u_i^z$  to  $u_{i'}^z$  implies that  $\Delta(\succ_z, \succ_i) \subseteq \Delta(\succ_z, \succ_{i'})$ , while a path from  $u_{i'}^z$  to  $u_i^z$  implies that  $\Delta(\succ_z, \succ_{i'}) \subseteq \Delta(\succ_z, \succ_i)$ . Thus, both paths cannot exist simultaneously because  $\succ_i \neq \succ_{i'}$ .

Now, an order of the vertices along a maximum-weight directed path corresponds to a subset  $\mathcal{S}' \subseteq \mathcal{S}(V)$  of preference orders, such that  $\mathcal{S}'$  is *single-crossing*, and the sum  $\sum_{\succ \in \mathcal{S}'} \#(\succ, V)$  is maximum: The second vertex on the maximum-weight path fixes the first preference order of the single-crossing order. Each successive vertex  $u_i^z$  on the path represents the successive preference order  $\succ_i$  in the single-crossing order (this is true by Lemma 1 and by the way we define an arc). The arc weights ensure that the sum of the weights on the path equals the total number of represented voters.

As to the running time analysis, we need  $O(n \cdot m)$  time to compute the set  $\mathcal{S}(V)$ . Then, for each two (not necessarily distinct) preference orders  $\succ, \succ' \in \mathcal{S}(V)$ , we compute  $\Delta(\succ, \succ')$ . This can be done by checking the relative order of each pair of alternatives in  $O(n^2 \cdot m^2)$  time. Further, we construct the directed graph in  $O(n^3 \cdot m^2)$  time. Finally, we compute the maximum-weight path in a directed acyclic graph with  $n^2$  vertices and  $n^3$  arcs in  $O(n^3)$  time. To achieve this, we first replace all positive weights  $w$  with  $-w$ , and then use the algorithm in the textbook of Cormen et al. [18, Sec 24.2] to find a minimum-weight path. In total, the running time is  $O(n^3 \cdot m^2)$ .  $\square$

Consider a profile  $\mathcal{P}$  with three alternatives  $a, b, c$  and four voters  $v_1, v_2, v_3, v_4$  whose preference orders are depicted in Figure 3(a). This profile is not single-crossing since it

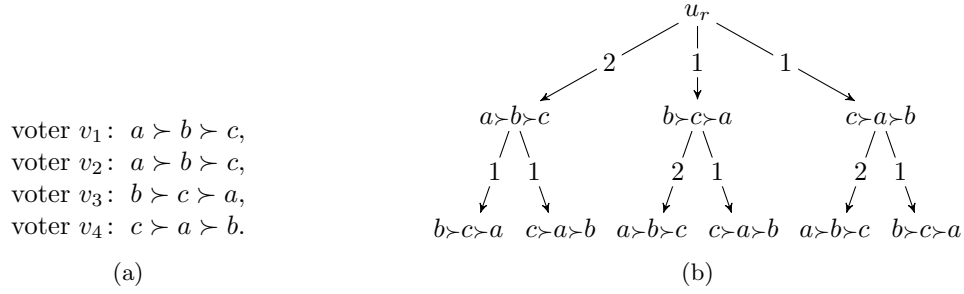


Figure 3: An example illustrating how to construct a weighted directed graph for a given profile. (a) A profile with four voters and three alternatives. Note that the first two voters have the same preference orders and that this profile is not single-crossing. (b) A weighted directed graph for the left profile. Note that we label each vertex with its corresponding preference order. The weight on an arc denotes the number of voters in the profile that have the preference order labeled in the source vertex. For instance, there is an arc from the root  $u_r$  to its left most “child”  $a \succ b \succ c$  with weight 2. This means that the left profile has two voters with preference order  $a \succ b \succ c$ .

contains a  $\gamma$ -configuration with regard to the alternatives  $a, c, a, b, a, b$  and voters  $v_1, v_3, v_4$ . The set of the preference orders of all voters is  $\{a \succ b \succ c, b \succ c \succ a, c \succ a \succ b\}$ . According to our algorithm of finding a single-crossing profile with maximum number of voters, we first construct a weighted directed graph as depicted in Figure 3(b). Then, we will find a maximum-weight path in the graph. We can verify that there are four maximum-weight paths, including this one  $u_r \rightarrow (a \succ b \succ c) \rightarrow (b \succ c \succ a)$  with weight three. A single-crossing profile with maximum number of voters has three voters. For instance, the profile with voters  $v_1, v_2, v_3$ .

## 6 Conclusion

In terms of computational complexity theory, this work is one of the starting points for preference profiles which are “close” to being nicely structured. We have shown that making a profile single-crossing by deleting as few voters as possible can be solved in polynomial time. In contrast, making a profile nicely structured by deleting at most  $k$  voters or at most  $k$  alternatives is NP-hard for all other considered cases. However, we mention in passing that all these problems become tractable when  $k$  is small: All considered properties are characterized by a fixed number of forbidden substructures. Thus, by branching over all possible voters (resp. alternatives) of each forbidden substructure in the profile one obtains a fixed-parameter algorithm [21, 23, 33, 43] that is efficient for small distances. One line of future research is to investigate more sophisticated and more efficient (fixed-parameter) algorithms to compute the distance of a profile to a nicely structured one [25].

A second line of research which was started by Erdélyi et al. [29] for single-peaked profiles is to study further distance measures such as “the number of pairs of alternatives to swap”. Besides the domain restrictions studied in this paper, there is also a very nicely structured property in the literature, the so-called 1-D Euclidean representation. It models the ability to place voters and alternatives onto a real line such that a voter prefers an alternative to another one if and only if the first one is closer to the voter. 1-D Euclidean profiles are necessarily single-peaked and single-crossing. The last two properties, however, are not sufficient to characterize the 1-D Euclidean profile [16, 17, 28]. In fact, Chen et al. [16]

show that the 1-D Euclidean profile *cannot* be characterized by finitely many forbidden substructures. Nevertheless, recognizing 1-D Euclidean profiles can be done in polynomial time [22, 24, 38]. The computational complexity of making a profile 1-D Euclidean using a minimum number of modifications remains unexplored.

A third line of research investigates whether and in which way tractability for nicely structured preference profiles transfers to profiles that are only close to being nicely structured (see also key questions 6 and 7 in [14]). It was started by Cornaz et al. [19, 20], Faliszewski et al. [32], Skowron et al. [48], Yang and Guo [52] who look into several notions of nearly nicely structured profiles which are different from, but related to ours. There are cases where the computational tractability of voting problems on nicely structured profiles transfers to nearly nicely structured profiles and cases where the vulnerability disappears even if the preference profile is extremely close to being nicely-structured.

**Acknowledgements.** Robert Brederick was supported by the German Research Foundation (DFG), research project PAWS, NI 369/10. Jiehua Chen was supported by the Studienstiftung des Deutschen Volkes. Gerhard J. Woeginger was supported by the Alexander von Humboldt Foundation.

## References

- [1] J. M. Abello and C. R. Johnson. How large are transitive simple majority domains? *SIAM Journal on Algebraic and Discrete Methods*, 5(4):603–618, 1984. → p. 6.
- [2] Ádám Galambos and V. Reiner. Acyclic sets of linear orders via the Bruhat orders. *Social Choice and Welfare*, 30(2):245–264, 2008. → p. 6.
- [3] K. J. Arrow. A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4):328–346, 1950. → p. 1.
- [4] N. Baigent. Metric rationalisation of social choice functions according to principles of social choice. *Mathematical Social Sciences*, 13(1):59–65, 1987. → p. 4.
- [5] M. Ballester and G. Haeringer. A characterization of the single-peaked domain. *Social Choice and Welfare*, 36(2):305–322, 2011. → pp. 5 and 7.
- [6] S. Barberà, F. Gul, and S. Ennio. Generalized median voter schemes and committees. *Journal of Economic Theory*, 61(2):262–289, 1993. → p. 3.
- [7] J. Bartholdi III and M. A. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986. → p. 1.
- [8] J. J. Bartholdi III, C. A. Tovey, and M. A. Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare*, 6(2):157–165, 1989. → p. 1.
- [9] N. Betzler, A. Slinko, and J. Uhlmann. On the computation of fully proportional representation. *Journal of Artificial Intelligence Research*, 47(1):475–519, 2013. → p. 4.
- [10] D. Black. On the rationale of group decision making. *Journal of Political Economy*, 56(1):23–34, 1948. → p. 1.
- [11] F. Brandt, M. Brill, E. Hemaspaandra, and L. A. Hemaspaandra. Bypassing combinatorial protections: Polynomial-time algorithms for single-peaked electorates. *Journal of Artificial Intelligence Research*, 53:439–496, 2015. → pp. 2 and 4.



- [12] R. Bredereck, J. Chen, and G. J. Woeginger. A characterization of the single-crossing domain. *Social Choice and Welfare*, 41(4):989–998, 2013. → pp. 2, 5, and 8.
- [13] R. Bredereck, J. Chen, and G. J. Woeginger. Are there any nicely structured preference profiles nearby? In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI '13)*, pages 62–68. AAAI Press, 2013. → p. 1.
- [14] R. Bredereck, J. Chen, P. Faliszewski, J. Guo, R. Niedermeier, and G. J. Woeginger. Parameterized algorithmics for computational social choice: Nine research challenges. *Tsinghua Science and Technology*, 19(4):358–373, 2014. → p. 23.
- [15] J. Chen, P. Faliszewski, R. Niedermeier, and N. Talmon. Combinatorial voter control in elections. In *Proceedings of the 39th International Symposium on Mathematical Foundations of Computer Science (MFCS '14)*, volume 8635 of *Lecture Notes in Computer Science*, pages 153–164. Springer, 2014. → p. 4.
- [16] J. Chen, K. Pruhs, and G. J. Woeginger. The one-dimensional euclidean domain: Finitely many obstructions are not enough, June 2015. [arXiv:1506.03838v1 \[cs.GT\]](#). → p. 22.
- [17] C. H. Coombs. *A Theory of Data*. John Wiley and Sons, 1964. → p. 22.
- [18] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms*. MIT Press, 2009. → p. 21.
- [19] D. Cornaz, L. Galand, and O. Spanjaard. Bounded single-peaked width and proportional representation. In *Proceedings of the 20th European Conference on Artificial Intelligence (ECAI '12)*, pages 270–275. IOS Press, 2012. → pp. 3 and 23.
- [20] D. Cornaz, L. Galand, and O. Spanjaard. Kemeny elections with bounded single-peaked or single-crossing width. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI '13)*, pages 76–82. AAAI Press, 2013. → pp. 3 and 23.
- [21] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015. → p. 22.
- [22] J. Doignon and J. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994. → pp. 1, 2, and 23.
- [23] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Springer, 2013. → p. 22.
- [24] E. Elkind and P. Faliszewski. Recognizing 1-Euclidean preferences: An alternative approach. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT '14)*, volume 8768 of *Lecture Notes in Computer Science*, pages 146–157. Springer, 2014. → p. 23.
- [25] E. Elkind and M. Lackner. On detecting nearly structured preference profiles. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI '14)*. AAAI Press, 2014. → pp. 4 and 22.
- [26] E. Elkind, P. Faliszewski, and A. M. Slinko. Clone structures in voters' preferences. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC '12)*, pages 496–513. ACM Press, 2012. → pp. 2 and 20.

- [27] E. Elkind, P. Faliszewski, and A. M. Slinko. Rationalizations of Condorcet-consistent rules via distances of hamming type. *Social Choice and Welfare*, 39(4):891–905, 2012. → p. 4.
- [28] E. Elkind, P. Faliszewski, and P. Skowron. A characterization of the single-peaked single-crossing domain. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI '14)*, pages 654–660. AAAI Press, 2014. → p. 22.
- [29] G. Erdélyi, M. Lackner, and A. Pfandler. Computational aspects of nearly single-peaked electorates. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI '13)*, pages 283–289. AAAI Press, 2013. → pp. 3, 4, 13, and 22.
- [30] B. Escoffier, J. Lang, and M. Öztürk. Single-Peaked Consistency and its Complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI '08)*, pages 366–370. IOS Press, 2008. → p. 1.
- [31] P. Faliszewski, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. The shield that never was: Societies with single-peaked preferences are more open to manipulation and control. *Information and Computation*, 209(2):89–107, 2011. → pp. 2 and 4.
- [32] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. The complexity of manipulative attacks in nearly single-peaked electorates. *Artificial Intelligence*, 207:69–99, 2014. → pp. 2, 4, and 23.
- [33] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006. → p. 22.
- [34] M. R. Garey and D. S. Johnson. *Computers and Intractability—A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979. → pp. 9 and 16.
- [35] K. Inada. A note on the simple majority decision rule. *Econometrica*, 32(32):525–531, 1964. → pp. 2 and 5.
- [36] K. Inada. The simple majority decision rule. *Econometrica*, 37(3):490–506, 1969. → pp. 2 and 5.
- [37] B. Klaus, H. Peters, and T. Storcken. Strategy-proof division of a private good when preferences are single-dipped. *Economics Letters*, 55(3):339–346, 1997. → p. 2.
- [38] V. Knoblauch. Recognizing one-dimensional Euclidean preference profiles. *Journal of Mathematical Economics*, 46(1):1–5, 2010. → p. 23.
- [39] T. Meskanen and H. Nurmi. Closeness counts in social choice. In *Power, Freedom, and Voting*, pages 289–306. Springer, 2008. → p. 4.
- [40] J. A. Mirrlees. An exploration in the theory of optimal income taxation. *Review of Economic Studies*, 38(2):175–208, 1971. → p. 2.
- [41] B. Monjardet. Acyclic domains of linear orders: A survey. In *The Mathematics of Preference, Choice and Order*, pages 139–160. Springer, 2009. → p. 6.
- [42] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980. → p. 2.
- [43] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006. → p. 22.

- [44] C. Puppe and A. M. Slinko. Condorcet domains, median graphs and the single crossing property, May 2015. [arXiv:1505.06982v1 \[cs.GT\]](#). → p. 6.
- [45] K. W. Roberts. Voting over income tax schedules. *Journal of Public Economics*, 8(3): 329–340, 1977. → p. 2.
- [46] A. Sen and P. K. Pattanaik. Necessary and sufficient conditions for rational choice under majority decision. *Journal of Economic Theory*, 1(2):178–202, 1969. → pp. 2 and 5.
- [47] A. K. Sen. A possibility theorem on majority decisions. *Econometrica*, 34(2):491–499, 1966. → pp. 2, 5, and 6.
- [48] P. Skowron, L. Yu, P. Faliszewski, and E. Elkind. The complexity of fully proportional representation for single-crossing electorates. *Theoretical Computer Science*, 569:43–57, 2015. → pp. 4 and 23.
- [49] X. Sui, A. Francois-Nienaber, and C. Boutilier. Multi-dimensional single-peaked consistency and its approximations. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI '13)*. AAAI Press, 2013. → p. 3.
- [50] T. N. Tideman. Independence of clones as a criterion for voting rules. *Social Choice and Welfare*, 4(3):185–206, 1987. → p. 3.
- [51] T. Walsh. Uncertainty in preference elicitation and aggregation. In *Proceedings of the 22nd Conference on Artificial Intelligence (AAAI '07)*, pages 3–8. AAAI Press, 2007. → pp. 2 and 4.
- [52] Y. Yang and J. Guo. The control complexity of  $r$ -approval: from the single-peaked case to the general case. In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS '14)*, pages 621–628. IFAAMAS, 2014. → pp. 3 and 23.
- [53] H. P. Young. Extending Condorcet’s rule. *Journal of Economic Theory*, 16(2):335–353, 1977. → p. 4.