On a make-to-stock production/mountain model with hysteretic control

On a Make-to-Stock Production/Mountain Model with Hysteretic Control

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Abstract

We consider a make-to-stock production-inventory model with one machine that produces stock in a buffer. The machine is subject to breakdowns. During up periods, the machine fills the buffer at a level-dependent rate $\alpha(x) > 0$. During down periods, the production rate is zero, and the demand rate is either $\beta(x) > 0$ or $\gamma(x) > 0$ when the inventory level is x; which of the two demand rates applies depends on a hysteretic control policy.

We determine the conditions under which the steady-state distribution of the inventory level exists, and we derive that distribution. Other performance measures under consideration are the number of switches from $\beta(\cdot)$ to $\gamma(\cdot)$ per busy period, the busy period distribution, and the overshoot above a particular hysteretic level.

1 Introduction

We consider a make-to-stock production-inventory model with one machine that produces stock in a buffer. The machine is subject to breakdowns. During up periods (machine working), the machine fills the buffer at a level-dependent rate $\alpha(x) > 0$. During down periods, the production rate is zero, and the demand rate is either $\beta(x) > 0$ or $\gamma(x) > 0$ when the inventory level is x. Which of the two demand rates applies depends on a hysteretic (a, b) control policy, with $0 < a < b$. If the inventory level is $x < a$, the demand rate during down periods is $\beta(x)$. If the inventory level is $x > b$, the demand

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rate during down periods is $\gamma(x)$. If the inventory level crosses a from above while the demand rate function is $\gamma(\cdot)$, it instantaneously switches to $\beta(\cdot)$. If the inventory level crosses b from below while the demand rate function is $\beta(\cdot)$, it instantaneously switches to $\gamma(\cdot)$. See Figure 1 below.

The hysteretic control is motivated as follows. Suppose that the controller has the choice to satisfy the demand of either one or two market destinations. Namely, he is able to sell the product in two market centers, each having a state-dependent demand rate. He then applies the following dynamic control policy. When the machine is working (i.e., during the up periods) he will always sell the product in both markets so that the input rate $\alpha(x)$ equals the production rate minus the accumulated demand rate in the two markets. When the machine is in repair (down period) and the content level is low (below level a) the controller sells the product only in the first market center in which the demand rate is $\beta(x)$ and temporarily abandons the second market. When the machine is in repair and the content level is high (above level b) the controller will sell the product in both market centers, with accumulated demand rate $\gamma(x)$. However, when the content $\in [a, b]$ the release rate is either $\beta(x)$ or $\gamma(x)$, according to the hysteretic control described above. A decision problem for the present model may be to determine a and b such that a particular objective function is minimized. That objective function will involve holding costs, production costs, and also switching costs. The reason why a hysteretic control policy may be attractive is that it balances holding costs and switching costs, and may in particular reduce switching costs. In this paper we focus on a probabilistic analysis; that analysis may be used subsequently for optimization and control purposes.

Related literature. There is a huge literature on production-inventory models and on dam/storage processes. We restrict ourselves here to a few strands of research which are close to the present study. One such strand is queueing/production/dam/storage models with state-dependent increments/decrements. We refer to Dshalalow [12] for a survey and bibliography on queueing models with state-dependent parameters. A more recent queueing study is [4], which considers queues with workload-dependent arrival and service rates; see also [18] for a study of exit times of $M/G/1$ type queues with a general workload-dependent service rate. Contrary to queues, in storage processes one may have up periods instead of upward jumps. The papers [5, 6, 21] consider the stationary buffer content of a storage process with an underlying background Markov process. In [5], the background process alternates between on and off, and accordingly the buffer content increases or decreases at some state-dependent rate. In [21] the background process can be in more than two different states, but the authors restrict themselves to a finite buffer content. The background process has three states in [6]; in one state, the buffer content increases in a workload-dependent way; in the other two states, the workload decreases linearly, with different rates.

A second strand of related literature concerns hysteretic control. In [19] the authors investigate a model that is similar to what we call the dam process (Figure 3). They derive expressions for the distribution of the process at the first upcrossing of level b

(see our Section 7) and use this result to specify and solve integro-differntial equations for the stationary distribution (our Section 3). We note that the approach in [19] differs from ours in that we study the more general mountain process (with level-dependent jumps for the dam process).

Dshalalow and co-authors have published several papers on queues with hysteretic control; see, e.g., [13]. Other interesting studies include [17] and [3]. In the latter paper, which has a quite general set-up, Bekker considers a reflected Lévy process that can be in two different states. The Lévy exponent is $\phi_i(\cdot)$ in state i. The control rule is specified by two levels $0 \leq m_2 \leq m_1$. When the Lévy process upcrosses m_1 while being in state 1, the state switches to 2; when the Lévy process downcrosses level m_2 while being in state 2, the state switches to 1. Hence, like in our study, when the process is between m_2 and m_1, the state can be either 1 or 2. This model contains several classical queueing models with some oscillating behaviour (see, e.g., [10] and [20]) and dam models where the input process is Brownian motion (see, e.g., [14] and [23]).

Finally, in the setting of production/inventory control, we mention Section 1.10 of [22] for a discussion of a production/inventory control model with variable production level and service level constraints, and the related papers [11, 16].

Main result. Our main result is the steady-state distribution of the inventory content level. It is obtained by first finding the distribution of the inventory content in down periods, which is done by splitting the down periods in 1-periods and 2-periods and deriving the inventory content distributions in both periods. Finally we express the density of the inventory content during up periods in its counterpart during down periods, using a level crossing argument. Other contributions of the paper include the distribution of the overshoot above hysteretic level b and, in the case of exponentially distributed up periods, the number of switches from $\beta(\cdot)$ to $\gamma(\cdot)$ per busy period, and the busy period distribution.

Organization of the paper. A detailed model description is presented in Section 2. In Section 3 we derive the steady-state distribution of the inventory content level. The conditions under which such a steady-state distribution exists are also discussed in this section. In Sections 4, 5 and 6 we assume that the up periods (machine working) are exponentially distributed. In that case, we subsequently obtain quite detailed results for the steady-state inventory content level distribution, the number of switches in a cycle, and the busy period distribution. Section 7 is devoted to a discussion of the overshoot distribution above level b, for the case of generally distributed up periods.

2 Model description

We consider a make-to-stock production/inventory model with one machine that produces stock into a buffer. The successive periods during which the machine is up (working) are independent and identically distributed (i.i.d.) with distribution function G and finite mean ν. The successive periods during which the machine is down (under repair) are independent and exponentially distributed with mean $1/\lambda$. Lengths of up and down periods are also independent. As mentioned in Section 1, the buffer level increases at rate $\alpha(x)$ when it is at level x and the machine is working. It decreases at rate $\beta(x)$ or $\gamma(x)$ when the machine is not working, dependent on a hysteretic (a, b) control policy. In the motivating example of Section 1, we have that $\gamma(x) \geq \beta(x)$ for all x. However, for the mathematical analysis this assumption is not necessary. A natural choice for $\beta(x)$ would be $\beta(x) = k_1 - \alpha(x)$, as this corresponds to constant demand k_1 and a production/filling rate of $\alpha(x)$, but we shall make no such restrictions. However, we require α, β and γ to be left-continuous with right limits.

As a result of the above, we distinguish between three types of periods. α-periods are the intervals of time in which the machine is working (the up periods). During an α-period, the rate of increment is $\alpha(x)$ if the inventory level is x. The β-periods are the intervals of time in which the machine is not working (under repair) and the buffer level decreases at rate $\beta(x)$; the γ-periods are the intervals of time in which the machine is not working and the buffer level decreases at rate $\gamma(x)$. It should be noted that the buffer level may reach 0 only during β-periods; it then stays at 0 until the machine has been repaired (an α-period starts).

We introduce the following definitions and assumptions:

$$A(x) = \int_0^x \frac{1}{\alpha(y)} dy < \infty, \quad x \geq 0, \tag{1}$$

$$B(x) = \int_0^x \frac{1}{\beta(y)} dy < \infty, \quad x \leq b, \tag{2}$$

$$C(x) = \int_a^x \frac{1}{\gamma(y)} dy < \infty, \quad x \geq a. \tag{3}$$

The interpretation of $A(x)$ is the following. It is the time it takes to reach level x, starting from level 0, when the machine does not break down in the meantime. $B(x)$ and $C(x)$ have similar interpretations. That is, $B(x)$ (for $x < b$) is interpreted as the time to reach level 0, starting from level x (where the demand rate is $\beta(\cdot)$) and $C(x)$ is interpreted as the time it takes to reach level a, starting from level x ($> a$), if the demand rate was $\gamma(\cdot)$. We assume that

$$\int_0^\infty \frac{1}{\alpha(y)} dy = \infty \tag{4}$$

to prevent explosions of the process and to obtain a proper distribution function for the jumps in the dam model presented in the following.

The two finiteness assumptions in (2) and (3) imply that level 0 can be reached from any level $x \leq b$ (whenever the output rate is $\beta(\cdot)$), and that level a can be reached from any level $x > a$ (whenever the output rate is $\gamma(\cdot)$), respectively.

3 Steady-state buffer level analysis

Let $\boldsymbol{X} = \{X(t) : t \geq 0\}$ be the fluid content level process (we also call $\boldsymbol{X}$ the mountain process, in line with a terminology that was already used for similar stochastic processes in, e.g., [7]), see Figure 1. $\boldsymbol{X}$ is a regenerative process with regeneration cycles starting at down-crossings of level a.

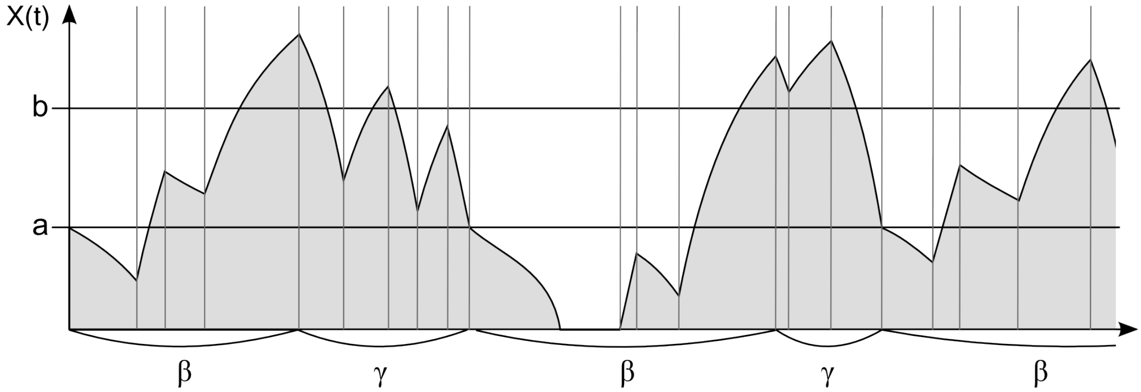

Fig.1: The mountain process $X(t)$ with decrease rates.

We call the process $\boldsymbol{X}$ stable if there is a stationary (or steady-state) distribution F_X and $\mathbb{P}(X(t) \le x)$ converges for $t \to \infty$ to $F_X(x)$ at all continuity points of F_X, i.e. $\boldsymbol{X}$ is ergodic. The following theorem provides a sufficient criterion for stability.

Theorem 1. *The process $\boldsymbol{X}$ is stable if there is a $z_0 \ge 0$ such that*

$$\sup_{z \ge z_0} \lambda \int_z^\infty \frac{\Gamma(z,u)}{\gamma(u)}\, du < 1, \tag{5}$$

where $\Gamma(x,y) = 1 - G(A(y) - A(x))$ is the probability that, when starting the process in x in an α-period, the next switch happens after the process crosses $y > x$.

Proof. Let us assume first that $\alpha(x) = 1$ and $\beta(x) = \gamma(x)$. We consider the dam process $\boldsymbol{D}$ which can be derived from $\boldsymbol{X}$ by deleting the α-periods and replacing them by upward jumps which equal the increments of the deleted α-periods (see Figure 3 below). Then $\boldsymbol{X}$ is stable iff $\boldsymbol{D}$ is stable. The process $\boldsymbol{D}$ has decrease rate $\gamma(x)$ and jumps with the following distribution. If the process jumps at time t then

$$\mathbb{P}(D(t+) - D(t-) \le y | D(t-) = x) = G(y),$$

independent of x (this is because we assume $\alpha(x) = 1$). It follows that the process $\boldsymbol{D}$ is a storage process as described in [9] and we may employ Proposition 11 there, stating that $\boldsymbol{D}$ is stable if there is a w_0 such that

$$\sup_{w \ge w_0} \lambda \int_0^\infty \int_w^{w+y} \frac{1}{\gamma(u)}\, du\, G(dy) < 1. \tag{6}$$

Now turn to the case where $\alpha(x)$ is arbitrary but still $\gamma(x) = \beta(x)$. We introduce the process $\boldsymbol{X}^*$ defined by $X^*(t) = A(X(t))$ and note that during up times

$$\frac{d}{dt} X^*(t) = \frac{d}{dt} A(X(t)) = \frac{1}{\alpha(X(t))} \frac{d}{dt} X(t) = \frac{1}{\alpha(X(t))} \alpha(X(t)) = 1,$$

so $X^*(t)$ has increase rate 1. Moreover, during decrease times we have

$$\frac{d}{dt}X^*(t) = \frac{d}{dt}A(X(t)) = \frac{1}{\alpha(X(t))}\frac{d}{dt}X(t) = \frac{\gamma(X(t))}{\alpha(X(t))},$$

so that the modified process $X^*(t)$ has decrease rate $\gamma^*(x) = \frac{\gamma(A^{-1}(x))}{\alpha(A^{-1}(x))}$. Using Condition (6) we conclude that $\boldsymbol{X}^*$ is stable if there is a w_0 such that

$$\sup_{w \geq w_0} \lambda \int_0^\infty \int_w^{w+y} \frac{\alpha(A^{-1}(u))}{\gamma(A^{-1}(u))} \, du \, G(dy) < 1. \tag{7}$$

The criterion of the theorem then follows from

$$\begin{aligned}
&\lambda \int_0^\infty \int_w^{w+y} \frac{\alpha(A^{-1}(u))}{\gamma(A^{-1}(u))} \, du \, G(dy) \\
&= \lambda \int_0^\infty \frac{\alpha(A^{-1}(u+w))}{\gamma(A^{-1}(u+w))}(1 - G(u)) \, du \\
&= \lambda \int_{A^{-1}(w)}^\infty \frac{1}{\gamma(u)}(1 - G(A(u) - w)) \, du,
\end{aligned}$$

using $w = A(z)$ and noting that, since stability depends on the decrease rate only for large x, the stability property will not change if we allow $\beta(x) \neq \gamma(x)$, $x < b$ and that $\boldsymbol{X}^*$ is stable iff $\boldsymbol{X}$ is stable. □

Corollary 2. *The process $\boldsymbol{X}$ is stable if there is a $z_0 \geq 0$ such that*

$$\sup_{z \geq z_0} \frac{\alpha(z)}{\gamma(z)} < \frac{1}{\lambda\nu}.$$

Proof. Recall the sufficient criterion (7). Then

$$\sup_{w \geq w_0} \lambda \int_0^\infty \int_w^{w+y} \frac{\alpha(A^{-1}(u))}{\gamma(A^{-1}(u))} \, du \, G(dy) \leq \sup_{w \geq w_0} \frac{\alpha(A^{-1}(w))}{\gamma(A^{-1}(w))} \lambda \int_0^\infty y \, G(dy) < 1.$$

Let $w = A(z)$ and $w_0 = A(z_0)$. □

From now on we assume that the steady-state distribution F_X exists. Let X be a random variable with distribution F_X. Note that the distribution F_X has an atom $\pi_X = \mathbb{P}(X = 0)$ at 0, but F_X is an absolutely continuous distribution for all $x > 0$; we denote the steady-state density by f_X. The fact that f_X exists can be shown by level crossing theory (LCT). Throughout we assume that all appearing steady-state densities are left-continuous.

Also by LCT, $f_X(x)$ can be interpreted as the long-run average number of up - and downcrossings of level x per time unit. Define the conditional content level densities during up periods (α-periods) and down periods (non α-periods) by f_P and f_D, respectively. Since the fraction of time up equals $\lambda\nu/(1 + \lambda\nu)$, we have:

$$f_X(x) = \frac{\lambda\nu}{1+\lambda\nu} f_P(x) + \frac{1}{1+\lambda\nu} f_D(x), \quad x > 0. \tag{8}$$

Note that $\int_0^\infty f_P(x)\mathrm{d}x = 1$; there is no atom at zero, because the inventory level is always positive during up periods. f_P can be interpreted as the density of the process which is obtained by deleting the down periods from the mountain $\boldsymbol{X}$ and gluing together the successive up periods (see Figure 2).

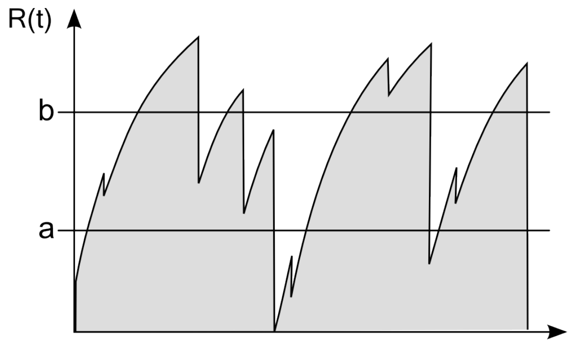

Fig.2: The production process $R(t)$ corresponding to the process $X(t)$ given in Figure 1.

Similarly, f_D can be interpreted as the density of the level of the dam which is generated by deleting the α-periods from the mountain $\boldsymbol{X}$ and gluing together the non α-periods, see Figure 3. Note that in the latter dam both the jumps and the release rates are state dependent. It is easily seen that

$$\pi_D = \mathbb{P}(X = 0|\text{non } \alpha\text{-period}) = \frac{\mathbb{P}(X = 0)}{\mathbb{P}(\text{non } \alpha\text{-period})} = (1 + \lambda\nu)\pi_X.$$

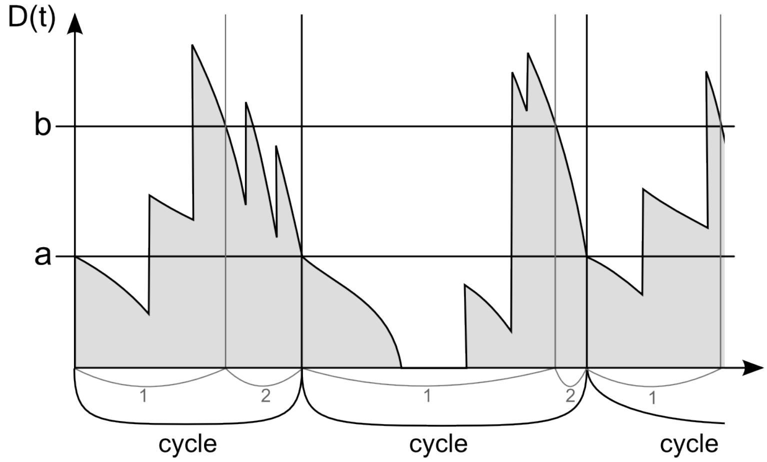

Fig.3: The dam process $D(t)$ corresponding to the process $X(t)$ given in Figure 1.

It follows from Equation (8) that $f_X(\cdot)$ is found once we have obtained $f_P(\cdot)$ and $f_D(\cdot)$. Later in this section we shall indicate how one may express $f_P(\cdot)$ in terms of $f_D(\cdot)$, so that it suffices to obtain $f_D(\cdot)$. Therefore we now restrict ourselves to non α-periods, taking a closer look at the density $f_D(\cdot)$. The dam process $\boldsymbol{D} = \{D(t) : t \geq 0\}$ whose steady-state distribution is given by

$$F_D(x) = \pi_D + \int_0^x f_D(y)\mathrm{d}y,$$

is a regenerative process. For the sake of convenience, we define a regenerative cycle as the time between two consecutive epochs at which the release rate switches from $\gamma(\cdot)$ to $\beta(\cdot)$. We further split the regenerative cycles into two sub-cycles, which are called the 1-periods and the 2-periods. 1-periods start with a downcrossing through a that gives rise to a switch from $\gamma(\cdot)$ to $\beta(\cdot)$; the release rate is $\beta(a)$ at the beginning of a 1-period. Each 1-period lasts until the first downcrossing through b. The process must have upcrossed b some time earlier in the 1-period, giving rise to a switch from $\beta(\cdot)$ to $\gamma(\cdot)$. Now we disassemble the process $\boldsymbol{D}$ into two regenerative processes: the successive 1-periods, which glued together yield $\boldsymbol{D_1} = \{D_1(t) : t \geq 0\}$ and the successive 2-periods, which glued together yield $\boldsymbol{D_2} = \{D_2(t) : t \geq 0\}$. See also Figure 4.

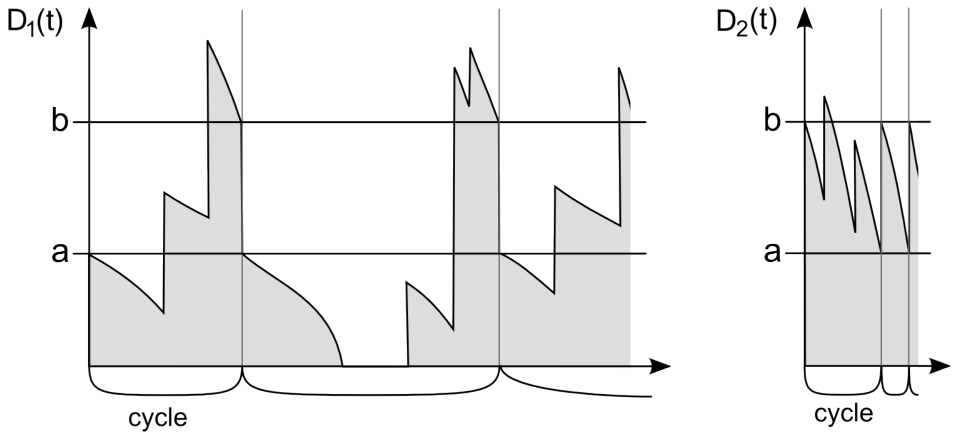

Fig.4: The Processes $D_1(t)$ and $D_2(t)$ corresponding to the process $X(t)$ given in Figure 1.

Let f_1 and f_2 be the steady-state densities of $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$, respectively (we denote the respective distributions by F_1 and F_2).

Obviously, the density f_D is a weighted sum of f_1 and f_2. The next lemma specifies the weight factors.

Lemma 3. *We have*

$$f_D(x) = \kappa f_1(x) + (1-\kappa)f_2(x), \quad x > 0, \tag{9}$$

where κ, the fraction of time the process is in a 1-period, given it is in a down period, equals

$$\kappa = \frac{\gamma(a+)f_2(a+)}{\gamma(a+)f_2(a+) + \gamma(b+)f_1(b+)}. \tag{10}$$

Proof. In steady state the long-run average number of switches from $\beta(\cdot)$ to $\gamma(\cdot)$ is equal to the long-run average number of switches from $\gamma(\cdot)$ to $\beta(\cdot)$. A switch from $\beta(\cdot)$ to $\gamma(\cdot)$ occurs in each 1-period at the first upcrossing of level b (recall that level b is upcrossed only once during a cycle of $\boldsymbol{D_1}$). Similarly, a switch from $\gamma(\cdot)$ to $\beta(\cdot)$ occurs each time that $\boldsymbol{D_2}$ downcrosses level a (recall that level a is downcrossed only once during a cycle of $\boldsymbol{D_2}$). We thus obtain

$$\kappa\gamma(b+)f_1(b+) = (1-\kappa)\gamma(a+)f_2(a+), \tag{11}$$

yielding (10). $\square$

Remark 1. A related, but slightly different, way of deriving (10) is based on the following observation. Since level b is downcrossed only once during a 1-period, it follows by LCT that the mean length of a 1-period equals $1/(\gamma(b+)f_1(b+))$. Similarly, level a is downcrossed only once during a 2-period, so that the mean length of a 2-period equals $1/(\gamma(a+)f_2(a+))$. As a result,

$$\kappa = \frac{1/(\gamma(b+)f_1(b+))}{1/(\gamma(b+)f_1(b+)) + 1/(\gamma(a+)f_2(a+))} = \frac{\gamma(a+)f_2(a+)}{\gamma(a+)f_2(a+) + \gamma(b+)f_1(b+)}.$$

We shall now employ LCT to obtain the densities $f_1(\cdot)$ and $f_2(\cdot)$, thus yielding $f_D(\cdot)$ via Lemma 3. Once we have those densities, LCT will be employed once more, to provide relations that determine $f_P(\cdot)$ in each of the three intervals $(0,a)$, (a,b) and (b,∞). Observing that the ratio of the lengths of α-periods and non α-periods equals $\nu : \frac{1}{\lambda}$, so that the process spends proportions of time $\frac{\lambda\mathbb{E}[G]}{1+\lambda\mathbb{E}[G]}$ and $\frac{1}{1+\lambda\mathbb{E}[G]}$ in α-periods and non α-periods, respectively, we have:

$$\lambda\nu\alpha(x)f_P(x) = \begin{cases} \beta(x)f_D(x) & ; \quad x < a, \\ \kappa\beta(x)f_1(x) + (1-\kappa)\gamma(x)f_2(x) & ; \quad a < x < b, \\ \gamma(x)f_D(x) & ; \quad x > b. \end{cases}$$

Recall that the probability that, when starting the process in x in an α-period, the next switch happens after the process crosses $y > x$, is given by $\Gamma(x,y) = 1 - G(A(y) - A(x))$.

Lemma 4. *Let $\pi_1 = 1/\kappa$ be the probability of an empty system, given that the process is in a 1-period. The following balance equations hold for f_1:*

$$\begin{aligned}
\beta(x)f_1(x) &= \lambda\int_0^x \Gamma(w,x)f_1(w)\,dw + \lambda\pi_1\Gamma(0,x), \quad 0 < x \le a, \\
\beta(x)f_1(x) &= \lambda\int_0^x \Gamma(w,x)f_1(w)\,dw + \lambda\pi_1\Gamma(0,x) - \gamma(b+)f_1(b+) \quad a < x \le b, \\
\gamma(x)f_1(x) &= \lambda\int_0^x \Gamma(w,x)f_1(w)\,dw + \lambda\pi_1\Gamma(0,x), \quad x > b.
\end{aligned}$$

The balance equations for f_2 are given by:

$$\gamma(x)f_2(x) = \begin{cases} \lambda \int_a^x \Gamma(w,x)f_2(w)\,dw + \gamma(a+)f_2(a+), & a < x \le b, \\ \lambda \int_a^x \Gamma(w,x)f_2(w)\,dw, & x > b. \end{cases} \tag{12}$$

Proof. (i) First consider f_1. Clearly, in the region $(0,a]$ the long-run average number of downcrossings is equal to the long-run average number of upcrossings. For $x \in (0,a]$, $\beta(x)f_1(x)$ is the long-run average number of downcrossings of level x by $\boldsymbol{D_1}$. Since level x can only be upcrossed by jumps, every upcrossing must be a jump. Condition on the state just before the jump. If it is $w > 0$, then a jump is an upcrossing if and only if the jump is greater than $A(x) - A(w)$. The probability of the latter event is $\Gamma(w,x)$. If it is $w = 0$, the jump is an upcrossing if and only if the jump is greater than $A(x)$. The latter event occurs with probability $\Gamma(0,x)$, and the probability of the event $\{w = 0\}$ is π_1.

When $x \in (a,b]$ the case is similar to the previous case $x \in (0,a]$ with a slight modification. Still, the release rate is $\beta(x)$, but in this case the long-run average number of downcrossings is **not** equal to the long-run average number of upcrossings. In fact, in each cycle of $\boldsymbol{D_1}$ the number of upcrossings minus the number of downcrossings is equal to 1. The number of upcrossings of level b is also equal to 1. Dividing by the mean of a 1-period, it follows that the downcrossing rate for $\boldsymbol{D_1}$ equals the upcrossing rate for $\boldsymbol{D_1}$ minus 1 divided by the mean 1-period. The latter ratio equals the upcrossing rate of level b in $\boldsymbol{D_1}$.

When $x \in (b,\infty)$ the case is again similar to the case $x \in (0,a]$ under another slight modification. Here the number of upcrossings and downcrossings is the same for each cycle, but the release rate is $\gamma(x)$, not $\beta(x)$.

(ii) Next consider f_2. For $x \in (a,b]$, $\gamma(x)f_2(x)$ is the long-run average number of downcrossings of level x by $\boldsymbol{D_2}$. In each cycle the number of downcrossings minus the number of upcrossings of level x is equal to 1. The number of downcrossings of level a in each cycle is also equal to 1. That means that the long-run average number of downcrossings of level x is equal to the long-run average number of upcrossings plus the long-run average number of downcrossings of level a. The latter number is equal to $\gamma(a+)f_2(a+)$.

Finally, for $x \in (b,\infty)$ the number of downcrossings in each cycle is equal to the number of upcrossings, which means that the long-run average number of downcrossings is equal to the long-run average number of upcrossings. $\square$

Below we shall indicate how one can determine $f_1(\cdot)$ and $f_2(\cdot)$ from the integral equations in Lemma 4. All those integral equations are Volterra integral equations of the second kind, the solution of which is straightforward. We refer to, e.g., Harrison and Resnick [15] for a detailed analysis of similar integral equations for queueing and storage processes with non-constant release rate, and for a discussion of convergence of the ensuing infinite sums. We first define

$$B(x,y) = \frac{\lambda\Gamma(y,x)}{\beta(x)}$$

and

$$C(x,y) = \frac{\lambda\Gamma(y,x)}{\gamma(x)}$$

for $x > y$.
(i) The first equation of Lemma 4 ($0 < x \le a$) becomes:

$$f_1(x) = \int_0^x B(x,w) f_1(w) dw + \pi_1 B(x,0), \tag{13}$$

which is a Volterra integral equation of the second kind on $(0,a]$. Such an equation is known to be uniquely solvable by a Neumann series (in the space of continuous functions), via a Picard iteration. Let us briefly indicate that iteration.

We define $B^{(1)}(x,y) = B(x,y)$, and

$$B^{(n+1)}(x,y) = \int_y^x B(x,z) B^{(n)}(z,y) dz, \qquad B^*(x,y) = \sum_{n=1}^{\infty} B^{(n)}(x,y). \tag{14}$$

In order to guarantee convergence of $B^*(x,0)$, we are going to show by induction that for $n \ge 1$

$$B^{(n)}(x,0) \le \frac{\lambda^n}{(n-1)!\beta(x)} B(x)^{n-1}. \tag{15}$$

For $n = 1$ this bound is obviously true. If the assertion were correct for $n-1$ then

$$\begin{aligned} B^{(n)}(x,0) &\le \frac{\lambda^n}{(n-2)!\beta(x)} \int_0^x \frac{\Gamma(z,x)\Gamma(0,z)}{\beta(z)} B(z)^{n-2} dz \\ &\le \frac{\lambda^n}{(n-1)!\beta(x)} \int_0^x \frac{(n-1)B(z)^{n-2}}{\beta(z)} dz \;=\; \frac{\lambda^n}{(n-1)!\beta(x)} B(x)^{n-1}. \end{aligned}$$

from which (15) follows. Hence $B^*(x,0)$ is well defined and $B^*(x,0) \le \frac{\lambda}{\beta(x)} e^{\lambda B(x)}$.

Iterating $f_1(\cdot)$ for $0 < x \le a$ we get

$$f_1(x) = \pi_1 B^*(x,0). \tag{16}$$

(ii) The second equation of Lemma 4 ($a < x \le b$) becomes:

$$f_1(x) = \int_0^a B(x,w) f_1(w) dw + \int_a^x B(x,w) f_1(w) dw + \pi_1 B(x,0) - \frac{k}{\beta(x)}, \tag{17}$$

where

$$k = \beta(b) \int_0^b B(b,w) f_1(w) dw + \pi_1 \beta(b) B(b,0). \tag{18}$$

Let

$$h_\beta(x) = \begin{cases} \int_0^a B(x,w) f_1(w)dw + \pi_1 B(x,0) - \frac{k}{\beta(x)}, & a < x \le b, \\ 0, & 0 < x \le a. \end{cases}$$

Then $h_\beta(\cdot)$ is a known function except for the constants π_1 and k; note that only given functions and parameters and $f_1(w)$ restricted to $(0,a]$ appear in the definition of $h_\beta(\cdot)$, so that

$$f_1(x) = h_\beta(x) + \int_a^x B(x,w) f_1(w)dw, \quad a < x \le b, \tag{19}$$

is another Volterra integral equation of the second kind. $f_1(x)$ for $x \in (a,b]$ can be written as

$$\begin{aligned} f_1(x) &= h_\beta(x) + \int_a^x B(x,w) \left[h_\beta(w) + \int_a^w B(w,y) f_1(y)dy \right] dw \\ &= h_\beta(x) + \int_a^x B(x,w) h_\beta(w)dw + \int_a^x B^{(2)}(x,y) f_1(y)dy \\ &= \dots \\ &= h_\beta(x) + \sum_{n=1}^{\infty} \int_a^x B^{(n)}(x,w) h_\beta(w)dw = h_\beta(x) + \int_a^x B^*(x,w) h_\beta(w)\mathrm{d}w. \end{aligned}$$

(iii) The third equation of Lemma 4 becomes:

$$f_1(x) = \int_0^x C(x,w) f_1(w)dw + \pi_1 C(x,0) = s(x) + \int_b^x C(x,w) f_1(w)dw,$$

where $s(x) = \int_0^b C(x,w) f_1(w)dw + \pi_1 C(x,0)$. Similarly to (13) we define $C^{(1)}(x,y) = C(x,y)$, and

$$C^{(n+1)}(x,y) = \int_y^x C(x,z) C^{(n)}(z,y)dz, \qquad C^*(x,y) = \sum_{n=1}^{\infty} C^{(n)}(x,y). \tag{20}$$

Note that similar to the derivation of (15) we can show that $C^*(x,y)$ is well defined. Then the solution of $f_1(x)$ for $x > b$ is

$$f_1(x) = s(x) + \int_b^x C^*(x,w) s(w)dw. \tag{21}$$

We have determined $f_1(x)$ except for the constants π_1 and k. These unknowns can now be computed from the two equations

$$f_1(b-) = 0$$

and

$$\int_0^\infty f_1(x)dx = 1 - \pi_1.$$

Summarizing, we have found:

Theorem 5. *The density $f_1(\cdot)$ of the inventory content level during 1-periods is given by*

$$f_1(x) = \begin{cases} \pi_1 B^*(x,0), & 0 < x \leq a, \\ h_\beta(x) + \int_a^x B^*(x,w) h_\beta(w) \mathrm{d}w, & a < x \leq b, \\ s(x) + \int_b^x C^*(x,w) s(w) \mathrm{d}w, & x > b, \end{cases} \tag{22}$$

where the unknown constants π_1 and k (which also feature in $h_\beta(x)$) are determined as indicated above.

Now we turn to the determination of $f_2(x)$. The fourth equation of Lemma 4 becomes:

$$f_2(x) = \int_a^x C(x,w) f_2(w) dw + g(x),$$

where

$$g(x) = \frac{\gamma(a+) f_2(a+)}{\gamma(x)}. \tag{23}$$

The solution of this equation is

$$f_2(x) = g(x) + \int_a^x C^*(x,w) g(w) dw. \tag{24}$$

Finally consider the fifth equation of Lemma 4, which becomes:

$$f_2(x) = \int_a^x C(x,w) f_2(w) dw = \int_a^b C(x,w) f_2(w) dw + \int_b^x C(x,w) f_2(w) dw.$$

Using (24) it follows that

$$\begin{aligned} f_2(x) &= \int_a^b C(x,w)[g(w) + \int_a^w C^*(w,y) g(y) dy] dw + \int_b^x C(x,w) f_2(w) dw \\ &= h_\gamma(x) + \int_b^x C(x,w) f_2(w) dw \end{aligned}$$

where

$$h_\gamma(x) = \int_a^b C(x,w)[g(w) + \int_a^w C^*(w,y) g(y) dy] dw, \quad x > b.$$

Thus we get

$$f_2(x) = h_\gamma(x) + \int_b^x C^*(x,w) h_\gamma(w) dw.$$

Note that the unknown constant $f_2(a)$ is included in $g(x)$. However, it can be computed from the normalizing condition

$$\int_a^\infty f_2(x) dx = 1.$$

Summarizing, we have found:

Theorem 6. *The density $f_2(\cdot)$ of the inventory content level during 2-periods is given by*

$$f_2(x) = \begin{cases} g(x) + \int_a^x C^*(x,w)g(w)\mathrm{d}w, & a < x \le b, \\ h_\gamma(x) + \int_b^x C^*(x,w)h_\gamma(w)\mathrm{d}w, & x > b. \end{cases} \tag{25}$$

Remark 2. In order to obtain a regular probability density f_1 one still needs to show that under the stability condition (5) the solution f_1 is integrable, i.e. that

$$\begin{aligned} \int_{w_0}^{\infty} f_1(x)\,dx &= \int_{w_0}^{\infty} s(x)\,dx \\ &\quad + \int_{w_0}^{\infty}\int_b^{w_0} C^*(x,w)s(w)dw\,dx + \int_{w_0}^{\infty}\int_{w_0}^{x} C^*(x,w)s(w)dw\,dx \end{aligned} \tag{26}$$

is finite. Assuming that (5) holds there is a $w_0 > 0$ such that

$$\sup_{w \ge w_0} \int_w^{\infty} C(x,w)\,dx = c < 1. \tag{27}$$

The first integral in the right-hand side of (26) can be written as

$$\int_{w_0}^{\infty} s(x)\,dx = \int_0^b f_1(w)\int_{w_0}^{\infty} C(x,w)\,dx\,dw + \pi_1 \int_{w_0}^{\infty} C(x,0)\,dx.$$

Note that by definition $C(x,y) = \lambda(1 - G(A(x) - A(y)))/\gamma(x)$ is non-decreasing in y, implying that

$$\begin{aligned} &\int_0^b f_1(w)\int_{w_0}^{\infty} C(x,w)\,dx\,dw + \pi_1 \int_{w_0}^{\infty} C(x,0)\,dx \\ &\le \int_0^b f_1(w)\int_{w_0}^{\infty} C(x,w_0)\,dx\,dw + \pi_1 \int_{w_0}^{\infty} C(x,w_0)\,dx, \end{aligned}$$

which is finite due to (27). The second integral in (26) is equal to

$$\int_b^{w_0} s(w)\int_{w_0}^{\infty} C^*(x,w)\,dx\,dw \le \int_b^{w_0} s(w)\int_{w_0}^{\infty} C^*(x,w_0)\,dx\,dw,$$

and hence is finite, too. Finally, for the third integral in (26) note that since

$$\int_w^{\infty} C^{(2)}(x,w)\,dx = \int_w^{\infty}\int_w^{x} C(x,z)C(z,w)\,dz\,dx = \int_w^{\infty} C(z,w)\int_z^{\infty} C(x,z)\,dx\,dz,$$

it follows that $\sup_{w\ge w_0}\int_w^{\infty} C^{(2)}(x,w)\,dx < c^2$ and more generally (by induction) that $\sup_{w\ge w_0}\int_w^{\infty} C^{(n)}(x,w)\,dx < c^n$, implying

$$\sup_{w\ge w_0}\int_w^{\infty} C^*(x,w)\,dx < \frac{c}{1-c}.$$

Consequently

$$\int_{w_0}^{\infty} s(w)\int_w^{\infty} C^*(x,w)\,dx\;dw < \frac{c}{1-c}\int_{w_0}^{\infty} s(w)\;dw < \infty,$$

showing that f_1 is indeed a valid probability density.

Theorem 7. *Suppose that $\beta(x)$ and $\gamma(x)$ are continuous at $a(x)$ and $b(x)$. Then the densities $f_1(x)$ and $f_D(x)$ are discontinuous at $x = a$ and at $x = b$, while $f_2(x)$ is discontinuous at $x = b$.*

Proof. We have by Lemma 4

$$\begin{aligned} f_1(a) &= \frac{\lambda}{\beta(a)} \int_0^a \Gamma(w,a) f_1(w)\, dw + \frac{\lambda}{\beta(a)} \pi_1 \Gamma(0,a), \\ f_1(a+) &= f_1(a) - \frac{\gamma(b)}{\beta(a)} f_1(b+), \\ f_1(b) &= 0 \end{aligned}$$

and

$$f_1(b+) = \frac{\lambda}{\gamma(b)} \int_0^b \Gamma(w,b) f_1(w)\, dw + \frac{\lambda}{\gamma(b)} \pi_1 \Gamma(0,b).$$

It follows that $f_1(x)$ is discontinuous at $x = a$ iff $f_1(x)$ is discontinuous at $x = b$ iff $f_1(b+) \neq 0$, which follows from the fact that b is downcrossed once every cycle and $\gamma(b) < \infty$. For f_2 we obtain

$$\begin{aligned} f_2(b) &= \lambda \int_a^b \Gamma(w,b) f_2(w)\, dw + \gamma(a) f_2(a+), \\ f_2(b+) &= \lambda \int_a^b \Gamma(w,b) f_2(w)\, dw, \end{aligned}$$

so that f_2 is continuous at b iff $f_2(a+) = 0$. This is always true since a is downcrossed once every cycle and $\gamma(a) < \infty$ (alternatively use (11) and the fact that $f_1(b+) \neq 0$). Finally note that $f_D(x)$ is a weighted sum of $f_1(x)$ and $f_2(x)$. □

4 Exponential α-period

In this section we assume that the up periods are exponentially distributed, so that $G(x) = 1 - e^{-\mu x}$, $x > 0$. In this case we can obtain more explicit expressions for the various buffer level densities. Substituting $\Gamma(w,x) = e^{-\mu(A(x)-A(w))}$ into the results of Lemma 4 we obtain the integro-differential equation

$$f_1(x) = \begin{cases} \dfrac{\lambda}{\beta(x)} e^{-\mu A(x)} \displaystyle\int_0^x e^{\mu A(w)} f_1(w)\, dw + \dfrac{\lambda \pi_1}{\beta(x)} e^{-\mu A(x)}; & 0 < x \leq a, \\ \dfrac{\lambda}{\beta(x)} e^{-\mu A(x)} \displaystyle\int_0^x e^{\mu A(w)} f_1(w)\, dw + \dfrac{\lambda \pi_1}{\beta(x)} e^{-\mu A(x)} - \dfrac{k}{\beta(x)}, & a < x \leq b, \\ \dfrac{\lambda}{\gamma(x)} e^{-\mu A(x)} \displaystyle\int_0^x e^{\mu A(w)} f_1(w)\, dw + \dfrac{\lambda \pi_1}{\gamma(x)} e^{-\mu A(x)}, & x > b, \end{cases}$$

where the (yet) unknown constant (see also (18)) $k = \lambda \int_0^b e^{-\mu(A(b)-A(w))} dF_1(w)$.

Theorem 8. *The stationary density f_1 is given by*

$$f_1(x) = \begin{cases} \dfrac{\lambda\pi_1}{\beta(x)} e^{\lambda B(x) - \mu A(x)}, & 0 < x \le a, \\ \dfrac{1}{\beta(x)} \zeta(x), & a < x \le b, \\ \dfrac{1}{\gamma(x)} e^{\lambda(C(x)-C(b)) - \mu(A(x)-A(b))} \left(\zeta(b) + k\right), & x > b, \end{cases}$$

where

$$\zeta(x) = e^{\lambda B(x) - \mu A(x)} \left(\lambda\pi_1 - k e^{\mu A(a) - \lambda B(a)} - k\mu \int_a^x \frac{e^{\mu A(w) - \lambda B(w)}}{\alpha(w)} \, dw \right).$$

The density f_2 is given by

$$\begin{aligned} f_2(x) &= f_2(a+) \frac{\gamma(a+)}{\gamma(x)} e^{\lambda C(x) - \mu A(x)} \\ &\times \begin{cases} e^{\mu A(a) - \lambda C(a)} + \mu \displaystyle\int_a^x \frac{e^{\mu A(w) - \lambda C(w)}}{\alpha(w)} \, dw, & a < x \le b, \\ e^{\mu A(a) - \lambda C(a)} - e^{\mu A(b) - \lambda C(b)} + \mu \displaystyle\int_a^b \frac{e^{\mu A(w) - \lambda C(w)}}{\alpha(w)} \, dw, & x > b, \end{cases} \end{aligned}$$

where the constant $f_2(a+)$ follows from the normalization $\int_a^\infty f_2(u)\, du = 1$.

Proof of Theorem 8. Before we start, notice that for $0 < x \le y$

$$\int_x^y \frac{\lambda e^{\lambda A(w)}}{\alpha(w)} \, dw = e^{\lambda(A(y) - A(x))}$$

and likewise $\int_x^y \frac{\lambda e^{\lambda B(w)}}{\beta(w)} \, dw = e^{\lambda(B(y)-B(x))}$ and $\int_x^y \frac{\lambda e^{\lambda C(w)}}{\gamma(w)} \, dw = e^{\lambda(C(y)-C(x))}$. Let $h(x) = \beta(x) e^{\mu A(x)} f_1(x)$. Then

$$h(x) = \begin{cases} \lambda \displaystyle\int_0^x \frac{h(w)}{\beta(w)} \, dw + \lambda\pi_1; & 0 < x \le a, \\ \lambda \displaystyle\int_0^x \frac{h(w)}{\beta(w)} \, dw + \lambda\pi_1 - k e^{\mu A(x)}, & a < x \le b, \\ \lambda \dfrac{\beta(x)}{\gamma(x)} \displaystyle\int_0^x \frac{h(w)}{\beta(w)} \, dw + \dfrac{\lambda\pi_1 \beta(x)}{\gamma(x)}, & x > b. \end{cases} \tag{28}$$

It follows that $h(x) = \lambda\pi_1 e^{\lambda B(x)}$ for $0 < x \le a$. For $a < x \le b$ we obtain after substituting the above expression for $h(x)$, $0 < x \le a$, into the second line of (28),

$$\begin{aligned} h(x) &= \lambda\pi_1 \int_0^a \frac{\lambda e^{\lambda B(w)}}{\beta(w)} \, dw + \lambda \int_a^x \frac{h(w)}{\beta(w)} \, dw + \lambda\pi_1 - k e^{\mu A(x)} \\ &= \lambda\pi_1 e^{\lambda B(a)} - k e^{\mu A(x)} + \lambda \int_a^x \frac{h(w)}{\beta(w)} \, dw. \end{aligned}$$

It follows that $h'(x) = -k\mu e^{\mu A(x)}/\alpha(x) + \lambda h(x)/\beta(x)$ and hence

$$
\begin{aligned}
h(x) &= e^{\lambda(B(x)-B(a))}\left(h(a+) - k\int_a^x \frac{\mu}{\alpha(w)} e^{-\lambda(B(w)-B(a))+\mu A(w)}\,dw\right) \\
&= e^{\lambda B(x)}\left(\lambda\pi_1 - ke^{\mu A(a)-\lambda B(a)} - k\int_a^x \frac{\mu}{\alpha(w)} e^{\mu A(w)-\lambda B(w)}\,dw\right).
\end{aligned}
$$

Finally, plugging this into the third equation in (28) leads, for $x > b$, to

$$
\begin{aligned}
\frac{\gamma(x)}{\beta(x)} h(x) &= \lambda\pi_1 e^{\lambda B(a)} + \left(\lambda\pi_1 - ke^{\mu A(a)-\lambda B(a)}\right)\int_a^b \frac{\lambda e^{\lambda B(w)}}{\beta(w)}\,dw \\
&\quad - \lambda k\int_a^b \frac{e^{\lambda B(w)}}{\beta(w)}\int_a^w \frac{\mu}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\,du\,dw + \lambda\int_b^x \frac{h(w)}{\beta(w)}\,dw \\
&= \lambda\pi_1 e^{\lambda B(a)} + \left(\lambda\pi_1 - ke^{\mu A(a)-\lambda B(a)}\right)(e^{\lambda B(b)} - e^{\lambda B(a)}) \\
&\quad - k\int_a^b \frac{\mu}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\int_u^b \frac{\lambda e^{\lambda B(w)}}{\beta(w)}\,dw\,du + \lambda\int_b^x \frac{h(w)}{\beta(w)}\,dw.
\end{aligned}
$$

Letting $K(x) = h(x)\gamma(x)/\beta(x) = \gamma(x)e^{\mu A(x)} f_1(x)$ for $x > b$, we obtain

$$
K(x) = K(b+) + \lambda\int_b^x \frac{K(w)}{\gamma(w)}\,dw, \quad x > b,
$$

with

$$
K(b+) = e^{\lambda B(b)}\left(\lambda\pi_1 + ke^{\mu A(b)-\lambda B(b)} - ke^{\mu A(a)-\lambda B(a)} - k\int_a^b \frac{\mu}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\,du\right).
$$

It follows that $K(x) = K(b+)e^{\lambda(C(x)-C(b))}$ which in turns yields the desired result for the density $f_1(x)$.

For f_2 we have, according to Lemma 4,

$$
\gamma(x) f_2(x) = \begin{cases} \lambda e^{-\mu A(x)}\displaystyle\int_a^x e^{\mu A(w)} f_2(w)\,dw + \gamma(a+) f_2(a+), & a < x \le b, \\ \lambda e^{-\mu A(x)}\displaystyle\int_a^x e^{\mu A(w)} f_2(w)\,dw, & x > b. \end{cases}
$$

Letting $m(x) = \gamma(x)e^{\mu A(x)} f_2(x)$ we obtain for $a < x \le b$

$$
m(x) = m(a+)e^{\mu(A(x)-A(a))} + \lambda\int_a^x \frac{m(w)}{\gamma(w)}\,dw
$$

leading to

$$
m(x) = m(a+)e^{\lambda C(x)}\left(e^{-\lambda C(a)} + e^{-\mu A(a)}\int_a^x \frac{\mu}{\alpha(w)} e^{\mu A(w)-\lambda C(w)}\,dw\right).
$$

which results in the asserted equation for $f_2(x)$ for $a < x \leq b$. It follows that for $x > b$

$$m(x) = \lambda \int_a^b \frac{m(a+)e^{\lambda C(w)}}{\gamma(w)} \left(e^{-\lambda C(a)} + e^{-\mu A(a)} \int_a^w \frac{\mu}{\alpha(u)} e^{\mu A(u) - \lambda C(u)} \, du \right) dw$$
$$+ \lambda \int_b^x \frac{m(w)}{\gamma(w)} \, dw$$
$$= m(a+)(e^{\lambda(C(b)-C(a))} - 1) + \lambda m(a+) e^{-\mu A(a)} \int_a^b \frac{\mu}{\alpha(u)} e^{\mu A(u) - \lambda C(u)} \int_u^b \frac{e^{\lambda C(w)}}{\gamma(w)} \, dw \, du$$
$$+ \lambda \int_b^x \frac{m(w)}{\gamma(w)} \, dw = \hat{c} e^{\lambda C(b) - \mu A(a)} m(a+) + \lambda \int_b^x \frac{m(w)}{\gamma(w)} \, dw$$

where $\hat{c} = e^{\mu A(a) - \lambda C(a)} - e^{\mu A(b) - \lambda C(b)} + \int_a^b \frac{\mu}{\alpha(u)} e^{\mu A(u) - \lambda C(u)} \, du$. Consequently $m(x) = \hat{c} e^{\lambda C(b) - \mu A(a)} m(a+) e^{\lambda(C(x) - C(b))}$ and hence

$$f_2(x) = \hat{c} e^{\lambda C(b) - \mu A(a)} \frac{m(a+)}{\gamma(x)} e^{\lambda(C(x)-C(b)) - \mu A(x)} = \hat{c} f_2(a+) \frac{\gamma(a)}{\gamma(x)} e^{\lambda C(x) - \mu A(x)}. \square$$

5 Number of switches in a busy cycle

An important performance measure in the hysteresis model is the number of switches, N, from release rate $\beta(\cdot)$ to release rate $\gamma(\cdot)$ – which equals the number of switches from release rate $\gamma(\cdot)$ to release rate $\beta(\cdot)$. Indeed, there could be costs involved in such switches, and it is not attractive to have a large number of such switches in a short time interval. In this section we shall determine the probability distribution of N, for the case of exponentially distributed α-periods. In the analysis, a key role is played by $\hat{D}_{max}$, the cycle maximum (i.e., the largest value of the workload process in a busy cycle) of a related dam process $\hat{\boldsymbol{D}}$. We shall derive its distribution in Theorem 9. The distribution of N, which turns out to be a modified geometric distribution, then follows in a straightforward manner; it is given in Theorem 10. We make the following preparatory observations.
(i) Under the stability condition of Theorem 1, each time the mountain exceeds b, it will return below b, and eventually below a. When it returns below a, the release rate switches back from $\gamma(\cdot)$ to $\beta(\cdot)$. Subsequently, there is a fixed probability that the mountain reaches b again, before returning to 0. That probability does not depend on $\gamma(\cdot)$, because all this time until either b or 0 is reached, the rate is $\beta(\cdot)$ or $\alpha(\cdot)$. Note that we also use the memoryless property of the exponential(λ) down periods, allowing us to ignore the history before the mountain process crossed a from above. The above implies that the distribution of N, given that $N \geq 1$, is geometric; it also implies that the distribution of N does not depend on $\gamma(\cdot)$. To determine the parameter of the geometric distribution, we can therefore work with a slightly simpler mountain process $\hat{\boldsymbol{X}}$ *without* hysteresis. The only difference between $\hat{\boldsymbol{X}}$ and the original mountain process $\boldsymbol{X}$ is that in $\hat{\boldsymbol{X}}$ the release rate during down periods is *always* equal to $\beta(x)$ when $x < b$; the release rate during down periods still equals $\gamma(x)$ when $x \geq b$ (hence the stability

condition of Theorem 1 is satisfied).
(ii) In determining the parameter of the geometric distribution of the number N of switches in a cycle, it is crucial to have the distribution of the cycle maximum of $\hat{\boldsymbol{X}}$. Here cycle maximum means: highest point of the process during a busy cycle. In particular, we need to know whether the cycle maximum exceeds b. In [7] we have studied the cycle maximum of the mountain process $\hat{\boldsymbol{X}}$ (i.e., without hysteresis, and with exponential up- and down-periods). In [7] we modified the mountain process $\hat{\boldsymbol{X}}$ into a dam process $\hat{\boldsymbol{D}}$ by replacing the increments during the exponential(α) up periods by state-dependent jumps. We subsequently observed that the cycle maxima of $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{D}}$ coincide. In the next theorem we shall determine the distribution of this cycle maximum, using results from [7].

In what follows we denote by $\mathbb{P}_x(\cdot)$ a probability under the condition that the process starts in level x and is going down. We write $\mathbb{P}_x^{\uparrow}(\cdot)$ for a probability under the condition that the process starts in level x, but with an upward jump. We further denote the cycle maximum of process $\hat{\boldsymbol{D}}$ (and hence of process $\hat{\boldsymbol{X}}$) by $\hat{D}_{max}$.

Theorem 9. *Suppose that the α-periods have an exponential distribution. For $y > z$ the probability distribution of the cycle maximum $\hat{D}_{max}$ is given by*

$$\mathbb{P}_z^{\uparrow}(\hat{D}_{max} > y) = \frac{1 + \mu \int_0^z \frac{1}{\alpha(u)} e^{\mu A(u) - \lambda B(u)}\, du}{1 + \mu \int_0^y \frac{1}{\alpha(u)} e^{\mu A(u) - \lambda B(u)}\, du} \tag{29}$$

and

$$\mathbb{P}_z(\hat{D}_{max} > y) = \frac{1 + \mu \int_0^z \frac{1}{\alpha(u)} e^{\mu A(u) - \lambda B(u)}\, du - e^{\mu A(z) - \lambda B(z)}}{1 + \mu \int_0^y \frac{1}{\alpha(u)} e^{\mu A(u) - \lambda B(u)}\, du}. \tag{30}$$

Proof. Observe that the cycle maximum of $\hat{\boldsymbol{D}}$ occurs at a moment of a jump and it must be the last record value of the cycle. In other words, level x is the cycle maximum of $\hat{\boldsymbol{D}}$ iff it is a record value and in addition, after reaching level x the dam $\hat{\boldsymbol{D}}$ will reach level 0 before upcrossing x again. Let $\theta(x)$ be the probability of the latter event.

We have translated the problem of finding the distribution of the cycle maximum in the mountain process $\hat{\boldsymbol{X}}$ to that of finding the distribution of the cycle maximum $\hat{D}_{max}$ in a dam process with exponential up- and down periods. In Section 2 of [7] the following results were obtained:
(i) Let $r(x)$ be the hazard rate function of $\hat{D}_{max}$ at x. Since the α-periods are $\exp(\mu)$ distributed, $\mu\, dx/\alpha(x)$ is the infinitesimal probability that an arbitrary record value of $\hat{\boldsymbol{D}}$ lands in $[x, x + dx)$. But $\hat{D}_{max} \in [x, x + dx)$ if and only if the latter record value is the last record value in the busy period and the probability of the latter event is $\theta(x)$. By the strong Markov property, we find $r(x)$ by taking the product of $\mu/\alpha(x)$ and $\theta(x)$ and obtain

$$r(x) = \mu \frac{\theta(x)}{\alpha(x)}. \tag{31}$$

(ii) $\theta(x)$ is given by the following expression:

$$\theta(x) = \frac{e^{\mu A(x)-\lambda B(x)}}{1+\mu\int_0^x \frac{e^{\mu A(u)-\lambda B(u)}}{\alpha(u)}\,du}.$$

By (31) it now follows that

$$r(x) = \frac{\mu e^{\mu A(x)-\lambda B(x)}/\alpha(x)}{1+\mu\int_0^x \frac{e^{\mu A(u)-\lambda B(u)}}{\alpha(u)}\,du}.$$

It is easily seen that the numerator of the above expression is the derivative of the denominator. Hence, defining

$$W(x) = 1+\mu\int_0^x \frac{e^{\mu A(u)-\lambda B(u)}}{\alpha(u)}\,du,$$

we have:

$$r(x) = \frac{W'(x)}{W(x)}.$$

Since the hazard rate function r of $\hat{D}_{max}$ is independent of the starting point, it follows that $\mathbb{P}_z^{\uparrow}(\hat{D}_{max} > y) = e^{-\int_z^y r(u)du} = e^{-\int_z^y \log(W(u))'\,du} = W(z)/W(y)$, which leads to (29). Equation (30) follows from the fact that

$$\begin{aligned}
\mathbb{P}_z(\hat{D}_{max} > y) &= (1-\theta(z))\mathbb{P}_z^{\uparrow}(\hat{D}_{max} > y) \\
&= \mathbb{P}_z^{\uparrow}(\hat{D}_{max} > y)\left(1 - \frac{e^{\mu A(z)-\lambda B(z)}}{1+\mu\int_0^z \frac{1}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\,du}\right) \\
&= \frac{1+\mu\int_0^z \frac{1}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\,du - e^{\mu A(z)-\lambda B(z)}}{1+\mu\int_0^y \frac{1}{\alpha(u)} e^{\mu A(u)-\lambda B(u)}\,du}. \qquad (32)
\end{aligned}$$

Here the first equality is seen as follows: when the process starts at z and is going down, then with probability $1-\theta(z)$ it returns to z from below before reaching 0, and the probability of subsequently exceeding y is equal to the probability of eventually exceeding y if the process starts with a jump upward from z (because of the memoryless property of the exponential α (up) periods). □

Theorem 10. *Suppose that the α-periods have an exponential distribution. Then the number of switches in a busy cycle has distribution*

$$\mathbb{P}(N=n) = \begin{cases} \mathbb{P}_0^{\uparrow}(\hat{D}_{max} \le b); & n = 0, \\ \mathbb{P}_0^{\uparrow}(\hat{D}_{max} > b)\mathbb{P}_a(\hat{D}_{max} > b)^{n-1}\mathbb{P}_a(\hat{D}_{max} \le b); & n = 1,2,\dots \end{cases} \qquad (33)$$

Proof. $N = 0$ if the cycle maximum stays below b. If the cycle maximum exceeds b, there is at least one switch. The strong Markov property implies that N, conditionally on $N \ge 1$, is geometricially distributed, with "success" probability the probability $\mathbb{P}_a(\hat{D}_{max} \le b)$ that, after going below a, the process does not exceed b again before reaching 0. □

6 Total Up Period and Busy Period Analysis

In this section we restrict the attention to the case that $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. We also assume that the up periods (which are also the α-periods) are exponential(μ_α), the β-periods are exponential(λ_β) and the γ-periods are exponential(λ_γ).

Let T_M be the busy period of the mountain; this is the interval of time in which the mountain is strictly positive. During T_M the mountain alternates among its three possible components: the α-periods, the β-periods and the γ-periods. During a busy period of the mountain, T_α designates the duration of the α-periods, T_β the duration of the β-periods and T_γ the duration of the γ-periods, so that $T_M = T_\alpha + T_\beta + T_\gamma$. To construct T_α from T_M just take $\beta = \gamma \simeq \infty$ so that the decreasing slopes become negative jumps. Alternatively, by taking $\alpha \simeq \infty$ we construct an hysteretic dam with alternating release rates β and γ.

Let

$$\mathbb{E}[e^{-\omega T_M}], \mathbb{E}[e^{-\omega T_\alpha}], \mathbb{E}[e^{-\omega(T_\beta+T_\gamma)}]$$

be the Laplace transforms (LSTs) of T_M, the total up period T_α during T_M, and the total down period $T_\beta + T_\gamma$, respectively. It should be noted that

$$\mathbb{E}[e^{-\omega T_M}] \neq \mathbb{E}[e^{-\omega T_\alpha}]\mathbb{E}[e^{-\omega(T_\beta+T_\gamma)}],$$

since T_α and $T_\beta + T_\gamma$ are not independent. If $\alpha \simeq \infty$ the mountain can be interpreted as an hysteretic dam whose *wet period* analysis (including the LST $\mathbb{E}[e^{-\omega(T_\beta+T_\gamma)}]$) has already been analyzed in Bar-Lev and Perry (1993). Accordingly, for completeness, in this study we focus on the analysis of $\mathbb{E}[e^{-\omega T_\alpha}]$ (Subsection 6.1) and $\mathbb{E}[e^{-\omega T_M}]$ (Subsection 6.2).

6.1 Analysis of the Total Up Period

Our goal in this subsection is to determine this LST. Taking $x = 0$ then gives us the LST of the total up period. As was mentioned above, each decreasing slope (of either rate β or γ) becomes a negative jump. The negative jump sizes, obtained by the above construction, are either exponential(λ_β/β) or exponential(λ_γ/γ), respectively. As a result of the above construction some negative jumps in the α-period that downcross level a (but not all of them) start as exponential(λ_γ/γ) jumps and after a downcrossing of level a the undershoots below level a change their law and become exponential(λ_β/β).

Consider the production process $\boldsymbol{R}$ from Section 3. For any $0 \leq x < a$ define the stopping time

$$L = \inf\{t > 0 : R(t) \leq 0 \text{ or } R(t) = b\}. \tag{34}$$

Note that up to time L the negative jumps are exponential(λ_β/β) and independent.

We now introduce an equation based on a renewal argument:

$$\mathbb{E}_x[e^{-\omega T_\alpha}] = \mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}] + \mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}]\mathbb{E}_b[e^{-\omega T_\alpha}], \tag{35}$$

where the first term on the right of (35) says that if the event $\{R(L) = 0\}$ occurred, the α-period is terminated, but if the event $\{R(L) = b\}$ occurred, the jump sizes of the α-period after time L become exponential(λ_γ/γ) and are independent of L by the strong Markov property.

We are done when we have determined the three components of the right hand side of (35). The first two are given in [2]:

$$\mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}] = \frac{e^{-\xi_0(\omega)(b-x)} - e^{-\xi_1(\omega)(b-x)}}{\frac{\lambda_\beta/\beta}{\lambda_\beta/\beta+\xi_0(\omega)}e^{-\xi_0(\omega)b} - \frac{\lambda_\beta/\beta}{\lambda_\beta/\beta+\xi_0(\omega)}e^{-\xi_1(\omega)b}} \tag{36}$$

and

$$\mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}] = e^{-\xi_0(\omega)(b-x)} - \frac{\lambda_\beta/\beta}{\lambda_\beta/\beta+\xi_0(\omega)}e^{-\xi_0(\omega)b}\mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}], \tag{37}$$

where

$$\begin{aligned}\xi_0(\omega) &= \frac{(\mu_a/\alpha - \lambda_\beta/\beta + \omega) + \sqrt{(\mu_a/\alpha - \lambda_\beta/\beta + \omega)^2 + 4\omega\lambda_\beta/\beta}}{2},\\ \xi_1(\omega) &= \frac{(\mu_a/\alpha - \lambda_\beta/\beta + \omega) - \sqrt{(\mu_a/\alpha - \lambda_\beta/\beta + \omega)^2 + 4\omega\lambda_\beta/\beta}}{2}.\end{aligned}$$

We now turn to the third term, viz., $\mathbb{E}_b[e^{-\omega T_\alpha}]$. The first component of the time T_α, from the moment b is upcrossed, is the time τ_α, which is defined as follows.

On the event $\{R(L) = b\}$ let $L + \tau_\alpha = \inf\{t > 0 : R(t) < a\}$. Then, since $\{R(t) : 0 \le t \le \tau_\alpha\}$ can be interpreted as the *Attained Waiting Time* process of the $M/M/1$ queue (see, e.g., [1]) lifted at the origin to level x, τ_α can be interpreted as the busy period of a special $M/M/1$ queue with arrival rate λ_γ/γ and service rate μ_α/α (with $\lambda_\gamma/\gamma < \mu_\alpha/\alpha$) in which the first service of the busy period is $\frac{b-a}{\alpha} + Z_1$ where Z_1 is a regular service time, namely, Z_1 is exponential(μ_α/α). By the strong Markov property the latter busy period is the convolution of a regular busy period of the $M/M/1$ queue (with arrival rate λ_γ/γ and service rate μ_α/α in which $\lambda_\gamma/\gamma < \mu_\alpha/\alpha$) and a special busy period of the same queue in which the first service is the constant $\frac{b-a}{\alpha}$. Accordingly, we get

$$\mathbb{E}[e^{-\omega\tau_\alpha}] = \varphi_\alpha(\omega)e^{-\frac{(b-a)}{\alpha}[\omega+(\lambda_\gamma/\gamma)(1-\varphi_\alpha(\omega))]}$$

where $\varphi_\alpha(\omega)$ is the LST of the latter $M/M/1$ queue. That is

$$\varphi_\alpha(\omega) = \frac{(\lambda_\gamma/\gamma + \mu_\alpha/\alpha + \omega) + \sqrt{(\lambda_\gamma/\gamma + \mu_\alpha/\alpha + \omega)^2 - 4\lambda_\gamma\mu_\alpha/(\gamma\alpha)}}{2\lambda_\gamma/\gamma}. \tag{38}$$

Next we have

$$\mathbb{E}_b[e^{-\omega T_\alpha}] = \mathbb{E}[e^{-\omega\tau_\alpha}]\Big[e^{-(\lambda_\beta/\beta)a} + \int_{y=0}^{a}(\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega T_\alpha}]dy\Big]. \tag{39}$$

To understand the idea behind (39) note that the left hand side of (39) is the LST of the time from the moment that level b is reached until the end of the α-period. It is equal to the time until level a is downcrossed plus the residual time until level 0 is downcrossed. However, with probability $e^{-(\lambda_\beta/\beta)a}$ a downcrossing of level a is also a downcrossing of level 0, and the latter residual time is equal to 0. If a downcrossing of level a is not a downcrossing of level 0, then the negative jump after downcrossing of level a lands at some level y according to the density $(\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}$ and from that moment the residual time until the end of the busy period is $\mathbb{E}_y[e^{-\omega T_\alpha}]$.

To solve for $\mathbb{E}_y[e^{-\omega T_\alpha}]$ we set

$$K = \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega T_\alpha}]dy.$$

Now multiply both sides of (35) by $(\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}$ and integrate with respect to y on $(0,a)$. We get in (35)

$$\begin{aligned} K &= \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_x[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}]dy \\ &\quad +\mathbb{E}_b[e^{-\omega T_\alpha}]\cdot \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}]dy \end{aligned} \tag{40}$$

and substituting (39) in (40) we get

$$\begin{aligned} K &= \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}]dy \\ &\quad +\mathbb{E}[e^{-\omega \tau_\alpha}]\left[e^{-(\lambda_\beta/\beta)a}+K\right]\cdot \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}]dy. \end{aligned} \tag{41}$$

We use the notation

$$c_0 = \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}]dy, \tag{42}$$

and

$$c_b = \int_{y=0}^{a} (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a-y)}\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}]dy, \tag{43}$$

to get

$$K = \frac{c_0 + \mathbb{E}[e^{-\omega\tau_\alpha}]e^{-(\lambda_\beta/\beta)a}c_b}{1-\mathbb{E}[e^{-\omega\tau_\alpha}]c_b}. \tag{44}$$

To compute K we need to find c_0, c_b; we already have determined $\mathbb{E}[e^{-\omega\tau_\alpha}]$. c_0 and c_b follow by integrating over the functionals $\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=0\}}]$ and $\mathbb{E}_y[e^{-\omega L}\mathbb{1}_{\{R(L)=b\}}]$, which are given in (36) and (37). By (42)

$$c_0 = \frac{\lambda_\beta/\beta}{\lambda_\beta/\beta+\xi_0(\omega)}\left[e^{-\xi_0(\omega)(b-a)} - e^{-(\lambda_\beta/\beta)a-\xi_0(\omega)b}\right]$$

and by (43) we express c_b in terms of c_0. After some elementary algebra we get

$$c_b = \frac{(\lambda_\beta/\beta)e^{-\xi_0(\omega)(b-a)} - (\lambda_\beta/\beta)e^{-(\lambda_\beta/\beta)(a+\xi_0(\omega))} - (\lambda_\beta/\beta)e^{-\omega b}c_0}{\lambda_\beta/\beta + \xi_0(\omega)}.$$

Having determined c_0 and c_b we have found K, which constitutes the missing component in the right hand side of (39); and thus we have found $\mathbb{E}_b[e^{-\omega T_\alpha}]$, the only term that remained to be determined in the right hand side of (35).

6.2 The busy period of the mountain process

We are now in position to determine the LST $\mathbb{E}[e^{-\omega T_M}]$ of the busy period of the mountain process $\boldsymbol{X} = \{X(t) : t \geq 0\}$. For this, we exploit the LST $\mathbb{E}_x[e^{-\omega T_\alpha}]$ as an important building block. Notice that this LST is given by (35), where the three building blocks of that expression are all specified in the previous subsection. Define the stopping time

$$\vartheta_y = \inf\{t > 0 : X(t) = 0 \text{ or } X(t) = y\},$$

allowing y to be infinite, so that $T_M = \vartheta_\infty$. We use the abbreviation $\mathbb{E}_x^\uparrow[\cdot] = \mathbb{E}[\cdot \mid X(0) = x, X'(0) = \alpha]$ and $\mathbb{E}_x^\downarrow[\cdot] = E(\cdot \mid X(0) = x, X'(0) = -\beta)$, where $\mathbb{E}_x^\uparrow[\cdot]$ $(\mathbb{E}_x^\downarrow[\cdot])$, means that the starting point is x and the direction of the mountain at time 0 is up [down], so that the LST of the busy period of the mountain is $\mathbb{E}[e^{-\omega T_M}] = \mathbb{E}_0^\uparrow[e^{-\omega T_M}]$. Our goal in this subsection is to determine that LST. We first write

$$\begin{aligned} \mathbb{E}_0^\uparrow[e^{-\omega T_M}] &= \mathbb{E}_0^\uparrow[e^{-\omega\vartheta_a}\mathbb{1}_{\{X(\vartheta_a)=0\}}] + \mathbb{E}_0^\uparrow[e^{-\omega\vartheta_a}\mathbb{1}_{\{X(\vartheta_a)=a\}}]\mathbb{E}_a^\uparrow[e^{-\omega T_M}] \\ &= \mathbb{E}_0[e^{-\omega(1+\frac{\alpha}{\beta})L}\mathbb{1}_{\{R(L)=0\}}] + \mathbb{E}_0[e^{-\omega((1+\frac{\alpha}{\beta})L-\frac{a}{\beta})}\mathbb{1}_{\{R(L)=a\}}]\mathbb{E}_a^\uparrow[e^{-\omega T_M}]. \qquad (45) \end{aligned}$$

The right hand side of the first step of (45) says that, on the event $\{X(\vartheta_a) = 0\}$, T_M is equal to ϑ_a, and on the event $\{X(\vartheta_a) = a\}$, T_M is equal to ϑ_a plus the residual period it takes to the end of the busy period, where the starting state of the residual extra period is a. Also, by the strong Markov property, the latter extra period that starts at state a is independent of the time until level a is reached from state 0. The second step of (45) follows from the fact that on the event $\{X(\vartheta_a) = 0\}$ we get by construction of $\boldsymbol{R}$ from $\boldsymbol{X}$ that $\vartheta_a = (1 + \frac{\alpha}{\beta})L$ (on the event $\{R(L) = 0\}$) and on the event $\{X(\vartheta_a) = a\}$ we have $\vartheta_a = (1 + \frac{\alpha}{\beta})L - \frac{a}{\beta}$ (on the event $\{R(L) = a\}$).

There are three terms that we need to determine in the rightmost side of (45). The two functionals $\mathbb{E}_0[e^{-\omega(1+\frac{\alpha}{\beta})L}\mathbb{1}_{\{R(L)=0\}}]$ and $\mathbb{E}_0[e^{-\omega((1+\frac{\alpha}{\beta})L-\frac{a}{\beta})}\mathbb{1}_{\{R(L)=a\}}]$ are already given in (36) and in (37), respectively, but under a special replacement of parameters; y is replaced by 0 and b is replaced by a. It remains to determine $\mathbb{E}_a^\uparrow[e^{-\omega T_M}]$. We first write:

$$\begin{aligned} \mathbb{E}_a^\uparrow[e^{-\omega T_M}] &= \mathbb{E}_a^\uparrow[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=0\}}] \\ &\qquad + \mathbb{E}_a^\uparrow[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=b\}}]\mathbb{E}_b^\uparrow[-\omega\vartheta_a]\mathbb{E}_a^\downarrow[e^{-\omega T_M}] \\ &= \mathbb{E}_a[e^{-\omega((1+\frac{\alpha}{\beta})L-\frac{a}{\alpha})}\mathbb{1}_{\{R(L)=0\}}] \\ &\qquad + \mathbb{E}_a[e^{-\omega((1+\frac{\alpha}{\beta})L-\frac{b-a}{\beta})}\mathbb{1}_{\{R(L)=b\}}]\mathbb{E}_b^\uparrow[e^{-\omega\vartheta_a}]\mathbb{E}_a^\downarrow[e^{-\omega T_M}]. \qquad (46) \end{aligned}$$

The first step of (46) says that if at the origin the starting state is a and the direction is up then the stochastic behavior of T_M can evolve according to two disjoint events. On the event $\{X(\vartheta_b) = 0\}$ the mountain reaches level 0 before it reaches level b and $T_M = \vartheta_b$. On the event $\{X(\vartheta_b) = b\}$ the mountain reaches level b before it returns to level 0. Then, after reaching level b it takes time ϑ_a to go from level b down to level a and later it takes time T_M to go from level a down to level 0, where the direction of the mountain after reaching level a is down.

For the second step of (46) we use the fact that on the event $\{X(\vartheta_b) = 0\}$ the stopping time ϑ_b can be expressed as $(1 + \frac{\alpha}{\beta})L - \frac{a}{\alpha}$ on the event $\{R(L) = 0\}$ and on the event $\{X(\vartheta_b) = b\}$ the stopping time ϑ_b can be expressed as $((1 + \frac{\alpha}{\beta})L - \frac{b-a}{\beta})$ on the event $\{R(L) = b\}$.

It remains to determine the four terms in the rightmost side of (46). For the two functionals with indicator functions in the rightmost side of (46), we again refer to (36) and (37).

To compute the functional $\mathbb{E}_b^{\uparrow}[e^{-\omega\vartheta_a}]$ we first note that if during ϑ_a the α-periods are deleted and the down periods are glued together the mountain becomes a dam with the constant release rate γ. Let ϑ_a^* be the modified version of ϑ_a after the deletion of the α-periods. Then, ϑ_a^* can be also interpreted as the busy period of a special $M/M/1$ queue with arrival rate λ_γ/γ and service rate μ_α/α in which the first service of the busy period is equal to $b - a + Z$ where $Z \sim \exp(\mu_\alpha/\alpha)$ is a generic service.

Lemma 11. $\mathbb{E}_b^{\uparrow}[e^{-\omega\vartheta_a}] = e^{\omega\frac{b-a}{\alpha}}\varphi_\alpha(\omega(1+\frac{\gamma}{\alpha}))e^{-\frac{b-a}{\alpha}(\omega(1+\frac{\gamma}{\alpha})+(\lambda_\gamma/\gamma)(1-\varphi_\alpha((\omega(1+\frac{\gamma}{\alpha}))))}.$

Proof. If the first service of the dam (after deleting the α-periods from the mountain) is $b - a + Z$ it follows by the strong Markov property that the busy period can be expressed as a convolution of two busy periods; the first is the regular busy period of the $M/M/1$ queue and the second is a busy period whose first service is the constant $b - a$. We thus have

$$\mathbb{E}[e^{-\omega\vartheta_a^*}] = \varphi_\alpha(\omega)e^{-(b-a)([\omega+(\lambda_\gamma/\gamma)(1-\varphi_\alpha(\omega))])}$$

where $\varphi_\alpha(\omega)$ is given in (38). However, by construction we have

$$\vartheta_a = (1 + \frac{\gamma}{\alpha})\vartheta_a^* - \frac{b-a}{\alpha}.$$

Thus,

$$\mathbb{E}_b^{\uparrow}[e^{-\omega\vartheta_a}] = \mathbb{E}[e^{-\omega[(1+\frac{\gamma}{\alpha})\vartheta_a^*-\frac{b-a}{\alpha}]}]$$

and the lemma is proven. □

The functional $\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}]$ is the only remaining unknown term in the rightmost side of (46). For its determination we construct the renewal equation (in terms of LSTs)

$$\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}] = \mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=0\}}] + \mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=b\}}]\mathbb{E}_b^{\uparrow}[e^{-\omega\vartheta_a}]\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}]. \quad (47)$$

The argument used in (47) is similar to that used before. If the event $\{X(\vartheta_b) = 0\}$ occurs we have $T_M = \vartheta_b$. But if the event $\{X(\vartheta_b) = b\}$ occurs the mountain regenerates itself at state b and from this point the LST of time until the end of the busy period is $\mathbb{E}_b^{\uparrow}[e^{-\omega\tau_M}]\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}]$. Solving for $\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}]$ in (47) we get

$$\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}] = \frac{\mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=0\}}]}{1 - \mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=b\}}]\mathbb{E}_b^{\uparrow}[e^{-\omega\vartheta_a}]}. \tag{48}$$

To find the functionals $\mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=0\}}]$ and $\mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=b\}}]$, let $\boldsymbol{V} = \{V(t) : t \geq 0)\}$ be the work process of the $M/M/1$ queue with arrival rate λ_β/β and service rate μ_α/α and define the stopping time $L^* = \inf\{t > 0 : V(t) = 0 \quad or \quad V(t) \geq b\}$. Then (see [2]):

$$\mathbb{E}_A[e^{-\omega L^*}\mathbb{1}_{\{V(L^*)>b\}}] = \frac{e^{-\hat{\xi}_0(\omega)(b-a)} - e^{-\hat{\xi}_1(\omega)(b-a)}}{\frac{\mu_\alpha/\alpha}{\mu_\alpha/\alpha+\xi_0(\omega)}e^{-\hat{\xi}_0(\omega)b} - \frac{\mu_\alpha/\alpha}{\mu_\alpha/\alpha+\xi_0(\omega)}e^{-\hat{\xi}_1(\omega)b}}$$

where

$$\begin{aligned}
\hat{\xi}_0(\omega) &= \frac{-(\mu_a/\alpha - \lambda_\beta/\beta - \omega) + \sqrt{(\mu_a/\alpha - \lambda_\beta/\beta - \omega)^2 + 4\omega\mu_a/\alpha}}{2}, \\
\hat{\xi}_1(\omega) &= \frac{-(\mu_a/\alpha - \lambda_\beta/\beta - \omega) - \sqrt{(\mu_a/\alpha - \lambda_\beta/\beta - \omega)^2 + 4\omega\mu_a/\alpha}}{2},
\end{aligned}$$

and

$$\mathbb{E}_a[e^{-\omega L^*}\mathbb{1}_{\{V(L^*)=0\}}] = e^{-\hat{\xi}_0(\omega)(b-a)} - \frac{\mu_\alpha/\alpha}{\mu_\alpha/\alpha + \xi_0(\omega)}e^{-\hat{\xi}_0(\omega)b}\mathbb{E}_a[e^{-\omega L^*}\mathbb{1}_{\{V(L^*)>b\}}].$$

Finally, by a similar argument used above we have

$$\mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=0\}}] = \mathbb{E}_a^{\downarrow}[e^{-\omega[(1+\frac{\alpha}{\beta})L^* - \frac{b-a}{\alpha}]}\mathbb{1}_{\{V(L^*)=0\}}]$$

and

$$\mathbb{E}_a^{\downarrow}[e^{-\omega\vartheta_b}\mathbb{1}_{\{X(\vartheta_b)=b\}}] = \mathbb{E}_a^{\downarrow}[e^{-\omega[(1+\frac{\alpha}{\beta})L^* - \frac{b-a}{\beta}]}\mathbb{1}_{\{V(L^*)>b\}}].$$

Substitution in (48) yields $\mathbb{E}_a^{\downarrow}[e^{-\omega T_M}]$. Substituting the result in (46), we finally obtain the law of the busy period of the mountain.

7 Overshoot above level *b*

In the previous section, we determined the LST of the busy period and of the total up period in a busy period. We made the simplifying assumption that the α-periods are exponentially distributed. In such a case, the overshoot above level b is also exponential. In the present section, we determine the distribution of the overshoot of the mountain above level b in the case of general (not necessarily exponential) α-periods. That overshoot distribution is of importance when one wants to study, e.g., the distribution of the time above level b, or the time from an upcrossing above b until for the first time level a is reached again. It is the first step towards determining the busy period LST for generally distributed α-periods; however, such a determination is extremely cumbersome, and falls outside the scope of the present paper.

By construction, the overshoot of the mountain above level b is equal to the (only) overshoot of $\boldsymbol{D_1}$ above the same level b in a 1-period. Accordingly, we define the stopping time $T = \inf\{t > 0 : D_1(t) \geq b\}$ and construct a modified regenerative process $\tilde{\boldsymbol{D}}_{\boldsymbol{1}} = \{\tilde{D}_1(t) : t \geq 0\}$ such that $\{\tilde{D}_1(t), 0 \leq t \leq T\} = \{D_1(t), 0 \leq t \leq T\}$, but after the upcrossing of level b by $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ the arrival process is stopped and the release rate is equal to 1. Then, $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ is a regenerative process whose cycle is terminated whenever $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ reaches level b. That is, up to time T the sample paths of $\boldsymbol{D_1}$ and $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ are the same, however, after T the process $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ decreases at rate one until it reaches level b, so that the cycle is $T + R$ where R is interpreted as the overshoot of $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$ above level b.

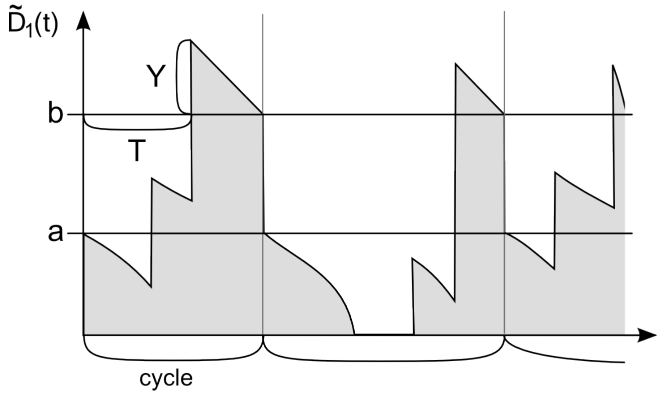

Fig.5: The process $\tilde{D}_1(t)$ corresponding to the process $X(t)$ given in Figure 1.

Let $\tilde{f}_1$ be the steady-state density of $\tilde{\boldsymbol{D}}_{\boldsymbol{1}}$, with distribution $\tilde{F}_1$. By LCT we obtain the following balance equations:

$$\beta(x)\tilde{f}_1(x) = \lambda \int_0^x \Gamma(w,x)\tilde{f}_1(w)\,dw + \lambda\tilde{\pi}_1\Gamma(0,x), \quad 0 < x \leq a,$$
$$\beta(x)\tilde{f}_1(x) = \lambda \int_0^x \Gamma(w,x)\tilde{f}_1(w)\,dw + \lambda\tilde{\pi}_1\Gamma(0,x) - \tilde{f}_1(b+), \quad a < x \leq b,$$
$$\tilde{f}_1(x) = \lambda \int_0^x \Gamma(w,x)\tilde{f}_1(w)\,dw + \lambda\tilde{\pi}_1\Gamma(0,x), \quad x > b,$$

where $\tilde{\pi}_1 = 1 - \int_0^\infty \tilde{f}_1(w)\,dw$.

Notice that the three above equations differ from those for $f_1(x)$ in Lemma 4 only marginally; another constant $\tilde{\pi}_1$ appears instead of π_1, and the equations for $x > b$ differ; $\gamma(x)$ is there replaced by 1. Hence we obtain almost identical formulas for $\tilde{f}_1(x)$

as for $f_1(x)$, for $x \leq b$. For $x > b$, the determination of $\tilde{f}_1(x)$ is slightly easier than that of $f_1(x)$. The function

$$\frac{\tilde{f}_1(x+b)}{\int_0^\infty \tilde{f}_1(x+b)\,dx}, \quad x > 0,$$

is the conditional density of $\tilde{\boldsymbol{D}}_{\mathbf{1}}$ given that $\tilde{\boldsymbol{D}}_{\mathbf{1}} > b$. Now, by deleting the time periods in which $\tilde{\boldsymbol{D}}_{\mathbf{1}} \leq b$ and gluing together the time periods in which $\tilde{\boldsymbol{D}}_{\mathbf{1}} > b$ we obtain a typical sample path of the *forward recurrence time* of a renewal process whose interrenewal times have the same distribution as Y. We designate the distribution of Y by H_Y. The proof of the next theorem makes use of the fact that

$$h_e(x) = \frac{1 - H_Y(x)}{\mathbb{E}[Y]}$$

where $h_e(\cdot)$ is interpreted as the equilibrium density associated with H_Y.

Theorem 12. *The distribution function of Y is given by*

$$H_Y(x) = 1 - \frac{\tilde{f}_1(x+b)}{\tilde{f}_1(b)}.$$

Proof. From the explanation above we have

$$\frac{\tilde{f}_1(x+b)}{\int_0^\infty \tilde{f}_1(u+b)du} = \frac{1 - H_Y(x)}{\mathbb{E}[Y]}. \tag{49}$$

Substituting $x = 0$ in (49) we get

$$\mathbb{E}[Y] = \frac{\int_0^\infty \tilde{f}_1(u+b)du}{\tilde{f}_1(b)}. \tag{50}$$

The theorem follows by substituting (50) into (49). □

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