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Stochastic inventory models

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Lecture Notes and Toolbox

for the course

Stochastic Operations Management

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Content:

1. Introduction

1.1 Why Lecture Notes?

This document is meant to be a supplement to standard textbooks on Inventory Management like Silver et al. (1998), Nahmias (2009), Cachon and Terwiesch (2012), Stevenson et al. (2011), Camm and Anderson (2011), Chopra and Meindl (2012) and Taylor (2012). Most standard textbooks make many simplifying assumptions when modeling an inventory system. These assumptions lead to relatively simple formulas, which can be easily implemented in Excel or which can be used in exercises and exams. The price to be paid for this simplicity is the fact that the assumptions are often not valid in practice, leading to the implementation of approximations and non-optimal solutions. The results presented in this document are based on fewer and more realistic assumptions when compared to standard textbooks. For different situations, for example when demand is discrete (integer numbers only) or when demand is continuous and follows a gamma or normal distribution, we derive exact expressions for key performance indicators (KPI's) of inventory systems. These expressions can be implemented in a spreadsheet using standard statistical functions in Excel. To support the implementation of these results we also provide the students of this course with an Excel-based toolbox.

Another reason for this document is the fact that inventory control systems in which ordering can only be done periodically (e.g., once every two weeks) and in (multiples of) a fixed batch-size are very common in practice but receive hardly any attention in standard textbooks. Most standard textbooks also focus on systems with continuous demand distributions (and sometimes predominantly on one distribution only: the normal distribution). The assumption of a continuous distribution is not appropriate if the average demand is very low (like in a spare parts environmnent) and the assumption of a normal distribution is not valid if the demand uncertainty is high.

In courses where the textbook of Nahmias (2009) is used, this document replaces sections 5.4 to 5.6. We suggest to read sections 5.1 to 5.3 from Nahmias (2009) before reading this reader to get a better perspective on Inventory Management in general and to get acquainted with the basic mathematical model in Stochastic Inventory Management: the newsboy model. This newsboy model considers a one-period inventory system, while in this document we focus on multi-period inventory systems.

1.2 Classification: (R, s, nQ), (R, s, S), (s, nQ) and (s, S)-control systems

A major classification of inventory control systems was introduced by Silver et al. (1998). To honour the fact that they explicitly recognized the so-called review period as a key decision variable we will use their classification and the associated notation throughout this document. Please note that this notation differs from the notation in Nahmias (2009) and other standard textbooks.

Before presenting their classification, we introduce an important concept called the inventory position. The inventory position in an inventory system is equal to the sum of the inventories in the system minus the backorders. Backorders are customer orders which could not be delivered from the inventory on hand and which will be delivered as soon as new inventory on hand becomes available. So the inventory position is negative if there is no inventory on hand in the system while there are backorders. The inventories in the system include both the scheduled receipts (replenishment orders which have been placed at the supplier in the past but that have not yet been delivered; the sum of all scheduled receipts in the system is also known as ‘goods in transit’ or ‘pipe-line inventory’) and the inventory on hand in the stock point.

Inventory control systems typically are based on the inventory position rather than on the inventory on hand. The reason for this is simple. If the inventory replenishment decision would be based on the inventory on hand and if it takes some periods before the replenishment order arrives (the so-called lead time), then after ordering the inventory on hand has not changed. Therefore a new replenishment order will be generated as soon as possible since the inventory on hand is still too low. By using the inventory position as the basis for the replenishment decision this error is prevented, since immediately after ordering the inventory position is raised with the quantity ordered. If the replenishment order arrives in the stock point, the inventory on hand is raised while the inventory position remains unchanged since the sum of all scheduled receipts decreased with the same quantity as the inventory on hand increased.

Silver et al. first differentiate inventory systems which are reviewed continuously to see whether a replenishment of the inventory is needed and systems which are reviewed only periodically. In practice in most situations a periodic review is applied, for example if delivery schedules are fixed (e.g., trucks visiting the warehouse only once every review period). The time between two moments when the inventory levels are reviewed is called the review period. Silver et al. use the capital letter R to denote the review period.

The second differentiation in inventory systems is based on the replenishment quantity. In some systems the inventory position is replenished in (integer multiples of) a fixed quantity, for example if items are delivered in case packs (e.g., a carton box containing 12 consumer units) or in full pallets. The capital letter Q is used to denote the fixed base replenishment quantity. In other systems the inventory position is replenished up to a fixed order-up-to level and as a result the replenishment quantity is variable, depending on the inventory position at the moment of ordering. Silver et al. use the capital letter S to denote the order-up-to level. In both systems with a fixed and a variable replenishment quantity the decision to replenish the inventory depends on whether the inventory position has dropped below a critical level called the reorder level, denoted by the small letter s.

Based on the two classification criteria discussed above, we have four basic inventory control systems, which are referred to as (R, s, nQ), (R, s, S), (s, nQ) and (s, S)-systems. If the system is reviewed continuously, the review period is zero and this is reflected in the notation by omitting the capital R (e.g., (s, S) versus (R, s, S)). In the (R, s, nQ) and the (s, nQ) control system the small letter n denotes the integer multiple of the fixed base replenishment quantity which is needed to raise the inventory position to or above the reorder level. These four control systems are depicted in the matrix in Figure 1 below:

	Periodic review	Continuous review
Fixed base replenishment quantity	(R, s, nQ)	(s, Q)
Variable replenishment quantity	(R, s, S)	(s, S)

Figure 1. Classification of inventory control systems.

To clarify the four control systems, below we will give a formal description of the replenishment logic used in the (R, s, nQ) and the (s, S) system.

The replenishment logic used in the (R, s, nQ) is as follows:

> *If at a review moment the inventory position is below the reorder level s, then n times Q units are ordered with n the minimum integer which is needed to bring the inventory position after ordering back to or above the reorder level s.*

The replenishment logic used in the (s, S) system is as follows:

> *As soon as the inventory position drops below the reorder level s, the amount of units is ordered which is needed to bring the inventory position after ordering back to the order-up-to level S.*

1.3 Focus of the Lecture Notes

In this document we focus on (R, s, nQ)-systems, since this type of system is encountered very often in practice. In practice frequently some type of coordination in replenishment of inventories is desirable. For example when a group of items is produced simultaneously to gain economies of scale, e.g., at the metal press factory at DAF Trucks items are produced in family groups, since changing production from one group to another requires very large set-up times while changing between items within the family group requires little set-up time. Or the coordination is desirable when items are delivered periodically due to a fixed transportation schedule. For example, supermarkets are typically having a fixed ‘ordering and delivery’ schedule to be able to balance workload in the DC and to make the workload in the supermarkets predictable so that managers can schedule and select the workforce on a medium term rather than on the short term and offer employees fixed working-hours. Also when items are shipped from the same supplier, when items are using the same transportation mode, or when a supplier offers a discount when the total volume ordered for a set of items at the same time exceeds a certain total volume, coordinated replenishment is preferable. The main advantage of continuous review systems is the fact that they require less safety stock to offer the same customer service. Yet this advantage is limited, since the review period in a periodic review system can be given a very small value to reap most of these benefits. As a matter of fact, by selecting an extremely small value for the review period when applying the exact results for periodic systems presented in this document, virtually exact approximations will be available for continuous systems.

The fact that ordering in the (R, s, nQ)-systems is done in (multiples of) a fixed base replenishment quantity is also in line with the actual practice in many companies and industries. In retail for example fixed case packs are used. This is done primarily to enable efficient handling of goods, but it also helps to protect the goods against damage and to reduce errors in the issuance or administration of goods. In warehouses goods are very often replenished in full pallets or in full layers of a pallet, again to make the handling of goods more efficient: handling a pallet with one carton box on it requires the same amount of time as handling a full pallet. Although it has been proven in inventory theory that order-up-to systems like the (s, S)-system result in lower total average costs than systems using fixed base replenishment quantities, it is important to recognize that these proofs are given for models that do not include the handling efficiencies described above in their objective function.

Since the objective function or constraints may be different in each situation in practice, we will not use a single objective function in this paper. Rather we will derive expressions for five Key Performance Indicators (KPI’s) which are important and flexible building blocks when making a model for a specific situation in practice. In this way the results in this document can be applied in a wide range of situations rather than only in situations where the single objective function would be applicable.

1.4 Definition of variables

Apart from the parameters specifying the inventory control systems (R, Q, n, s and S), we will use the following variables in this document:

***D*(*t*₁, *t*₂)**

The demand during time interval $(t_1, t_2]$. If demand is stationary (i.e. time-invariant), then the demand during time interval $(t_1, t_2]$ is also denoted by D_t.

L

The lead time is the time which elapses from the moment a replenishment order is created until the moment the replenishment order arrives in the stock point.

There are different types of inventory that are relevant when analyzing an inventory system. Below we give the notation and the definition for each of them. When analyzing an inventory system we will typically consider the behavior of the system during a single arbitrary review period. We denote the review moment at the beginning of this review period as τ. If a replenishment order is generated at this review moment, this order will arrive L (the lead time) periods later, so at time $\tau + L$, in the stockpoint and will then be added to the inventory on hand (the physical inventory in the stockpoint). The first review moment after time τ will be at time $\tau + R$. If a replenishment order is placed at this review moment, this order will arrive in the stockpoint at time $\tau + R + L$. For now, we assume all orders have a deterministic constant lead time. This assumption will be relaxed later on in this document.

We define the **review cycle** as the time interval $(\tau, \tau + R)$ and the **potential delivery cycle** as the time interval $(\tau + L, \tau + R + L)$. The word potential is added since not every review period a replenishment order will be placed. Especially if Q is much larger than the average demand during a review period, there will be many review moments when the inventory position is well above the reorder level and no replenishment order is generated.

Figure 2 shows a sample path of a (R, s, nQ)-system having a relatively small base replenishment quantity Q. At time τ, the first review moment occurs. Because the inventory position at time τ is (strictly) below the reorder level $s = 22$, a replenishment order is placed. The order size is 12, because one batch is sufficient to bring the inventory position at or above the reorder level. Because the lead time is 1 time unit, this order is delivered at time $\tau + 1$, when it becomes available as inventory on hand. Between time τ and time $\tau + 1$, the inventory position is higher than the inventory on hand, because it includes these scheduled receipts. Although the inventory position at time $\tau + 1$ and time $\tau + 2$ is below the reorder level, no orders are placed because these are no review moments. In the interval $(\tau + 2, \tau + 3]$ more demand occurs than there is inventory on hand left, leading to outstanding backorders. At time $\tau + 3$, a new review moment occurs. The minimum number of batches needed to get the inventory position at or above the minimum level is 3; in total 36 items are ordered. This is immediately reflected in the inventory position. When the scheduled receipts are delivered at time $\tau + 4$, the outstanding customer demand is met first, after which the remainder of the ordered items becomes available as inventory on hand.

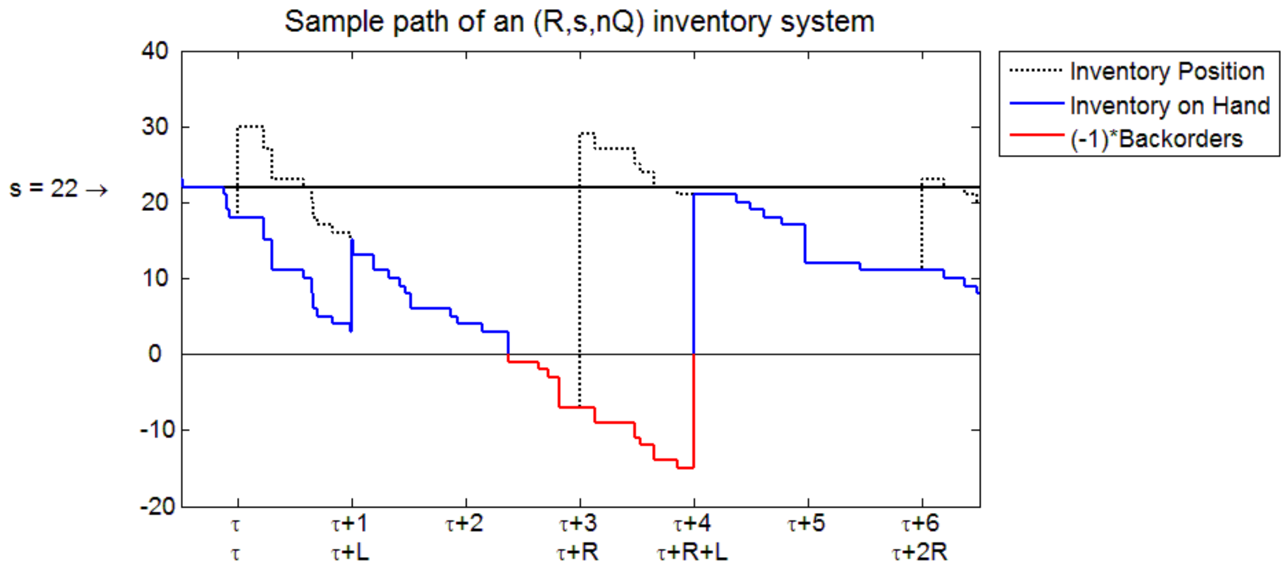


Figure 2. A sample path of the inventory position, the inventory on hand, and the backorders for an item under a (R, s, nQ) inventory system, where the parameters are given by $R = 3$, $s = 22$, and $Q = 12$. For this item, the demand during an interval $(t, t + 1]$ follows a discrete uniform distribution on $\{0, \ldots, 15\}$ and the lead time is $L = 1$.

If the lead time is equal to an integer multiple of the review period, at some points in time a delivery and review moment will coincide. For those situations we have to make assumptions on the **order of events** in the system. In those cases we will assume that demand occurs during the review period, the replenishment decision takes place at the end of the review period and the delivery is made immediately after the replenishment decision. This sequencing of events (delivery after the replenishment decision) enables us to also model a situation with zero lead time.

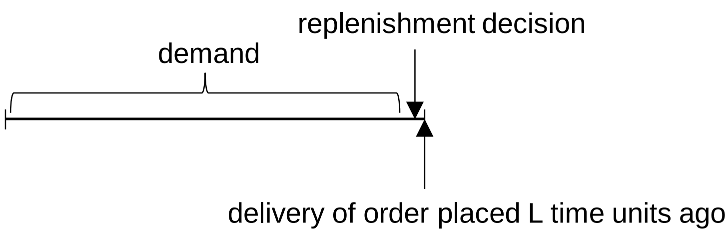


Figure 3. The order of events if the end of the period is a review moment.

Sometimes an item is stored at multiple locations, e.g. in the DC and multiple stores. When analyzing the inventory typically an item at a single specific location is considered. To uniquely identify the combination of an item and the location the term

Stock Keeping Unit (SKU) is being used. A Stock Keeping Unit is defined (Silver et al. (1998)) as 'an item of stock that is completely specified as to function, style, size, color, and, usually, location'. For convenience we will mostly be using the term item rather than SKU in these lecture notes if there is no danger of misinterpretation.
We define the following inventory-related variables:

$\boldsymbol{I^{OH}(t)}$
The inventory on hand in the stock point at time *t*. In the analyses later on in this document we will typically consider the inventory on hand levels at two moments: at the beginning of the potential delivery cycle just *after* a potential delivery (when the inventory on hand is at the highest level during the cycle) and at the end of the potential delivery cycle just *before* a potential delivery (when the inventory on hand is at the lowest level during the cycle). We will denote these two inventory on hand levels with the variables $I^{OH}(\tau + L)$ and $I^{OH}(\tau + R + L)$.

$\boldsymbol{IP(t)}$
The inventory position of the system at time *t*. The inventory position is equal to the sum of all inventories in the system minus the backorders. The inventories in the system include both the scheduled receipts (replenishment orders which have been placed at the supplier in the past but which have not yet been delivered) and the inventory on hand in the stock point. Again we consider the inventory position at two moments: at the beginning of the review cycle just *after* a potential replenishment order has been placed at the supplier and at the end of the review cycle just *before* a potential replenishment order has been placed. We will denote these two inventory position levels with the variables $IP(\tau)$ and $IP(\tau + R)$.

$\boldsymbol{BO(t)}$
The backorders outstanding at time *t*. Backorders outstanding (in consumer units) are equal to the customer demand which has not been delivered yet due to an out of stock situation. We will consider the number of backorders at two moments: at the beginning of the potential delivery cycle just *after* a potential delivery (when the backorders are at the lowest level during the cycle) and at the end of the potential delivery cycle just *before* a potential delivery (when the backorders are at the highest level during the cycle). We will denote these two backorder levels with the variables $BO(\tau + L)$ and $BO(\tau + R + L)$.

Above both the inventory on hand and the backorders are defined at a particular point in time (*t*). In the analysis below we will also be interested in the average inventory on hand during the potential delivery cycle and the number of new backorders generated during the potential delivery cycle. So these two variables are defined for a time interval rather than at a specific point in time. A notation which clearly reflects the relevant time interval could have been $I^{OH}(\tau + L, \tau + R + L)$ and

$BO(\tau + L, \tau + R + L)$, but for ease of notation we will use the notation I^{OH} and BO for these two variables, i.e. *without* a time variable attached to it.

When analyzing an inventory system we will consider five Key Performance Indicators (KPI's) which are relevant in practice: the expected inventory on hand, the expected number of orderlines per review period, the expected order size, the fill rate and the discrete ready rate. We use the following notation and definitions for these five KPI's:

$\boldsymbol{E}[\boldsymbol{I^{OH}}(\boldsymbol{\tau} + \boldsymbol{t})]$
The expected inventory on hand at the beginning of an arbitrary potential delivery cycle just after a potential delivery (if $t = L$) or at the end of an arbitrary potential delivery cycle just before a potential delivery (if $t = L + R$). Traditionally in the scientific literature on periodic review systems the expected inventory on hand at $t = L + R$ is used as the KPI for the inventory on hand (see Äxsater (2010)). However, especially when the review period is very large compared to Q, it may be better to also take into account the expected inventory on hand at the beginning of the delivery cycle, for example by taking the average of $E[I^{OH}(\tau + L)]$ and $E[I^{OH}(\tau + R + L)]$ as the KPI for the inventory on hand. It should be noted that in this case the resulting KPI is an approximation for the average inventory on hand (when measured on a continuous time scale) although this approximation is very good if the fill rate is high.

$\boldsymbol{E}[\boldsymbol{OL}]$
The expected number of order lines per review period.

$\boldsymbol{E}[\boldsymbol{OS}]$
The expected order size, which is the long term average quantity ordered (averaged over all replenishment orders).

$\boldsymbol{P_2}$
The long-term fraction of demand delivered immediately from stock, also known as the fill rate. This is in many situations the preferred way to define the customer service. While this definition is very useful to set reorder levels in many situations, it should be noted that in environments where demand is not registered but only sales (like in retail), the fill rate which is actually achieved in practice cannot be measured.

$\boldsymbol{P_3^{discrete}}$
The probability that we have positive inventory on hand just before a potential delivery moment, hereafter referred to as the discrete ready rate. One way to measure the customer service, in supermarkets for example, is to register for every stock keeping unit (SKU) whether the inventory on hand is equal to zero just before a potential delivery moment. Silver et al. (1998) define a measure which is similar to the discrete ready rate. While the discrete ready rate used in this document

measures the availability at specific discrete moments in time, their measure (the so-called ready rate P_3, which is defined as the fraction of time the inventory on hand is positive) measures the product availability continuously over time. Due to inaccuracy of inventory data, retailers prefer to do a physical check on empty shelves periodically rather than to rely on the continuous monitoring of inventory data in their computer systems when they want to assess the true customer service they provide.

Silver et al. (1998) introduce another product availability measure, called P_1, which is defined as the probability that the inventory on hand is positive just before an actual delivery takes place. The reason that this measure is cited in almost any textbook is due to the nice and simple analytical expressions for the reorder level which can be derived when using this measure. Two major drawbacks of using this measure are that in most situations it does not reflect the product availability as perceived by the customers and it does not take into account the positive effect of large lot-sizes or large review periods on the product availability. To see that P_1 is not a good reflection of the product availability as perceived by customers, and to better understand why this measure will not be discussed in this document, consider an item with the following inventory and demand data (see Table 1): starting inventory on hand at the beginning of week 1, just after a replenishment order arrived at the end of week 0, is equal to 99 units. Demand per week is equal to 20 units and the next replenishment order will arrive at the end of week 5.

Table 1: An example to illustrate the difference between service measures.

Weeknr.	Demand	End of week inventory before potential delivery
1	20 units	79 units
2	20 units	59 units
3	20 units	39 units
4	20 units	19 units
5	20 units	0 units; 1 unit backorder

Based on these data the fill rate (P_2) is 99% (99 units out of 100 units were delivered immediately from the inventory on hand), while in this example P_1, the probability that the inventory on hand is positive just before an actual delivery takes place, is equal to zero. Clearly, by no means does this reflect the true product availability perceived by the customers. Especially in situations with large lot-sizes we find that the fill rate P_2 is larger than P_1. In those situations the lot-sizing inventory partially takes over the role of the safety stock. Note that the discrete ready rate $P_3^{discrete}$ is similar to P_1 when the review period is so large that a replenishment order is generated each review period. In situations with large lot-sizes but small review periods (e.g. if in the example above the review period would be one day), the discrete $P_3^{discrete}$ is more similar to the fill rate than P_1.

1.5 Generic expressions for KPI's

To find an expression for the five KPI's introduced in Section 1.4, we will analyze the inventory system during an arbitrary review period. We will make three assumptions which will be relaxed later on:

1. The lead time is constant and deterministic.
2. Demand which is not met from inventory on hand immediately is backordered.
3. Products are non-perishable.

An important building block for the analyses made in this document is the result found by Hadley and Whitin in 1963, who derived the probability density function for the inventory position at an arbitrary review period. Although with this result the inventory position at the arbitrary review moment τ is known, this information is not sufficient to determine the inventory on hand at that moment: part of the inventory position may be scheduled receipts and part may be inventory on hand. However, we can find an equation for the inventory on hand when we look L periods later: at that moment ($\tau + L$) just after the potential delivery (whether there is an actual delivery at time $\tau + L$ depends on whether or not in period τ a replenishment order has been placed) all inventory which was in transit at time τ has entered the warehouse and is now part of the inventory on hand. In the meantime however, the inventory also decreased due to the demand during time interval $(\tau, \tau + L)$. When we assume that demand, which cannot be satisfied from the inventory on hand immediately, is backordered, we then have the following expression for the inventory on hand at time $\tau + L$

$$I^{OH}(\tau + L) = \max(IP(\tau) - D(\tau, \tau + L), 0) = (IP(\tau) - D(\tau, \tau + L))^+.$$

For ease of notation we will use the notation x^+ to denote $\max(0, x)$ throughout this paper.

Since the distribution function for IP as well as D is known, the expression above fully specifies the distribution function for the inventory on hand just after a potential delivery. Likewise we can derive an expression for the inventory on hand just *before* the next potential delivery moment, at time $\tau + R + L$. Note that the first review moment after τ is at time $\tau + R$ and an order placed at this time will not have arrived in the warehouse at time $\tau + R + L$, if we look at the system just before this potential delivery. Since no extra orders have arrived in the warehouse after time $\tau + L$ and up to the moment just before $\tau + R + L$, while there may have been demand during the interval $(\tau + L, \tau + R + L)$ we know that

$I^{OH}(\tau + R + L) = (I^{OH}(\tau + L) - D(\tau + L, \tau + R + L))^+ = (IP(\tau) - D(\tau, \tau + R + L))^+$.

So the following expression holds for $t = L$ as well as for $t = L + R$

$$I^{OH}(\tau + t) = (IP(\tau) - D(\tau, \tau + t))^+. \quad (1.1)$$

Note that the inventory on hand is highest at the beginning of the potential delivery cycle, $I^{OH}(\tau + L)$, and lowest at the end of this cycle, $I^{OH}(\tau + R + L)$. See Figure 2.

Now the expected inventory on hand at time t is simply equal to

$$E[I^{OH}(\tau + t)] = E[(IP(\tau) - D(\tau, \tau + t))^+]. \quad (1.2)$$

It is left to the reader as an exercise to derive the following formulas for the number of backorders at time $\tau + L$ and at time $\tau + R + L$ using exactly the same line of reasoning as used to derive the formulas for the expected inventory on hand.

$$BO(\tau + L) = \max(D(\tau, \tau + L) - IP(\tau), 0) = (D(\tau, \tau + L) - IP(\tau))^+$$
$$BO(\tau + R + L) = (D(\tau, \tau + R + L) - IP(\tau))^+ .$$

So we have for $t = L$ and $t = R + L$

$$BO(\tau + t) = (D(\tau, \tau + t) - IP(\tau))^+ . \quad (1.3)$$

For BO, the number of extra backorders in the system during the time interval $(\tau + L, \tau + R + L)$, i.e. the backorders which were not yet present at time $\tau + L$ but which were present at time $\tau + R + L$, we can derive the following expression

$$BO = BO(\tau + R + L) - BO(\tau + L).$$

The expressions for $I^{OH}(t)$ and $BO(t)$ derived above can be used to determine the fill rate P_2 and the discrete ready rate P_3^{discr}. The fill rate is equal to the fraction of demand delivered from stock immediately. So $1 - P_2$ is equal to the fraction of demand which has been backordered. When considering a potential delivery cycle this is equal to the expected extra backorders in the time interval $(\tau + L, \tau + R + L)$ divided by the average demand during this interval

$$P_2 = 1 - \frac{E[BO]}{E[D(\tau + L, \tau + R + L)]} = 1 - \frac{E[BO(\tau + R + L)] - E[BO(\tau + L)]}{E[D(\tau + L, \tau + R + L)]} \quad (1.4)$$

$$= 1 - \frac{E[\{D(\tau, \tau + R + L) - IP(\tau)\}^+] - E[\{D(\tau, \tau + L) - IP(\tau)\}^+]}{E[D(\tau + L, \tau + R + L)]}.$$

The discrete ready rate is equal to the probability that at the end of a potential delivery cycle just before delivery the inventory on hand is positive. So

$$P_3^{discrete} = P(I^{OH}(\tau + R + L) > 0) = P(\{IP(\tau) - D(\tau, \tau + R + L)\}^+ > 0)$$

$$= P(IP(\tau) > D(\tau, \tau + R + L)). \qquad (1.5)$$

The expected number of orderlines per review cycle is simply equal to the probability that a replenishment order is generated at a review moment. This is equal to the probability that at period $\tau + R$ the inventory position just *before* a replenishment decision is less than the reorder level. This inventory position equals the inventory position at period τ just *after* the replenishment decision minus the demand during the review period. As a result we have

$$E[OL] = P(IP(\tau) - D(\tau, \tau + R) < s). \qquad (1.6)$$

Since in the long run the expected supplied quantity should be equal to the expected demand during a review period, and since the expected supply quantity is equal to the expected order size per replenishment order times the probability a replenishment order is generated during a review period, we have

$$E[D(\tau, \tau + R)] = E[OS] \cdot E[OL]$$

and therefore

$$E[OS] = \frac{E[D(\tau, \tau + R)]}{E[OL]}. \qquad (1.7)$$

In this chapter we have derived generic expressions for all five KPI's in terms of the stochastic variables IP and/or D. These expressions for an inventory control system with backordering are generic since they hold for any demand probability distribution and for any periodic review replenishment logic (so not only for a (R, s, nQ)-control system, but also for (R, s, S)- and other control systems). We can make these expressions more specific when making assumptions on the demand distribution and the replenishment logic used in the inventory control system. These more specific expressions will depend on whether demand distributions are assumed to be discrete or continuous. Chapter 2 will focus on systems with discrete demand distributions, while chapter 3 will focus on systems with continuous demand distributions. In both chapters it is assumed that a (R, s, nQ)-control system is applied. Alternative control systems are discussed in section 4.3.

2. Discrete demand distribution

In this chapter specific expressions for the five KPI's are derived for systems in which demand and supply quantities can only be integers. In other words, we only consider discrete demand distributions in this chapter.

2.1 Expressions for KPI's when demand is discrete

Throughout this paper we assume that in a (R, s, nQ)-system a replenishment order is generated if at a review moment the inventory position is strictly below the reorder level ($IP < s$ rather than $IP \leq s$). The reason for this definition of a reorder level is that this definition is aligned with terminology used in practice. In practice sometimes companies work with a so-called min-max rule. The terms min and max are used to indicate that the inventory position should be equal to at least the min level and at most the max level. This implies that they will only order if the inventory position is strictly below the min level. Note that in commercial software for inventory control sometimes it is assumed that an order is generated when the inventory position is at or below a reorder level s. In those cases the reorder levels derived in this document should be decreased by one unit (if demand and supply are discrete) when comparing them with reorder levels used in commercial software.

Given this definition of the reorder level and the replenishment logic of the (R, s, nQ) control system, the inventory position just after a potential ordering moment is always larger than $s - 1$ and less than or equal to $s - 1 + Q$ if demand and supply are in discrete numbers. Hadley and Whitin (1963) have proven that for a (R, s, nQ)-system with backordering the inventory position at an *arbitrary* review moment just after the potential ordering moment follows a discrete uniform distribution on the set $\{s, s + 1, \ldots, s - 1 + Q\}$ if demand and supply is in discrete numbers.

Note that Hadley and Whitin's result does not hold for lost sales systems. This can be shown easily with a counterexample: consider a lost sales (R, s, nQ)-system with s=3, Q=20, L=0 days, R=1 day and demand per day equal to 25 with probability 0.4 and equal to 26 with probability 0.6. In this system the inventory position at every review moment (after demand has occurred and before a replenishment order is placed) will be equal to 0. So the inventory position at every review period after a replenishment order is placed will be equal to 20.

The derivation of the results below assumes the reader is capable to determine the expectation of the function of one or two stochastic variables. For those who are less familiar with this part of stochastic theory, the short summary and explanation given in Appendix 1 is strongly recommended.

Let's first derive an expression for the expected inventory on hand at time t, with $t = L$ or $t = L + R$. From (1.2) in Section 1.5 we know

$$E[I^{OH}(\tau + t)] = E[(IP(\tau) - D(\tau, \tau + t))^+].$$

Next, we first determine the expectation of the function of the two stochastic variables IP and D. Since $IP \sim u(s, s-1+Q)$, we know that

$P(IP = s+i) = \frac{1}{Q}$ for $i = 0,1,\ldots,Q-1$ and zero elsewhere.

This results in

$$E[I^{OH}(\tau+t)] = E[\{IP(\tau) - D(\tau,\tau+t)\}^{+}]$$

$$= \sum_{k=-\infty}^{\infty} P(IP = k) E[\{k - D(\tau,\tau+t)\}^{+}] \qquad \text{(substitute } i = k-s\text{)}$$

$$= \frac{1}{Q}\sum_{i=0}^{Q-1} E\,[\{s+i-D(\tau,\tau+t)\}^{+}]$$

$$= \frac{1}{Q}\sum_{i=0}^{Q-1}\sum_{d=0}^{\infty}\{s+i-d\}^{+} P(D_t = d)$$

$$= \frac{1}{Q}\sum_{i=0}^{Q-1}\sum_{d=0}^{s+i-1}\{s+i-d\} P(D_t = d). \qquad (2.1)$$

Note that for ease of notation we use the stochastic variable D_t to denote the demand during t periods.

Likewise we can derive the expressions for P_2 and P_3^{discr}, using (1.4) and (1.5)

$$P_2 = 1 - \frac{E[\{D(\tau,\tau+R+L) - IP(\tau)\}^{+}] - E[\{D(\tau,\tau+L) - IP(\tau)\}^{+}]}{E[D(\tau+L,\tau+R+L)]}$$

$$= 1 - \frac{\sum_{k=-\infty}^{\infty} P(IP = k)\left(E[\{D(\tau,\tau+R+L) - k\}^{+}] - E[\{D(\tau,\tau+L) - k\}^{+}]\right)}{E[D(\tau+L,\tau+R+L)]}$$

$$= 1 - \frac{\frac{1}{Q}\sum_{i=0}^{Q-1}\sum_{d=s+i+1}^{\infty}[P(D_{L+R} = d) - P(D_L = d)]\ \{d-s-i\}}{E[D_R]}. \qquad (2.2)$$

$$P_3^{discrete} = P\big(IP(\tau) > D(\tau,\tau+R+L)\big) = \sum_{k=-\infty}^{\infty} P(IP = k) P(D_{L+R} < k)$$

$= \frac{1}{Q}\sum_{k=s}^{s+Q-1} P(D_{L+R} < k) = \frac{1}{Q}\sum_{k=s}^{s+Q-1}\sum_{d=0}^{k-1} P(D_{L+R} = d)$ if $s \geq 1$, zero otherwise (2.3)

Using (1.6) we can derive the expression for the expected number of orderlines per review period, $E[OL]$

$$E[OL] = P(IP(\tau) - D(\tau,\tau + R) < s) = \sum_{k=-\infty}^{\infty} P(IP = k)P(D_R > k - s)$$

$$= \frac{1}{Q}\sum_{i=0}^{Q-1} P(D_R > i) = \frac{1}{Q}\sum_{i=0}^{Q-1}\left\{1 - \sum_{d=0}^{i} P(D_R = d)\right\}. \qquad (2.4)$$

When combining this formula with formula (1.7) we get the expression for the expected order size, $E[OS]$

$$E[OS] = \frac{E[D_R]}{\frac{1}{Q}\sum_{i=0}^{Q-1}\{1-\sum_{d=0}^{i} P(D_R=d)\}}. \qquad (2.5)$$

2.2 Calculating the KPI's for a given discrete demand distribution

In this section we will use an example to show how to determine the convolution of stochastic discrete variables, and given the convolution how to determine the five KPI's.

The formulas in section 2.1 all depend on one or more of the stochastic variables D_L, D_R and/or D_{L+R} .The probability distribution for these variables can be derived in two ways: 1. By using the fact that the probability distribution of the demand during t periods ($t = L,\ R,\ L + R$) is the t-fold convolution of the probability distribution of the single period demand and 2. By first calculating or measuring the mean μ_t and the standard deviation σ_t of the demand during t periods ($t = L,\ R,\ L + R$) and then fitting a discrete probability distribution on μ_t and σ_t. These two options will be discussed in more detail below.

Option 1. Deriving the t-fold convolution of the single period probability distribution

This approach assumes that the single period probability distribution is known. It may be estimated from empirical data. If for example demand is registered during 50 weeks and in 20 weeks demand was equal to 3 units, while in the other 30 weeks demand was equal to 4 units, then we estimate (with $D_1$the single period demand)

for $\boldsymbol{t = 1}$

$P(D_1 = 3) = \frac{20}{50} = 0.4$ and

$$P(D_1 = 4) = \frac{30}{50} = 0.6$$

Assume $L = 1$ and $R = 2$, so $L + R = 3$. In order to determine the five KPI's we then need the probability distributions of D_1, D_2 and D_3. The probability distribution of D_t can be derived using the equation

$$P(D_t = k) = \sum_{j+i=k} P(D_{t-1} = j \; and \; D_1 = i)$$

If D_{t-1} and D_1 are independent stochastic variables, this can be written as

$$P(D_t = k) = \sum_{j+i=k} P(D_{t-1} = j) \cdot P(D_1 = i)$$

With this equation we can determine the t-fold convolution. In our example we then have

for $\boldsymbol{t = 2}$

Since D_1 is equal to 3 or 4 units, D_2 can vary between 6 and 8 units.

$$P(D_2 = 6) = P(D_1 = 3 \; and \; D_1 = 3) = 0.4 * 0.4 = 0.16$$

$$P(D_2 = 7) = P(D_1 = 3 \; and \; D_1 = 4) + P(D_1 = 4 \; and \; D_1 = 3)$$

$$= 0.4 * 0.6 + 0.6 * 0.4 = 0.48$$

$$P(D_2 = 8) = P(D_1 = 4 \; and \; D_1 = 4) = 0.6 * 0.6 = 0.36$$

for $\boldsymbol{t = 3}$

Since D_1 is equal to 3 or 4 units and D_2 can vary between 6 and 8 units, D_3 will vary between 9 and 12 units.

$$P(D_3 = 9) = P(D_2 = 6 \; and \; D_1 = 3) = 0.16 * 0.4 = 0.064$$

$$P(D_3 = 10) = P(D_2 = 6 \; and \; D_1 = 4) + P(D_2 = 7 \; and \; D_1 = 3)$$

$$= 0.16 * 0.6 + 0.48 * 0.4 = 0.288$$

$$P(D_3 = 11) = P(D_2 = 7 \; and \; D_1 = 4) + P(D_2 = 8 \; and \; D_1 = 3)$$

$$= 0.48 * 0.6 + 0.36 * 0.4 = 0.432$$

$$P(D_3 = 12) = P(D_2 = 8 \; and \; D_1 = 4) = 0.36 * 0.6 = 0.216$$

With these probability distributions the five KPI's can be derived. For example, when $s = 10$ and $= 2$, formula (2.2) gives

$$P_2 = 1 - \frac{\frac{1}{2}(\sum_{d=s+1}^{\infty}[P(D_3 = d) - P(D_1 = d)]\{d - s\} + \sum_{d=s+2}^{\infty}[P(D_3 = d) - P(D_1 = d)]\{d - s - 1\})}{2 * E[D_1]}$$

Since $E[D_1] = \sum_{d=0}^{\infty} d.P(D_1 = d) = 3 * 0.4 + 4 * 0.6 = 3.6$ and

$P(D_1 = d) = 0$ for $d = 9,\ 10,\ 11,\ 12$

the fill rate P_2 is equal to

$$P_2 = 1 - \frac{\frac{1}{2}\{P(D_3 = 11) * 1 + P(D_3 = 12) * 2 + P(D_3 = 12) * 1\}}{7.2}$$

$$= 1 - \frac{\frac{1}{2}\{0.432 * 1 + 0.216 * 2 + 0.216 * 1\}}{7.2} = 0.925$$

With (2.4) we determine the expected number of orderlines per review period.

$$E[OL] = \frac{1}{2}\{1 - P(D_2 = 0) + 1 - P(D_2 = 0) - P(D_2 = 1)\} = 1.$$

Note that since the single period demand is always larger than Q in this example, every review period a replenishment order is generated.

Likewise $E[I^{OH}(\tau + L)]$, $E[I^{OH}(\tau + L + R)]$ and $P_3^{discrete}$ can be calculated.

While above we assumed that demand in consecutive periods is independent, this assumption is not needed if we determine the probability distribution from demand in t periods directly from empirical data (using a time series of observations for D_t).

Option 2. Fitting a discrete probability distribution on μ_t and σ_t.

The second option to determine the probability distribution function is in fact an approximation. By first calculating or measuring the mean μ_t and the standard deviation σ_t of the demand during t periods ($t = L,\ R,\ L + R$) and then fitting a theoretical discrete probability distribution on μ_t and σ_t, an approximative probability distribution is found. Adan et al. (1995) developed a simple and effective fitting procedure based on μ_t and σ_t. In their procedure they consider four discrete probability distributions: the geometric (GEO), negative binomial (NB), Poisson ($POIS$), and binomial (BIN) distribution. A $GEO(p)$ random variable has probability distribution $p_i = (1 - p)p^i$, $i = 0,1,2, \ldots$, and an $NB(k, p)$ variable is the sum of k independent $GEO(p)$ variables. A $POIS(\lambda)$ random variable is Poisson distributed with mean λ, and a $BIN(k,p)$ variable is binomially distributed, where k is the number of trials and p the success probability.

To choose one of these four probability distributions, they consider the variable a which is defined as

$$a = \frac{\sigma^2/\mu - 1}{\mu}$$

They choose the binomial distribution when $-1 < a < 0$, the Poisson distribution when $a = 0$, the negative binomial distribution when $0 < a < 1$, and the geometric distribution when $a \geq 1$. This is visualized in Figure 4. For certain combinations of a small value for the Variance to Mean and a small value for the Mean they prove that those combinations do not exist if demand is discrete. These combinations are reflected by the area below the red line in Figure 4. For the exact determination of the parameters of the distribution (like the parameters k and p for the negative binomial distribution) we refer to Adan et al.(1995).

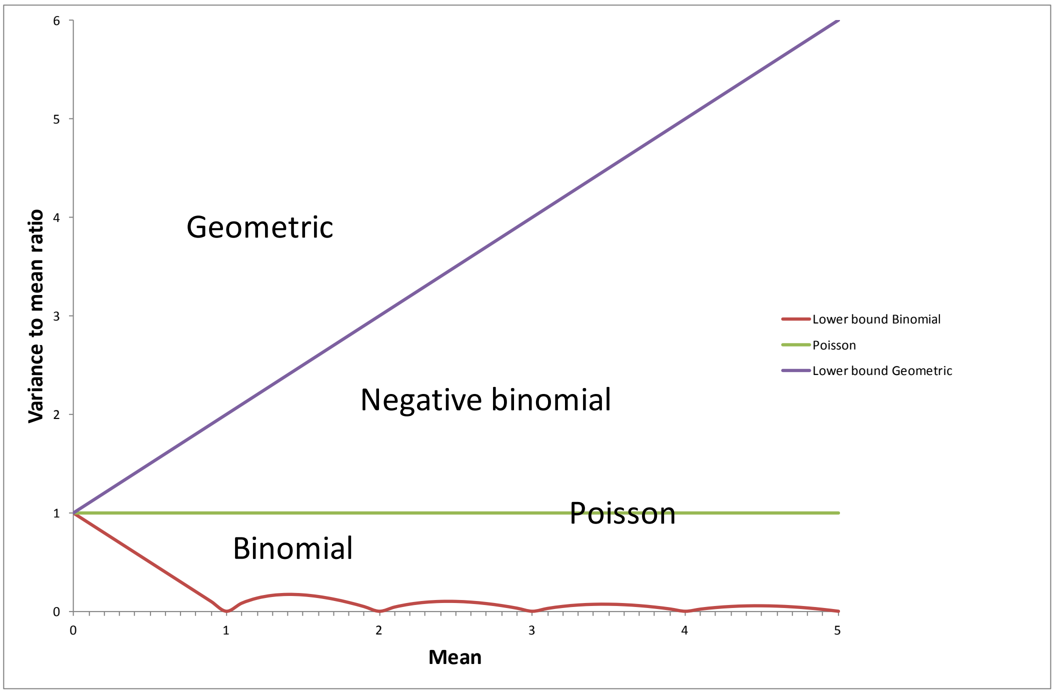


Figure 4. The probability distributions selected in the fitting procedure of Adan et al. as a function of the mean and the Variance to mean (VTM).

With the probability distribution selected based on the fitting procedure of Adan et al., the five KPI's can be determined again using (2.1)-(2.5).

The advantage of the fitting procedure of Adan et al. is the fact that only two numbers, μ_t and σ_t, are required to fully specify the probability distribution function. The advantage of using the full empirical distribution functions, as described above when discussing the first option, is the fact that it gives exact results. However, this is under the assumption that the empirical distribution is known. The reality is that often this empirical distribution is based on a limited set of historical data. As such it is an estimate of the true probability distribution. As new information becomes available, this distribution function will change again. So with limited historical data the fitting procedure of Adan et al. may be more robust: when new demand information becomes available the estimates of the mean and standard deviation will change also, but the KPI's are likely to change less.

The fitting procedure of Adan et al. is implemented in the Excel-based DoBr tool.

3. Continuous demand distribution

In this section demand and supply are assumed to be real numbers rather than integers. In other words, we will consider continuous demand distributions here. In Section 3.1 we will first derive formulas for the KPI's assuming demand is continuously distributed but without making any further assumption on the demand distribution. The formulas derived in section 3.1 can be expressed in terms of functions that can be easily evaluated in Excel when demand is either gamma or normally distributed. Section 3.2 derives the expressions when demand is gamma distributed and Section 3.3 derives them when demand is normally distributed.

3.1 Expressions for KPI's when demand is continuous

For the (R, s, nQ) inventory control system with backordering and demand being a continuous stochastic variable, a replenishment order is triggered as soon as the inventory position drops below the reorder level s. For this system Hadley and Whitin have proven that the inventory position at an arbitrary review moment, just after the replenishment order is placed, is (continuously) uniformly distributed between s and $s + Q$.
Note the differences with Hadley and Whitin's result for the system with discrete demand: the continuous uniform pdf versus the discrete uniform pdf and the interval $(s, \boldsymbol{s+Q})$ versus the set $\{s, s+1, \ldots, \boldsymbol{s-1+Q}\}$.

- **The fill rate**

If we assume demand is identically and independently distributed (i.i.d.) in disjoint time intervals we can derive the expression for the fill rate. This derivation starts with formula (1.4)

$$P_2 = 1 - \frac{E[BO(\tau+R+L)] - E[BO(\tau+L)]}{E[D_R]}.$$

Next we will derive a general expression for $E[BO(\tau+t)]$ with $t = R + L$ or $= L$. Hereto we use the fact that $IP(\tau) \sim u(s, s+Q)$ and we introduce the continuous stochastic variable $\Delta = IP(\tau) - s$, so $\Delta \sim u(0, Q)$. Finally we introduce the notation for two probability density functions (pdf's): $f_t(.)$ is the pdf for D_t, the demand during t periods, and $g(.)$ is the pdf for the stochastic variable Δ.

Using (1.3) we have

$$E[BO(\tau+t)] = E[\{D(\tau,\tau+t) - IP(\tau)\}^+] = E[(D_t - s - \Delta)^+]$$

$$= \int_{-\infty}^{\infty}\int_{0}^{Q} (x - s - \delta)^+ f_t(x) g(\delta) d\delta dx$$

$$= \int_{-\infty}^{s}\int_{0}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx + \int_{s}^{s+Q}\int_{0}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx + \int_{s+Q}^{\infty}\int_{0}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx$$

$$= \int_{-\infty}^{s}\int_{0}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx + \int_{s}^{s+Q}\int_{0}^{x-s}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx + \int_{s}^{s+Q}\int_{x-s}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx + \int_{s+Q}^{\infty}\int_{0}^{Q}(x-s-\delta)^{+}f_t(x)g(\delta)d\delta dx.$$

The integration over x starts with $-\infty$ since in our analyses we also include probability distribution functions, like the normal distribution, having a positive probability that demand is negative. In reality of course demand is not negative.

Note that in the first integral $(x-s-\delta)^{+}=0$ since $x \leq s$. Likewise in the third integral $(x-s-\delta)^{+}=0$ since $\delta \geq x-s$. Therefore both the first and third integral are equal to zero. Given the values for x and δ in the second and fourth integral, the function $(x-s-\delta)^{+}$ can be replaced by $(x-s-\delta)$.

So, also knowing that $g(\delta)=1/Q$, we have

$$E[BO(\tau+t)] = \frac{1}{Q}\int_{s}^{s+Q}\int_{0}^{x-s}(x-s-\delta)f_t(x)d\delta dx + \frac{1}{Q}\int_{s+Q}^{\infty}\int_{0}^{Q}(x-s-\delta)f_t(x)d\delta dx.$$

Simply integrating over δ gives us

$$E[BO(\tau+t)]$$

$$= \frac{1}{Q}\int_{s}^{s+Q}[(x-s)\delta - \frac{1}{2}\delta^2]_{\delta=0}^{\delta=x-s} f_t(x)dx + \frac{1}{Q}\int_{s+Q}^{\infty}[(x-s)\delta - \frac{1}{2}\delta^2]_{\delta=0}^{\delta=Q} f_t(x)dx$$

$$= \frac{1}{Q}\int_{s}^{s+Q}\frac{1}{2}(x-s)^2 f_t(x)dx + \frac{1}{Q}\int_{s+Q}^{\infty}[(x-s)Q - \frac{1}{2}Q^2] f_t(x)dx$$

$$= \frac{1}{Q}\int_{s}^{s+Q}\frac{1}{2}(x-s)^2 f_t(x)dx + \int_{s+Q}^{\infty}x f_t(x)dx - (s+\frac{Q}{2})\int_{s+Q}^{\infty}f_t(x)dx. \qquad (3.1)$$

- **The number of order lines and the order size**

To determine the expected number of order lines per review period we start with (1.6). When demand is continuous, we can derive the following formula

$$E[OL] = P(IP(\tau) - D_R < s) = P(s + \Delta - D_R < s) = P(D_R > \Delta)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_{\{x>\delta\}} f_R(x) g(\delta) d\delta dx$$

$$= \int_0^{\infty}\int_0^{\min(Q,x)} f_R(x) g(\delta) d\delta dx = \frac{1}{Q}\int_0^{\infty} f_R(x)\min(Q,x)dx$$

$$= \frac{1}{Q}\int_0^{Q} x f_R(x)dx + \frac{1}{Q}\int_Q^{\infty} Q f_R(x)dx$$

$$= 1 - F_R(Q) + \frac{1}{Q}\int_0^{Q} x f_R(x)dx. \quad (3.2)$$

with $1_{\{x>\delta\}}$ the indicator function which is equal to one if $x > \delta$ and zero otherwise.

According to (1.7) the expected order size $E[OS]$ is equal to

$$E[OS] = \frac{E[D_R]}{E[OL]}. \quad (3.3)$$

- **The inventory on hand**

To determine the expected inventory on hand we start with (1.2). The expected inventory hand at time $\tau + t$, denoted by $E[I^{OH}(\tau + t)]$, is equal to

$$E[I^{OH}(\tau + t)] = E[(IP(\tau) - D_t)^+] = E[(s + \Delta - D_t)^+]$$

$$= \int_{-\infty}^{\infty}\int_0^{Q} (s + \delta - x)^+ f_t(x) g(\delta) d\delta dx$$

$$= \int_{-\infty}^{\infty}\int_0^{Q} [s + \delta - x + (x - s - \delta)^+] f_t(x) g(\delta) d\delta dx$$

$$= \int_{-\infty}^{\infty}\int_0^{Q} (s - x + \delta) f_t(x) g(\delta) d\delta dx + \int_{-\infty}^{\infty}\int_0^{Q} (x - s - \delta)^+ f_t(x) g(\delta) d\delta dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{Q}\left[(s - x)Q + \frac{1}{2}Q^2\right] f_t(x)dx + E[(D_t - s - \Delta)^+]$$

$$= s + \frac{Q}{2} - E[D_t] + E[BO(\tau + t)]\,. \quad (3.4)$$

Note that with this formulas we have shown a direct link between the expected inventory on hand and the expected number of backorders at time t.

In case $E[BO(\tau+t)]$, with $t=L$ and $t=L+R$, is close to zero (i.e. if the fill rate is close to 100%), if the safety stock ss is defined as

$$ss := s - E[D_{L+R}],$$

and if demand is i.i.d., then the expected inventory on hand at time $\tau+L$ resp. $\tau+R+L$ is approximately equal to

$$E[I^{OH}(\tau+L)] \approx \frac{Q}{2} + E[D_R] + ss$$

$$E[I^{OH}(\tau+R+L)] \approx \frac{Q}{2} + ss.$$

Recall that these quantities represent the expected inventory on hand at the beginning and end of an arbitrary review period and not, as in some other textbooks, the expected inventory on hand at the beginning and end of a replenishment cycle (defined as the time interval starting from the moment a replenishment order has arrived and ending at the moment the next replenishment order is about to arrive).

- **The discrete ready rate**

For the discrete ready rate $P_3^{discrete}$ we use (1.5) to derive the following result

$$P_3^{discrete} = P(IP(\tau) > D_{L+R}) = P(D_{L+R} < s + \Delta)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_{\{x<s+\delta\}} f_{L+R}(x) g(\delta) d\delta dx$$

$$= \int_{-\infty}^{s+Q}\int_{\max(0,x-s)}^{Q} f_{L+R}(x) g(\delta) d\delta dx = \frac{1}{Q}\int_{-\infty}^{s+Q} f_{L+R}(x)[Q - \max(0, x-s)]dx$$

$$= F_{L+R}(s+Q) - \frac{1}{Q}\int_{s}^{s+Q}(x-s) f_{L+R}(x)dx$$

$$= \frac{s+Q}{Q}F_{L+R}(s+Q) - \frac{s}{Q}F_{L+R}(s) - \frac{1}{Q}\int_{s}^{s+Q} x f_{L+R}(x)dx. \qquad (3.5)$$

3.2 Gamma distribution

In case D_t is **gamma** distributed with mean $\alpha\beta$ and variance $\alpha\beta^2$, then the pdf is equal to

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-x/\beta}$$

with $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$.

With $F(x|\alpha,\beta)$ being the cdf of the gamma distribution and using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ we can derive the following general expressions

$$\int_a^b f(x|\alpha,\beta)dx = F(b|\alpha,\beta) - F(a|\alpha,\beta). \tag{3.6}$$

$$\begin{aligned}\int_a^b xf(x|\alpha,\beta)dx &= \int_a^b \frac{x^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}dx \\ &= \alpha\beta \int_a^b \frac{x^{\alpha}}{\Gamma(\alpha+1)\beta^{\alpha+1}} e^{-\frac{x}{\beta}}dx \\ &= \alpha\beta[F(b|\alpha+1,\beta) - F(a|\alpha+1,\beta)]\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_a^b x^2 f(x|\alpha,\beta)dx &= \int_a^b \frac{x^{\alpha+1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}dx \\ &= (\alpha+1)\alpha\beta^2 \int_a^b \frac{x^{\alpha+1}}{\Gamma(\alpha+2)\beta^{\alpha+2}} e^{-\frac{x}{\beta}}dx \\ &= (\alpha+1)\alpha\beta^2[F(b|\alpha+2,\beta) - F(a|\alpha+2,\beta)]\end{aligned} \tag{3.8}$$

Below these expressions will be used to derive expressions for the KPI's.

- **The fill rate**

The fill rate P_2 can be determined by combining (1.4) with

$$E[BO(\tau+t)] = \frac{1}{Q}\int_s^{s+Q} \frac{1}{2}(x-s)^2 f_t(x)dx + \int_{s+Q}^{\infty} xf_t(x)dx - \left(s+\frac{Q}{2}\right)\int_{s+Q}^{\infty} f_t(x)dx,$$

where $f_t(.)$ is the probability density function for the stochastic variable D_t, the demand during t periods.

Using expressions (3.6)-(3.8), we can rewrite this expression for $E[BO(\tau+t)]$.

$$E[BO(\tau+t)]$$
$$=\frac{1}{Q}\int_{s}^{s+Q}\frac{1}{2}(x-s)^2 f(x|\alpha,\beta)dx+\int_{s+Q}^{\infty} xf(x|\alpha,\beta)dx-(s+\frac{Q}{2})\int_{s+Q}^{\infty} f(x|\alpha,\beta)dx$$

$$=\frac{(\alpha+1)\alpha\beta^2}{2Q}[F(s+Q|\alpha+2,\beta)-F(s|\alpha+2,\beta)]$$
$$-\left(\frac{s+Q}{Q}\right)\alpha\beta F(s+Q|\alpha+1,\beta)+\frac{s\alpha\beta}{Q}F(s|\alpha+1,\beta)$$
$$+\frac{(s+Q)^2}{2Q}F(s+Q|\alpha,\beta)-\frac{s^2}{2Q}F(s|\alpha,\beta)+\alpha\beta-\left(s+\frac{Q}{2}\right). \qquad (3.9)$$

Since D_t is gamma distributed with mean $\alpha\beta$ and variance $\alpha\beta^2$, α and β in the equation above can be solved from $\alpha\beta=t\mu$ and $\alpha\beta^2=t\sigma^2$ when demand data are available to estimate μ and σ or when μ and σ are given. This yields

$\beta=\frac{\sigma^2}{\mu}$ and $\alpha=t\frac{\mu^2}{\sigma^2}$

When combining (1.4) and (3.9), we have an exact expression for the P_2 in an (R,s,nQ)-system with gamma distributed demand and backordering.

- **The number of order lines and the order size**

According to (3.2), when demand is continuous the expected number of orderlines per review period is equal to

$$E[OL]=1-F_R(Q)+\frac{1}{Q}\int_0^Q xf_R(x)dx.$$

In case demand during the review period is gamma distributed with parameters α_R and β_R, we get (by using (3.7))

$$E[OL]=1-F(Q|\alpha_R,\beta_R)+\frac{1}{Q}\int_0^Q xf(x|\alpha_R,\beta_R)dx$$
$$=1-F(Q|\alpha_R,\beta_R)+\frac{\alpha_R\beta_R}{Q}F(Q|\alpha_R+1,\beta_R). \qquad (3.10)$$

The expected order size $E[OS]$ can be determined using (3.3) and (3.10).

- **The inventory on hand**

If demand is gamma distributed, the expected inventory on hand follows from combining equation (3.9) with (3.4).

- **The discrete ready rate**

In case demand is gamma distributed, formula (3.5)

$$P_3^{discrete} = \frac{s+Q}{Q} F_{L+R}(s+Q) - \frac{s}{Q} F_{L+R}(s) - \frac{1}{Q}\int_s^{s+Q} x f_{L+R}(x)dx$$

can be rewritten, using formula (3.7) into

$$P_3^{discrete} = \frac{s+Q}{Q} F(s+Q|\alpha_{L+R}, \beta_{L+R}) - \frac{s}{Q} F(s|\alpha_{L+R}, \beta_{L+R}) \\ - \frac{\alpha_{L+R}\beta_{L+R}}{Q}[F(s+Q|\alpha_{L+R}+1, \beta_{L+R}) - F(s|\alpha_{L+R}+1, \beta_{L+R})]. \quad (3.11)$$

3.3 Normal distribution

In this section demand is assumed to be normally distributed. So the probability distribution function $f(x)$ is equal to

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We will derive expressions for the five KPI's that can be easily implemented in Excel. To this end the following results, which are derived in Appendix 2, will be used to transform expressions for a stochastic variable X with a non-standard normal pdf $f(x)$, mean μ and standard deviation σ into expressions for a stochastic variable V with standard normal pdf $\varphi(v)$and cdf $\Phi(v)$ with $\varphi(v) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})$.

$$\int_a^b f(x)dx = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi(\frac{a-\mu}{\sigma}) \quad (3.12)$$

$$\int_a^b x f(x)dx = \sigma\varphi\left(\frac{a-\mu}{\sigma}\right) - \sigma\varphi\left(\frac{b-\mu}{\sigma}\right) + \mu\Phi\left(\frac{b-\mu}{\sigma}\right) - \mu\Phi\left(\frac{a-\mu}{\sigma}\right) \quad (3.13)$$

$$\int_a^b x^2 f(x)dx = \sigma(\mu+a)\varphi\left(\frac{a-\mu}{\sigma}\right) - \sigma(\mu+b)\varphi\left(\frac{b-\mu}{\sigma}\right) \\ + (\sigma^2+\mu^2)\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right] \quad (3.14)$$

- **The fill rate**

The fill rate P_2 can be determined by combining (1.4) with

$$E[BO(\tau+t)] = \frac{1}{Q}\int_{s}^{s+Q} \frac{1}{2}(x-s)^2 f_t(x)dx + \int_{s+Q}^{\infty} x f_t(x)dx - (s+\frac{Q}{2})\int_{s+Q}^{\infty} f_t(x)dx.$$

where $f_t(.)$ is the probability density function for the stochastic variable D_t, the demand during t periods with $\mu_t = t\mu$ and $\sigma_t = \sqrt{t}\sigma$ if demand is i.i.d..

For ease of notation below we leave out the subindices t for μ and σ when deriving formula (3.15). By using (3.12)-(3.14) the expression for $E[BO(\tau+t)]$ can be rewritten into

$$\begin{aligned}
E[BO(\tau+t)] = {} & \frac{\sigma}{2Q}(\mu+s)\varphi\left(\frac{s-\mu}{\sigma}\right) - \frac{\sigma}{2Q}(\mu+s+Q)\varphi\left(\frac{s+Q-\mu}{\sigma}\right) \\
& + \frac{1}{2Q}(\sigma^2+\mu^2)\left[\Phi\left(\frac{s+Q-\mu}{\sigma}\right) - \Phi\left(\frac{s-\mu}{\sigma}\right)\right] \\
& - \frac{s}{Q}\left[\sigma\varphi\left(\frac{s-\mu}{\sigma}\right) - \sigma\varphi\left(\frac{s+Q-\mu}{\sigma}\right) + \mu\Phi\left(\frac{s+Q-\mu}{\sigma}\right) - \mu\Phi\left(\frac{s-\mu}{\sigma}\right)\right] \\
& + \frac{s^2}{2Q}\Phi\left(\frac{s+Q-\mu}{\sigma}\right) - \frac{s^2}{2Q}\Phi\left(\frac{s-\mu}{\sigma}\right) \\
& + \sigma\varphi\left(\frac{s+Q-\mu}{\sigma}\right) + \mu - \mu\Phi\left(\frac{s+Q-\mu}{\sigma}\right) \\
& - \left(s+\frac{Q}{2}\right) + (s+\frac{Q}{2})\Phi\left(\frac{s+Q-\mu}{\sigma}\right) \\
= {} & \frac{\sigma}{2Q}(s+Q-\mu)\varphi\left(\frac{s+Q-\mu}{\sigma}\right) - \frac{\sigma}{2Q}(s-\mu)\varphi\left(\frac{s-\mu}{\sigma}\right) \\
& + \frac{\sigma^2+(s+Q-\mu)^2}{2Q}\Phi\left(\frac{s+Q-\mu}{\sigma}\right) - \frac{\sigma^2+(s-\mu)^2}{2Q}\Phi\left(\frac{s-\mu}{\sigma}\right) \\
& - \left(s+\frac{Q}{2}-\mu\right) \qquad (3.15)
\end{aligned}$$

When combining (1.4) and (3.15), we have an exact expression for the P_2 in an (R,s,nQ)-system with normally distributed demand and backordering.

When $t = R + L$ and $s = E[D_{L+R}] + ss$, this expression can be written as

$$\begin{aligned}
E[BO(\tau+R+L)] = {} & \frac{\sigma_{L+R}}{2Q}(ss+Q)\varphi\left(\frac{ss+Q}{\sigma_{L+R}}\right) - \frac{\sigma_{L+R}}{2Q}(ss)\varphi\left(\frac{ss}{\sigma_{L+R}}\right) \\
& + \frac{\sigma_{L+R}^2+(ss+Q)^2}{2Q}\Phi\left(\frac{ss+Q}{\sigma_{L+R}}\right) - \frac{\sigma_{L+R}^2+(ss)^2}{2Q}\Phi\left(\frac{ss}{\sigma_{L+R}}\right) \\
& - \left(ss+\frac{Q}{2}\right)
\end{aligned}$$

For the readers who are familiar with the theory in Zipkin's handbook (2000), an alternative way to derive the expressions for the fill rate for gamma or normal demand is given in Appendix 4.

- **The number of order lines and the order size**

In case demand is normally distributed, the expression for $E[OL]$ can be rewritten and expressed in the standard normal distribution functions, using (3.2) and (3.13). The expression for $E[OL]$ can then be rewritten into:

$$E[OL] = 1 - F_R(Q) + \frac{1}{Q}\int_0^Q x f_R(x)dx$$

$$= 1 - \Phi(\frac{Q-\mu_R}{\sigma_R}) + \frac{\sigma_R}{Q}\left[\varphi\left(\frac{-\mu_R}{\sigma_R}\right) - \varphi\left(\frac{Q-\mu_R}{\sigma_R}\right)\right] + \frac{\mu_R}{Q}\left[\Phi\left(\frac{Q-\mu_R}{\sigma_R}\right) - \Phi\left(\frac{-\mu_R}{\sigma_R}\right)\right]. \quad (3.16)$$

The expected order size $E[OS]$ can then be determined by combining (3.3) and (3.16).

- **The inventory on hand**

If demand is normally distributed, the expected inventory on hand follows from combining equation (3.4) with (3.15).

- **The discrete ready rate**

In case demand is normally distributed, formula (3.5)

$$P_3^{discrete} = \frac{s+Q}{Q}F_{L+R}(s+Q) - \frac{s}{Q}F_{L+R}(s) - \frac{1}{Q}\int_s^{s+Q} x f_{L+R}(x)dx$$

can be rewritten (again using (3.13)) and expressed in the standard normal distribution functions

$$P_3^{discrete} = \frac{s+Q}{Q}\Phi\left(\frac{s+Q-\mu_{L+R}}{\sigma_{L+R}}\right) - \frac{s}{Q}\Phi\left(\frac{s-\mu_{L+R}}{\sigma_{L+R}}\right)$$

$$+\frac{\sigma_{L+R}}{Q}\left[\varphi\left(\frac{s+Q-\mu_{L+R}}{\sigma_{L+R}}\right) - \varphi\left(\frac{s-\mu_{L+R}}{\sigma_{L+R}}\right)\right] - \frac{\mu_{L+R}}{Q}[\Phi\left(\frac{s+Q-\mu_{L+R}}{\sigma_{L+R}}\right) - \Phi\left(\frac{s-\mu_{L+R}}{\sigma_{L+R}}\right)]$$

When $s = \mu_{L+R} + ss$, we have

$$P_3^{discrete} = \frac{ss+Q}{Q}\Phi\left(\frac{ss+Q}{\sigma_{L+R}}\right) - \frac{ss}{Q}\Phi\left(\frac{ss}{\sigma_{L+R}}\right) + \frac{\sigma_{L+R}}{Q}\left[\varphi\left(\frac{ss+Q}{\sigma_{L+R}}\right) - \varphi\left(\frac{ss}{\sigma_{L+R}}\right)\right]. \quad (3.17)$$

4. Relaxation of assumptions

In standard textbooks typically many assumptions are made, like

- demand is always normally distributed,
- immediately after a delivery the backorders are always equal to zero,
- the order size is always equal to exactly one base replenishment quantity,
- at the moment of ordering the inventory position is exactly equal to the reorder level.

In some papers, Lecture Notes or textbooks, including De Kok (1991b), De Kok (2012) and Tempelmeier (2011) it is recognized that this last assumption often does not hold, i.e. at the moment of ordering the inventory position may be strictly less than the reorder level. This difference between the reorder level and the inventory position at the moment of ordering is called the undershoot. The first two moments of the undershoot are typically determined using approximations from Tijms (1986). Tijms notes that these approximations are good as long as the base replenishment quantity is more than 1.5 times the average demand during the review period.

The results derived so far in this document do not depend on any of the assumptions and approximations mentioned above. Still some assumptions were made. We assumed items are non-perishable, demand which is not satisfied in out-of-stock situations is backordered, lead times are deterministic, demand is stationary, a (R,s,nQ)-control system is applied and subsequent replenishment orders cannot pass (i.e. an order B for a certain item placed later than order A for the same item cannot arrive earlier). In this chapter we will discuss how to relax all these assumptions except for the last one. The analysis of the model becomes intractable if the last assumption is relaxed. In practice this assumption is generally valid since an item is typically ordered at a single supplier who has no reason to handle and ship the orders in a sequence which is different from the sequence in which he received the orders. Only in case it is possible to place both regular orders and emergency orders, having a regular and a shorter average lead time, another type of replenishment policy (like the dual index policy) is needed. See the review paper by Minner (2003).

4.1 Stochastic lead times.

Up to now we assumed that the lead time is constant, deterministic and equal to L. In this section we show how to determine the KPI's in case the lead time is stochastic.

All expressions for the KPI's derived in chapters 1 to 3 depend on $D(\tau, \tau + L)$, $D(\tau, \tau + R)$ and/or $D(\tau, \tau + R + L)$. This implies, if we assume demand in consecutive periods is identically and independently distributed (i.i.d.), that all KPI's depend on D_L, D_R and/or D_{L+R}. If the lead time is a stochastic variable we first have to calculate the mean and variance of D_L, D_R and/or D_{L+R}. Next we have to make an assumption about the probability distribution of the demand and then we can determine the KPI's again using the expressions derived in chapters 2 and 3.

If the lead time is a *discrete* stochastic variable with mean $E[L]$ and variance $var[L]$, demand in one period is stochastic with mean $E[D_1]$ and variance $var[D_1]$, and the review period is equal to an *integer* value R, then we can determine the mean and variance of D_L, D_R and D_{L+R} using the equations

$$E[D_R] = RE[D_1] \quad \text{and} \quad var[D_R] = Rvar[D_1] \tag{4.1}$$

$$E[D_{L+R}] = E[D_L] + RE[D_1] \quad \text{and} \quad var[D_{L+R}] = var[D_L] + Rvar[D_1] \tag{4.2}$$

$$E[D_L] = E[L] \cdot E[D_1] \quad \text{and} \quad var[D_L] = E[L] \cdot var[D_1] + E^2[D_1] \cdot var[L] \tag{4.3}$$

The derivation of these formulas is similar to the derivation of formula (4.3) in De Kok (2012). Below we derive formula (4.1). The derivation of formulas (4.2) and (4.3) is given in Appendix 3.

If we assume the review period is equal to an integer number of periods, say K, and demand is i.i.d. we have

$$E[D(\tau, \tau + R)](= E[D_R])$$

$$= E[\textstyle\sum_{k=1}^{K} D(\tau + k - 1, \tau + k)]$$

$$= \textstyle\sum_{k=1}^{K} E[D(\tau + k - 1, \tau + k)]$$

$$= \sum_{k=1}^{K} E[D_1]$$

$$= KE[D_1]$$

and

$$var[D(\tau, \tau + R)](= var[D_R])$$

$$= var\left[\sum_{k=1}^{K} D(\tau + k - 1, \tau + k)\right] =$$

$$= var\left[\sum_{k=1}^{K} D(\tau + k - 1, \tau + k)\right] =$$

$$= \sum_{k=1}^{K} var[D(\tau + k - 1, \tau + k)]$$

$$= \sum_{k=1}^{K} var[D_1]$$

$$= Kvar[D_1] \tag{4.4}$$

While the variable R has the dimension 'time', the constant K is dimensionless. For convenience we define here R too as a number of periods. Then $R = K$ and (4.4) results in (4.1).

4.2 The (R, S), (R, s, S), (s, nQ) and (s, S)-system

The (R, S)-system is a special case of the (R, s, nQ)-system: it is a (R, s, nQ)-system with $Q = 1$ when demand is discrete and it is virtually identical (i.e. can be approximated extremely accurately) to a (R, s, nQ)-system with $Q = \varepsilon$ (with ε a real number close to zero) when demand is continuous. So all results derived in the previous chapters also apply for the (R, S)-system. A special feature of the (R, S)-system is the fact that the reorder quantity is always equal to the demand in the most recent review period (assuming demand is stationary). Early exact analyses for this system were provided by De Kok (1991a). It is left as an exercise to determine the exact formulas for the (R, S)-system with discrete and continuous demand using the expressions in chapter 1 in combination with the fact that the inventory position at any review period just after potential ordering is exactly equal to S.

The (R, s, S)-system assumes that there is a minimum order quantity MOQ (equal to $S - s + 1$ for discrete demand and $S - s$ for continuous demand) rather than a base replenishment quantity Q. In case demand is discrete, the expressions for the probability distribution of the inventory position as derived by Zheng and Federgruen (1991) together with the formulas in section 2.1 can be used to find exact expressions for the (R, s, S)-system. Zheng and Chen (1992) compare the performance between (R, s, S)- and (R, s, nQ)-systems and show that for many situations the performance is very close. For the special case of Erlang distributed demand (gamma demand with integer shape parameter), Moors and Strijbosch (2002) have derived exact expressions for the fill rate in an (R, s, S)-system.

The (s, nQ) and (s, S)-systems are continuous review systems. They can be interpreted as periodic review systems with extremely small review periods. So the results derived for the periodic systems can be applied here too in a straightforward manner.

4.3 Non-stationary demand

In reality often demand is not stationary e.g. due to trends, seasonal patterns and/or product life cycles. As a result the average demand may increase or decrease as time goes by. In those situations an exact analysis of time-varying and stochastic demand is far too complicated for routine use in practice, according to Silver et. al. (1998). For these cases heuristic approaches are suggested in the literature to determine the reorder level. Rather than determining a single reorder level based on the mean and the standard deviation of the demand during $L + R$ periods, now every review moment a new reorder level is determined and it is based on the forecasted demand for the next $L + R$ periods and the standard deviation of the forecast error for these $L + R$ periods. The reorder level at time t (s_t) is equal to the demand forecast made in period t for the demand for the next $L + R$ periods ($D^{fcst}_{t,t+L+R}$) plus a safety stock (ss_t), which depends on the standard deviation of the forecast error for the next L periods and for the next $L + R$ periods, so

$$s_t = D^{fcst}_{t,t+L+R} + ss_t.$$

A key issue then is how to estimate this standard deviation of the forecast error for the next L periods and the next $L + R$ periods. If e.g. demand follows a positive trend then measuring the standard deviation of the forecast error in the periods preceding the current review moment will tend to underestimate the standard deviation of the forecast error for the $L(+R)$ periods *after* the current review moment. The reason behind this is that if demand increases, then the standard deviation (of demand as well as) of the forecast error increases. To counter this effect, Silver et al. (1998, p.341-343) suggest to derive a relationship between the standard deviation of the forecast error for the next $L(+R)$ periods and the demand forecast for the next $L(+R)$ periods. They assume this relationship holds for a group of SKU's and uses regression to estimate the relationship, cf. the method described earlier in their book (p.114-116 and 126-127).

It can also be proven that demand forecasting increases the variance if it is applied to a stationary demand process. See e.g. Nahmias (2009, p.110-112 and Appendix 2A). This effect increases with the lead time (plus review period) and can be taken into account when estimating the standard deviation of the forecast error, as shown in Strijbosch et al. (2011).

The comparison of the different approaches to estimate the standard deviation of forecast errors over a lead time (plus review period) is beyond the scope of this text. We like to stress however the importance of using the standard deviation of the forecast error rather than the standard deviation of demand when determining reorder levels and KPI's for systems with non-stationary demand. The best motivation for this can be seen from an item having deterministic demand following a strong seasonal pattern or a trend. The standard deviation of demand for this item is very large, while the standard deviation of the forecast error is zero for this item. Since demand is deterministic for this item, no safety stock is needed if the reorder level is set to the demand forecast for the next $L(+R)$ periods.

4.4 Lost sales

In some environments, like supermarkets, customers are not willing to wait for their item if the item is temporarily out-of-stock. Rather they substitute to another item in the same store, buy the item elsewhere or do not buy the item. In all these cases unmet demand is lost for the particular item. In a lost sales model the average sales will be lower than in a model in which unmet demand is backordered. As a result the average inventory on hand and the service level in a lost sales model will be higher than in a similar backordering system. In this section we present three approximations for the fill rate in a lost sales system as well as approximations for the expected number of orderliness and the expected order size.

Tijms and Groenevelt (1984) suggest approximating the fill rate in a lost sales model with a (s, S)-policy by first calculating the fill rate for a similar backordering system (P_2^{BO}) and then solving the fill rate for the lost sales system (P_2^{LS1}) from the equation

$$\frac{1-P_2^{LS1}}{P_2^{LS1}} = 1 - P_2^{BO} \Leftrightarrow P_2^{LS1} = \frac{1}{2-P_2^{BO}} \qquad (4.5)$$

Silver et al. (1998, footnote on page 268) applied the same idea to the continuous review (s, Q) system with normal distributed demand assuming there is no undershoot. The first approximation for the fill rate in a (R, s, nQ) system with lost sales is to solve (4.5) using the fact that P_2^{BO} is known from the expressions for the fill rate in a (R, s, nQ) system with backordering, derived in chapter 2 or 3 depending on the demand distribution. This approximation is called P_2^{LS1}.

This basic approximation for the fill rate can be further refined by using an iterative procedure to determine the fill rate in a lost sales system. This second approximation is based on the notion that the main reason for a higher fill rate in a lost sales system compared to the otherwise identical backordering system is due to the fact that in a lost sales system sales is less than demand. To capture this effect, the sales is estimated iteratively (using the fact that sales is equals to the fill rate times demand) and used as input in the expression for the fill rate in an otherwise identical backordering system. This leads to the following iterative procedure, where D represents the demand

Step 1. Set $i = 0$ and $P_{2,0}^{LS2}(\cdots, D, \cdots) = P_2^{BO}(\cdots, D, \cdots)$ using the fill rate expression for the backordering system.

Step 2. Set $i := i + 1$. Update the mean and variance-to-mean ratio of D_i' with $\mu_i' = P_{2,i-1}^{LS2} \cdot \mu$ and $VTM(D_i') = 1 + P_{2,i-1}^{LS2} \cdot (VTM(D) - 1)$.

Step 3. Calculate $P_{2,i}^{LS2}(\cdots, D, \cdots) = P_2^{BO}(\cdots, D_i', \cdots)$ using the fill rate expression for the backordering system.

Step 4. If $i < i^{MIN}$ or ($\left|P_{2,i}^{LS2} - P_{2,i-1}^{LS2}\right| > \varepsilon$ and $i < i^{MAX}$) then continue with Step 2, else Stop.

Note that Van Donselaar & Broekmeulen (2013) kept the variance to mean ratio constant in their iterative procedure. The adjustment proposed here guarantees a feasible fit for the mixed discrete distribution according to the procedure of Adan et al. (1995).

The third approximation for the fill rate in a lost sales system improves the accuracy of the first two approximations using regression. First two important factors are identified which have a large impact on the performance of a lost sales system. These two factors are the extent to which the demand during the lead-time plus review period is uncertain and the number of orders outstanding. The first can be easily measured via the coefficient of variation of the demand during the lead-time plus review period which, in case demand is identically and independently distributed, is simply equal to:

$$c_{L+R} = \frac{\sigma}{\mu\sqrt(L+R)}$$

Already in 1963, Hadley and Whitin (p. 197) noted that for the lost sales system it is necessary to take explicit account of the number of orders outstanding and the times at which they were placed. The simplest single measure to take into account the number of outstanding orders is the variable nOO, which is defined as the ratio between the expected demand during the lead-time $L \cdot \mu$ and $Max(Q, R \cdot \mu)$, which is a simple proxy for the expected order size in a (R, s, nQ)-system:

$$nOO = \frac{L \cdot \mu}{Max(Q, R \cdot \mu)}$$

After identifying these two key variables, Van Donselaar and Broekmeulen (2013) used regression to improve the accuracy of the first two approximations (P_2^{LS1} and P_2^{BO}). This lead to the third approximation for the fill rate in a lost sales system, P_2^{LS3}

$$P_2^{LS3} = \begin{cases} \frac{\left(P_2^{LS2} + 0.062 \cdot nOO - 0.128\right)}{0.062 \cdot nOO + 0.87} & nOO < 5 \\ \frac{\left(P_2^{BO} - 1.0172 - 1.3218 \cdot (c_{L+R})^{-0.552}\right)}{1.3218 \cdot (c_{L+R})^{-0.552}} & nOO \geq 5 \end{cases} \quad (4.6)$$

This third approximation indeed gives very accurate estimates for the fill rate in a lost sales system as can be seen from Figure 5, where the fill rate of a lost sales system (on the horizontal axis) is compared with four different approximations (on the vertical axis).

The KPI's $E[I^{OH}(\tau + t)]$ (for $t = L$ and $t = L + R$), $E[OL)]$ and $P_3^{discrete}$ for a lost sales system are determined in the same way as described for the fill rate above, i.e. using the expressions for these KPI's derived in chapters 2 and 3 in combination with an iterative procedure. In this iterative procedure the demand in the expressions for the KPI's is replaced by the estimated fill rate in the (i-1)-th iteration times the demand. The expected order size is then determined using the equation $E[OS] * E[OL] = P_2^{LS2} * E[D_R]$.

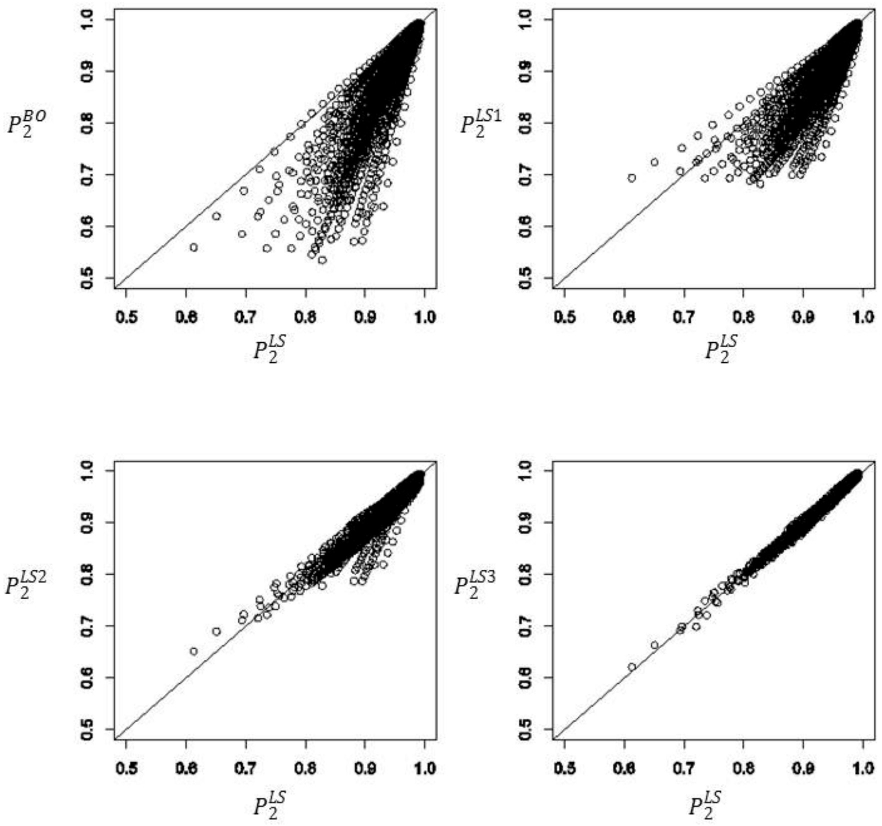


Figure 5. Four different approximations for the fill rate of a lost sales system (from Van Donselaar & Broekmeulen, 2013).

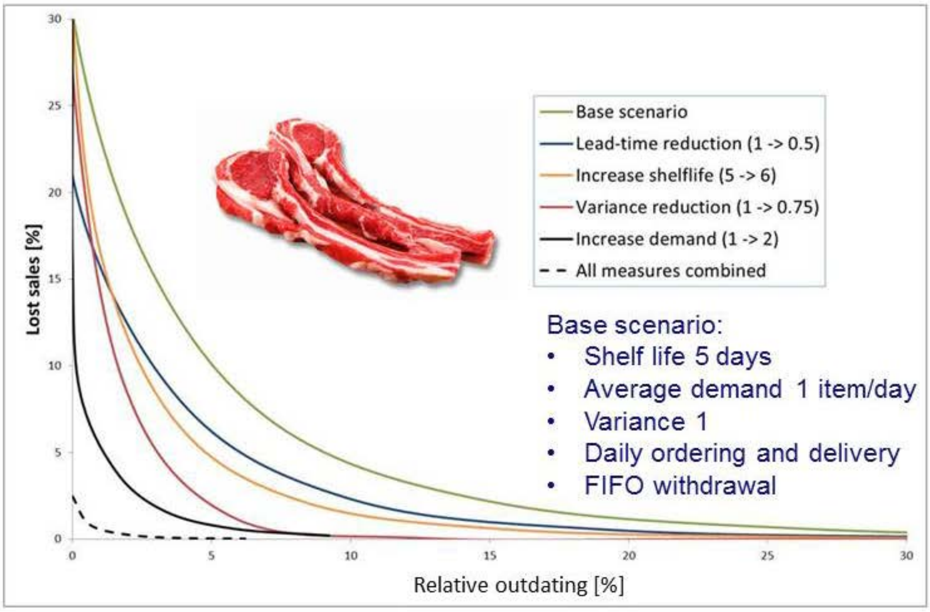
4.5 Perishable items

Perishable items are items with a fixed lifetime. The fixed lifetime is defined here as the number of periods the item can be sold, starting from the moment the item is delivered and added to the inventory on hand until the expiration date is exceeded. Just like lost sales systems, systems for perishable items are notoriously hard to analyze and so we have to rely on approximations in this case as well. Most approximations in the literature for perishable items hold for specific cases only, e.g. if the lead time is one period, the base replenishment quantity is one unit, the fixed lifetime is equal to 2 periods or demand is Poisson distributed. Recently Broekmeulen and Van Donselaar (2009) developed a new replenishment logic specifically for perishable items and this logic has the nice feature that the safety stock needed to achieve a target fill rate is virtually independent of the fixed lifetime. This implies that all results for the fill rate derived in chapters 1 to 3 also can be applied for perishable items without major restrictions on the system parameters or the demand distribution.

The new replenishment logic, called the EWA logic, is a modified (R, s, nQ)-logic and simple to implement if the age of inventories in the system is known. The only difference with the replenishment logic in a (R, s, nQ)-system is the determination of the inventory position. In the EWA-logic the inventory position is reduced with the expected outdating during the next $L + R - 1$ periods, before a replenishment decision is made. This is called the modified inventory position. If the modified inventory position is below the reorder level s, a replenishment order is generated with size $n \cdot Q$ with n the minimal integer value to bring the modified inventory position back to or above the reorder level. The expected outdating is determined by assuming that demand in the next $L + R - 1$ periods is simply equal to the average demand (or forecasted demand if demand is non-stationary).

Apart from the fill rate, another key KPI for perishable items is the relative outdating, defined as the percentage of demand which is outdated since the expiration date is exceeded before it is sold. Van Donselaar and Broekmeulen (2012) derived approximations for the relative outdating using a mix of analytically derived expressions, simulation and regression.

The approximations for the fill rate and the relative outdating enable the trade-off between these two KPI's, for example if a change in the system parameters is evaluated. By varying the (safety stock or) the reorder level this trade-off can be visualized with graphs like in Figure 6. In this Figure a base scenario for lamb chops is compared with alternative scenarios in which for example the lead time is reduced from one day to 0.5 day or the shelf life (fixed lifetime) is reduced from 6 to 5 days.



5. Determining reorder levels

The expressions for the KPI's derived in chapters 2 and 3 can be used directly to determine reorder levels. If for example the commercial or inventory manager requires that each item should meet a given target fill rate, formula (2.2) in combination with a search procedure will provide the right reorder level. In the DoBr-tool a simple function is available which only requires the system parameters and the target fill rate as an input and then returns the reorder level. Many inventory control systems in practice set reorder levels in a similar way rather than using cost parameters. This may be due to its simplicity (reducing a potential multi-item problem to a single-item problem), the lack of accurate cost data (e.g. how to determine the cost of a backorder-situation), or the fact that customers do not accept that fill rates differ strongly between different items (having the perception that an item should either be included in the assortment and be available with a high probability or not be included in the assortment at all).

The expressions for the KPI's derived in earlier chapters can also be used to build optimization models with objective functions and/or restrictions for a single item or a group of items. Section 5.1 deals with the single item problem when the objective is to minimize the expected costs. Section 5.2 deals with the multi-item problem in the presence of a budget constraint. In section 5.3 other implementation issues are discussed when determining the reorder level.

5.1 The single item problem

If the base replenishment quantity is exogenous, e.g. determined by the supplier, and the inventory holding costs and the backordering costs are linear and we are dealing with a single item, then the objective function is

$$\min\ \Pi = 0.5h\{E[I^{OH}(\tau + L)] + E[I^{OH}(\tau + L + R)]\} + b\{E[BO(\tau + L + R)] - E[BO(\tau + L)]\} \tag{5.1}$$

In this objective function we approximate the average expected inventory as the simple average of the inventory at the beginning and end of an arbitrary potential delivery cycle.

with

Π = the expected total relevant costs per review period

h_k = the inventory holding cost per unit of inventory on hand for item k

b_k = the backordering cost per unit backordered for item k

To get familiar with the behavior of this objective function in different parameter settings, the reader is invited to implement this function in the DoBr-tool and to visualize the value of the objective function as a function of the reorder level. Implementation of the objective function in the DoBr-tool is straightforward using the functions DoBr_EIOH_Begin, DoBr_EIOH_End and DoBr_FillRate (and using (1.4) to express the expected backordering costs per review period in terms of the fill rate).

When demand is continuous, formulas (3.1) and (3.4) can be used to transform this objective function into

$$\min\ \Pi = h\{s + 0.5Q - 0.5E[D_L] - 0.5E[D_{L+R}]\} +$$
$$(0.5h - b)E[BO(\tau + L)] + (0.5h + b)E[BO(\tau + L + R)].$$

The optimal reorder level which minimizes this objective function can be found by setting the derivative equal to zero

$$\frac{d\Pi}{ds} = h + (0.5h - b)\frac{dE[BO(\tau + L)]}{ds} + (0.5h + b)\frac{dE[BO(\tau + L + R)]}{ds} = 0$$

Intermezzo: *According to Leibnitz's rule, if we have an integral of the form*

$$I(s) = \int_{y_1(s)}^{y_s(s)} f(s, y)dy,$$

then the derivative of this integral is equal to

$$\frac{dI(s)}{ds} = \int_{y_1(s)}^{y_2(s)} \frac{df(s,y)}{ds} dy + f(s, y_2)\frac{dy_2(s)}{ds} - f(s, y_1)\frac{dy_1(s)}{ds}.$$

When applying Leibnitz's rule to formula (3.1)

$$E[BO(\tau + t)] = \frac{1}{Q}\int_{s}^{s+Q} \frac{1}{2}(x - s)^2 f_t(x)dx + \int_{s+Q}^{\infty} x f_t(x)dx - \left(s + \frac{Q}{2}\right)\int_{s+Q}^{\infty} f_t(x)dx,$$

we have

$$\frac{dE[BO(\tau + t)]}{ds} =$$

$$= \frac{1}{Q}[-\int_{s}^{s+Q}(x - s)f_t(x)dx + \frac{1}{2}Q^2 f_t(s + Q) - 0]$$

$$+[0 + 0 - (s + Q)f_t(s + Q)] - \{\left(s + \tfrac{Q}{2}\right)\cdot\left(-f_t(s + Q)\right) + \int_{s+Q}^{\infty} f_t(x)dx\}$$

$$= \frac{-1}{Q}\int_{s}^{s+Q}(x-s)f_t(x)dx - [1 - \int_{-\infty}^{s+Q} f_t(x)dx\,] \qquad (5.2)$$

This implies (see the derivation of formula (3.5))

$$\frac{dE[BO(\tau+L+R)]}{ds} = P_3^{discrete} - 1. \qquad (5.3)$$

So we have

$$\frac{d\Pi}{ds} = h + (0.5h - b)\{P_3^{discrete}(s;R=0) - 1\} + (0.5h + b)\{P_3^{discrete}(s;R) - 1\}$$

with $P_3^{discrete}(s;R=0)$ being equal to formula (3.5) with the review period equal to zero ($R = 0$).

The second derivative of $E[BO(\tau+t)]$ is equal to (using again Leibnitz's rule)

$$\frac{d^2E[BO(\tau+t)]}{ds^2} = \frac{1}{Q}\int_{s}^{s+Q} f_t(x)dx \qquad (5.4)$$

So the second derivative of the objective function is equal to

$$\frac{d^2\Pi}{ds^2} = \frac{0.5h - b}{Q}\int_{s}^{s+Q} f_L\,(x)dx + \frac{0.5h + b}{Q}\int_{s}^{s+Q} f_{L+R}(x)dx.$$

If this second derivative is non-negative (e.g. when $f_{L+R}(x) \geq f_L(x)$, which holds e.g. for the normal probability density function if $x \geq \mu_{L+R}$), the objective function is convex in s and therefore the optimal reorder level can be determined by setting the first derivative of the objective function equal to zero, i.e. to solve

$$h + (0.5h - b)\{P_3^{discrete}(s;R=0) - 1\} + (0.5h + b)\{P_3^{discrete}(s;R) - 1\} = 0 \qquad (5.5)$$

A simple bi-section search procedure can be used for this.

In case the base replenishment quantity Q is not exogenous, an additional term is added to the objective function representing the expected ordering costs. If the fixed cost per order is equal to K, the expected costs per review period are equal to

$$\min\ \Pi = KE[OL] + h\{s + 0.5Q - 0.5E[D_L] - 0.5E[D_{L+R}]\} +$$
$$(0.5h - b)E[BO(\tau+L)] + (0.5h + b)E[BO(\tau+L+R)] \qquad (5.6)$$

Note we do not need to include the variable ordering (or purchasing) costs, since the expected demand and sales are not dependent on the reorder level and the base replenishment quantity due to the backordering assumption.

The optimal reorder level and the optimal base replenishment quantity should then be solved by the equations which emerge from setting the two partial derivatives of the objective function equal to zero. In combination with (5.5) we then also have

$$\frac{\partial \Pi}{\partial Q} = K\frac{\partial E[OL]}{\partial Q} + 0.5h + (0.5h - b)\frac{\partial BO(\tau + L)}{\partial Q} + (0.5h + b)\frac{\partial BO(\tau + L + R)}{\partial Q} = 0 \quad (5.7)$$

with

$$\frac{\partial E[OL]}{\partial Q} = \frac{-1}{Q^2}\int_0^Q x f_R(x)dx$$

$$\frac{\partial BO(\tau + t)}{\partial Q} = \frac{-1}{2Q^2}\int_s^{s+Q}(x - s)^2 f_t(x)dx - \frac{1}{2}\int_{s+Q}^{\infty} f_t(x)dx$$

The results for the partial derivates are derived using Leibnitz's rule, (3.1) and (3.2).

Then the optimal reorder level and optimal base replenishment quantity can be found by iteratively solving (5.5) and (5.7).

In practice, the base replenishment quantity is often limited to be chosen from a finite set of values. For example, handling in a warehouse/distribution centre is typically more efficient if it is restricted to handling an integer number of layers on a pallet, half a pallet (rounded to the nearest integer number of layers), three quarters of a full pallet (rounded to the nearest integer number of layers) or in full pallets. If the ordering costs mainly consist of material handling costs, then these ordering costs may be non-linear in the expected number of orderlines: handling layers often implies additional handling time (for putting the layer(s) on a pallet) compared to handling full pallets. Due to the limited set of potential optimal values for Q and due to the potential non-linear expected ordering costs, it may be better in these situations to determine the optimal reorder level and the optimal base replenishment quantity using (5.5) to determine the optimal reorder level for each potential base replenishment quantity and then evaluate the expected total relevant costs per review period using (5.6) for each potential base replenishment quantity. Finally the optimal reorder level and base replenishment quantity are selected based on the lowest expected total relevant costs.

5.2 The multi item problem with budget constraint

In this section we consider a multi-item problem. More specifically we consider the following problem: Given a budget (in euro's) which is available to invest in inventories, how to set the reorder levels for a group of items in such a way that the costs for backorders per review period are minimized?

If we approximate the average expected inventory by taking the simple average of $E[I^{OH}(\tau+L)]$ and $E[I^{OH}(\tau+R+L)]$, this problem can be formulated as

$$\min \sum_{k=1}^{n}(1-P_{2,k}(s_k))\cdot E\left[D_{R,k}\right]\cdot b_k$$

$$\text{s.t.} \quad \sum_{k=1}^{n} 0.5\{E[I_k^{OH}(\tau+L)]+E[I_k^{OH}(\tau+L+R)]\}\cdot g_k = B$$

$$s_k \geq 0, k=1,\dots,n$$

with

b_k = the cost per unit backordered for item k

g_k = the value per unit of inventory on hand for item k

B = the budget available to invest in inventories

n = the number of items in the group

Strictly speaking, the budget restriction is a less-than-or-equal-to restriction. However, if we assume demand is continuous and the costs per unit backordered are positive, it is optimal to allocate the total budget and hence to use an equal-to restriction. Note that in this problem we do not include the ordering costs in the objective function since we assume that the base replenishment quantity is an exogenous variable. Also the inventory holding costs are not included in the objective function, which is correct if the inventory holding costs are equal to a constant (which is independent of the item) times the value of a unit. To see this, note that in this case the budget determines the total value of the inventories and hence the inventory holding costs are fixed if the budget is fixed.

Since often in practice the backordering costs are not known, a simplification of this model may be used by setting b_k equal to 1. In this case the objective function is to maximize the expected total demand per review period which is not backordered given the budget for inventories. This is identical to maximizing the aggregate fill rate for this group of items, since the average demand per review period for all items is a constant. This simplified objective function is in line with current practice where it is observed that sometimes companies report the overall performance of their inventory system in terms of two indicators: the total value of the inventories and some kind of aggregate fill rate, where the latter typically does not take into account the value or costs of the item.

The restriction can be included in the objective function using a Lagrange-multiplier λ.

The objective function then becomes

$$\min \sum_{k=1}^{n}(1-P_{2,k}(s_k))E\left[D_{R,k}\right]b_k$$

$$+\lambda[\sum_{k=1}^{n} 0.5 g_k\{E[I_k^{OH}(\tau+L)] + E[I_k^{OH}(\tau+L+R)]\} - B]$$

s.t. $s_k \geq 0, k = 1, \ldots, n$

When demand is continuous, formulas (1.4) and (3.4) can be used to transform this objective function into

$$\min\ \Pi = \sum_{k=1}^{n}\{E[BO_k(\tau+L+R)] - E[BO_k(\tau+L)]\}b_k\ +\ \lambda\,[-B +$$

$$\sum_{k=1}^{n} g_k\{s_k + 0.5(Q_k - E\left[D_{L,k}\right] - E\left[D_{L+R,k}\right] + E[BO_k(\tau+L+R)] + E[BO_k(\tau+L)])\}]$$

Taking the partial derivatives we get

$$\frac{\partial \Pi}{\partial s_k} = \frac{\partial E[BO_k(\tau+L+R)]}{\partial s_k}\{b_k + 0.5\lambda g_k\} - \frac{\partial E[BO_k(\tau+L)]}{\partial s_k}\{b_k - 0.5\lambda g_k\} + \lambda g_k = 0$$

$$\frac{\partial \Pi}{\partial \lambda} = 0$$

$$= \sum_{k=1}^{n} g_k\left\{s_k + 0.5(Q_k - E\left[D_{L,k}\right] - E\left[D_{L+R,k}\right] + E[BO_k(\tau+L+R)] + E[BO_k(\tau+L)])\right\} - B \quad (5.8)$$

So we have

$$\frac{\partial \Pi}{\partial s_k} = \left[P_3^{discrete}(s_k; R=0) - 1\right]\{b_k + 0.5\lambda g_k\}$$

$$-\left[P_3^{discrete}(s_k; R) - 1\right]\{b_k - 0.5\lambda g_k\} + \lambda g_k = 0$$

with $P_3^{discrete}(s_k; R=0)$ being equal to formula (3.5) with the review period equal to zero ($R = 0$).

This can be rewritten as:

$$\frac{P_3^{discrete}(s_k; R=0)}{P_3^{discrete}(s_k; R)} = \frac{b_k + 0.5\lambda g_k}{b_k - 0.5\lambda g_k} \qquad (5.9)$$

This equation can be solved if $0 < \lambda < \frac{2b_k}{g_k}$ for all k

($\lambda > 0$ since $P_3^{discrete}(s_k; R=0) > P_3^{discrete}(s_k; R)$ if $R > 0$).

So the optimal reorder levels for a group of items with a restriction on the total available budget can be determined by iteratively solving equations (5.8) and (5.9). Lau and Lau (1995) show for a similar (but single-period) problem that if demand can never be zero, or has a very long left tail, it may be impossible to numerically solve an equation like (5.9) for certain values of λ. For those situations the reader is referred to the methodology in their paper, which is developed to provide a solution for any demand distribution.

The solution from (5.8) and (5.9) can be used, by varying the budget, to present exchange curves to the commercial, operations and financial managers showing how the aggregate fill rate increases if the budget for inventories increases. This type of exchange curve is an effective decision support tool for setting parameters at the senior management level.

5.3 Implementation issues

When implementing inventory control theory in practice, several implementation issues may arise. We will discuss some of these issues below.

- Selection of the demand distribution function

There are several ways to decide which demand distribution to use. If in reality demand is discrete, the most obvious choice is to use either the empirical distribution or to fit a theoretical distribution function on this empirical distribution function (as discussed in chapter 2). Especially for items with low or very low demand (like in retail or spare parts settings) this will give much better results than using a continuous distribution. However, if the average demand during lead time plus review period is very large (say >2000 units) and at the same time reorder levels have to be recalculated for more than 100,000 items on a daily basis, then the computational time will become rather large when using a discrete distribution function. At the same time, the inaccuracy of using a continuous distribution function will decrease when the average demand during the lead time plus review period increases. Therefore, if the computation time becomes prohibitive we suggest to use a continuous distribution function for these items.

- Adjusting reorder levels when using continuous demand models in discrete demand environments

When the KPI's are determined using continuous demand models while in reality demand is discrete, Zipkin (2000, p. 210) suggests to correct the reorder level with 0.5 in the formulas for continuous demand models. In the DoBr-tool we assume that demand in reality is continuous when a gamma or normal distribution is applied and therefore the reorder level is not adjusted for this. When the continuous distribution is applied in situations with a very large reorder level, then the impact of having a correction with 0.5 or not will be very small.

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Appendix 1. The expectation of the function of a stochastic variable

How to determine the expectation of the function of a stochastic variable depends on whether the variable is discrete or continuous. These two cases will be discussed separately.

A1.1 The expectation of the function of a discrete stochastic variable

Let X be a discrete stochastic variable. The realization (in a random experiment) of this variable X is equal to an integer value with a certain probability. With $P(X = k)$) we refer to the probability that the realization of the stochastic variable X takes on the value k. For discrete stochastic variables it is known that by definition $\sum_{k=0}^{\infty} P(X = k) = 1$. In this formula we assumed that X can only take on non-negative values (0, 1, 2,…).

For this course it is very important to understand the meaning of 'the expectation of the function of a stochastic variable' and how to calculate/determine this expectation. The simplest expectation is the expectation of X itself, $E[X]$. This can be calculated as follows

$$E[X] = \sum_{k=0}^{\infty} kP(X = k) \qquad \text{(A1.1)}$$

To determine the variance of a (discrete or continuous) stochastic variable we only need the expectation of X and X^2, since $\mathrm{var}[X] = E[X^2] - (E[X])^2$. The expectation of X^2 can be calculated as follows

$$E[X^2] = \sum_{k=0}^{\infty} k^2P(X = k) \qquad \text{(A1.2)}$$

Rather than memorizing for every function of X how to determine its expectation, it is much easier and more insightful to know how to determine for any function of a stochastic variable its expectation. Let $h(X)$ be an arbitrary function of the discrete stochastic variable X (with domain k=0, 1, 2, …). The expectation of this function is simply the weighted sum of the value of this function for a given realization of the stochastic variable ($h(k)$), where the weights are simply equal to the probability that the stochastic variable X takes on this value k. So we have

$$E[h(X)] = \sum_{k=0}^{\infty} h(k)P(X = k) \qquad \text{(A1.3)}$$

Note that for $h(X) = X$ and $h(X) = X^2$ this simply results in formulas (A1.1) resp. (A1.2)!

This expression for the expectation of the function of a stochastic variable is not only relevant for this course Stochastic Operations Management, but will also be used in multiple other courses in the Bachelor and Master program. From here on students are expected to be able to determine these expectations!

Slightly more complex is the determination of the expectation of the function of two discrete stochastic variables. In this case we take the weighted average over both stochastic variables, so we have

$$E[h(X,Y)] = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} h(k,j)\,P(X = k)P(Y = j) \qquad \text{(A1.4)}$$

A1.2 The expectation of the function of a continuous stochastic variable

For a continuous stochastic variable Z there is no use in working with the probability that this variable takes on a specific value z, (i.e. $\mathrm{P}(\mathrm{Z}=\mathrm{z})$), since this probability cannot be distinguished from zero. It is common to use $\mathrm{f}(\mathrm{z})$ and $\mathrm{F}(\mathrm{z})$ rather than with $\mathrm{P}(\mathrm{Z}=\mathrm{z})$ and $(\mathrm{Z} \leq \mathrm{z})$, where $\mathrm{f}(.)$ f is the probability density function and $\mathrm{F}(.)$ the cumulative probability distribution function.
Again we know by definition that all probability densities add up to 1: $\int_{-\infty}^{\infty} \mathrm{f}(\mathrm{z})\mathrm{dz} = 1$.
Here we assumed that Z can take on both negative and non-negative values. Furthermore the following relation holds for $\mathrm{F}(\mathrm{z})$

$$\mathrm{F}(\mathrm{z}) = \int_{-\infty}^{\mathrm{z}} \mathrm{f}(\mathrm{r})\mathrm{dr}.$$

The expectation of a continuous stochastic variable is determined analogous to the expectation of a disrete stochastic variable:

$$\mathrm{E}(\mathrm{Z}) = \int_{-\infty}^{\infty} \mathrm{z}\mathrm{f}(\mathrm{z})\mathrm{dz} \qquad \text{(A1.5)}$$

The more general expression for the expectation of the function $\mathrm{h}(\mathrm{Z})$ of a continuous stochastic variable Z is equal to

$$\mathrm{E}[\mathrm{h}(\mathrm{Z})] = \int_{-\infty}^{\infty} \mathrm{h}(\mathrm{z})\mathrm{f}(\mathrm{z})\mathrm{dz} \qquad \text{(A1.6)}$$

And for the expectation of a function of two stochastic variables Z and W with $\mathrm{f}(.)$ the probability density function for Z and $\mathrm{g}(.)$ the probability density function for W we have:

$$\mathrm{E}[\mathrm{h}(\mathrm{Z},\mathrm{W})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{h}(\mathrm{z},\mathrm{w})\,\mathrm{f}(\mathrm{z})\mathrm{g}(\mathrm{w})\mathrm{dzdw} \qquad \text{(A1.7)}$$

Note the similarity with the expressions for the discrete stochastic variables where the probabilities are replaced by the probability densities and where the summation signs are replaced by the integrals.

Appendix 2. Derivation of formulas for standard normal demand

In this appendix we will show how to transform expressions for a non-standard normally distributed variable X with pdf $f(x)$, mean μ and standard deviation σ into expressions for a standard normally distributed variable V with pdf $\varphi(v)$and cdf $\Phi(v)$

For this purpose it will be very helpful to first consider the relationships between the standard (or unit) normal probability distribution function (pdf) $\varphi(v)$ and its first and second order derivatives $\varphi^{'}(v)$ and $\varphi^{''}(v)$ and the standard normal cumulative distribution function (cdf) $\Phi(v)$. To this end we first derive the first and second order derivatives of $\varphi(v)$

$$\varphi(v) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{v^2}{2})$$

$$\varphi^{'}(v) = \frac{-v}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right) = -v\varphi(v)$$

$$\varphi''(v) = \frac{(v^2-1)}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right) = (v^2-1)\varphi(v)$$

With these equations and knowing that $\Phi'(v) = \varphi(v)$ we can derive the following results

$$\int_a^b \varphi(v)dv = \Phi(b) - \Phi(a)$$

$$\int_a^b v\varphi(v)dv = \int_a^b -\varphi'(v)dv = \varphi(a) - \varphi(b)$$

$$\begin{aligned}\int_a^b v^2\varphi(v)dv &= \int_a^b \varphi''(v)dv + \int_a^b \varphi(v)dv = \varphi'(b) - \varphi'(a) + \Phi(b) - \Phi(a)\\ &= a\varphi(a) - b\varphi(b) + \Phi(b) - \Phi(a).\end{aligned}$$

The following results can be used to transform expressions for a non-standard normally distributed variable X with pdf $f(x)$, mean μ and standard deviation σ into expressions for a standard normally distributed variable V with pdf $\varphi(v)$and cdf $\Phi(v)$

$$\int_a^b f(x)dx = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi(\frac{a-\mu}{\sigma})$$

$$\begin{aligned}\int_a^b xf(x)dx &= \int_a^b \sigma\frac{(x-\mu+\mu)}{\sigma}f(x)dx\\ &= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \sigma\left(v+\frac{\mu}{\sigma}\right)\frac{1}{\sigma}\varphi(v)d(\sigma v)\\ &= \sigma\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} v\varphi(v)dv + \mu\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma}\varphi(v)dv\end{aligned}$$

$$= \sigma\varphi\left(\frac{a-\mu}{\sigma}\right) - \sigma\varphi\left(\frac{b-\mu}{\sigma}\right) + \mu\Phi\left(\frac{b-\mu}{\sigma}\right) - \mu\Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$\begin{aligned}
\int_a^b x^2 f(x)dx &= \int_a^b \sigma^2 (\frac{x-\mu+\mu}{\sigma})^2 f(x)dx \\
&= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \sigma^2 (v+\frac{\mu}{\sigma})^2 \frac{1}{\sigma}\varphi(v)d(\sigma v) \\
&= \sigma^2 \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} v^2\varphi(v)dv + 2\mu\sigma \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} v\varphi(v)dv + \mu^2 \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \varphi(v)dv \\
&= \sigma^2[\frac{a-\mu}{\sigma}\varphi\left(\frac{a-\mu}{\sigma}\right) - \frac{b-\mu}{\sigma}\varphi\left(\frac{b-\mu}{\sigma}\right) + \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)] \\
&\quad + 2\mu\sigma[\varphi\left(\frac{a-\mu}{\sigma}\right) - \varphi\left(\frac{b-\mu}{\sigma}\right)] \\
&\quad + \mu^2[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi(\frac{a-\mu}{\sigma})] \\
&= \sigma(\mu+a)\varphi\left(\frac{a-\mu}{\sigma}\right) - \sigma(\mu+b)\varphi\left(\frac{b-\mu}{\sigma}\right) \\
&\quad + (\sigma^2+\mu^2)\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right].
\end{aligned}$$

Appendix 3. Demand during the lead time (plus review period) when the lead time is uncertain

In this Appendix we derive formulas (4.2) and (4.3), i.e. the formulas for the mean and variance of

1. the demand during the lead time and
2. the demand during the lead time plus review period.

We assume both R and L are equal to an integer number of periods, say K and M. R is a deterministic constant and L a discrete stochastic variable with mean $E[L]$ and variance $var[L]$. Then M is also a discrete stochastic variable with mean $E[M]$ and variance $var[M]$.

Note that L has the dimension "time" whereas M is a dimensionless number. Using the fact that demand is i.i.d. we have

$$E[D(\tau,\tau+R+L)](=E[D_{L+R}])$$

$$= E[\textstyle\sum_{k=1}^{K+M} D(\tau+k-1,\tau+k)]$$

$$= \textstyle\sum_{k=1}^{K} E[D(\tau+k-1,\tau+k)] + E[\sum_{k=K+1}^{K+M} D(\tau+k-1,\tau+k)]$$

$$= \textstyle\sum_{k=1}^{K} E[D(\tau+k-1,\tau+k)] + E[D(\tau+K,\tau+K+M)]$$

$$= KE[D_1] + E[D_M]$$

and

$$var[D(\tau,\tau+R+L)](=var[D_{L+R}])$$

$$= var\left[\sum_{k=1}^{K+M} D(\tau+k-1,\tau+k)\right] =$$

$$= var\left[\sum_{k=1}^{K} D(\tau+k-1,\tau+k)\right] + var[\sum_{k=K+1}^{K+M} D(\tau+k-1,\tau+k)]$$

$$= var\left[\sum_{k=1}^{K} D(\tau+k-1,\tau+k)\right] + var[D(\tau+K,\tau+K+M)]$$

$$= Kvar[D_1] + var[D_M]$$

For convenience we define L and R too as a *number* of periods. Then $L = M$ and $R = K$ and the equations above result in formula (4.2).

Next, we determine $E[D_M]$ and $var[D_M]$ with M a discrete stochastic variable.

$$E[D_M] = E[\sum_{k=1}^{M} D(\tau + k - 1, \tau + k)]$$

Since this is an expectation of the function of a discrete stochastic variable M (and a stochastic variable D) we determine this expectation using (A1.3) from Appendix 1

$$E[h(X)] = \sum_{k=0}^{\infty} h(k)P(X = k) \qquad \text{(A1.3)}$$

$$E[D_M] = E[\sum_{k=1}^{M} D(\tau + k - 1, \tau + k)]$$

$$= \sum_{m=0}^{\infty} E[\sum_{k=1}^{m} D(\tau + k - 1, \tau + k)]P(M = m)$$

$$= \sum_{m=0}^{\infty} m \cdot E[D_1] \cdot P(M = m)$$

$$= E[D_1] \sum_{m=0}^{\infty} m \cdot P(M = m)$$

$$= E[M] \cdot E[D_1]$$

To determine $var[D_M]$, we use the fact that $var[D_M] = E[D_M^2] - E^2[D_M]$.

Hence we first determine

$$E[D_M^2] = E[\{\sum_{k=1}^{M} D(\tau + k - 1, \tau + k)\}^2]$$

$$= \sum_{m=0}^{\infty} E[\{\sum_{k=1}^{m} D(\tau + k - 1, \tau + k)\}^2]P(M = m)$$

$$\left(\text{Since } \sigma^2(X) = E(X^2) - E^2(X) \text{ we have } E(X^2) = \sigma^2(X) + E^2(X), \text{ so}\right)$$

$$= \sum_{m=0}^{\infty} \{var[\sum_{k=1}^{m} D(\tau + k - 1, \tau + k)] + E^2[\sum_{k=1}^{m} D(\tau + k - 1, \tau + k)]\}P(M = m)$$

$$= \sum_{m=0}^{\infty} \{m \cdot var[D_1] + (m \cdot E[D_1])^2\}P(M = m)$$

$$= E[M\} \cdot var[D_1] + E^2[D_1] \cdot E[M^2]$$

and then we have

$$var[D_M] = E[D_M^2] - E^2[D_M]$$

$$= E[M\} \cdot var[D_1] + E^2[D_1] \cdot E[M^2] - E^2[M] \cdot E^2[D_1]$$

$$= E[M\} \cdot var[D_1] + E^2[D_1] \cdot var[M].$$

For convenience we define L too as a *number* of periods. Then $L = M$ and the equations above result in formula (4.3).

Note that in order to keep dimensions correct, it is essential to interpret L as a *number* of periods and not as delivery *time*.

Appendix 4. Alternative formulas for $E[BO(\tau+t)]$

Zipkin (Foundations in Inventory Management (2000)) introduced the ccdf (F^0), the first order loss function (F^1), and the second order loss function (F^2) for any continuous demand distribution f(x), with:

$$F^0(y) = \int_y^\infty f(x)dx$$
$$F^1(y) = \int_y^\infty (x-y)f(x)dx$$
$$F^2(y) = \int_y^\infty (x-y)^2 f(x)dx$$

Then the generic expression for $E[BO(\tau+t)]$ can be expressed in these functions very easy:

$$E[BO(\tau+t)] = \frac{1}{Q}[F^2(s) - F^2(s+Q)] + F^1(s+Q) + \frac{Q}{2}F^0(s+Q)$$

Alternative derivation of BO-formula for gamma demand:

According to Zipkin (2000, p.457) the following equations hold for gamma distributed demand with the pdf $f(x|\alpha,\beta)$ and with cdf $F(x|\alpha,\beta)$:

$$F^0(y) = 1 - F(y|\alpha,\beta)$$

$$F^1(y) = (\alpha\beta - y)F^0(y) + \beta y f(y|\alpha,\beta)$$

$$F^2(y) = \frac{1}{2}[(\alpha\beta - y)^2 + \alpha\beta^2]F^0(y) + \frac{1}{2}\beta\big((\alpha+1)\beta - y\big)yf(y|\alpha,\beta)$$

With these functions and some straightforward algebra the generic expression for $E[BO(\tau+t)]$ can be rewritten into the following formula for **gamma** distributed demand (note that 1/(2Q) holds for all 4 parts of the expression!):

$$E[BO(\tau+t)] = \frac{1}{2Q}\{\ F^0(s)[(\alpha\beta - s)^2 + \alpha\beta^2]$$

$$+ F^0(s+Q)[-2Q^2 + 4(\alpha\beta - s)Q - (\alpha\beta - s)^2 - \alpha\beta^2]$$

$$+ f(s|\alpha,\beta)[(\alpha+1)\beta^2 s - \beta s^2]$$

$$+ f(s+Q|\alpha,\beta)\big[3\beta Q^2 + \beta Q\big(4s - \beta(\alpha+1)\big) + \beta s^2 - \beta^2 s(\alpha+1)\big]\ \}$$

Since D_t is gamma distributed with mean $\alpha\beta$ and variance $\alpha\beta^2$, α and β in the equation above can be solved from $\alpha\beta = t\mu$ and $\alpha\beta^2 = t\sigma^2$.

Note: Zipkin (2000, p. 457) also mentions that one can fit α and β to any positive mean and variance.

Alternative derivation of BO-formula for normal demand:

Before rewriting this expression let us first consider the derivates of $\varphi(v)$ and $\varphi'(v)$:

$$\varphi(v) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{v^2}{2})$$

$$\varphi'(v) = \frac{-v}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right) = -v\varphi(v)$$

$$\varphi''(v) = \frac{(v^2-1)}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right) = (v^2-1)\varphi(v)$$

With these results we can rewrite the three integrals in (2):

$$\int_k^{k+c} (v-k)^2\,\varphi(v)dv =$$
$$= \int_k^{k+c}(v^2-1)\varphi(v)dv - 2k\int_k^{k+c} v\varphi(v)dv + (k^2+1)\int_k^{k+c}\varphi(v)dv$$
$$= \varphi'(k+c) - \varphi'(k) + 2k\big(\varphi(k+c)-\varphi(k)\big) + (k^2+1)(\Phi(k+c)-\Phi(k))$$
$$= -(k+c)\varphi(k+c) + k\varphi(k) + 2k\big(\varphi(k+c)-\varphi(k)\big) + (k^2+1)(\Phi(k+c)-\Phi(k))$$
$$= (k-c)\varphi(k+c) - k\varphi(k) + (k^2+1)(\Phi(k+c)-\Phi(k))$$

$$\int_{k+c}^{\infty}(v-k-c)\varphi(v)dv = \int_{k+c}^{\infty} v\varphi(v)dv - (k+c)\int_{k+c}^{\infty}\varphi(v)dv$$
$$= \varphi(k+c) - (k+c)(1-\Phi(k+c))$$

$$\int_{k+c}^{\infty}\varphi(v)dv = 1-\Phi(k+c)$$

In Section 3.1 we derived the following expressions for the fill rate P_2

$$P_2 = 1 - \frac{E[BO(\tau+R+L)] - E[BO(\tau+L)]}{E[D_R]} \qquad (A4.1)$$

with

$$E[BO(\tau+t)] = \frac{1}{Q}\int_s^{s+Q}\frac{1}{2}(x-s)^2 f(x)dx + \int_{s+Q}^{\infty}(x-s-Q)f(x)dx + \frac{Q}{2}\int_{s+Q}^{\infty}f(x)dx$$

and $f(.)$ the probability density function for the stochastic variable D_t, the demand during t periods.

In case x is **normally** distributed, we can rewrite this into the following expression, by using the substitutions $v = \frac{x-\mu}{\sigma}$, $k = \frac{s-\mu}{\sigma}$ and $c = \frac{Q}{\sigma}$ with μ and σ the mean resp. standard deviation of x, with $\varphi(v)$ and $\Phi(v)$ the standard (or unit) normal probability density function (pdf) and cumulative distribution function (cdf).

$$E[BO(\tau + t)] = \frac{\sigma}{2c}\int_k^{k+c}(v-k)^2\,\varphi(v)dv + \sigma\int_{k+c}^{\infty}(v-k-c)\varphi(v)dv + \frac{c\sigma}{2}\int_{k+c}^{\infty}\varphi(v)dv$$
(2)

By using (3.30)-(3.32) this can be rewritten into

$$E[BO(\tau + t)] = \frac{\sigma}{2c}[(k-c)\varphi(k+c) - k\varphi(k) + (k^2+1)\big(\Phi(k+c) - \Phi(k)\big)] +$$
$$\sigma\left(\varphi(k+c) - (k+c)\big(1-\Phi(k+c)\big)\right) + \frac{c\sigma}{2}(1-\Phi(k+c))$$
$$= \varphi(k+c)\left(\frac{k}{c}+1\right)\frac{\sigma}{2} - \varphi(k)\frac{k\sigma}{2c} + \Phi(k+c)\left((k^2+1)\frac{\sigma}{2c} + (k+c)\sigma - \frac{c\sigma}{2}\right) -$$
$$\Phi(k)(k^2+1)\frac{\sigma}{2c} - \sigma(k+\frac{c}{2}). \qquad \text{(A4.2)}$$

Note that in (A4.2) σ represents σ_t (for ease of notation we left out the subindices t so far), k represents $k_t = \frac{s-\mu_t}{\sigma_t}$ and c represents $c_t = \frac{Q}{\sigma_t}$ with $\mu_t = t\mu$ and $\sigma_t = \sqrt{t}\sigma$ if demand is i.i.d.

When combining (A4.1) and (A4.2), we have an exact expression for the P_2 in an (R,s,nQ)-system with normally distributed demand and backordering. Expression (A4.2) can be rewritten into expression (3.15).

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