

Generalized Hopf fibration and geometric $SO(3)$ reduction of the 4DOF harmonic oscillator

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Generalized Hopf fibration and geometric $SO(3)$ reduction
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by

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Generalized Hopf fibration and geometric $SO(3)$ reduction of the 4DOF harmonic oscillator

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Abstract

It is shown that the generalized Hopf map $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{R} \times \mathbb{R}$ in quaternion formulation can be interpreted as an $SO(3)$ orbit map for a symplectic $SO(3)$ action. As a consequence the generalized Hopf fibration $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ appears in the $SO(3)$ geometric symplectic reduction of the 4DOF isotropic harmonic oscillator.

Key Words: Hopf map, Hopf fibration, symplectic reduction, harmonic oscillator.

AMS Subject Classification: primary: 53D20, 37J15, secondary: 55R10.

1 Introduction

It is well known that one may reduce the phase space of the 2DOF harmonic oscillator by using the symmetry given by the \mathbb{S}^1 -action generated by the flow of the Hamiltonian. Restricting the reduction mapping to the three spheres given by the energy surfaces, the orbit mapping becomes a reduction of the energy surface \mathbb{S}^3 to the reduced phase space \mathbb{S}^2 , which mapping is exactly the Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{S}^2$. See [3]. In section 2. we will make this precise starting from the classical description of the Hopf fibration in complex coordinates. In the following sections we generalize these ideas to the 4DOF isotropic harmonic oscillator. For this purpose we will introduce quaternions in section 3. and the generalized generalized Hopf map in terms of quaternions in section 4. In section 5 we will show that, after choosing an appropriate real representation, this Hopf map can be considered as an orbit map for an appropriately chosen symplectic $SO(3)$ action, and that restriction to an energy surface for the harmonic oscillator, gives the generalized Hopf fibration $\mathbb{S}^7 \rightarrow \mathbb{S}^4$. In the final section we will say something about how the twistor fibration, that appears in the diagram for the generalized Hopf fibration, fits in the scheme of symplectic reduction.

2 Hopf fibration in 2DOF

In this section we will start with the common complex description of the Hopf map and show how this leads to an orbit map and reduction map for a 2DOF harmonic oscillator.

For the ambient space of the classical Hopf fibration, consider the complex vector space \mathbb{C}^2 , together with the usual hermitian inner product. Let $z \in \mathbb{C}^2$ and $w \in \mathbb{C}^2$, where $z = (z_1, z_2)$, $w = (w_1, w_2)$ and $z_1, z_2, w_1, w_2 \in \mathbb{C}$, then

$$\langle z, w \rangle = \bar{z}w = \bar{z}_1w_1 + \bar{z}_2w_2 .$$

Thus \mathbb{S}^3 can be shown as a subset in \mathbb{C}^2 given by $\mathbb{S}^3 = \{z \in \mathbb{C}^2 \mid \langle z, z \rangle = 1\}$. Define an equivalence relation on \mathbb{S}^3 by $z \sim \omega$ iff $\omega = \lambda z$, $\lambda \in \mathbb{S}^1$. We get that

$$\mathbb{S}^3 / \sim \cong P(\mathbb{C}^2) = \mathbb{C}P^1 .$$

Therefore, by means of the stereographic projection $\mathbb{C}P^1 \rightarrow \mathbb{S}^2$, the classic Hopf fibration $\Pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is built in a constructive process that we may describe by means of the commutative diagram in figure (1).

$$\begin{array}{ccc}
 \mathcal{A}: \quad \mathbb{C}^2 - \{0\} & \xrightarrow{\quad\quad\quad} & \mathbb{C} \times \mathbb{R} \times \mathbb{R} - \{0\} \\
 z = (z_1, z_2) & \cdots\cdots\cdots\rightarrow & (z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2), \frac{1}{2}(|z_1|^2 + |z_2|^2)) \\
 \vdots \downarrow & & \downarrow \\
 \frac{r}{|z|}(z_1, z_2) & \xrightarrow{P_r} & \mathbb{S}^3(r) \\
 \vdots \downarrow & & \downarrow P_{1,2,3} \\
 \frac{r}{|z|}(z_1, z_2) & \xrightarrow{P_\sim} & \mathbb{C}P^1 \\
 \vdots \downarrow & & \downarrow \\
 [(z_1, z_2)] & \xrightarrow{\quad\quad\quad} & \mathbb{S}^2(\frac{r^2}{2}) \\
 & \xrightarrow{\quad\quad\quad} & (z_1\bar{z}_2, 1/2(|z_1|^2 - |z_2|^2))
 \end{array}$$

Figure 1: Hopf map and classic Hopf fibration diagram.

Here Π is the stereographic projection, P_r is the map from $\mathbb{C}^2 - \{0\}$ to $\mathbb{S}^3(r)$ that matches each semi-ray through the origin with the corresponding element of module equal to r , P_\sim from $\mathbb{S}^3(r)$ into $\mathbb{C}P^1$ identifies each point in the sphere to its corresponding class of equivalence, $P_{1,2,3}$ is the elimination of the fourth component and \mathcal{A} is the Hopf map. The Hopf fibration \mathcal{F}_r from $\mathbb{S}^3(r)$ to $\mathbb{S}^2(\frac{r^2}{2})$ is given by composing Π and P_\sim , it can also

be obtained by means of the restriction of \mathcal{A} , the Hopf map, to $\mathbb{S}^3(r)$ and composing it with $P_{1,2,3}$.

Notice that $\Pi : \mathbb{C}P^1 \longrightarrow \mathbb{S}^2(\frac{r}{2})$ is based on the classic stereographic projection from $\mathbb{S}^2(\frac{r}{2}) - N$ onto the real plane, in this case the north pole is covered by the infinite point $[(1, 0)]$.

Moving to cartesian coordinates, and taking into account the natural relation between \mathbb{C} and \mathbb{R}^2 , by setting $z = (z_1, z_2) = (q_1 + iq_2, Q_1 + iQ_2)$ we get $\mathcal{A}_{\mathbb{R}} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given by

$$\mathcal{A}_{\mathbb{R}}(q_1, q_2, Q_1, Q_2) = (\omega_1(q, Q), \omega_2(q, Q), \omega_3(q, Q), \omega_4(q, Q)) ,$$

where

$$\begin{aligned} \omega_1 &= q_1 Q_2 - q_2 Q_1, & \omega_3 &= \frac{1}{2}(q_1^2 + q_2^2 - Q_1^2 - Q_2^2), \\ \omega_2 &= q_1 Q_1 + q_2 Q_2, & \omega_4 &= \frac{1}{2}(q_1^2 + q_2^2 + Q_1^2 + Q_2^2) , \end{aligned}$$

and the following relation between the ω'_i s holds

$$\mathcal{C}(q_1, q_2, Q_1, Q_2) = \omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_4^2 = 0 . \quad (1)$$

Notice that the image of $\mathbb{C}^2 - \{0\}$ by the Hopf map is given by the algebraic manifold defined in (1) by \mathcal{C} , it is a 3-dimensional cone with vertex in the coordinate origin. If we consider \mathbb{R}^4 together with the standard symplectic form then the polynomials ω_i span a Lie algebra with bracket being the Poissonbracket induced by the symplectic form. ω_1 is in the center of this Lie algebra, ie. we have $\{\omega_1, \omega_i\} = 0$ for $i = 2, 3, 4$. Thus this Hopf map is an orbit map for the symplectic S^1 -action generated by the Hamiltonian ω_1 with respect to the standard symplectic form on \mathbb{R}^4 . Restriction to $\omega_4^{-1}(c)$, i.e. to levelsets of ω_4 , then gives a Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$, where the fibre is the orbit for the ω_1 -action. Although this restriction of the Hopf map does correspond to a reduction of the manifold $\omega_4^{-1}(c)$ for the ω_1 -action, it is not a reduction with respect to the ω_4 -action.

However, one can give a construction of the Hopf map such that it will be an orbit map for the ω_4 -action and such that the reduction map will be a Hopf fibration with the orbit of the ω_4 -action as a fibre, as will be shown below.

Consider \mathbb{R}^4 together with the standard symplectic form. The above given real representation of the Hopf map is not an orbit map for the symplectic action of ω_4 . To obtain a real representation which is an orbit map consider the real representation obtained by setting $z = (z_1, z_2) = (q_1 - iQ_1, q_2 - iQ_2)$, which gives

$$\tilde{\mathcal{A}}_{\mathbb{R}}(q_1, q_2, Q_1, Q_2) = (\tau_1(q, Q), \tau_2(q, Q), \tau_3(q, Q), \tau_4(q, Q)) ,$$

where

$$\begin{aligned}\tau_1 &= q_1 Q_2 - q_2 Q_1, & \tau_3 &= \frac{1}{2}((q_1^2 + Q_1^2) - (q_2^2 + Q_2^2)), \\ \tau_2 &= q_1 q_2 + Q_1 Q_2, & \tau_4 &= \frac{1}{2}(q_1^2 + q_2^2 + Q_1^2 + Q_2^2) = \omega_4,\end{aligned}$$

and the following relation between the τ_i 's holds

$$\mathcal{C}(q_1, q_2, Q_1, Q_2) = \tau_1^2 + \tau_2^2 + \tau_3^2 - \tau_4^2 = 0. \quad (2)$$

Then τ_4 is the Hamiltonian of the harmonic oscillator. The invariants for the symplectic \mathbb{S}^1 -action generated by the flow of the Hamiltonian vector field corresponding to τ_4 with respect to the standard symplectic form are τ_1, τ_2, τ_3 , and τ_4 . Consequently we have that $\tilde{\mathcal{A}}_{\mathbb{R}}$ is the orbit map for this \mathbb{S}^1 -action. Restriction of $\mathcal{A}_{\mathbb{R}}$ to level surfaces $\tau_4 = c$ then gives a map $\mathbb{S}^3 \rightarrow \mathbb{S}^2$, which is a reduction map with the 2-sphere given by $\tau_1^2 + \tau_2^2 + \tau_3^2 = c^2$ as a reduced phase space.

3 Preliminaries on quaternions

The generalization of the Hopf map that we give next is based upon the generalization from the complex numbers to the quaternions. For that reason, we provide here a brief description of the division ring \mathbb{H} , that also will set up a fixed notation through this note.

The hyper-complex numbers of rank 4 were invented by W. R. Hamilton [11], he gave them the name of *quaternion*, it is a division ring \mathbb{H} , the elements of which are denoted by $\mathbf{q} = (q_1, q) \in \mathbb{H}$, and may be regarded as a real part q_1 plus the imaginary vector part $q = (q_2, q_3, q_4) \in \mathbb{R}^3$. Quaternions having zero scalar part are called *pure quaternions* and can be regarded both as a vector in \mathbb{R}^3 and as a quaternion. Therefore there is a bijective identification between \mathbb{R}^3 and the pure quaternions.

The quaternions, together with the operations of addition (+) by components and scalar multiplication (\cdot), may be identified with \mathbb{R}^4 as a vector space. The quaternionic multiplication (\circ) provide a division ring structure to \mathbb{H} .

It is customary to use the notation $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for a basis in $(\mathbb{H}, +, \mathbb{H})$, that is, $\mathbf{1} = (1, 0, 0, 0)$, $\mathbf{i} = (0, 1, 0, 0)$, $\mathbf{j} = (0, 0, 1, 0)$ and $\mathbf{k} = (0, 0, 0, 1)$, thus elements in \mathbb{H} may be expressed as follows

$$\mathbb{H} := \{\mathbf{q} = q_1 \mathbf{1} + q_2 \mathbf{i} + q_3 \mathbf{j} + q_4 \mathbf{k}, q_1, q_2, q_3, q_4 \in \mathbb{R}\}.$$

Quaternionic multiplication is performed in the usual manner, like polynomial multiplication, taking the following relations into account. Note that the relations given in table 1.

\circ	$\mathbf{1}$	\mathbf{i}	\mathbf{j}	\mathbf{k}
$\mathbf{1}$	$\mathbf{1}$	\mathbf{i}	\mathbf{j}	\mathbf{k}
\mathbf{i}	\mathbf{i}	$-\mathbf{1}$	\mathbf{k}	$-\mathbf{j}$
\mathbf{j}	\mathbf{j}	$-\mathbf{k}$	$-\mathbf{1}$	\mathbf{i}
\mathbf{k}	\mathbf{k}	\mathbf{j}	$-\mathbf{i}$	$-\mathbf{1}$

Table 1: Multiplication table for basis vectors

for the multiplication may be deduced directly from the ones that Hamilton gave initially, namely $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$.

Also we can think of $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as the four roots of unity. An alternative way of defining the quaternionic product making use of the “dot” and “cross” in \mathbb{R}^3 is given by

$$\mathbf{q} \circ \mathbf{Q} = (q_1 Q_1 - q \cdot Q, q_1 Q + Q_1 q + q \times Q). \quad (3)$$

For the sake of a cleaner notation, we will drop the symbols (\cdot) and (\circ) , they will be used just in case of possible confusion.

Definition (3) is, of course, directly deduced from the relations given above and may be written explicitly in term of coordinates as follows

$$\begin{aligned} \mathbf{qQ} &= q_1 Q_1 - (q_2 Q_2 + q_3 Q_3 + q_4 Q_4) \\ &\quad + (q_2 Q_1 + q_1 Q_2 + q_4 Q_3 - q_3 Q_4) \mathbf{i} \\ &\quad + (q_3 Q_1 - q_4 Q_2 + q_1 Q_3 + q_2 Q_4) \mathbf{j} \\ &\quad + (q_4 Q_1 + q_3 Q_2 - q_2 Q_3 + q_1 Q_4) \mathbf{k}. \end{aligned} \quad (4)$$

In addition, every quaternion $\mathbf{q} = (q_1, q)$ has a *conjugate* $\bar{\mathbf{q}} = (q_1, -q)$, that is, the real numbers are fixed by the conjugation and $\bar{\mathbf{i}} = -\mathbf{i}$, $\bar{\mathbf{j}} = -\mathbf{j}$, $\bar{\mathbf{k}} = -\mathbf{k}$. Note that $\overline{\mathbf{qQ}} = \bar{\mathbf{Q}}\bar{\mathbf{q}}$.

The usual *hermitian inner product* is defined in \mathbb{H} as

$$\langle \mathbf{q}, \mathbf{Q} \rangle = \mathbf{q}\bar{\mathbf{Q}}, \quad (5)$$

such a inner product is extended in a natural way to \mathbb{H}^2 , the vector space made of column vectors $(\mathbf{q}, \mathbf{Q})^T$, where $\mathbf{q}, \mathbf{Q} \in \mathbb{H}$

$$\langle (\mathbf{q}, \mathbf{Q}), (\mathbf{p}, \mathbf{P}) \rangle = \mathbf{q}\bar{\mathbf{Q}} + \mathbf{p}\bar{\mathbf{P}}. \quad (6)$$

Note that, for the hermitian product defined above, a different choice could be made instead of (5), that is,

$$\langle \mathbf{q}, \mathbf{Q} \rangle = \bar{\mathbf{q}}\mathbf{Q}, \quad (7)$$

also defines a hermitian inner product in \mathbb{H} .

The *norm* of a quaternion is denoted by $|\mathbf{q}|$, sometimes called the length of \mathbf{q} , it is the scalar defined by

$$|\mathbf{q}| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} = \sqrt{\mathbf{q}\bar{\mathbf{q}}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2},$$

notice that this definition is the same that for the length of a vector in \mathbb{R}^4 , or equivalently the Euclidean norm. This definition of norm is independent of the choice in the definition of the hermitian product (5) *versus* (7).

Finally we give the expression for the *inverse*, for every $\mathbf{q} \neq 0$ there exists another quaternion \mathbf{q}^{-1} , which will be noted as the inverse and is given by

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{|\mathbf{q}|^2}.$$

4 Generalized Hopf map

In this section we generalize the classical Hopf fibration from \mathbb{C}^2 to the quaternion space \mathbb{H}^2 . See figure (FibracionCuaternionica).

$$\begin{array}{ccc}
 \mathcal{A}_{\mathbb{H}}: \mathbb{H}^2 - \{0\} & \xrightarrow{\quad\quad\quad} & \mathbb{H} \times \mathbb{R} \times \mathbb{R} - \{0\} \\
 \Sigma = (q, Q) & \cdots\cdots\cdots\rightarrow & (q\bar{Q}, \frac{1}{2}(|q|^2 - |Q|^2), \frac{1}{2}(|q|^2 + |Q|^2)) \\
 \vdots & & \vdots \\
 \frac{r}{|\Sigma|} (q, Q) & \begin{array}{c} \downarrow P_{\mathbb{H},r} \\ \mathbb{S}^7(r) \end{array} & \begin{array}{c} \downarrow P_{\mathbb{H}} \\ \mathbb{S}^4(\frac{r^2}{2}) \end{array} \\
 \vdots & \begin{array}{c} \downarrow P_{\mathbb{H}\sim} \\ \mathbb{C}\mathbb{P}^3 \end{array} & \downarrow \\
 [(q, Q)] & \xrightarrow{\quad\quad\quad \Pi_{\mathbb{H}} \quad\quad\quad} & (q\bar{Q}, 1/2(|q|^2 - |Q|^2)) \\
 & \cdots\cdots\cdots\rightarrow &
 \end{array}$$

$\mathcal{F}_{\mathbb{H}}$ (diagonal arrow from $\mathbb{S}^7(r)$ to $\mathbb{S}^4(\frac{r^2}{2})$)

Figure 2: Generalized Hopf map and Hopf fibration.

Introducing real coordinates $\mathbf{q} = (q_1, q_2, q_3, q_4)$, through the natural equivalence $\mathbb{H} \equiv \mathbb{R}^4$, for the first copy of \mathbb{H} , and $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$ for the second copy of \mathbb{H} , we obtain a real representation of the Hopf map

$$\begin{aligned}
\mathcal{A}_{\mathbb{R}}(\mathbf{q}, \mathbf{Q}) &= (\langle \mathbf{q}, \mathbf{Q} \rangle, \frac{1}{2}(|\mathbf{q}|^2 - |\mathbf{Q}|^2), \frac{1}{2}(|\mathbf{q}|^2 + |\mathbf{Q}|^2)) \\
&= (\omega_1(\mathbf{q}, \mathbf{Q}), \omega_2(\mathbf{q}, \mathbf{Q}), \omega_3(\mathbf{q}, \mathbf{Q}), \omega_4(\mathbf{q}, \mathbf{Q}), \omega_5(\mathbf{q}, \mathbf{Q}), \omega_6(\mathbf{q}, \mathbf{Q})),
\end{aligned} \tag{8}$$

where $\langle \mathbf{q}, \mathbf{Q} \rangle$ is the hermitian inner product defined by (5). Therefore, the components of $\mathcal{A}_{\mathbb{R}}$ are defined as follows

$$\begin{aligned}
\omega_1(\mathbf{q}, \mathbf{Q}) &= q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4, \\
\omega_2(\mathbf{q}, \mathbf{Q}) &= -(q_1 Q_2 - q_2 Q_1) - (q_3 Q_4 - q_4 Q_3), \\
\omega_3(\mathbf{q}, \mathbf{Q}) &= -(q_1 Q_3 - q_3 Q_1) + (q_2 Q_4 - q_4 Q_2), \\
\omega_4(\mathbf{q}, \mathbf{Q}) &= -(q_1 Q_4 - q_4 Q_1) - (q_2 Q_3 - q_3 Q_2), \\
\omega_5(\mathbf{q}, \mathbf{Q}) &= \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 - Q_1^2 - Q_2^2 - Q_3^2 - Q_4^2), \\
\omega_6(\mathbf{q}, \mathbf{Q}) &= \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2).
\end{aligned} \tag{9}$$

It is easy to check that assuming $q_3 = q_4 = Q_3 = Q_4 = 0$, we get the classic Hopf map. On the other hand the *Hopf variables* ω_i , $i = 1 \dots, 6$, satisfy

$$\mathcal{C}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 - \omega_6^2 = 0, \quad \omega_6 \geq 0, \tag{10}$$

Even more, considering the algebraic manifold given by \mathcal{C} and by eliminating the vertex $\mathcal{C}^* = \mathcal{C} - \{0\}$, it can be proven that $\mathcal{A}_{\mathbb{H}}(\mathbb{H}^2 - \{0\}) = \mathcal{C}^*$. Now if we restrict $\mathcal{A}_{\mathbb{H}}$ to the 7-dimensional sphere

$$\mathbb{S}^7(r) = \{(\mathbf{q}, \mathbf{Q}) \in \mathbb{R}^8 \mid q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = r^2 = 2\omega_6\} \tag{11}$$

we obtain the generalized Hopf fibration

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}}(\mathbf{q}, \mathbf{Q}) : \quad \mathbb{S}^7(r) &\longrightarrow \mathbb{S}^4\left(\frac{r^2}{2}\right) \\
(\mathbf{q}, \mathbf{Q}) &\rightsquigarrow (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)
\end{aligned} \tag{12}$$

where $\mathbb{S}^4(\frac{r^2}{2})$ is the 4-dimensional sphere.

A different choice of the hermitian inner product in the definition of $\mathcal{A}_{\mathbb{R}}$, as defined by (7), leads to an analogous scheme. Under this condition the components $\omega_1, \omega_5, \omega_6$ of $\mathcal{A}_{\mathbb{R}}$ remain the same and $\omega_2, \omega_3, \omega_4$ become the following three new functions

$$\begin{aligned}
G_1(q, Q) &= ((q_1 Q_2 - q_2 Q_1) - (q_3 Q_4 - q_4 Q_3)) , \\
G_2(q, Q) &= ((q_1 Q_3 - q_3 Q_1) + (q_2 Q_4 - q_4 Q_2)) , \\
G_3(q, Q) &= ((q_1 Q_4 - q_4 Q_1) - (q_2 Q_3 - q_3 Q_4)) .
\end{aligned}$$

The diagram given in Figure (2) keeps its validity with the slight changes on the definition of $\mathcal{A}_{\mathbb{R}}$ given above, even more, relation (10) is transformed into

$$\mathcal{C}(\omega_1, G_1, G_2, G_3, \omega_5, \omega_6) = \omega_1^2 + G_1^2 + G_2^2 + G_3^2 + \omega_5^2 - \omega_6^2 = 0, \quad \omega_6 \geq 0. \quad (13)$$

5 Reduction of the 4DOF harmonic oscillator

Consider on \mathbb{R}^8 , with co-ordinates $(q, Q) = (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4)$, and standard symplectic form ω , the isotropic harmonic oscillator with Hamiltonian

$$H_2(q, Q) = \frac{1}{2} \|q\|^2 + \frac{1}{2} \|Q\|^2 .$$

To get a convenient ordering we will rename the functions obtained so far. Let

$$\sigma_7(q, Q) = G_1(q, Q), \quad \sigma_8(q, Q) = G_2(q, Q), \quad \sigma_9(q, Q) = G_3(q, Q). \quad (14)$$

The Hamiltonian flows corresponding to $\sigma_7, \sigma_8, \sigma_9$ generate a group G , which is a symmetry group for the system $(H_2, \omega, \mathbb{R}^8)$, because $\{\sigma_7, H_2\} = \{\sigma_8, H_2\} = \{\sigma_9, H_2\} = 0$, where $\{, \}$ is the Poisson bracket associated to the standard symplectic form ω . Furthermore $\{G_i, G_j\} = \varepsilon_{ijk} G_k$, thus these functions form a basis for the Lie algebra $\mathfrak{so}(3)$ and $G = SO(3)$.

Next consider the functions

$$\begin{aligned} \sigma_1(q, Q) &= \omega_2(q, Q) = -((q_1 Q_2 - q_2 Q_1) + (q_3 Q_4 - q_4 Q_3)) , \\ \sigma_2(q, Q) &= \omega_3(q, Q) = -((q_1 Q_3 - q_3 Q_1) - (q_2 Q_4 - q_4 Q_2)) , \\ \sigma_3(q, Q) &= \omega_4(q, Q) = -((q_1 Q_4 - q_4 Q_1) + (q_2 Q_3 - q_3 Q_2)) , \\ \sigma_4(q, Q) &= \omega_1(q, Q) = \langle q, Q \rangle , \\ \sigma_5(q, Q) &= \omega_5(q, Q) = \frac{1}{2} \|q\|^2 - \frac{1}{2} \|Q\|^2 , \\ \sigma_6(q, Q) &= \omega_6(q, Q) = \frac{1}{2} \|q\|^2 + \frac{1}{2} \|Q\|^2 . \end{aligned}$$

Note that these same polynomials also play a role in the reduction of the Kepler system [4] and the reduction of the 4DOF isotropic harmonic oscillator [5, 6, 7].

The polynomials $\sigma_1, \sigma_2, \sigma_3$ span a Lie algebra isomorphic to $\mathfrak{so}(3)$. Furthermore $\{\sigma_4, \sigma_5\} = -2\sigma_6$, $\{\sigma_4, \sigma_6\} = -2\sigma_5$, $\{\sigma_5, \sigma_6\} = 2\sigma_4$, that is, $\sigma_4, \sigma_5, \sigma_6$ span a Lie algebra isomorphic to $\mathfrak{su}(1, 1) \cong \mathfrak{sl}(2, \mathbb{R})$. In addition $\sigma_7, \sigma_8, \sigma_9, \sigma_1, \sigma_2, \sigma_3$ span a Lie algebra isomorphic to $\mathfrak{so}(3) \times \mathfrak{so}(3) \cong \mathfrak{so}(4)$ and $\{\sigma_{i+6}, \sigma_j\} = \{\sigma_i, \sigma_j\} = 0$ for $i = 1, 2, 3$, and $j = 4, 5, 6$.

Note that $\mathfrak{sl}(2, \mathbb{R})$ generated by $\sigma_4, \sigma_6 + \sigma_5, \sigma_6 - \sigma_5$ gives the full linear Lie algebra invariant under the $SO(4)$ action generated by $\sigma_7, \sigma_8, \sigma_9, \sigma_1, \sigma_2, \sigma_3$ (see [12]). If we

consider $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$, with the first copy of $\mathfrak{so}(3)$ generated by $\sigma_7, \sigma_8, \sigma_9$ and the second copy of $\mathfrak{so}(3)$ generated by $\sigma_1, \sigma_2, \sigma_3$ then the full linear Lie algebra of invariants for the first copy of $\mathfrak{so}(3)$ is $\mathfrak{so}(3) \times \mathfrak{sl}(2, \mathbb{R})$ generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$, and the full linear Lie algebra of invariants for the second copy of $\mathfrak{so}(3)$ is $\mathfrak{so}(3) \times \mathfrak{sl}(2, \mathbb{R})$ generated by $\sigma_7, \sigma_8, \sigma_9, \sigma_4, \sigma_5, \sigma_6$. Thus polynomials σ_i are invariants for the G -action and are generators for the space of G -invariant polynomials. Consequently the orbit map for the G -action is

$$\rho : \mathbb{R}^8 \rightarrow \mathbb{R}^6; (q, Q) \rightarrow (\sigma_1, \dots, \sigma_6) .$$

Consequently, ρ is equivalent to the real representation of the Hopf map given in (8).

Let $S_{ij}(q, Q) = q_i Q_j - q_j Q_i$ Then

$$\sum_{1 \leq i < j \leq 4} S_{ij}^2 = \|q\|^2 \|Q\|^2 - \langle q, Q \rangle^2 = -\sigma_5^2 + \sigma_6^2 - \sigma_4^2 , \quad (15)$$

and

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0 . \quad (16)$$

Thus

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = -\sigma_5^2 + \sigma_6^2 - \sigma_4^2 , \quad (17)$$

and

$$\sigma_7^2 + \sigma_8^2 + \sigma_9^2 = -\sigma_5^2 + \sigma_6^2 - \sigma_4^2 . \quad (18)$$

Thus the orbit space $\rho(\mathbb{R}^8)$ for the above defined $SO(3)$ representation is five dimensional and is determined by (17). The reduced phase spaces for the G -action on the one hand are the co-adjoint orbits for the action of $SO(3)$ on $\mathfrak{so}(3) \times \mathfrak{so}(3)$, and on the other hand are determined by the $\mathfrak{so}(3)$ Casimir $C(q, Q) = \sigma_7^2 + \sigma_8^2 + \sigma_9^2$. Therefore the reduced phase spaces are $\rho(C^{-1}(c))$, i.e. the subset of \mathbb{R}^6 determined by

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 + \sigma_3^2 &= c , \\ -\sigma_5^2 + \sigma_6^2 - \sigma_4^2 &= c . \end{aligned} \quad (19)$$

Thus, if $c \neq 0$, the reduced phase space is the four dimensional product of a two-sphere and a two-sheeted hyperboloid. If $c=0$ we have a critical two dimensional reduced phase space which is a cone times a point.

When we consider the harmonic oscillator Hamiltonian then $H_2^{-1}(h)$ is $\mathbb{S}^7 \subset \mathbb{R}^8$ and

$$\rho(H_2^{-1}(h)) : \mathbb{S}^7 \rightarrow \mathbb{S}^4 ,$$

is a generalized Hopf map giving the generalized Hopf fibration of the seven-sphere (see [2]). The \mathbb{S}^4 is given by

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 = h^2 .$$

This four-sphere intersects the $SO(3)$ -reduced phase spaces $\rho(C^{-1}(c))$ in the sets

$$\begin{aligned}\sigma_1^2 + \sigma_2^2 + \sigma_3^2 &= c, \\ \sigma_4^2 + \sigma_5^2 &= h^2 - c, \quad h \leq c \leq 0,\end{aligned}\tag{20}$$

which are topologically $\mathbb{S}^2 \times \mathbb{S}^1$. The sets given by (20) are not reduced phase spaces for a combined reduction with respect to the $SO(3)$ action by G and the \mathbb{S}^1 action generated by H_2 , because the induced H_2 -action on $\rho(C^{-1}(c)) \cap \rho(H_2^{-1}(h))$ is an \mathbb{S}^1 action which is trivial on $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = c$ and which rotates along $\sigma_4^2 + \sigma_5^2 = h - c$. An additional reduction with respect to the H_2 -action will therefore result in a two dimensional reduced phase space defined by $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = c$.

One should be aware that the above construction depends on the choices of the real coordinates for the quaternion representation. An other choice will alter the way the Lie algebra structure of the quaternions is represented and change the role of the harmonic oscillator Hamiltonian with respect to the Poisson bracket induced by the standard symplectic form.

6 The twistor fibration

In the above we have shown that a generalized Hopf map is present in the dynamics of the isotropic harmonic oscillator on \mathbb{R}^8 by explicit construction. Actually we constructed some representation of the map $\mathcal{A}_{\mathbb{H}} \circ P_{\mathbb{H}}$ in the diagram in figure (2), which is a restriction of an orbit map for a symplectic group action. We know that the map $\Pi_{\mathbb{H}}$ in this diagram represents the twistor fibration (see [1]) or the Calabi-Hopf-Penrose fibration $\mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^3 \rightarrow S^4$ which is obtained via a quotient of two Hopf maps [8]:

$$\begin{array}{ccc} \mathbb{C}^4 - \{0\} & \longrightarrow & \mathbb{H}^2 - \{0\} \\ \text{Hopf}_{\mathbb{C}} \downarrow & & \downarrow \text{Hopf}_{\mathbb{H}} \\ \mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^3 & \longrightarrow & \mathbb{H}\mathbb{P}^1 \approx S^4 \end{array}$$

Figure 3: Twistor fibration

In this section we will show how the twistor fibration can be obtained by factorizing the orbit map for the symplectic $SO(3)$ -action through the orbit map for a symplectic $SO(2)$ -action.

One should notice that the linear Lie algebra of $SO(3)$ invariants will be noncompact due to the presence of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra. Because the linear Lie algebra of invariants for

the H_2 action is isomorphic to the compact Lie algebra $\mathfrak{su}(4)$, it is impossible to perform the above constructed $SO(3)$ reduction in a way that factors through the H_2 reduction.

If in the diagram in figure(2) the map $P_{\mathbb{H},r} \circ P_{\mathbb{H},\sim}$ would describe an H_2 reduction, than this would imply an $SO(3)$ reduction factoring through the H_2 reduction. Consequently the second map $P_{\mathbb{H},\sim}$ is not an H_2 -reduction, although it is known that the reduced phase space for the H_2 -action is $\mathbb{C}\mathbb{P}^3$ (see [10]).

Choose σ_7 from the representation of the $SO(3)$ group action. Then $\{\sigma_7, H_2\} = 0$ and we can identify \mathbb{H}^2 with \mathbb{C}^4 in such a way that the action of σ_7 is given by $e^{it}z$, $z \in \mathbb{C}^4$. Let ρ_{σ_7} be the orbit map for the flow of σ_7 . Then $\rho_{\sigma_7}(H_2^{-1}(h)) \cong \mathbb{C}\mathbb{P}^3$. More details on the action of σ_7 can be found in [9]. Consequently the orbit map for the symplectic σ_7 -action, when restricted to $H_2(q, Q) = h$, coincides with the map $\text{Hopf}_{\mathbb{C}}$ in the diagram given in figure (3) and with $P_{\mathbb{H},r} \circ P_{\mathbb{H},\sim}$ in figure(2). Now the σ_7 -action is a subgroup of the $SO(3)$ -action giving $SO(3)/SO(2) \cong S^2 \cong S^3/S^1 \cong \mathbb{C}\mathbb{P}^1$ as the fiber of the twistor fibration, where S^3/S^1 is the Hopf fibration. It is now clear that the map $\Pi_{\mathbb{H}}$ in figure(2) cannot be an orbit map for a symplectic group action, i.e. $SO(3)$ reduction cannot be performed by symplectic reduction in stages.

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