

# Network synchronization of time-delayed coupled nonlinear systems using predictor-based diffusive dynamic couplings

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## Network synchronization of time-delayed coupled nonlinear systems using predictor-based diffusive dynamic couplings

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# Network synchronization of time-delayed coupled nonlinear systems using predictor-based diffusive dynamic couplings

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We study the problem of controlled network synchronization of coupled *semipassive systems* in the case when the outputs (the coupling variables) and the inputs are subject to constant time-delay (as it is often the case in a networked context). *Predictor-based dynamic output feedback controllers* are proposed to interconnect the systems on a given network. Using Lyapunov-Krasovskii functional and the notion of semipassivity, we prove that under some mild assumptions, the solutions of the interconnected systems are globally ultimately bounded. Sufficient conditions on the systems to be interconnected, on the network topology, on the coupling dynamics, and on the time-delays that guarantee global state synchronization are derived. A local analysis is provided in which we compare the performance of our predictor-based control scheme against the existing *static diffusive couplings* available in the literature. We show (locally) that the time-delay that can be induced to the network may be increased by including the predictors in the loop. The results are illustrated by computer simulations of coupled Hindmarsh-Rose neurons. © 2015 AIP Publishing LLC.

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This manuscript focuses on controlled synchronization of identical nonlinear systems interacting on networks with general topologies and interconnected through *predictor-based diffusive dynamic couplings*. The systems are said to be *diffusively coupled*, if they interact through weighted differences of the form  $\gamma(y_j - y_i)$  with some positive constant  $\gamma$  called the *coupling strength* and  $y_i, y_j$  denoting the outputs of the  $i$ th and  $j$ th systems. An important element of our control scheme is the use of a communication network. Network communication is necessary in the study of synchronization to transmit and receive measurement and control data among the systems. Because of the time needed to transmit data over the network, the use of networked communication to exchange information results in unavoidable time-delays. This networked-induced delays are undesirable because they may lead to the loss of synchrony. Hence, when studying synchronization among dynamical systems with networked communication, it is important to design control algorithms which are robust with respect to time-delays. The results presented here follow the same research line as Refs. 1 and 2, where sufficient conditions for synchronization of diffusively interconnected nonlinear systems with and without time-delays are derived. In order to derive their results, the authors assume that the individual systems are *semipassive*<sup>3</sup> with respect to the coupling variable  $y_i$ , and their corresponding internal dynamics have some desired stability properties (*convergent internal dynamics*)<sup>4</sup>. In particular, in Ref. 2, the authors study the problem of network synchronization of *diffusively time-*

*delayed coupled semipassive systems*. They prove that under some mild assumptions, there always exists a region  $S$  in the parameter space (coupling strength  $\gamma$  versus time-delay  $\tau$ ), such that if  $(\gamma, \tau) \in S$ , the systems synchronize. Nevertheless, it is important to note that for this class of systems, once the network topology is specified, the region  $S$  is fixed. In other words, the time-delay that may be induced to the network without compromising the synchronous behavior is limited by the network topology.<sup>5</sup> Here, we show that by including predictors in the couplings, we may increase the time-delay that can be induced. We propose *predictor-based diffusive dynamic couplings* based on the concept of *anticipating synchronization*<sup>6</sup> that on the one hand estimate future values  $y_i(t + \tau)$  of the outputs  $y_i(t)$ , and on the other hand interconnect the systems through these time-ahead estimated signals. By including the predictors in the loop, a new parameter  $\kappa$  comes into play. This  $\kappa$  plays the role of the *predictor gain*, i.e., it is a parameter of the predictors that can be tuned to make the *prediction error dynamics* converge to the origin. We derive sufficient conditions for global state synchronization of the interconnected systems. In particular, it is proved that under some assumptions, there always exists a region in the *extended parameter space* (coupling strength  $\gamma$ , time-delay  $\tau$ , and *predictor gain*  $\kappa$ ), such that if  $\gamma$ ,  $\tau$ , and the new parameter  $\kappa$  belong to this region, the systems synchronize. Finally, we provide a *local analysis* in which the performance of our predictor-based control scheme is compared against the existing *static diffusive couplings* available in the literature. It is shown (locally) that the amount of time-delay that can be induced to the network may be increased by including the predictors in the loop.

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## I. INTRODUCTION

Network synchronization of dynamical systems has attracted attention of many researchers over the last decades. This is because synchronization is a quite common phenomenon in nature, science, and engineering. For instance, in biology, it is well known that thousands of fireflies light up simultaneously<sup>7</sup> and that groups of Japanese tree frogs (*Hyla japonica*) may show synchronous behavior in their calls.<sup>8</sup> In medicine and neuroscience, clusters of synchronized pacemaker neurons regulating our heartbeat,<sup>9</sup> synchronized neurons in the olfactory bulb that allow us to detect and distinguish between odors,<sup>10</sup> and our circadian rhythm, which is synchronized to the 24-h day-night cycle<sup>11,12</sup> are clear examples. In engineering, one of the most commonly cited examples of network synchronization is the problem of coordinated motion of individual mobile agents.<sup>13–17</sup> Particularly, in Refs. 14, 16, and 17, the authors address the *platooning problem*, i.e., the problem of designing intelligent vehicle/highway systems that can significantly increase safety and highway capacity. The general objective of the platooning problem is to pack the driving vehicles together as tightly as possible in order to increase traffic flow while preventing amplification of disturbances throughout the string. There are also several research groups studying synchronization in robotics, where multiple robots carry out tasks that cannot be achieved by a single one. For instance, in Ref. 18, the author proposes a dynamic output controller that solves the synchronization problem of two (or more) robot manipulators, under a master-slave scheme. In Ref. 19, the formation control problem for unicycle mobile robots interacting on symmetric graphs is studied. The range of engineering examples of network synchronization reaches way beyond coordinated motion. For instance, synchronization in power networks,<sup>20</sup> control of the directional sensitivity of smart antennas,<sup>21</sup> and synchronization of microelectromechanical systems (MEMS), which has promising applications such as neurocomputing<sup>22</sup> and improvements of signal-to-noise ratios.<sup>23</sup> Several more examples of synchronous behavior in physics, biology, and engineering can be found in, for instance, Refs. 7, 12, 24, and 25.

Some of the first technical results regarding synchronization of chaotic nonlinear systems are presented in Refs. 26 and 27. In these papers, it is shown that coupled chaotic systems may synchronize in spite of their high sensitivity to initial conditions. After these results, considerable interest in the notion of synchronization of general nonlinear systems has arisen. In this manuscript, we study the problem of controlled network synchronization of nonlinear systems interconnected through *diffusive time-delayed couplings*. This type of couplings arises naturally for interconnected systems since the transmission of signals can be expected to take some time. There already exist some results in this direction. For instance, the authors in Ref. 28 give sufficient conditions for network synchronization in terms of Linear Matrix Inequalities (LMIs) for a class of nonlinear systems interconnected through *Pyragas-type*<sup>29</sup> time-delayed couplings. *Adaptive control methods* have been often used to address the network synchronization problem in the presence of time-delays. In Refs. 30 and 31, the authors propose *diffusive*

*adaptive couplings* to solve the synchronization problem for nonidentical dynamical systems described by Euler-Lagrange equations and subject to transmission time-delays. The authors in Ref. 32 consider networks of diffusively time-delayed coupled systems. They apply the *speed-gradient method* to derive adaptive algorithms for the automatic adjustment of coupling strengths and time-delays such that the coupled systems synchronize. In the same spirit, in Ref. 33, the authors use the speed-gradient method to solve the synchronization problem in networks of time-delayed coupled *Stuart-Landau oscillators*. However, in all these papers, the authors impose strong conditions on the systems, i.e., they have to be *fully actuated* and/or the *complete state* must be available for feedback. In Refs. 1, 34, and 35, the authors study synchronous behavior in delay-free *diffusively coupled* networks as a consequence of the inherent *dissipation* in the subsystems and the couplings. Moreover, they do not necessarily assume *complete state feedback* and *fully actuated* dynamics, but they also allow for *output feedback* controllers and *underactuated* systems. In Ref. 2, the authors consider the problem of network synchronization of diffusively time-delayed coupled *semipassive systems*. They prove that under some conditions, there exists a region  $\mathcal{S}$  in the  $(\gamma, \tau)$ -parameter space such that if  $(\gamma, \tau) \in \mathcal{S}$ , the systems synchronize. However, as mentioned before, for this class of systems, once the network topology is specified, the region  $\mathcal{S}$  is fixed.<sup>5</sup>

Here, we show that by using *predictor-based output feedback controllers* based on the concept of *anticipating synchronization*,<sup>6</sup> the amount of time-delay that can be induced to the network without compromising the synchronous behavior may be increased. In Ref. 36, we have started studying these ideas for a class of passive LTI systems. Note that for LTI systems the *separation principle* holds, i.e., the predictor dynamics and the coupling structure can be designed independently. However, the analysis becomes more involved for nonlinear systems, since in general, the separation principle does not hold; in this case, there is a strong nonlinear relation between the predictor dynamics and the coupling structure.

The remainder of the paper is organized as follows. In Sec. II, the notions of *semipassivity*, *convergent systems*, and some basic terminology of *graph theory* are introduced. The system description and the problem statement are introduced in Sec. III. The predictor structure is given in Sec. IV before introducing the proposed *predictor-based diffusive dynamic coupling* in Sec. V. In Secs. VI and VII, we present the main results on global boundedness and network synchronization. Moreover, a local analysis in which we explain the “mechanism of action” behind our predictor-based couplings is also presented. In Sec. VIII, simulation results of coupled Hindmarsh-Rose neurons are presented. Finally, conclusions are stated in Sec. IX.

## II. PRELIMINARIES

Throughout this paper, the following notation is used: the symbol  $\mathbb{R}_{>0}$  ( $\mathbb{R}_{\geq 0}$ ) denotes the set of positive (non-negative) real numbers. The Euclidian norm in  $\mathbb{R}^n$  is denoted simply as  $|\cdot|$ ,  $|x|^2 = x^T x$ , where  $^T$  defines transposition. The notation  $\text{col}(x_1, \dots, x_n)$  stands for the column vector composed of the elements  $x_1, \dots, x_n$ . This notation will be also

used in case the components  $x_i$  are vectors. The induced norm of a matrix  $A \in \mathbb{R}^{n \times n}$ , denoted by  $\|A\|$ , is defined as  $\|A\| = \max_{x \in \mathbb{R}^n, |x|=1} |Ax|$ . The  $n \times n$  identity matrix is denoted by  $I_n$  or simply  $I$ , if no confusion can arise. Likewise, the  $n \times m$  matrices composed of only ones and only zeros are denoted as  $\mathbf{1}_{n \times m}$  and  $\mathbf{0}_{n \times m}$ , respectively. A function  $V : \mathcal{X} \rightarrow \mathbb{R}_{>0}$  defined on a neighborhood  $\mathcal{X}$  of  $\mathbb{R}^n$  with  $0 \in \mathcal{X}$  is positive definite (negative definite) if  $V(x) > 0$  ( $V(x) < 0$ ) for all  $0 \in \mathcal{X} \setminus \{0\}$  and  $V(0) = 0$ . It is radially unbounded if  $\mathcal{X} = \mathbb{R}^n$ . If  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , then  $V(\cdot)$  is called proper. If a quadratic form  $x^T P x$  with a symmetric matrix  $P = P^T$  is positive definite, then  $P$  is called positive definite. For positive definite matrices, we use the notation  $P > 0$ . The spectrum of a matrix  $A$  is denoted by  $\text{spec}(A)$ . For any two matrices  $A$  and  $B$ , the notation  $A \otimes B$  (the Kronecker product<sup>37</sup>) stands for the matrix composed of submatrices  $A_{ij} B$ , where  $A_{ij}$ ,  $i, j = 1, \dots, n$ , stands for the  $ij$ th entry of the  $n \times n$  matrix  $A$ . Let  $\mathcal{X} \subset \mathbb{R}^n$  and  $\mathcal{Y} \subset \mathbb{R}^m$ . The space of continuous functions from  $\mathcal{X}$  to  $\mathcal{Y}$  is denoted by  $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ . If the functions are (at least)  $r \geq 0$  times continuously differentiable, then it is denoted by  $\mathcal{C}^r(\mathcal{X}, \mathcal{Y})$ . If the derivatives of a function of all orders ( $r = \infty$ ) exist, the function is called smooth and if the derivatives up to a sufficiently high order exist the function is named sufficiently smooth. Time-delayed signals are denoted as  $x(t)^\tau = x(t - \tau)$  with time-delay  $\tau \in \mathbb{R}_{>0}$ . For simplicity of notation, we often suppress the explicit dependence of time  $t$ .

### A. Communication graphs

Given a set of interconnected systems, the communication topology is encoded through a communication graph. The convention is that system  $i$  receives information from system  $j$  if and only if there is a directed link from node  $j$  to node  $i$  in the communication graph. Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$  denote a weighted digraph (directed graph), where  $\mathcal{V} = \{v_1, v_2, \dots, v_k\}$  is the set of nodes,  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the set of edges, and  $A$  is the weighted adjacency matrix with nonnegative elements  $a_{ij}$ . The neighbors of  $v_i$  is the set of directed edges to a node  $v_i$  and it is denoted as  $\mathcal{E}_i$ . If the graph does not contain self-loops, it is called simple. Throughout this manuscript, it is assumed that the communication graph is *strongly connected*, i.e., for every two nodes  $(i, j) \in \mathcal{V}$ , there is at least one path connecting  $i$  and  $j$ . If two nodes have a directed edge in common, they are called *adjacent*. Assume that the network consists of  $k$  nodes, then the *adjacency matrix*  $A \in \mathbb{R}^{k \times k} := a_{ij}$  with  $a_{ij} > 0$ , if  $\{i, j\} \in \mathcal{E}$  and  $a_{ij} = 0$  otherwise. Finally, we introduce the *degree matrix*  $D \in \mathbb{R}^{k \times k} := \text{diag}\{d_1, \dots, d_k\}$  with  $d_i = \sum_{j \in \mathcal{E}_i} a_{ij}$  and  $L := D - A$ , which is called the *Laplacian matrix* of the graph  $\mathcal{G}$ , see Ref. 37 for further details.

### B. Semipassive systems

Consider the system

$$\dot{x} = f(x, u), \tag{1a}$$

$$y = h(x), \tag{1b}$$

with state  $x \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^m$ , output  $y \in \mathbb{R}^m$ , sufficiently smooth functions  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ , and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

**Definition 1** (Ref. 3). *The dynamical system (1) is called  $\mathcal{C}^r$ -semipassive if there exists a nonnegative function  $V \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ ,  $x \mapsto V(x)$ , called the storage function, such that  $\dot{V}(x, u) \leq y^T u - H(x)$ , where the function  $H \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$  is nonnegative outside some ball, i.e.,  $\exists \varphi > 0$  s.t.  $|x| \geq \varphi \rightarrow H(x) \geq \varrho(|x|)$ , for some continuous nonnegative function  $\varrho(\cdot)$  defined for  $|x| \geq \varphi$ . If the function  $H(\cdot)$  is positive outside some ball, then the system (1) is said to be strictly  $\mathcal{C}^r$ -semipassive.*

**Remark 1** *System (1) is  $\mathcal{C}^r$ -passive (strictly  $\mathcal{C}^r$ -passive) if it is  $\mathcal{C}^r$ -semipassive (strictly  $\mathcal{C}^r$ -semipassive) with  $H(\cdot)$  being positive semidefinite (positive definite).*

In light of Remark 1, a (strictly)  $\mathcal{C}^r$ -semipassive system behaves like a (strictly) passive system for large  $|x(t)|$ . From a physical point of view, one may think of a semipassive system as a passive system with a limited amount of free energy. The class of strictly semipassive systems includes, e.g., the chaotic Lorenz system,<sup>1</sup> and many models that describe the action potential dynamics of individual neurons.<sup>38</sup>

### C. Convergent systems

Consider the system (1a) and suppose  $f(\cdot)$  is Lipschitz in  $x$ ,  $u(\cdot)$  is piecewise continuous in  $t$  and takes values in some set  $u \in U \subseteq \mathbb{R}^m$ .

**Definition 2** (Ref. 39). *System (1a) is said to be convergent if and only if for any bounded signal  $u(t)$  defined on the whole interval  $(-\infty, +\infty)$  there is a unique bounded globally asymptotically stable solution  $\bar{x}_u(t)$  defined in the same interval for which it holds that,  $\lim_{t \rightarrow \infty} |x(t) - \bar{x}_u(t)| = 0$  for all initial conditions.*

For a *convergent system*, the limit solution is solely determined by the external excitation  $u(t)$  and not by the initial condition. A sufficient condition for convergence obtained in Ref. 39 and later extended in Ref. 4 is presented in the following proposition.

**Proposition 1** (Refs. 4 and 39). *If there exists a positive definite symmetric matrix  $P \in \mathbb{R}^{n \times n}$  such that all the eigenvalues  $\lambda_i(Q)$  of the symmetric matrix*

$$Q(x, u) = \frac{1}{2} \left( P \left( \frac{\partial f}{\partial x}(x, u) \right) + \left( \frac{\partial f}{\partial x}(x, u) \right)^T P \right) \tag{2}$$

*are negative and separated from zero, i.e., there exists a constant  $c \in \mathbb{R}_{>0}$  such that  $\lambda_i(Q) \leq -c < 0$ , for all  $i \in \{1, \dots, n\}$ ,  $u \in U$ , and  $x \in \mathbb{R}^n$ , then system (1a) is globally exponentially convergent. Moreover, for any pair of solutions  $x_1(t), x_2(t) \in \mathbb{R}^n$  of (1a), the following is satisfied*

$$\frac{d}{dt} ((x_1 - x_2)^T P (x_1 - x_2)) \leq -\alpha |x_1 - x_2|^2,$$

*with constant  $\alpha := \frac{c}{\lambda_{\max}(P)}$  and  $\lambda_{\max}(P)$  being the largest eigenvalue of the symmetric matrix  $P$ .*

### III. SYSTEM DESCRIPTION

Consider  $k$  identical nonlinear systems of the form

$$\dot{\zeta}_i = q(\zeta_i, y_i), \tag{3}$$

$$\dot{y}_i = a(\zeta_i, y_i) + Bu_i, \tag{4}$$

with  $i \in \mathcal{I} := \{1, \dots, k\}$ , state  $x_i := \text{col}(\zeta_i, y_i) \in \mathbb{R}^n$ , internal state  $\zeta_i \in \mathbb{R}^{n-m}$ , output  $y_i \in \mathbb{R}^m$ , input  $u_i \in \mathbb{R}^m$  sufficiently smooth functions  $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$ ,  $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ , and matrix  $B \in \mathbb{R}^{m \times m}$  being similar to a positive definite matrix. For the sake of simplicity it is assumed that  $B = I_m$  (results for the general case with  $B$  being similar to a positive definite matrix can be easily derived). The systems (3),(4) are assumed to be *strictly  $C^1$ -semipassive* and the internal dynamics (3) are supposed to be *convergent*. In Ref. 2, the authors derive sufficient conditions for network synchronization of *diffusively time-delayed coupled semipassive systems*, i.e., the systems (3),(4) interconnected through weighted differences of the form

$$u_i(t) = \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t - \tau) - y_i(t - \tau)), \tag{5}$$

where  $\tau \in \mathbb{R}_{>0}$  denotes the time-delay,  $y_j(t - \tau)$  and  $y_i(t - \tau)$  are the time-delayed outputs of the  $j$ th and  $i$ th systems,  $\gamma \in \mathbb{R}_{>0}$  denotes the coupling strength,  $a_{ij} \leq 0$  are the weights of the interconnections, and  $\mathcal{E}_i$  is the set of neighbors of  $v_i$ . Moreover, since the coupling strength is encompassed in the constant  $\gamma$ , then it is assumed without loss of generality that  $\max_{i \in \mathcal{I}} \sum_{j \in \mathcal{E}_i} a_{ij} = 1$ . The authors in Ref. 2 prove that the systems (3)–(5) asymptotically synchronize provided that  $\gamma$  is sufficiently large and the product of the coupling strength and the time-delay  $\gamma\tau$  is sufficiently small. Then, there exists a region  $\mathcal{S}$  in the parameter space, such that if  $(\gamma, \tau) \in \mathcal{S}$ , the systems synchronize. Nevertheless, it is important to notice that in the closed loop system (3)–(5), once the interconnections  $a_{ij}$  are specified, the region  $\mathcal{S}$  is fixed. Hence, the amount of time-delay that may be induced to the network is limited by the network topology.<sup>5</sup> In this manuscript, we propose *predictor-based diffusive dynamic couplings* in order to enhance robustness against time-delays in the network, i.e., by including some dynamics in the coupling, we may expand the synchronization region  $\mathcal{S}$ . The time-delay  $\tau$  that is being induced in coupling (5) could be realized as the sum of measurement and transmission time-delays. In this paper, it is necessary to make a clear distinction among these delays. The measurement time-delay  $\tau_1 \in \mathbb{R}_{>0}$  affects the outputs of the systems  $y_i(t)$ , resulting in time-delayed outputs  $y_i(t - \tau_1)$  being available for control purposes. The transmission time-delays are encompassed in  $\tau_2 \in \mathbb{R}_{>0}$ . It affects the control inputs  $u_i(t)$ , resulting in the time-delayed control signals  $u_i(t - \tau_2)$  being applied to the systems, see Figure 1. Notice that the total time-delay  $\tau$  in (5) is simply given by the sum of the individual delays, i.e.,  $\tau := \tau_1 + \tau_2$ . Therefore, the interconnected systems (3)–(5) could be realized as individual systems with input time-delay  $\tau_2$  as follows:

$$\dot{\zeta}_i = q(\zeta_i, y_i), \tag{6}$$

$$\dot{y}_i = a(\zeta_i, y_i) + u_i(t - \tau_2), \tag{7}$$

$$x_i = \phi(t), \quad t \in [-\tau_2, 0], \tag{8}$$

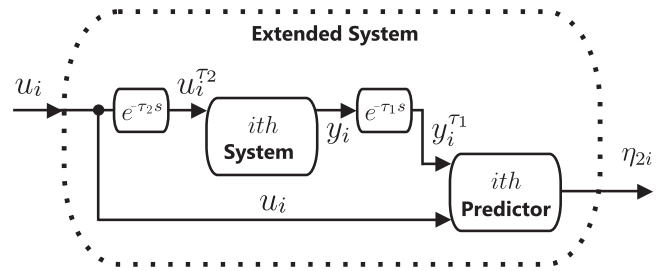


FIG. 1. Configuration of the prediction scheme.

with time-delayed input  $u_i^{\tau_2} \in \mathbb{R}^m$  and continuous function  $\phi : [-\tau_2, 0] \rightarrow \mathbb{R}^n$  specifying the initial history, in closed-loop with the following diffusive time-delayed coupling:

$$u_i(t) = \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t - \tau_1) - y_i(t - \tau_1)). \tag{9}$$

However, if the future value  $y_i(t + \tau_2)$  of  $y_i(t)$  could be obtained, then by applying the controller

$$u_i(t) = \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t + \tau_2) - y_i(t + \tau_2)), \tag{10}$$

the interconnected systems (6)–(8), (10) would be given by

$$\dot{\zeta}_i = q(\zeta_i, y_i), \tag{11}$$

$$\dot{y}_i = a(\zeta_i, y_i) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{12}$$

which is the delayed-free closed-loop system. From this point of view, we propose a control scheme, in which a predictor is used to estimate the future values  $y_i(t + \tau_2)$  from measurements of the available time-delayed output  $y_i(t + \tau_1)$ . Then, the output of the predictor is used to interconnect the systems, see Figure 3.

### IV. SYNCHRONIZATION-BASED PREDICTOR

In this section, we introduce the state predictor based on synchronization that is used to estimate  $y_i(t + \tau_2)$  from measurements of  $y_i(t - \tau_1)$ . In the first contribution concerning synchronization-based predictors,<sup>40</sup> the author studies the following coupled Ikeda equations:

$$\dot{\rho} = -\alpha\rho - \beta \sin(\rho^\tau), \tag{13}$$

$$\dot{z} = -\alpha z - \beta \sin(\rho), \tag{14}$$

with states  $\rho, z \in \mathbb{R}$ ,  $\rho^\tau(t) = \rho(t - \tau)$ , and constants  $\alpha, \beta, \tau \in \mathbb{R}_{>0}$ . Notice that the dynamics of the prediction error  $e(t) := z(t - \tau) - \rho(t)$  is simply given by  $\dot{e}(t) = -\alpha e(t)$ ; therefore, a necessary and sufficient condition for  $e(t)$  to converge to the origin is that  $\alpha > 0$ . Thus, the solution of (14) asymptotically synchronizes with the future solution of (13) at time instant  $t + \tau$ ; hence, (14) anticipates the dynamics of (13). This idea has been generalized into general multidimensional systems, in for instance, Refs. 6 and 41. Following these ideas, we propose a predictor based on synchronization for the class of systems under study. Consider  $k$  identical systems of the form

$$\dot{\eta}_{1i} = q(\eta_{1i}, \eta_{2i}), \tag{15}$$

$$\dot{\eta}_{2i} = a(\eta_{1i}, \eta_{2i}) + u_i + \kappa(y_i(t - \tau_1) - \eta_{2i}(t - \tau)), \tag{16}$$

$$\eta_i = \eta_{0i} \in \mathbb{R}^n, \quad t \in [-\tau, 0], \tag{17}$$

with measurement time-delay  $\tau_1 \in \mathbb{R}_{\geq 0}$ , total time-delay  $\tau = \tau_1 + \tau_2 \in \mathbb{R}_{\geq 0}$ , transmission time-delay  $\tau_2 \in \mathbb{R}_{\geq 0}$ ,  $i \in \mathcal{I} = \{1, \dots, k\}$ , state  $\eta_i := \text{col}(\eta_{1i}, \eta_{2i}) \in \mathbb{R}^n$ , internal state  $\eta_{1i} \in \mathbb{R}^{n-m}$ , actuated state  $\eta_{2i} \in \mathbb{R}^m$ , input  $u_i \in \mathbb{R}^m$ , smooth vectorfields  $q(\cdot)$  and  $a(\cdot)$  as in (3),(4), initial history  $\eta_{0i}$ , and gain  $\kappa \in \mathbb{R}_{>0}$ . The system (15)–(17) is called a predictor for system (6)–(8) if and only if

$$\begin{aligned} & \lim_{t \rightarrow \infty} (x_i(t - \tau_1) - \eta_i(t - \tau)) \\ & \equiv \lim_{t \rightarrow \infty} (x_i(t + \tau_2) - \eta_i(t)) = 0. \end{aligned}$$

Notice that if  $\kappa = u_i(t) = 0$ , the predictor dynamics (15),(16) is the same as the individual subsystems dynamics (6),(7) with  $u^{\tau_2} = 0$ . We construct the predictor in this way in order to take advantage of the stability properties of (6),(7), namely, *semipassivity* and *convergence*. Moreover, each system (6)–(8) together with the predictor (15)–(17) could be interpreted as an extended new system with input  $u_i$ , new output  $\eta_{2i}$ , and internal delays  $\tau_1$  and  $\tau_2$ , see Figure 1. Define the prediction error  $\epsilon_i = \text{col}(\epsilon_{1i}, \epsilon_{2i}) := x_i - \eta_i^{\tau_2}$ . Then, the prediction error dynamics is given by

$$\dot{\epsilon}_{1i} = q(\zeta_i, y_i) - q(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}), \tag{18}$$

$$\dot{\epsilon}_{2i} = a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}) - \kappa \epsilon_{2i}^{\tau}. \tag{19}$$

It follows that the system (15)–(17) is a predictor for system (6)–(8) if the zero solution of (18),(19) is asymptotically stable. In the following lemma, we give sufficient conditions for asymptotic stability of the origin of (18),(19). In particular, we prove that under some mild assumptions, there always exists a region  $\mathcal{S}_1$  in the parameter space (predictor gain  $\kappa$  and total time-delay  $\tau$ ), such that if  $(\kappa, \tau) \in \mathcal{S}_1$ , then the system (15)–(17) anticipates the dynamics (6)–(8). Moreover, it is also proved that the region  $\mathcal{S}_1$  is bounded by a *unimodal function*  $\varphi(\kappa)$  defined on some set  $\mathcal{J} \subset \mathbb{R}$ .

**Definition 3** The function  $\varphi : \mathcal{J} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\kappa \mapsto \varphi(\kappa)$  is called *unimodal* if for some value  $\kappa^* \in \mathcal{J}$ , it is monotonically increasing for  $\kappa \leq \kappa^*$  and monotonically decreasing for  $\kappa \geq \kappa^*$ . Hence, the maximum value of  $\varphi(\kappa)$  is given by  $\varphi(\kappa^*)$  and there are no other maxima.

**Lemma 1** Consider the  $k$  nonlinear systems (6)–(8), (15)–(17) and suppose that for every input signal  $u_i$ , and any finite time-delay  $\tau$  and predictor gain  $\kappa$ , the solutions of the systems are ultimately bounded (in Sec. VI, Lemma 2, we give sufficient conditions for ultimate boundedness of the closed-loop system for the class of inputs under study). In addition assume that

(H4.1) The internal dynamics (6) is convergent, i.e., there is a positive definite matrix  $P = P^T \in \mathbb{R}^{(n-m) \times (n-m)}$  such that the eigenvalues of the symmetric matrix

$$\frac{1}{2} \left( P \left( \frac{\partial q}{\partial \zeta_i}(\zeta_i, y_i) \right) + \left( \frac{\partial q}{\partial \zeta_i}(\zeta_i, y_i) \right)^T P \right), \tag{20}$$

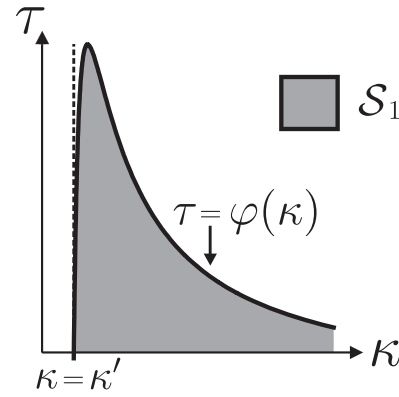


FIG. 2. Prediction region  $\mathcal{S}_1$ .

are uniformly negative and bounded away from zero for all  $\zeta_i \in \mathbb{R}^{n-m}$  and  $y_i \in \mathbb{R}^m$ .

Then, there exist a positive constant  $\kappa' \in \mathbb{R}_{>0}$  and a unimodal function  $\varphi : \mathcal{J} := [\kappa', \infty) \rightarrow \mathbb{R}_{\geq 0}$ ,  $\kappa \mapsto \varphi(\kappa)$  with  $\varphi(\kappa') = \lim_{\kappa \rightarrow \infty} \varphi(\kappa) = 0$ , such that if  $(\kappa, \tau) \in \mathcal{S}_1$  with  $\mathcal{S}_1 := \{\kappa, \tau \in \mathbb{R}_{\geq 0} | \kappa > \kappa', \tau < \varphi(\kappa)\}$ , then the systems (15)–(17) are global predictors for systems (6)–(8); and therefore,  $\lim_{t \rightarrow \infty} x_i(t + \tau_2) - \eta_i(t) = 0$ .

The proof of Lemma 1 can be found in the Appendix. The result stated in Lemma 1 amounts to the following. If the solutions of (6)–(8),(15)–(17) exist and are ultimately bounded, the zero solution of the prediction error dynamics (18),(19) is asymptotically stable provided that the predictor gain  $\kappa$  is sufficiently large and the total time delay  $\tau$  is smaller than some unimodal function  $\varphi(\kappa)$ , see Figure 2. Hence, there exists a region  $\mathcal{S}_1$  (gray area in Figure 2) such that if  $(\kappa, \tau) \in \mathcal{S}_1$  the system (15)–(17) asymptotically anticipates the dynamics (6)–(8).

### V. PREDICTOR-BASED DIFFUSIVE COUPLING

Let the  $k$  systems (6)–(8) be interconnected through a *Diffusive Dynamic Coupling* (DDC) of the form

$$\dot{\eta}_{1i} = q(\eta_{1i}, \eta_{2i}), \tag{21}$$

$$\dot{\eta}_{2i} = a(\eta_{1i}, \eta_{2i}) + u_i + \kappa(y_i(t - \tau_1) - \eta_{2i}(t - \tau)), \tag{22}$$

$$u_i = \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(\eta_{2j} - \eta_{2i}), \tag{23}$$

$$\eta_i = \eta_{0i} \in \mathbb{R}^n, \quad t \in [-\tau, 0], \tag{24}$$

with coupling strength  $\gamma \in \mathbb{R}_{\geq 0}$ , predictor gain  $\kappa \in \mathbb{R}_{\geq 0}$ , and interconnection weights  $a_{ij} = a_{ji} \geq 0$ . Since the coupling strength is encompassed in the constant  $\gamma$ , then it can be assumed without loss of generality that  $\max_{i \in \mathcal{I}} \sum_{j \in \mathcal{E}_i} a_{ij} = 1$ . The dynamic coupling (21)–(24) is the combination of the nonlinear predictor (15)–(17) and an estimated version of the time-ahead output feedback controller (10), see Figure 3. Then, the closed-loop system is given by

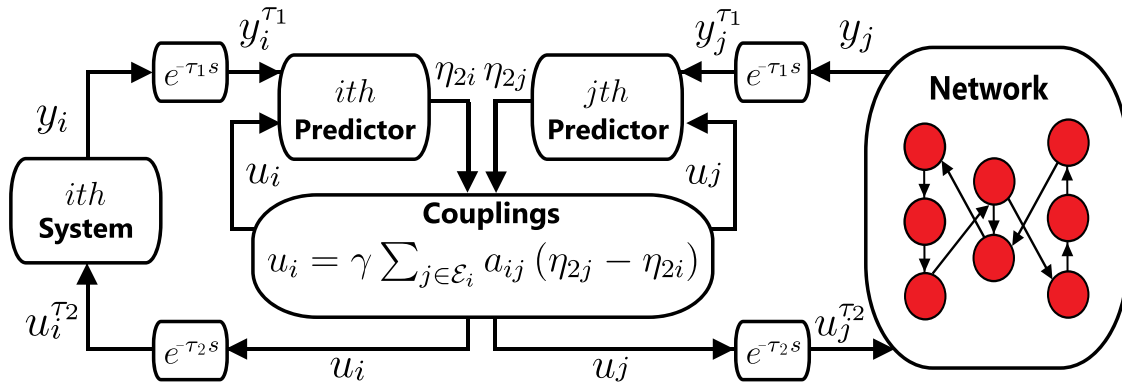


FIG. 3. Configuration of the proposed control scheme.

$$\begin{aligned} \dot{\zeta}_i &= q(\zeta_i, y_i), \\ \dot{y}_i &= a(\zeta_i, y_i) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} (\eta_{2j}^{\tau_2} - \eta_{2i}^{\tau_2}), \\ \dot{\eta}_{1i} &= q(\eta_{1i}, \eta_{2i}), \\ \dot{\eta}_{2i} &= a(\eta_{1i}, \eta_{2i}) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} (\eta_{2j} - \eta_{2i}) + \kappa (y_i^{\tau_1} - \eta_{2i}^{\tau_1}), \end{aligned}$$

with initial history (8),(24). Alternatively, since  $\epsilon_i = x_i - \eta_i^{\tau_2}$ , the closed-loop system can be written in terms of the prediction errors as follows:

$$\dot{\zeta}_i = q(\zeta_i, y_i), \tag{25}$$

$$\dot{y}_i = a(\zeta_i, y_i) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} (y_j - y_i + \epsilon_{2i} - \epsilon_{2j}), \tag{26}$$

$$\dot{\epsilon}_{1i} = q(\zeta_i, y_i) - q(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}), \tag{27}$$

$$\dot{\epsilon}_{2i} = a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}) - \kappa \epsilon_{2i}^{\tau_1}, \tag{28}$$

which is the delay-free closed-loop system (11),(12) perturbed by the prediction errors  $\epsilon_{2i}$ . Then, given the result in Ref. 1, it is intuitive to think that the systems may synchronize provided that  $\gamma$  is sufficiently large and the prediction errors converge sufficiently fast to the origin. However, before we start thinking about network synchronization, it is necessary to ensure that the solutions of the closed-loop systems (6)–(8),(21)–(24) are well defined, i.e., the solutions exist and are bounded.

**Remark 2** In the following sections, we present results about boundedness and synchronization of the solutions of the interconnected systems (6)–(8),(21)–(24). These results are given in terms of the coupling strength  $\gamma$ , the predictor gain  $\kappa$ , and the total time-delay  $\tau$ . By definition, the total time delay is given by the sum of the measurement time-delay  $\tau_1$  and transmission time-delay  $\tau_2$ , i.e.,  $\tau := \tau_1 + \tau_2$ . It follows that if boundedness and synchronization of the solutions is guaranteed for  $\tau \leq \bar{\tau} \in \mathbb{R}_{>0}$ ; then boundedness and synchronization is guaranteed for all  $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$  such that  $\tau_1 + \tau_2 \leq \bar{\tau}$ .

### VI. BOUNDEDNESS OF THE COUPLED SYSTEMS

In this section, we give sufficient conditions for ultimate boundedness of the solutions of closed-loop system (6)–(8),(21)–(24) interacting on simple strongly connected graphs.

**Lemma 2** Consider  $k$  identical systems (6)–(8) interconnected through the predictor-based DDC (21)–(24) with coupling strength  $\gamma \in \mathbb{R}_{\geq 0}$ , predictor gain  $\kappa \in \mathbb{R}_{\geq 0}$ , and total time-delay  $\tau \in \mathbb{R}_{\geq 0}$  on a simple strongly connected graph. Assume that

**(H6.1)** Each system (6),(7) is strictly  $C^1$ -semipassive with input  $u_i^{\tau_2}$ , output  $y_i$ , radially unbounded storage function  $V(x_i)$ , and the functions  $H(x_i)$  are such that there exist constants  $R, \delta \in \mathbb{R}_{>0}$  such that  $|x_i| > R$  implies that  $H(x_i) - \delta|y_i|^2 > 0$ .

Let  $\bar{\delta}$  be the largest  $\delta$  that satisfies (H6.1), then the solutions of the coupled systems (6)–(8),(21)–(24) are ultimately bounded for any finite  $\tau \in \mathbb{R}_{\geq 0}$  and  $(\gamma, \kappa) \in \mathcal{N}$  with  $\mathcal{N} := \{\gamma, \kappa \in \mathbb{R}_{\geq 0} \mid \frac{3\kappa}{2} + \gamma \leq \bar{\delta}\}$ .

The proof of Lemma 2 can be found in the Appendix.

**Remark 3** The result stated in Lemma 2 is independent of the time-delay. Therefore, if the conditions stated in Lemma 2 are satisfied, the solutions of the closed-loop system (6)–(8),(21)–(24) are ultimately bounded for arbitrary large time-delays.

### VII. NETWORK SYNCHRONIZATION

In this section, we give sufficient conditions for network synchronization of the interconnected systems. Define  $x := \text{col}(x_1, \dots, x_k)$  and the synchronization manifold  $\mathcal{M} := \{x \in \mathbb{R}^{kn} \mid x_i = x_j, \forall i, j \in \mathcal{I}\}$ . The systems (6)–(8), (21)–(24) are said to fully synchronize, or simply synchronize, if the synchronization manifold  $\mathcal{M}$  contains an asymptotically stable subset.

#### A. Global result

In the following theorem, we give sufficient conditions for the existence of an asymptotically stable subset of the synchronization manifold. In particular, we prove that under some mild assumptions, there always exists a region in the parameter space (coupling strength  $\gamma$ , predictor gain  $\kappa$ , and total time-delay  $\tau$ ), such that if  $\gamma, \kappa$ , and  $\tau$  belong to this region, the systems synchronize. Moreover, it is also proved that this region is bounded by a concave function  $\bar{\varphi} : \mathcal{K} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}, (\kappa, \gamma) \mapsto \bar{\varphi}(\kappa, \gamma)$ . The function  $\bar{\varphi}(\kappa, \gamma)$  has a unique maximum on  $\mathcal{K}$  and it has no other extrema.



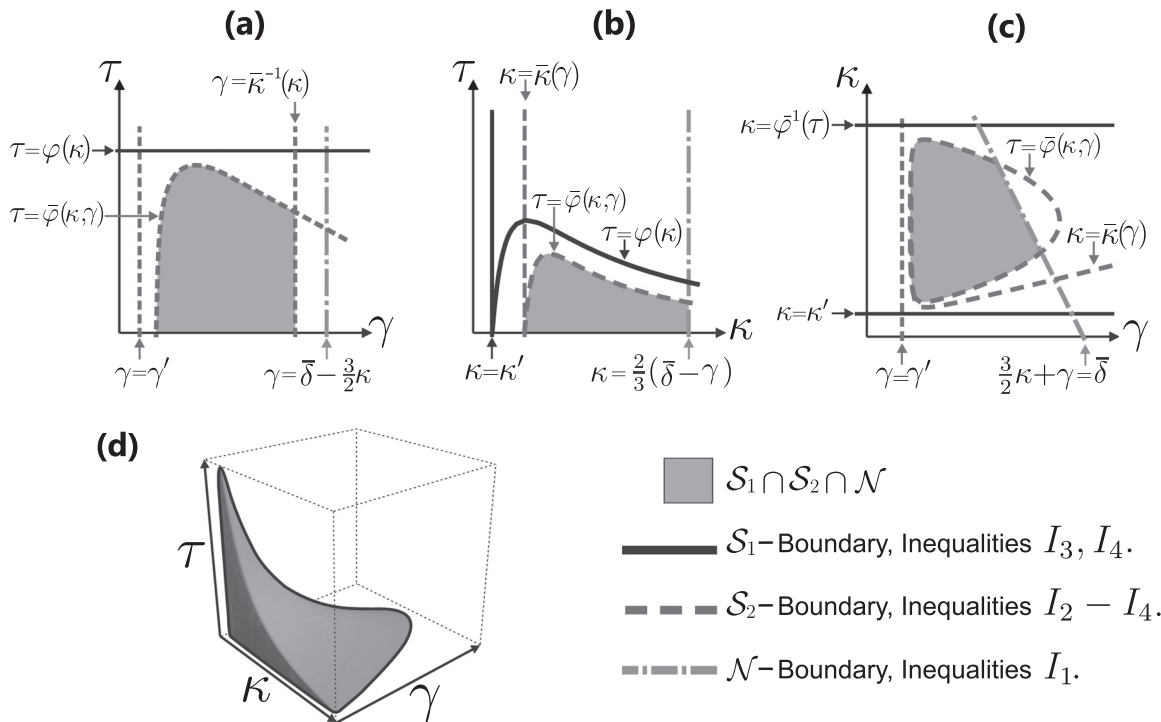


FIG. 4. Projections and a three dimensional sketch of the synchronization region  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$ .

**Theorem 1** Consider  $k$  identical systems (6)–(8) coupled through the predictor-based DDC (21)–(24) with coupling strength  $\gamma \in \mathbb{R}_{\geq 0}$ , predictor gain  $\kappa \in \mathbb{R}_{> 0}$ , and total time-delay  $\tau \in \mathbb{R}_{> 0}$  on a simple strongly connected graph. Assume that the conditions of Lemma 1 and Lemma 2 are satisfied. Then, there exist constants  $\gamma', \sigma', \kappa' \in \mathbb{R}_{> 0}$ , a function  $\bar{\kappa}(\gamma) := \kappa' + \frac{\sigma' \gamma^2}{\gamma - \gamma'}$ , and a concave function  $\bar{\varphi} : \mathcal{K} \rightarrow \mathbb{R}_{> 0}$ ,  $(\kappa, \gamma) \mapsto \bar{\varphi}(\kappa, \gamma)$ , with  $\bar{\varphi}(\bar{\kappa}(\gamma), \gamma) = \lim_{\kappa \rightarrow \infty} \bar{\varphi}(\kappa, \gamma) = 0$ , such that if  $(\gamma, \kappa, \tau) \in \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  with  $\mathcal{S}_2 := \{\gamma, \kappa, \tau \in \mathbb{R}_{\geq 0} \mid \gamma > \gamma', \kappa > \bar{\kappa}(\gamma), \tau < \varphi(\kappa, \gamma)\}$ ,  $\mathcal{S}_1$  the prediction set defined in Lemma 1, and  $\mathcal{N}$  the set where solutions are ensured to be bounded defined in Lemma 2, then the solutions of the closed-loop system (6)–(8),(21)–(24) are ultimately bounded and there exists a globally asymptotically stable subset of the synchronization manifold  $\mathcal{M}$ .

The proof of Theorem 1 is presented in the Appendix. The result stated in Theorem 1 amounts to the following. The interconnected systems asymptotically synchronize provided that the following inequalities are simultaneously satisfied:

$$\frac{3\kappa}{2} + \gamma \leq \bar{\delta}, \tag{I1}$$

$$\gamma > \gamma' = \frac{\kappa'}{\lambda_2}, \tag{I2}$$

$$\kappa > \bar{\kappa}(\gamma) = \kappa' + \frac{\sigma' \gamma^2}{\gamma - \gamma'}, \tag{I3}$$

$$\tau < \bar{\varphi}(\kappa, \gamma) < \varphi(\kappa), \tag{I4}$$

with  $\bar{\delta}$  the largest  $\delta$  that satisfies (H6.1). The constants  $\gamma', \sigma', \kappa'$ , and the unimodal functions  $\varphi(\kappa)$  and  $\bar{\varphi}(\kappa, \gamma)$  are derived in the proofs of Lemma 1 and Theorem 1, see (A8),

(A11), (A36), (A38), and (A39). Geometrically, the intersection of the inequalities (I1)–(I4) could be realized as a three-dimensional region in the parameter space. Hereafter, we refer to this region as the *synchronization region* and it is denoted as  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  as stated in Theorem 1. Indeed, it is not easy to visualize how the *synchronization region* looks like in the parameter space. Using inequalities (I1)–(I4), in Figure 4, we present sketches of the projections of  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  on the three planes and a three-dimensional sketch of the *synchronization region*.

### B. Discussion

So far, we have proved that the  $k$  systems (6)–(8) interconnected through the predictor-based dynamic diffusive coupling (21)–(24) asymptotically synchronize provided that the conditions stated in Theorem 1 are satisfied. However, we have not shown in what sense the *synchronization region*  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  may be greater than the *synchronization region*  $\mathcal{S}$  that would be obtained when the systems are coupled through the static diffusive coupling (9). The results presented in Theorem 1 are meant to prove existence of the *synchronization region*; therefore, the estimate of  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  given by the intersection of (I1)–(I4) may be conservative. This is because the approach taken in this manuscript is Lyapunov-based, i.e., we use Lyapunov-Razumikhin functions and Lyapunov-Krasovskii functionals to derive the results. It follows that the conditions stated in Lemma 1, Lemma 2, and Theorem 1 are sufficient but certainly not necessary. Hence, if both regions  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  and  $\mathcal{S}$  are obtained using these Lyapunov methods, it may be hard to extract quantitative insights out of them. Thus, a direct comparison between these conservative regions to evaluate the performance of the couplings would be meaningless. In the following

section, we provide a local analysis to illustrate the “mechanism of action” behind our predictor-based couplings. We compare (locally) the *synchronization regions* obtained with both controllers without using the mentioned Lyapunov methods. In particular, the provided analysis is related to the *Master Stability Function* (MSF) approach,<sup>27</sup> in the sense that the conditions for local synchronization follow from the stability properties of linear variational systems.

**C. Local analysis**

The  $k$  systems (6)–(8) can be written in the following compact form:

$$\dot{x}_i = f(x_i) + \mathcal{B}u_i(t - \tau_2), \tag{29}$$

$$y_i = \mathcal{C}x_i, \tag{30}$$

$$x_i = \phi(t), \quad t \in [-\tau_2, 0], \tag{31}$$

with  $i \in \mathcal{I} = \{1, \dots, k\}$ , and

$$x_i = \text{col}(\zeta_i, y_i), \quad f(x_i) := \text{col}(q(\zeta_i, y_i), a(\zeta_i, y_i)),$$

$$\mathcal{C} := (\mathbf{0}_{m \times (n-m)} \quad I_m), \quad \mathcal{B} := \mathcal{C}^T.$$

Then, the closed-loop stacked system (6)–(9) can be written as

$$\dot{x} = F(x) - \gamma(L \otimes \mathcal{BC})x(t - \tau), \tag{32}$$

with  $x := \text{col}(x_1, \dots, x_k)$ , Laplacian matrix  $L \in \mathbb{R}^{k \times k}$ , and  $F(x) := \text{col}(f(x_1), \dots, f(x_k))$ . Assume that:

**(H7.1)** The solutions of the coupled systems (6)–(9) are ultimately bounded, i.e., there exists a constant  $M \in \mathbb{R}_{>0}$  such that  $|x_i(t)| < M$  for all  $t \in [-\tau, \infty)$  and  $i \in \mathcal{I}$ . We refer the reader to Ref. 42, section 2, where sufficient conditions for boundedness of the solutions of the interconnected systems (6)–(9) are derived. The communication graph is *strongly connected* and  $a_{ij} = a_{ji}$  by assumption. Then, the Laplacian matrix is symmetric and its eigenvalues are real. Moreover, the matrix  $L$  has an algebraically simple eigenvalue  $\lambda_1 = 0$  and  $\mathbf{1} = \text{col}(1, \dots, 1) \in \mathbb{R}^k$  is the corresponding eigenvector.<sup>37</sup> Applying Gerschgorin’s disc theorem,<sup>43</sup> it can be concluded that the eigenvalues of  $L$  are nonnegative, i.e., the matrix  $L$  is positive semi definite. It follows that  $L$  has eigenvalues  $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$  ordered by increasing parts:  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$ . Since  $L$  is symmetric, then there exists a nonsingular matrix  $T \in \mathbb{R}^{k \times k}$  so that  $\Lambda := T^{-1}LT$ , where  $\Lambda$  denotes an upper block-triangular matrix with the eigenvalues of  $L$  on its diagonal. It can be proved that the matrix  $T$  can be chosen to satisfy

$$T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{1}_{k \times 1}, \quad (T^{-1})^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \nu, \tag{33}$$

for some vector  $\nu \in \mathbb{R}^{k \times 1}$  satisfying  $\nu^T \mathbf{1}_{k \times 1} = 1$  and  $\nu^T L = 0$ . It follows that the first column of  $T$  is  $\mathbf{1}_{k \times 1}$  and the first row of  $T^{-1}$  equals  $\nu^T$ . Introduce the change of

coordinates  $x := (T \otimes I_n)\bar{x}$ , then the closed-loop system in the new coordinates is given by

$$\dot{\bar{x}} = (T^{-1} \otimes I_n)F((T \otimes I_n)\bar{x}) - \gamma(\Lambda \otimes \mathcal{BC})\bar{x}(t - \tau). \tag{34}$$

Notice that  $\bar{x}_1 = \sum_{i=1}^k \nu_i x_i =: \xi$  with  $\nu^T \mathbf{1}_{k \times 1} = 1$ . Moreover,  $\bar{x}_j = \mathbf{0}_{n \times 1}, j = 2, \dots, k$  implies that  $x_i = \bar{x}_1 = \xi$  for all  $i \in \mathcal{I}$ , i.e., the coupled systems are synchronized. Linearizing (34) around  $\bar{x} = \text{col}(\xi, \mathbf{0}_{n \times 1}, \dots, \mathbf{0}_{n \times 1})$  yields

$$\dot{\bar{x}} = (I_k \otimes J_f(\xi))\bar{x} - \gamma(\Lambda \otimes \mathcal{BC})\bar{x}(t - \tau), \tag{35}$$

with  $J_f(\xi)$  denoting the Jacobian matrix of the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  evaluated along  $\xi = \sum_{i=1}^k \nu_i x_i \in \mathbb{R}^n$ . Smoothness of the vectorfield  $f(\cdot)$  and boundedness of the solutions imply that the Jacobian matrix  $J_f(\xi)$  is well defined and uniformly bounded. Moreover, since the system (35) is linear, then asymptotic stability of its zero solution  $\bar{x}_j = \mathbf{0}_{n \times 1}, j = 2, \dots, k$  amounts to asymptotic stability of the following equations:

$$\dot{z}_i = J_f(\xi)z_i - \gamma\lambda_i \mathcal{BC}z_i(t - \tau), \quad i = 2, 3, \dots, k. \tag{36}$$

Therefore, the  $k$  diffusively time-delayed coupled systems (6)–(9) locally synchronize provided that the coupling strength  $\gamma$  and the total time-delay  $\tau$  are such that the zero solution of the  $(k - 1)$  linear equations (36) are asymptotically stable uniformly in  $\xi(t)$ . Next, consider the closed-loop system (6)–(8),(21)–(24). Using the same compact form (32), the interconnected systems can be written as follows:

$$\dot{x} = F(x) - \gamma(L \otimes \mathcal{BC})(x - \epsilon), \tag{37}$$

$$\dot{\epsilon} = F(x) - F(x - \epsilon) - \kappa(I_k \otimes \mathcal{BC})\epsilon(t - \tau), \tag{38}$$

with prediction error  $\epsilon_i := \text{col}(\epsilon_{1i}, \epsilon_{2i})$  and stacked error  $\epsilon := \text{col}(\epsilon_1, \dots, \epsilon_k)$ . Assume that:

**(H7.2)** The conditions stated in Lemma 2 are satisfied.

Therefore, the solutions of the interconnected systems (6)–(8),(21)–(24) are uniformly ultimately bounded for all  $t \in [-\tau, \infty)$ . Inducing again the change of coordinates  $x = (T \otimes I_n)\bar{x}$  and  $\epsilon := (T \otimes I_n)\bar{\epsilon}$  with  $T$  as in (33), the closed-loop system is written as

$$\dot{\bar{x}} = (T^{-1} \otimes I_n)F((T \otimes I_n)\bar{x}) - \gamma(\Lambda \otimes \mathcal{BC})\bar{x} + \gamma(\Lambda \otimes \mathcal{BC})\bar{\epsilon}, \tag{39}$$

$$\dot{\bar{\epsilon}} = (T^{-1} \otimes I_n)F((T \otimes I_n)\bar{x}) - \kappa(I_k \otimes \mathcal{BC})\bar{\epsilon}^T - (T^{-1} \otimes I_n)F((T \otimes I_n)(\bar{x} - \bar{\epsilon})). \tag{40}$$

Linearizing (39),(40) around  $\bar{x} = \text{col}(\xi, \mathbf{0}_{n \times 1}, \dots, \mathbf{0}_{n \times 1})$  and  $\bar{\epsilon} = \text{col}(\mathbf{0}_{n \times 1}, \dots, \mathbf{0}_{n \times 1})$  yields

$$\dot{\bar{x}} = (I_k \otimes J_f(\xi))\bar{x} - \gamma(\Lambda \otimes \mathcal{BC})\bar{x} + \gamma(\Lambda \otimes \mathcal{BC})\bar{\epsilon}, \tag{41}$$

$$\dot{\bar{\epsilon}} = (I_k \otimes J_f(\xi))\bar{\epsilon} - \kappa(I_k \otimes \mathcal{BC})\bar{\epsilon}(t - \tau), \tag{42}$$

with  $J_f(\xi)$  the Jacobian matrix of  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  evaluated along  $\xi = \sum_{i=1}^k \nu_i x_i$ . The system (41),(42) is linear, then asymptotic stability of its zero solution  $\bar{x}_j = \mathbf{0}_{n \times 1}, j = 2, \dots, k$ , and  $\bar{\epsilon}_i = \mathbf{0}_{n \times 1}, i = 1, \dots, k$  implies asymptotic stability of the following equations:

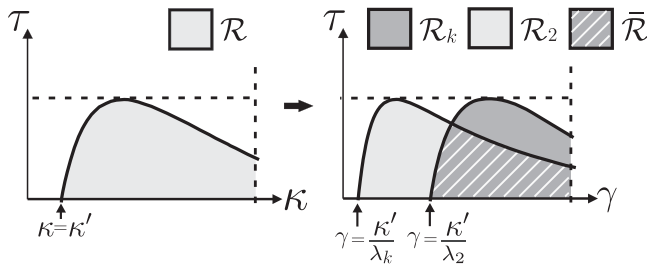


FIG. 5. Synchronization regions  $\mathcal{R}$ ,  $\mathcal{R}_2$ ,  $\mathcal{R}_k$ , and  $\bar{\mathcal{R}}$ .

$$\dot{z}_j = (J_f(\xi) - \gamma \lambda_j \mathcal{BC})z_j, \quad j = 2, 3, \dots, k, \quad (43)$$

$$\dot{\bar{\epsilon}}_i = J_f(\xi)\bar{\epsilon}_i - \kappa \mathcal{BC}\bar{\epsilon}_i(t - \tau), \quad i = 1, 2, \dots, k. \quad (44)$$

Hence, the  $k$  systems (6)–(8) interconnected through the predictor-based coupling (21)–(24) locally synchronize provided that the coupling strength  $\gamma$ , the predictor gain  $\kappa$ , and the total time-delay  $\tau$  are such that the zero solution of the  $(2k - 1)$  linear equations (43),(44) are asymptotically stable uniformly in  $\xi(t)$ . Summarizing, local synchronization of the coupled systems (6)–(9) and (6)–(8),(21)–(24) amounts to asymptotic stability of the origin of the systems (36) and (43),(44), respectively. Assume that:

**(H7.3)** The conditions stated in Lemma 1 are satisfied.

The  $k$  systems (44) are the linearization of the prediction error dynamics (18),(19) in the coordinates  $\bar{\epsilon} = (T \otimes I_n) \epsilon$ . Therefore, from Lemma 1, it follows that there exist a positive constant  $\kappa'$  and a unimodal function  $\varphi : [\kappa', \infty) \rightarrow \mathbb{R}_{\geq 0}$  with  $\varphi(\kappa') = \lim_{\kappa \rightarrow \infty} \varphi(\kappa) = 0$ , such that if  $(\kappa, \tau) \in \mathcal{R} := \{\kappa, \tau \in \mathbb{R}_{\geq 0} | \kappa > \kappa', \tau < \varphi(\kappa)\}$ , the zero solution of system (44) is asymptotically stable, see Figure 5. Notice that the dynamics (36), (43), and (44) share a similar structure. System (44) has the same dynamics as system (43) if  $\kappa = \gamma \lambda_j$ ,  $j = 2, \dots, k$ , and  $\tau = 0$ . Similarly, system (36) has the same dynamics as system (44) if  $\gamma \lambda_j = \kappa$ ,  $j = 2, \dots, k$ . Therefore, the existence of the region  $\mathcal{R}$  implies that:

**(P7.1)** The zero solutions of the  $(k - 1)$  systems (43) are asymptotically stable if  $\gamma > \frac{\kappa'}{\lambda_2}$ .

**(P7.2)** The zero solutions of the  $(k - 1)$  systems (36) are asymptotically stable if

$$(\gamma, \tau) \in \bar{\mathcal{R}} := \bigcap_{j=2}^k \mathcal{R}_j, \quad (45)$$

with  $\mathcal{R}_j := \{(\gamma, \tau) | (\kappa = \gamma \lambda_j, \tau) \in \mathcal{R}\}$ . Moreover, it can be proved that given unimodality of the function  $\varphi(\cdot)$ , the region  $\bar{\mathcal{R}}$  is simply given by  $\bar{\mathcal{R}} = \mathcal{R}_2 \cap \mathcal{R}_k$ , see section 5.3 in Ref. 44 for further details. In Figure 5, we provide a graphical interpretation of the statements given in Propositions (P7.1) and (P7.2). From (P7.1) and (P7.2), it follows that the coupled systems (6)–(9) locally synchronize if  $(\gamma, \tau) \in \bar{\mathcal{R}}$  and the coupled systems (6)–(8),(21)–(24) locally synchronize if  $(\kappa, \tau) \in \mathcal{R}$  and  $\gamma > \frac{\kappa'}{\lambda_2}$ . Notice that by introducing the predictor-based coupling, we have shifted the effect of the time-delay from the synchronization error dynamics to the prediction error dynamics. That is, if the

systems are coupled through the *static diffusive coupling* (9), the time-delay appears explicitly in the synchronization error dynamics (36) and it is directly linked to the network topology through the nonzero eigenvalues of the Laplacian matrix. On the other hand, if they interact through the *predictor-based coupling*, the time-delay appears in the prediction error dynamics (44), but not in the synchronization error dynamics (43); and therefore, in this case, the effect of the time-delay is not influenced by the network topology. Finally, from (P7.1) and (P7.2), we can immediately conclude the following:

- (a) If  $\lambda_2 < \lambda_k$ , then  $\text{area}(\mathcal{R}) > \text{area}(\bar{\mathcal{R}})$ .
- (b) If  $\lambda_2 = \lambda_k > 1$ , then  $\text{area}(\mathcal{R}) > \text{area}(\bar{\mathcal{R}})$ .
- (c) If  $\lambda_2 = \lambda_k = 1$ , then  $\text{area}(\mathcal{R}) = \text{area}(\bar{\mathcal{R}})$ .
- (d) If  $\lambda_2 = \lambda_k < 1$ , then  $\text{area}(\mathcal{R}) < \text{area}(\bar{\mathcal{R}})$ ,

with  $\lambda_2$  and  $\lambda_k$  being the smallest nonzero and the largest eigenvalues of the Laplacian matrix, and  $\text{area}(\mathcal{R})$  and  $\text{area}(\bar{\mathcal{R}})$  denoting the area of the regions  $\mathcal{R}$  and  $\bar{\mathcal{R}}$ , respectively. Therefore, locally, the predictor-based coupling would lead to greater or equal synchronization regions in cases (a)–(c). It is worth noting that for a given strongly connected graph,  $\lambda_2 = \lambda_k$ , if the network topology is *all-to-all*, i.e., each system in the network receives information from all the remaining systems.  $\square$

## D. On robustness of the control-scheme

The results presented in the previous sections are derived for networks of coupled *identical systems*. However, in practical situations, the dynamics of the systems cannot be expected to be perfectly identical. Moreover, the vectorfields  $q(\cdot)$  and  $a(\cdot)$  of the dynamics (6)–(8) must be exactly *known* to be able to construct the predictor-based coupling (21)–(24). This is unrealistic in practical situations, where there may be parametric uncertainties and/or unmodeled dynamics in the available models. In this situation, the best that can be done is to construct the couplings with the *known* part of the dynamics, which, hereafter, is referred to as *the nominal dynamics*. Hence, because of all these practical issues, we can not expect that the systems perfectly synchronize under the proposed control scheme. In the best case, if the uncertainties are sufficiently small (in some appropriate sense) it can be expected that the synchronization errors are bounded by a small constant  $\mu \in \mathbb{R}_{>0}$ , which, of course, needs to be small enough in order to consider that the systems are “*practically synchronized*.” Let the  $k$  systems (6)–(8) be the *nominal dynamics* of the following *perturbed* systems:

$$\dot{\zeta}_i = q(\zeta_i, y_i) + \Delta q_i(\zeta_i, y_i), \quad (46)$$

$$\dot{y}_i = a(\zeta_i, y_i) + u_i(t - \tau_2) + \Delta a_i(\zeta_i, y_i), \quad (47)$$

$$x_i = \phi(t), \quad t \in [-\tau_2, 0], \quad (48)$$

with  $i = 1, \dots, k$ , state  $x_i := \text{col}(\zeta_i, y_i) \in \mathbb{R}^n$ , internal state  $\zeta_i \in \mathbb{R}^{n-m}$ , output  $y_i \in \mathbb{R}^m$ , input  $u_i \in \mathbb{R}^m$ , sufficiently smooth *known* functions  $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$  and  $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ , and sufficiently smooth *unknown*

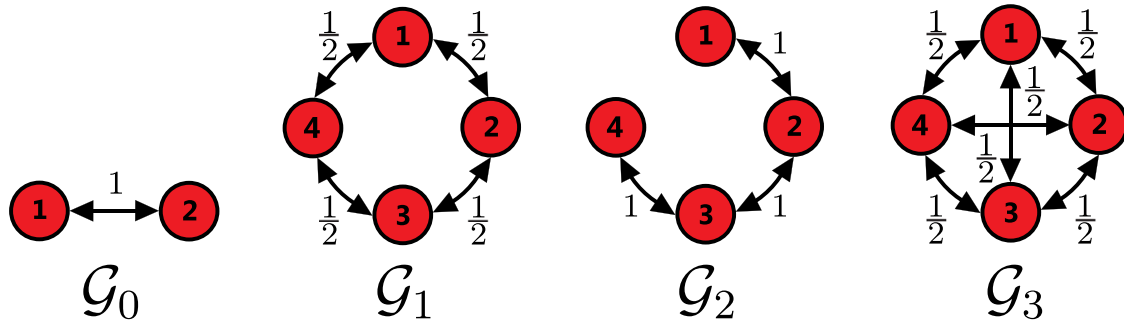


FIG. 6. Network Topologies.

functions  $\Delta q_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$  and  $\Delta a_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ . The vector-fields  $q(\cdot)$  and  $a(\cdot)$  are the *known* part of the dynamics, while the terms  $\Delta q_i(\zeta_i, y_i)$  and  $\Delta a_i(\zeta_i, y_i)$  represent parametric uncertainties and/or unmodeled dynamics. In this setting, the predictor based coupling (21)–(24) is constructed using the *known* vectorfields  $q(\cdot)$  and  $a(\cdot)$  of the nominal dynamics. Then, at this point, one may wonder under what conditions the solutions of the coupled perturbed systems (46)–(48),(21)–(24) are ultimately bounded and practically synchronize. First, we address the boundedness part in the following lemma, which is a slight modification of Lemma 2.

**Lemma 3** Consider  $k$  perturbed systems (46)–(48) interconnected through the predictor-based coupling (21)–(24) with coupling strength  $\gamma \in \mathbb{R}_{\geq 0}$ , predictor gain  $\kappa \in \mathbb{R}_{\geq 0}$ , and total time-delay  $\tau \in \mathbb{R}_{\geq 0}$  on a simple strongly connected graph. Assume that

(H7.4) Each system (46),(47) is strictly  $\mathcal{C}^1$ -semipassive with input  $u_i^{\tau_2}$ , output  $y_i$ , radially unbounded storage function  $V_i(x_i)$ , and the functions  $H_i(x_i)$  are such that there exist constants  $R_i, \delta_i \in \mathbb{R}_{> 0}$  such that  $|x_i| > R_i$  implies that  $H_i(x_i) - \delta_i |y_i|^2 > 0$ .

Let  $\bar{\delta}_i$  be the largest  $\delta_i$  that satisfies (H7.4) and define  $\bar{\delta}_{\min} := \min(\bar{\delta}_1, \dots, \bar{\delta}_k)$ , then the solutions of the coupled systems (46)–(48),(21)–(24) exist and are ultimately bounded for any finite  $\tau \in \mathbb{R}_{\geq 0}$  and  $(\gamma, \kappa) \in \mathcal{N}$  with  $\mathcal{N} := \{\gamma, \kappa \in \mathbb{R}_{\geq 0} \mid \frac{3\kappa}{2} + \gamma \leq \bar{\delta}_{\min}\}$ .

The proof of Lemma 3 follows the same lines as the proof of Lemma 2 and it is omitted here. The result stated in Lemma 3 implies that ultimate boundedness of the solutions of the coupled perturbed systems can still be guaranteed as long as each perturbed system (46),(47) is strictly  $\mathcal{C}^1$ -semipassive in the presence of the perturbation terms  $\Delta q_i(\zeta_i, y_i)$  and  $\Delta a_i(\zeta_i, y_i)$ . This is not hard to satisfy when the perturbations are due to parametric uncertainties; then, semipassivity of the nominal system may imply semipassivity of the perturbed one if the uncertainties are sufficiently small. The next step would be to show that under some conditions the coupled perturbed systems “practically synchronize.” Practical synchronization means that the differences between the states of the systems converge to some compact invariant set in finite time, and this set is bounded by a constant  $\mu \in \mathbb{R}_{> 0}$  which has to be small enough to consider that the systems are still synchronized. However, the formal study of practical

synchronization goes beyond the scope of this paper. The general purpose of this manuscript is to gain insights of the synchronization mechanisms for the class of systems and couplings under study. Particularly, we focus on the stability analysis of the synchronization manifold  $\mathcal{M}$  with respect to the *coupled unperturbed systems*. The practical implications of the control-scheme are not considered here and are left for future research.

### VIII. SIMULATION EXAMPLE

#### A. Network topology, convergence, and semipassivity

Consider a network of  $k$  systems coupled according to the graphs depicted in Figure 6. The networks are strongly connected and undirected. Each system in the network is assumed to be a Hindmarsh-Rose neuron<sup>45</sup> of the form

$$\begin{cases} \dot{z}_{1i} = 1 - 5y_i^2 - z_{1i}, \\ \dot{z}_{2i} = 0.005(4y_i + 6.472 - z_{2i}), \\ \dot{y}_i = -y_i^3 + 3y_i^2 + z_{1i} - z_{2i} + 3.25 + u_i^{\tau_2}, \end{cases} \quad (49)$$

with output  $y_i \in \mathbb{R}$ , internal states  $z_{1i}, z_{2i} \in \mathbb{R}$ , state  $x_i = \text{col}(z_{1i}, z_{2i}, y_i) \in \mathbb{R}^3$ , delayed input  $u_i^{\tau_2} \in \mathbb{R}$ , transmission time-delay  $\tau_2 \in \mathbb{R}_{\leq 0}$ , and  $i \in \mathcal{I} = \{1, \dots, k\}$ . It is well known that the Hindmarsh-Rose neuron (49) has a chaotic attractor<sup>45</sup> for  $u_i = 0$ . Furthermore, in Ref. 38, it is proved that the Hindmarsh-Rose neuron is strictly  $\mathcal{C}^1$ -semipassive with quadratic storage function  $V(z_{1i}, z_{2i}, y_i) := \frac{1}{2}y_i^2 + \sigma z_{1i}^2 + 25z_{2i}^2$ , constants  $\varsigma_1, \varsigma_2 \in (0, 1), 0 < \sigma < \frac{4\varsigma_1(1-\varsigma_2)}{25}$ , and

$$\begin{aligned} H(z_{1i}, z_{2i}, y_i) = & \varsigma_1 y_i^4 - 3y_i^3 - \frac{1}{4\sigma(1-\varsigma_2)} y_i^2 \\ & + \left( \sigma \varsigma_2 - \frac{25\sigma^2}{4(1-\varsigma_1)} \right) z_{1i}^2 + \frac{1}{4} z_{2i}^2 - 1.618 z_{2i} \\ & + \sigma(1-\varsigma_2) \left( z_{1i} - \frac{1}{2\sigma(1-\varsigma_2)} y_i \right)^2 - \sigma z_{1i} \\ & + (1-\varsigma_1) \left( y_i^2 + \frac{5\sigma}{2(1-\varsigma_1)} z_{1i} \right)^2 - 3.25 y_i. \end{aligned} \quad (50)$$

Moreover, the  $(z_{1i}, z_{2i})$ -dynamics (the internal dynamics) is *convergent* (in the sense of Definition 2), i.e., it satisfies the

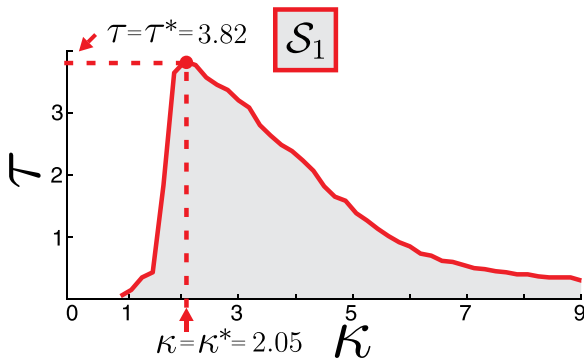


FIG. 7. Prediction region  $\mathcal{S}_1$ .

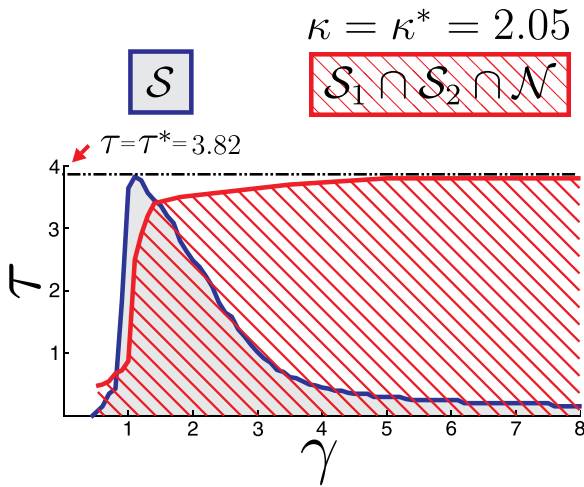


FIG. 8. Synchronization regions  $\mathcal{S}$  and  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  for  $\kappa = \kappa^*$  and  $\mathcal{G}_0$ , i.e., two coupled systems.

Demidovich condition (20) with  $P = I_2$  hence, assumption (H4.1) is satisfied.

**B. Predictor-based diffusive dynamic coupling**

Associated with systems (49), the dynamic couplings (21)–(24) take the following form:

$$\begin{cases} \dot{\eta}_{1i} = 1 - 5\eta_{3i}^2 - \eta_{1i}, \\ \dot{\eta}_{2i} = 0.005(4\eta_{3i} + 6.472 - \eta_{2i}), \\ \dot{\eta}_{3i} = -\eta_{3i}^3 + 3\eta_{3i}^2 + \eta_{1i} - \eta_{2i} \\ \quad + 3.25 + u_i + \kappa(y_i^{\tau_1} - \eta_{3i}^{\tau_1}), \\ u_i = \gamma \sum_{j \in \mathcal{E}_i} a_{ij}(\eta_{3j} - \eta_{3i}), \end{cases} \quad (51)$$

with predictor state  $\eta_i = \text{col}(\eta_{1i}, \eta_{2i}, \eta_{3i}) \in \mathbb{R}^3$ , measurement time-delay  $\tau_1 \in \mathbb{R}_{\geq 0}$ , total time-delay  $\tau \in \mathbb{R}_{\geq 0}$ , coupling strength  $\gamma \in \mathbb{R}_{\geq 0}$ , and predictor gain  $\kappa \in \mathbb{R}_{\geq 0}$ . As previously mentioned, each system (49) is strictly  $\mathcal{C}^1$ -semipassive with  $H(x_i)$  as in (50). It can be shown that the function  $H(x_i)$  satisfies the bound-edness assumption (H6.1) stated in Lemma 2 for arbitrary large coupling strength  $\gamma$  and predictor gain  $\kappa$ , i.e.,  $\mathcal{N} = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ . Therefore, by Lemma 2, the solutions of the closed-loop system (49),(51) always exist and are ultimately bounded. Moreover, since the internal dynamics is convergent, (H4.1) is satisfied; hence, by Lemma

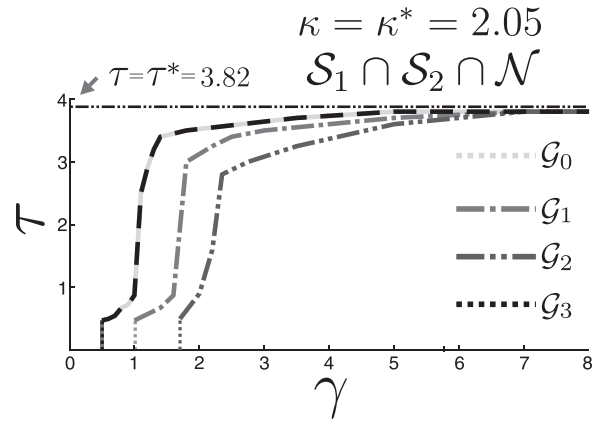


FIG. 9. Synchronization region  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  for different topologies  $\mathcal{G}_j, j = 1, 2, 3, 4$ , and  $\kappa = \kappa^*$ .

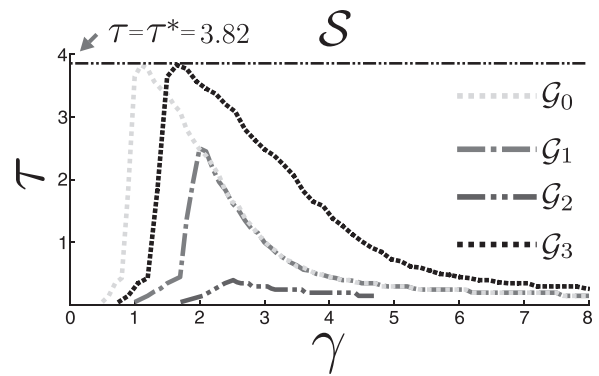


FIG. 10. Synchronization region  $\mathcal{S}$  for different topologies  $\mathcal{G}_j, j = 1, 2, 3, 4$ .

1, there exists a region  $\mathcal{S}_1 \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$  (as depicted in Figure 2), such that if  $(\kappa, \tau) \in \mathcal{S}_1$ , the predictor state  $\eta_i$  asymptotically anticipates the dynamics (49), i.e.,  $\lim_{t \rightarrow \infty} x_i(t + \tau_2) - \eta_i(t) = 0$ . Finally, by Theorem 1, there exists a nonempty set  $\mathcal{S}_2 \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$  (as depicted in Figure 4), such that if  $(\gamma, \kappa, \tau) \in \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$ , the systems synchronize.

**C. Simulation results**

In Figures 7–10, we show the results obtained through extensive computer simulations for  $\tau_1 = \tau_2 = \frac{\tau}{2}$ . Figure 7 depicts the prediction region  $\mathcal{S}_1$  introduced in Lemma 1. This region is clearly bounded by a unimodal function; and therefore, there is an optimal predictor gain  $\kappa = \kappa^* := 2.05$  which leads to the maximum time-delay  $\tau = \tau^* = 3.82$  that can be induced to the predictor. This maximum time-delay depends directly on the dynamics of the systems, but not on the network topology (see the proof of Lemma 1). In Figure 8, for  $\mathcal{G}_0$ , we show the synchronization region  $\mathcal{S}$  obtained when the two neurons are coupled via the static diffusive coupling (9) and a projection of the synchronization region  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  on the  $(\gamma, \tau)$ -plane obtained through the predictor-based coupling (51) for  $\kappa = \kappa^*$ . For this particular topology and  $\kappa$ , both couplings lead to approximately the same maximum time-delay  $\tau^*$ . This can be explained (locally) by the statement (b) of the local analysis since  $\lambda_2(\mathcal{G}_0) = \lambda_k(\mathcal{G}_0) = 2$ . However, the asymptotic behavior is

quite different. The upper bound of  $\mathcal{S}$  decreases asymptotically to zero as the coupling strength increases. On the other hand, the projection of the *synchronization region*  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  has an upper bound that converges to  $\tau^*$  asymptotically as  $\gamma$  is increased. Hence, in the latter case (for large  $\gamma$ ), the maximum time-delay is determined by the predictor gain  $\kappa$ , see Figure 7. Finally, in Figures 9 and 10, we show the regions  $\mathcal{S}$  and the projections of  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  for all the topologies depicted in Figure 6. It is clear that the regions  $\mathcal{S}$  in Figure 10 are strongly influenced by the network topology.<sup>42</sup> Conversely, the regions  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  in Figure 9 are influenced by the network topology only for small coupling strength. The upper bounds of the synchronization regions converge asymptotically to  $\tau^*$  independently of the network topology as  $\gamma$  is increased.

**IX. CONCLUSIONS**

We have presented a result on network synchronization in the case when the measurements of the available outputs and the transmission of the controllers are subject to different constant time-delays. We have shown that the time-delay that can be induced to the network may be increased by the proposed predictor-based dynamic diffusive couplings. Using the notion of semipassivity, we have provided sufficient conditions which guarantee existence and ultimate boundedness of the solutions of the closed-loop system. Sufficient conditions that guarantee (global) state synchronization have also been derived. We have provided a local analysis to illustrate the “mechanism of action” behind our predictor-based couplings. Finally, we have presented an illustrative example that shows that indeed it is possible to extend the synchronization regions with the proposed control scheme. While the regions  $\mathcal{S}$  obtained through the *static diffusive coupling* (9) are strongly influenced by the network topology, the regions  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  obtained with the predictor-based coupling are influenced by the network topology only for small coupling strength. As  $\gamma$  is increased, the upper bounds of  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  are determined by the prediction set  $\mathcal{S}_1$ , i.e., for a fixed  $\kappa$  and its corresponding maximum time-delay  $\tau^*$ , see Figure 7, the upper bounds of  $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{N}$  converge asymptotically to  $\tau^*$  independently of the network topology.

**APPENDIX: PROOFS**

**1. Proof of Lemma 1**

The boundedness assumption in Lemma 1 and smoothness of the right-hand side of the closed loop system imply that the solutions of (6)–(8),(15)–(17) exist and are unique. The prediction error is defined as  $\epsilon_i = x_i - \eta_i^{\tau_2}$  and the prediction error dynamics is given by (18),(19). Assumption (H4.1), Proposition 1, smoothness of the vectorfields, and boundedness of the solutions imply the existence of a positive definite function  $V_0 : \mathbb{R}^{n-m} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\dot{V}_0(\epsilon_{1i}, \epsilon_{2i}) \leq -\alpha|\epsilon_{1i}|^2 + c_0|\epsilon_{1i}||\epsilon_{2i}|, \tag{A1}$$

for some constants  $\alpha, c_0 \in \mathbb{R}_{>0}$ , see section 5 in Ref. 1 for further details. Using Leibniz’s rule  $\epsilon_{2i}^{\tau_2}$  can be written as

$$\epsilon_{2i}^{\tau_2} = \epsilon_{2i} - \int_{-\tau}^0 \dot{\epsilon}_{2i}(t+s)ds, \tag{A2}$$

it follows that the prediction error dynamics (19) can be written as

$$\begin{aligned} \dot{\epsilon}_{2i} &= a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}) \\ &\quad - \kappa\epsilon_{2i} + \kappa \int_{-\tau}^0 \dot{\epsilon}_{2i}(t+s)ds, \end{aligned} \tag{A3}$$

substitution of (19) in (A3) yields

$$\begin{aligned} \dot{\epsilon}_{2i} &= a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i}) \\ &\quad - \kappa\epsilon_{2i} - \kappa^2 \int_{-\tau}^0 \epsilon_{2i}(t+s-\tau)ds \\ &\quad + \kappa \int_{-\tau}^0 (a(\zeta_i, y_i)(t+s) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i})(t+s))ds. \end{aligned} \tag{A4}$$

Consider the function  $V_1(\epsilon_{2i}) = \frac{1}{2}\epsilon_{2i}^T \epsilon_{2i}$ . Then

$$\begin{aligned} \dot{V}_1 &\leq \epsilon_{2i}^T (a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i})) \\ &\quad - \kappa|\epsilon_{2i}|^2 - \kappa^2 \epsilon_{2i}^T \int_{-\tau}^0 \epsilon_{2i}(t+s-\tau)ds \\ &\quad + \kappa \epsilon_{2i}^T \int_{-\tau}^0 (a(\zeta_i, y_i)(t+s) \\ &\quad - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i})(t+s))ds. \end{aligned} \tag{A5}$$

Ultimate boundedness of the solutions and smoothness of the function  $a(\cdot)$  imply that

$$\epsilon_{2i}^T (a(\zeta_i, y_i) - a(\zeta_i - \epsilon_{1i}, y_i - \epsilon_{2i})) \leq c_1|\epsilon_{2i}|^2 + c_2|\epsilon_{2i}||\epsilon_{1i}|,$$

for some constants  $c_1, c_2 \in \mathbb{R}_{>0}$ . Let the function  $\mathcal{V}_1(\epsilon_{1i}, \epsilon_{2i}) := V_0(\epsilon_{1i}) + V_1(\epsilon_{2i})$  be a Lyapunov-Razumikhin function such that if  $\mathcal{V}_1(\epsilon_i(t)) > \kappa^2 \mathcal{V}_1(\epsilon_i(t+\theta))$  for  $\theta \in [-2\tau, 0]$  and some constant  $\kappa > 1$ , then

$$\begin{aligned} \dot{\mathcal{V}}_1 &\leq -\alpha|\epsilon_{1i}|^2 - (\kappa - c_1 - \kappa\kappa^2\tau - \kappa\kappa^2c_1)|\epsilon_{2i}|^2 \\ &\quad + (c_0 + c_2 + \kappa\kappa^2c_2)|\epsilon_{1i}||\epsilon_{2i}|. \end{aligned} \tag{A6}$$

The constant  $\kappa$  can be arbitrarily close to one as long as it is greater than one. Then, for the sake of simplicity, we take  $\kappa$  on the boundary  $\kappa = 1$  for the rest of the analysis. Some simple algebra shows that (A6) is negative definite if

$$(\kappa - \kappa') - \left( \kappa + \frac{\bar{c}_1}{\bar{c}_2} \right) \kappa\tau - \frac{1}{2\bar{c}_2} (\kappa\tau)^2 > 0, \tag{A7}$$

with

$$\kappa' := \frac{(c_0 + c_2)^2}{4\alpha} + c_1, \tag{A8}$$

$$\bar{c}_1 := \frac{2\alpha c_1 + c_0 c_2 + c_2^2}{c_2^2}, \quad \bar{c}_2 := \frac{2\alpha}{c_2^2}. \tag{A9}$$

All the constants in (A7) are positive by construction and  $\kappa$  and  $\tau$  are nonnegative by definition. Then, a necessary condition for (A7) to be satisfied is that  $\kappa > \kappa'$ . After some straightforward computations (A7) can be rewritten as follows:

$$\tau < -\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right) \pm \sqrt{\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right)^2 + 2\bar{c}_2 \frac{(\kappa - \kappa')}{\kappa^2}}. \quad (\text{A10})$$

The total time-delay  $\tau$  is nonnegative by definition. Hence, in order to satisfy (A10), it is sufficient to consider the possible positive values of the right-hand side of (A10), i.e., the positive square root. Then, inequality (A10) boils down to

$$\tau < \varphi(\kappa) := -\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right) + \sqrt{\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right)^2 + 2\bar{c}_2 \frac{(\kappa - \kappa')}{\kappa^2}}. \quad (\text{A11})$$

We are only interested in possible values of  $\kappa, \tau \in \mathbb{R}_{\geq 0}$  such that (A11) is satisfied. Then, we restrict the function  $\varphi(\kappa)$  to the set  $\mathcal{J} := [\kappa', \infty)$ . Next, we prove that the function  $\varphi : \mathcal{J} \rightarrow \mathbb{R}_{\geq 0}$  is unimodal. The function  $\varphi(\cdot)$  is continuous and real-valued on  $\mathcal{J}$ . Moreover, it is strictly positive on the interior of  $\mathcal{J}$ , it has a root at  $\kappa = \kappa'$ , i.e.,  $\varphi(\kappa') = 0$ , and

$$\begin{aligned} & \lim_{\kappa \rightarrow \infty} \varphi(\kappa) \\ &= \lim_{\kappa \rightarrow \infty} \frac{2\bar{c}_2 \left(\frac{1}{\kappa} - \frac{\kappa'}{\kappa^2}\right)}{\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right) + \sqrt{\left(\bar{c}_2 + \frac{\bar{c}_1}{\kappa}\right)^2 + 2\bar{c}_2 \left(\frac{1}{\kappa} - \frac{\kappa'}{\kappa^2}\right)}} \\ &= \frac{2\bar{c}_2(0)}{(\bar{c}_2 + 0) + \sqrt{(\bar{c}_2 + 0)^2 + 2\bar{c}_2(0)}} = \frac{0}{2\bar{c}_2} = 0. \end{aligned}$$

The function  $\varphi(\cdot)$  is differentiable on  $\mathcal{J}$ , then we can compute its local extrema by computing its critical points. It is easy to verify that  $\frac{\partial \varphi(\kappa)}{\partial \kappa}$  only for  $\kappa = \kappa_1^*$  and  $\kappa = \kappa_2^*$  with

$$\kappa_1^* = \left(1 + \frac{1}{1 + 2\bar{c}_1}\right) \kappa' + \frac{\sqrt{2\bar{c}_1^2 \bar{c}_2 \kappa' (1 + 2\bar{c}_1 + 2\bar{c}_2 \kappa')}}{\bar{c}_2 (1 + 2\bar{c}_1)}, \quad (\text{A12})$$

$$\kappa_2^* = \left(1 + \frac{1}{1 + 2\bar{c}_1}\right) \kappa' - \frac{\sqrt{2\bar{c}_1^2 \bar{c}_2 \kappa' (1 + 2\bar{c}_1 + 2\bar{c}_2 \kappa')}}{\bar{c}_2 (1 + 2\bar{c}_1)}. \quad (\text{A13})$$

Then,  $\kappa = \kappa_1^*$  and  $\kappa = \kappa_2^*$  are the critical points of  $\varphi(\kappa)$  and  $\varphi(\kappa_1^*)$  and  $\varphi(\kappa_2^*)$  are the corresponding global extrema. Notice that  $\kappa_1^* > \kappa'$ ; therefore,  $\kappa_1^*$  belongs to the interior of  $\mathcal{J}$ . It is difficult to visualize from (A13) whether  $\kappa_2^*$  is contained in  $\mathcal{J}$ . Then, we rewrite (A13) in a more suitable manner. After some algebra (A13) can be written as follows:

$$\kappa_2^* = \frac{2\bar{c}_2 \kappa' - \bar{c}_1^2}{\bar{c}_2 \left(1 + \bar{c}_1 + \bar{c}_1 \sqrt{1 + \frac{1 + 2\bar{c}_1}{\bar{c}_2 \kappa'}}\right)}. \quad (\text{A14})$$

Notice that the denominator of (A14) is strictly positive, then the sign of  $\kappa_2^*$  is solely determined by the numerator. Substitution of (A8) and (A9) in the numerator of (A14) yields

$$2\bar{c}_2 \kappa' - \bar{c}_1^2 = -\frac{4\alpha c_1 (c_0 c_2 + \alpha c_1)}{c_2^2}, \quad (\text{A15})$$

which is strictly negative. It follows that  $\kappa_2^*$  strictly negative as well; in consequence,  $\kappa_2^*$  is not contained on  $\mathcal{J}$ , i.e.,  $\kappa_2^* \notin \mathcal{J}$ . Then, the function  $\varphi(\cdot)$  has a unique extremum on  $\mathcal{J}$  and it is

given by  $\varphi(\kappa_1^*)$ . Finally, given that  $\varphi(\kappa') = 0, \lim_{\kappa \rightarrow \infty} \varphi(\kappa) = 0, \varphi(\kappa)$  is strictly positive on the interior of  $\mathcal{J}$ , and  $\varphi(\kappa_1^*)$  is the unique extremum on  $\mathcal{J}$ , it follows that  $\varphi(\kappa_1^*)$  is a unique local maximum on  $\mathcal{J}$ ; therefore, it can be concluded that the function  $\varphi(\cdot)$  is a *unimodal function* in the sense of Definition 3. Hence, (A6) is negative definite if  $\kappa > \kappa'$  and  $\tau < \varphi(\kappa)$ . Then, ultimate boundedness of the solutions and the Lyapunov-Razumikhin theorem imply that the set  $\{\epsilon_i = 0\}$  is a global attractor for  $\kappa > \kappa'$  and  $\tau < \varphi(\kappa)$ .  $\square$

**2. Proof of Lemma 2**

By assumption each system (6),(8) is strictly  $\mathcal{C}^1$ -semipassive with input  $u_i^{\tau_2}$ , output  $y_i$ , and radially unbounded function  $V(x_i)$ . Define the function  $W_1(x) := \sum_{i=1}^k \nu_i V(x_i)$ , where  $x = \text{col}(x_1, \dots, x_k)$  and the constants  $\nu_i$  denote the entries of the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix  $L$ , i.e.,  $\nu = (\nu_1, \dots, \nu_k)^T$  and  $\nu^T L = \nu^T (D - A) = 0$ . Note that  $L$  is singular by construction. Moreover, since it is assumed that the graph is strongly connected, then the zero eigenvalue is simple. Using the Perron-Frobenius theorem,<sup>37</sup> it can be shown that the vector  $\nu$  has strictly positive real entries, i.e.,  $\nu_i > 0$  for all  $i$ . Then, by assumption

$$\dot{W}_1(x) \leq \sum_{i=1}^k \nu_i \left(y_i^T u_i^{\tau_2} - H(x_i)\right). \quad (\text{A16})$$

Consider the term

$$\sum_{i=1}^k \nu_i y_i^T u_i^{\tau_2} = \gamma \sum_{i=1}^k \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} y_i^T (\eta_{2j}^{\tau_2} - \eta_{2i}^{\tau_2}), \quad (\text{A17})$$

using Young’s inequality it follows that

$$\sum_{i=1}^k \nu_i y_i^T u_i^{\tau_2} \geq \frac{\gamma}{2} \sum_{i=1}^k \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} \left(2|y_i|^2 + |\eta_{2j}^{\tau_2}|^2 + |\eta_{2i}^{\tau_2}|^2\right).$$

Notice that if  $\kappa = u_i = 0$ , the predictor dynamics (21),(22) is the same as (6),(7) with  $u^{\tau_2} = 0$ . Therefore, strict  $\mathcal{C}^1$ -semipassivity of (6),(7) implies strict  $\mathcal{C}^1$ -semipassivity of (21),(22) with radially unbounded function  $V(\eta_i)$ , output  $\eta_{2i}$ , and input  $w_i = u_i + \kappa y_i^{\tau_1} - \kappa \eta_{2i}^{\tau_1}$ . Define the functional

$$\begin{aligned} W_2(\eta_t(\theta)) &= \sum_{i=1}^k \nu_i \left( V(\eta_i) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} \int_{-\tau_2}^0 |\eta_{2i}(t+s)|^2 ds \right. \\ &\quad \left. + \frac{\kappa}{2} \int_{-\tau_1}^0 |y_i(t+s)|^2 ds + \frac{k}{2} \int_{-\tau}^0 |\eta_{2i}(t+s)|^2 ds \right), \end{aligned}$$

with  $\eta = \text{col}(\eta_1, \dots, \eta_k), \eta_t(\theta) = \eta(t + \theta) \in \mathcal{C}, \theta \in [-\tau, 0]$ , and  $\mathcal{C} = [-\tau, 0] \rightarrow \mathbb{R}^{kn}$  the Banach space of continuous functions mapping the interval  $[-\tau, 0]$  into  $\mathbb{R}^{kn}$ . Then, by assumption

$$\begin{aligned} \dot{W}_2 &= \sum_{i=1}^k \nu_i \left( \eta_{2i}^T w_i - H(\eta_i) + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} \left( |\eta_{2i}|^2 - |\eta_{2i}^{\tau_2}|^2 \right) \right. \\ &\quad \left. + \frac{\kappa}{2} (|y_i|^2 - |y_i^{\tau_1}|^2 + |\eta_{2i}|^2 - |\eta_{2i}^{\tau_1}|^2) \right). \quad (\text{A18}) \end{aligned}$$

Consider the term

$$\sum_{i=1}^k \nu_i \eta_{2i}^T w_i = \sum_{i=1}^k \nu_i \left( \gamma \sum_{j \in \mathcal{E}_i} a_{ij} \eta_{2i}^T (\eta_{2j} - \eta_{2i}) + \kappa \eta_{2i}^T y_i^{\tau_1} - \kappa \eta_{2i}^T \eta_{2i}^{\tau_2} \right),$$

using Young’s inequality it follows that

$$\sum_{i=1}^k \nu_i \eta_{2i}^T w_i \leq \sum_{i=1}^k \nu_i \left( \frac{\gamma}{2} \sum_{j \in \mathcal{E}_i} a_{ij} (|\eta_{2j}|^2 - |\eta_{2i}|^2) + \kappa |\eta_{2i}|^2 + \frac{\kappa}{2} |y_i^{\tau_1}|^2 + \frac{\kappa}{2} |\eta_{2i}^{\tau_2}|^2 \right).$$

Finally, define the functional  $W(x_i(\theta), \eta_i(\theta)) := W_1 + W_2$ , with  $x_i(\theta) = x(t + \theta) \in \mathcal{C}$  and  $\theta \in [-\tau, 0]$ . Then, combining the previous results

$$\begin{aligned} \dot{W} &\leq \sum_{i=1}^k \nu_i \left( -H(x_i) - H(\eta_i) + \frac{\kappa}{2} |y_i|^2 + \gamma \sum_{j \in \mathcal{E}_i} a_{ij} |y_i|^2 \right. \\ &\quad \left. + \frac{3\kappa}{2} |\eta_{2i}|^2 + \frac{\gamma}{2} \sum_{j \in \mathcal{E}_i} a_{ij} (2|\eta_{2i}|^2 + |\eta_{2j}|^2 - |\eta_{2i}|^2) \right. \\ &\quad \left. + \frac{\gamma}{2} \sum_{j \in \mathcal{E}_i} a_{ij} (|\eta_{2j}^{\tau_2}|^2 - |\eta_{2i}^{\tau_2}|^2) \right), \end{aligned} \tag{A19}$$

using  $\nu^T(D - A) = 0$  and  $\max_{i \in \mathcal{I}} \sum_{j \in \mathcal{E}_i} a_{ij} = 1$ , it follows that

$$\begin{aligned} \dot{W} &\leq \sum_{i=1}^k \left( -H(x_i) + \left( \gamma + \frac{3\kappa}{2} \right) |y_i|^2 \right) \\ &\quad + \sum_{i=1}^k \nu_i \left( -H(\eta_i) + \left( \gamma + \frac{3\kappa}{2} \right) |\eta_{2i}|^2 \right). \end{aligned} \tag{A20}$$

The function  $H(\cdot)$  is strictly positive if its argument is sufficiently large. Moreover, by assumption (H6.1), there exists a positive constant  $R \in \mathbb{R}_{>0}$  such that  $|x_i| > R$  implies that  $H(x_i) - \delta |y_i|^2 > 0$  for some  $\delta \in \mathbb{R}_{>0}$ . Let  $\bar{\delta}$  be the largest  $\delta$  that satisfies (H6.1), then for  $(\gamma, \kappa)$  satisfying  $\gamma + \frac{3\kappa}{2} \leq \bar{\delta}$  and for sufficiently large  $|\varsigma|$  with  $\varsigma := \text{col}(x, \eta)$ , it follows that  $\dot{W} < 0$ . The functional  $W$  is radially unbounded and positive definite by construction. Hence, there exists a constant  $\sigma \in \mathbb{R}_{>0}$  such that  $\dot{W}(\varsigma) < 0$  for  $\sigma$  and  $\varsigma$  satisfying  $W(\varsigma) \leq \sigma$ . Then, solutions starting in the set  $\{W(\varsigma) \leq \sigma^*\}$  will remain there for future time since  $W$  is negative on the boundary  $W(\varsigma) = \sigma$ . Moreover, for any  $\varsigma$  in the set  $\{W(\varsigma) \geq \sigma^*\}$  with  $\sigma^* > \sigma$ , the function  $\dot{W}(\varsigma)$  is strictly negative, which implies that, in this set,  $W(\varsigma)$  will decrease monotonically until the solutions enter the set  $\{W(\varsigma) \leq \sigma\}$  again. Therefore, it can be concluded that the solutions of the closed loop system (6),(7),(21)–(24) exist and are ultimately bounded for any finite  $\tau \geq 0$  and  $(\gamma, \kappa)$  satisfying  $\gamma + \frac{3\kappa}{2} \leq \bar{\delta}$ . ■

### 3. Proof of Theorem 1

The existence and uniqueness of the solutions follows from smoothness of the right-hand side of the closed-loop system. By Lemma 2, the solutions exist for all  $t \in [-2\tau, +\infty]$  and are ultimately bounded. Let  $\zeta = \text{col}(\zeta_1, \dots, \zeta_k) \in \mathbb{R}^{k(n-m)}$ ,  $y = \text{col}(y_1, \dots, y_k) \in \mathbb{R}^{km}$ ,  $\epsilon_1 = \text{col}(\epsilon_{11}, \dots, \epsilon_{1k}) \in \mathbb{R}^{k(n-m)}$ , and  $\epsilon_2 = \text{col}(\epsilon_{21}, \dots, \epsilon_{2k}) \in \mathbb{R}^{km}$ . Define  $M \in \mathbb{R}^{(k-1) \times k}$  as

$$M := (\mathbf{1}_{(k-1) \times 1} - I_{k-1}). \tag{A21}$$

Introduce the set of coordinates  $\tilde{\zeta} = (M \otimes I_{n-m})\zeta$ ,  $\tilde{y} = (M \otimes I_m)y$ ,  $\tilde{\epsilon}_1 = (M \otimes I_{n-m})\epsilon_1$ , and  $\tilde{\epsilon}_2 = (M \otimes I_m)\epsilon_2$ . Note that,  $\tilde{y}_1 = y_1 - y_2, \dots, \tilde{y}_{k-1} = y_{k-1} - y_k$ ,  $\tilde{\zeta}_1 = \zeta_1 - \zeta_2, \dots, \tilde{\zeta}_{k-1} = \zeta_{k-1} - \zeta_k$ ,  $\tilde{\epsilon}_{11} = \epsilon_{11} - \epsilon_{12}, \dots, \tilde{\epsilon}_{1(k-1)} = \epsilon_{11} - \epsilon_{1k}$ , and  $\tilde{\epsilon}_{21} = \epsilon_{21} - \epsilon_{22}, \dots, \tilde{\epsilon}_{2(k-1)} = \epsilon_{12} - \epsilon_{2k}$ . Then, it follows that  $\tilde{y} = \tilde{\zeta} = 0$  implies that the systems are synchronized. Assumption (H4.1), Proposition 1, smoothness of the vectorfields, and boundedness of the solutions imply the existence of a positive definite function  $V_2 : \mathbb{R}^{(k-1)(n-m)} \rightarrow \mathbb{R}_{\geq 0}, \tilde{\zeta} \mapsto V_2(\tilde{\zeta})$  such that

$$\dot{V}_2(\tilde{\zeta}, \tilde{y}) \leq -\alpha |\tilde{\zeta}|^2 + c_0 |\tilde{\zeta}| |\tilde{y}|, \tag{A22}$$

for some constants  $\alpha, c_0 \in \mathbb{R}_{>0}$ , see section 5 in Ref. 1 for further details. Notice that

$$\tilde{M} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -I_{k-1} \end{pmatrix} \rightarrow \tilde{M} \tilde{L} \tilde{M}^{-1} = \begin{pmatrix} \mathbf{0} & * \\ \mathbf{0} & \tilde{L} \end{pmatrix}, \tag{A23}$$

where  $\mathbf{1}$  and  $\mathbf{0}$  are all ones and all zeros vectors of appropriate dimensions,  $L$  denotes the Laplacian matrix. By assumption, the communication graph is *strongly connected* and the interconnections are mutual, i.e.,  $a_{ij} = a_{ji}$ . Then, the Laplacian matrix is symmetric and its eigenvalues are real. Moreover, the matrix  $L$  has an algebraically simple eigenvalue  $\lambda_1 = 0$  and  $\mathbf{1} = \text{col}(1, \dots, 1) \in \mathbb{R}^k$  is the corresponding eigenvector.<sup>37</sup> Applying Gerschgorin’s theorem<sup>43</sup> about localization of eigenvalues, it can be concluded that the eigenvalues of  $L$  are non-negative, i.e.,  $L$  is positive semidefinite. Since  $\text{spec}(\tilde{L}) = \text{spec}(L) \setminus \{0\}$ , it follows that the matrix  $\tilde{L} \in \mathbb{R}^{(k-1) \times (k-1)}$  has eigenvalues  $\lambda_2, \dots, \lambda_k \in \mathbb{R}_{>0}$ , with  $0 < \lambda_2 \leq \dots \leq \lambda_k$ . The stacked prediction errors are given by  $\epsilon_1 = \zeta - \eta_1^{\tau_2}$  and  $\epsilon_2 = y - \eta_2^{\tau_2}$  with  $\eta_1 = \text{col}(\eta_{11}, \dots, \eta_{1k}) \in \mathbb{R}^{k(n-m)}$  and  $\eta_2 = \text{col}(\eta_{21}, \dots, \eta_{2k}) \in \mathbb{R}^{km}$ . Then, the controller (23) can be written in matrix form as follows:

$$\begin{aligned} u(t) &= -\gamma(L \otimes I_m)\eta_2(t) \\ &= -\gamma(L \otimes I_m)y(t + \tau_2) + \gamma(L \otimes L_m)\epsilon_2(t + \tau_2), \end{aligned} \tag{A24}$$

where  $u = \text{col}(u_1, \dots, u_k) \in \mathbb{R}^{km}$ . Moreover, denote  $\tilde{u} := \text{col}((u_1 - u_2), \dots, (u_{k-1} - u_k))$ , it follows that

$$\tilde{u}(t) = -\gamma(\tilde{L} \otimes I_m)\tilde{y}(t + \tau_2) + \gamma(\tilde{L} \otimes L_m)\tilde{\epsilon}_2(t + \tau_2), \tag{A25}$$

with  $\tilde{L}$  as in (A23). Then, in the new coordinates, the closed loop system is given by

$$\dot{\tilde{\zeta}} = \tilde{q}(\tilde{y}, \tilde{\zeta}, y_1, \zeta_1), \tag{A26}$$

$$\dot{\tilde{y}} = \tilde{a}(\tilde{y}, \tilde{\zeta}, y_1, \zeta_1) - \gamma(\tilde{L} \otimes L_m)\tilde{y}(t) + \gamma(\tilde{L} \otimes I_m)\tilde{\epsilon}_2(t), \tag{A27}$$

where

$$\tilde{a}(\tilde{y}, \tilde{\zeta}, y_1, \zeta_1) = \begin{pmatrix} a(y_1, \zeta_1) - a(y_1 - \tilde{y}_1, \zeta_1 - \tilde{\zeta}_1) \\ \vdots \\ a(y_1, \zeta_1) - a(y_1 - \tilde{y}_{k-1}, \zeta_1 - \tilde{\zeta}_{k-1}) \end{pmatrix}, \tag{A28}$$



and

$$\tilde{q}(\tilde{y}, \tilde{\zeta}, y_1, \zeta_1) = \begin{pmatrix} q(y_1, \zeta_1) - q(y_1 - \tilde{y}_1, \zeta_1 - \tilde{\zeta}_1) \\ \vdots \\ q(y_1, \zeta_1) - q(y_1 - \tilde{y}_{k-1}, \zeta_1 - \tilde{\zeta}_{k-1}) \end{pmatrix}. \tag{A29}$$

Since  $L$  is symmetric, then there exists a nonsingular matrix  $U \in \mathbb{R}^{(k-1) \times (k-1)}$  such that  $\|U\| = 1$  and  $U\tilde{L}U^{-1} = \Lambda$ , where  $\Lambda$  denotes a diagonal matrix with the nonzero eigenvalues of  $L$  as entries. Introduce new coordinates  $\bar{y} = (U \otimes I_m)\tilde{y}$  and for consistency of notation  $\bar{\zeta} = \tilde{\zeta}$ . In the new coordinates, the closed loop system can be written as

$$\dot{\bar{\zeta}} = \bar{q}(\bar{y}, \bar{\zeta}, y_1, \zeta_1), \tag{A30}$$

$$\dot{\bar{y}} = \bar{a}(\bar{y}, \bar{\zeta}, y_1, \zeta_1) - \gamma(\Lambda \otimes I_m)\bar{y}(t) + \gamma(\Lambda \otimes I_m)\bar{e}_2(t), \tag{A31}$$

where  $\bar{q}(\bar{y}, \bar{\zeta}, y_1, \zeta_1) := \tilde{q}((U^{-1} \otimes I_m)\bar{y}, \bar{\zeta}, y_1, \zeta_1)$ ,  $\bar{a}(\bar{y}, \bar{\zeta}, y_1, \zeta_1) := (U \otimes I_m)\tilde{a}((U^{-1} \otimes I_m)\bar{y}, \bar{\zeta}, y_1, \zeta_1)$ , and  $\bar{e}_2 = (U \otimes I_m)\tilde{e}_2$ . Notice that  $\bar{y} = \bar{\zeta} = 0$  implies that the systems are synchronized because  $U$  is nonsingular. Since stability is invariant under a change of coordinates and  $\|U\| = 1$ , then from (A22), it follows that there exists a positive definite function  $\bar{V}_2 : \mathbb{R}^{(k-1)(n-m)} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\bar{\zeta} \mapsto \bar{V}_2(\bar{\zeta})$  such that

$$\dot{\bar{V}}_2(\bar{\zeta}, \bar{y}) \leq -\alpha|\bar{\zeta}|^2 + c_0|\bar{\zeta}||\bar{y}|, \tag{A32}$$

for some constants  $\alpha, c_0 \in \mathbb{R}_{>0}$ . Consider the function  $V_3(\bar{y}) = \frac{1}{2}\bar{y}^T\bar{y}$ . Then

$$\dot{V}_3 \leq -\gamma\lambda_2|\bar{y}|^2 + \bar{y}^T(\bar{a}(\bar{y}, \bar{\zeta}, y_1, \zeta_1) + \gamma(\Lambda \otimes I_m)\bar{e}_2). \tag{A33}$$

Ultimate boundedness of the solutions and smoothness of the function  $a(\cdot)$  imply that

$$\begin{aligned} \bar{y}^T\bar{a}(\bar{y}, \bar{\zeta}, y_1, \zeta_1) &\leq c_1|\bar{y}|^2 + c_2|\bar{y}||\bar{\zeta}|, \\ \gamma\bar{y}^T(\Lambda \otimes I_m)\bar{e}_2 &\leq \gamma\lambda_k|\bar{y}||\bar{e}_2|, \end{aligned}$$

for some positive constants  $c_1, c_2 \in \mathbb{R}_{>0}$ . Consider the function  $\mathcal{V}_2(\bar{\zeta}, \bar{y}) = \bar{V}_2(\bar{\zeta}) + V_3(\bar{y})$ , then

$$\dot{\mathcal{V}}_2 \leq -(\gamma\lambda_2 - c_1)|\bar{y}|^2 + (c_2 + c_0)|\bar{y}||\bar{\zeta}| - \alpha|\bar{\zeta}|^2 + \gamma\lambda_k|\bar{y}||\bar{e}_2|.$$

Next, from the proof of Lemma 1, consider the function

$$\mathcal{V}_3(\epsilon_1, \epsilon_2) = \sum_{i=1}^k \mathcal{V}_1(\epsilon_{1i}, \epsilon_{2i}) = \sum_{i=1}^k V_0(\epsilon_{1i}) + V_1(\epsilon_{2i}), \tag{A34}$$

with  $V_0(\cdot)$  and  $V_1(\cdot)$  from (A1) and (A5), respectively. Let the function  $\mathcal{V}_3(\epsilon_1, \epsilon_2)$  be a Lyapunov-Razumikhin function such that if  $\mathcal{V}_3(\epsilon_1(t), \epsilon_2(t)) > \kappa^2\mathcal{V}(\epsilon_1(t+\theta), \epsilon_2(t+\theta))$  for  $\theta \in [-2\tau, 0]$  and some constant  $\kappa > 1$ , then from the proof of Lemma 1, it follows that

$$\begin{aligned} \dot{\mathcal{V}}_3 &\leq -\alpha|\epsilon_1|^2 - (\kappa - c_1 - \kappa\kappa^2\tau - \kappa\kappa\tau c_1)|\epsilon_2|^2 \\ &\quad + (c_0 + c_2 + \kappa\kappa\tau c_2)|\epsilon_1||\epsilon_2|. \end{aligned}$$

Finally, consider the function  $\mathcal{V}(\bar{x}, \epsilon) = \mathcal{V}_2(\bar{\zeta}, \bar{y}) + \mathcal{V}_3(\epsilon_1, \epsilon_2)$  with  $\bar{x} = \text{col}(\bar{\zeta}, \bar{y})$  and  $\epsilon = \text{col}(\epsilon_1, \epsilon_2)$ . Then, using the fact that  $|\bar{e}_2| \leq \|M\||\epsilon_2| = \sqrt{k}|\epsilon_2|$ , taking  $\kappa$  on the boundary  $\kappa = 1$ , and combining the previous results, it follows that

$$\begin{aligned} \dot{\mathcal{V}} &\leq -\alpha|\bar{\zeta}|^2 - (\gamma\lambda_2 - c_1)|\bar{y}|^2 + (c_0 + c_2)|\bar{y}||\bar{\zeta}| \\ &\quad -\alpha|\epsilon_1|^2 - (k - c_1 - \kappa^2\tau - \kappa\tau c_1)|\epsilon_2|^2 \\ &\quad + (c_0 + c_2 + \kappa\tau c_2)|\epsilon_1||\epsilon_2| + \gamma\sqrt{k}\lambda_k|\bar{y}||\epsilon_2|. \end{aligned} \tag{A35}$$

Some straightforward algebra shows that (A35) is negative definite if the following inequalities are satisfied:

$$\gamma > \gamma' := \frac{1}{\lambda_2} \left( c_1 + \frac{(c_0 + c_2)^2}{4\alpha} \right) = \frac{\kappa'}{\lambda_2}, \tag{A36}$$

$$(\kappa - \bar{\kappa}(\gamma)) - \left( \kappa + \frac{\bar{c}_1}{\bar{c}_2} \right) \kappa\tau - \frac{1}{\bar{c}_2} (\kappa\tau)^2 > 0, \tag{A37}$$

with constants  $\kappa', \bar{c}_1, \bar{c}_2 \in \mathbb{R}_{>0}$  from the proof of Lemma 1, defined in (A8) and (A9), and

$$\begin{cases} \bar{\kappa}(\gamma) := \kappa' + \frac{\sigma'\gamma^2}{\gamma - \gamma'}, \\ \sigma' := \frac{k\lambda_k^2}{4\lambda_2}. \end{cases} \tag{A38}$$

Since the constants in (A36) and (A37) are positive by construction and  $\kappa, \tau$ , and  $\gamma$  are nonnegative by definition, then a necessary condition for (A37) to be satisfied is that  $\kappa > \bar{\kappa}(\gamma)$ . We are only interested in possible values of  $\kappa, \tau, \gamma \in \mathbb{R}_{\geq 0}$  such that (A36) and (A37) are satisfied. Then, we restrict the function  $\bar{\kappa}(\gamma)$  to the set  $\Gamma := (\gamma', \infty)$ . It is easy to verify that the function  $\bar{\kappa} : \Gamma \rightarrow [\bar{\kappa}(2\gamma'), \infty)$  is strictly positive, continuous, and real-valued on  $\Gamma$ . Notice that the inequality (A37) has the same structure as (A7), which is the inequality that has to be satisfied to render the origin of the prediction error dynamics (18),(19) asymptotically stable. The only difference between them is that the delay-free term in (A7) depends solely on  $\kappa$  while in (A37) depends on both  $\kappa$  and  $\gamma$ . Then, as in the proof of Lemma 1, the inequality (A37) can be rewritten as

$$\tau < \bar{\varphi}(\kappa, \gamma) := -\left( \bar{c}_2 + \frac{\bar{c}_1}{\kappa} \right) + \sqrt{\left( \bar{c}_2 + \frac{\bar{c}_1}{\kappa} \right)^2 + 2\bar{c}_2 \frac{(\kappa - \bar{\kappa}(\gamma))}{\kappa^2}}. \tag{A39}$$

Again, we are only interested in possible values of  $\kappa, \tau, \gamma \in \mathbb{R}_{\geq 0}$  such that (A36) and (A37) are satisfied. Then, we restrict the function  $\bar{\varphi}(\kappa, \gamma)$  to the set where  $\kappa > \bar{\kappa}(\gamma)$  and  $\gamma > \gamma'$ , i.e., restricted to  $\mathcal{K} := \{\kappa, \gamma \in [\bar{\kappa}(2\gamma'), \infty) \times \Gamma | \kappa > \bar{\kappa}(\gamma)\}$ . The function  $\bar{\varphi} : \mathcal{K} \rightarrow \mathbb{R}_{\geq 0}$  is continuous and real-valued on  $\mathcal{K}$ . Moreover, it is strictly positive on the interior of  $\mathcal{K}$ , it is zero on the curve  $\kappa = \bar{\kappa}(\gamma)$ , i.e.,  $(\bar{\varphi} \circ \bar{\kappa})(\gamma) = 0$ , and  $\lim_{\kappa \rightarrow \infty} \bar{\varphi}(\kappa, \gamma) = 0$  for all  $\gamma \in \Gamma$ . The function  $\varphi(\cdot)$  is differentiable on  $\mathcal{K}$ , then we can compute its local extrema by computing its critical points. It is easy to verify that  $\frac{\partial \bar{\varphi}(\kappa, \gamma)}{\partial \kappa} = 0$  only for  $\kappa = \bar{\kappa}_3^*(\gamma)$  and  $\kappa = \bar{\kappa}_4^*(\gamma)$  with

$$\bar{\kappa}_3^* = \left(1 + \frac{1}{1 + 2\bar{c}_1}\right)\bar{\kappa}(\gamma) + \frac{\sqrt{2\bar{c}_1^2\bar{c}_2\bar{\kappa}(\gamma)(1 + 2\bar{c}_1 + 2\bar{c}_2\bar{\kappa}(\gamma))}}{\bar{c}_2(1 + 2\bar{c}_1)}, \tag{A40}$$

$$\bar{\kappa}_4^* = \left(1 + \frac{1}{1 + 2\bar{c}_1}\right)\bar{\kappa}(\gamma) - \frac{\sqrt{2\bar{c}_1^2\bar{c}_2\bar{\kappa}(\gamma)(1 + 2\bar{c}_1 + 2\bar{c}_2\bar{\kappa}(\gamma))}}{\bar{c}_2(1 + 2\bar{c}_1)}. \tag{A41}$$

Likewise,  $\frac{\partial \bar{\varphi}(\kappa, \gamma)}{\partial \gamma} = 0$  only for  $\gamma = 0$  and  $\gamma = 2\gamma' = \frac{2\kappa'}{\lambda_2}$ . Notice that  $\bar{\kappa}_3^*(\gamma) > \bar{\kappa}(\gamma)$  for all  $\gamma \in \Gamma$ ; therefore,  $\bar{\kappa}_3^*(\gamma)$  belongs to the interior of  $\mathcal{K}$ . It is difficult to visualize from (A41) whether  $\bar{\kappa}_4^*(\gamma)$  is contained in  $\mathcal{K}$ . Then, we rewrite (A41) in a more suitable manner. After some algebra (A41) can be written as  $\bar{\kappa}_4^*(\bar{\kappa}) = A(\bar{\kappa})\bar{\kappa}(\gamma)$  with

$$A(\bar{\kappa}) := \left(1 + \frac{1}{1 + 2\bar{c}_1}\right) - \frac{\bar{c}_1\sqrt{4 + \frac{2}{\bar{c}_2\bar{\kappa}(\gamma)} + \frac{4\bar{c}_1}{\bar{c}_2\bar{\kappa}(\gamma)}}}{1 + 2\bar{c}_1}. \tag{A42}$$

Hence, if  $A(\bar{\kappa}) < 1$  for all  $\bar{\kappa} \in [\bar{\kappa}(2\gamma'), \infty)$ , it follows that  $\bar{\kappa}_4^*(\bar{\kappa}) = A(\bar{\kappa})\bar{\kappa}(\gamma) < \bar{\kappa}(\gamma)$  for all  $\bar{\kappa} \in \Gamma$  and therefore  $\bar{\kappa}_4^*(\bar{\kappa}) \notin \mathcal{K}$ . It is easy to check that the function  $A(\bar{\kappa})$  does not have any critical points,  $\lim_{\bar{\kappa} \rightarrow 0^+} A(\bar{\kappa}) = -\infty$ , and  $\lim_{\bar{\kappa} \rightarrow \infty^+} A(\bar{\kappa}) = \frac{2}{1 + 2\bar{c}_1}$ . Then, the function  $A(\bar{\kappa})$  does not have any local maxima on the interval  $(0, \infty)$  and its greatest value occurs at infinity. It follows that  $A(\bar{\kappa}) < 0$  for all  $\bar{\kappa} \in [\bar{\kappa}(2\gamma'), \infty)$  if  $\frac{2}{1 + 2\bar{c}_1} < 1$ , which is trivially true from the definition of  $\bar{c}_1$  in (A9). Hence, it can be concluded that the function  $\bar{\varphi}(\kappa, \gamma)$  has a unique extremum on  $\mathcal{K}$  and it is given by  $\bar{\varphi}(\bar{\kappa}_3^*(2\gamma'), 2\gamma')$ . Finally, given that  $\bar{\varphi}(\bar{\kappa}(\gamma), \gamma) = 0$ ,  $\lim_{\kappa \rightarrow \infty} \bar{\varphi}(\bar{\kappa}, \gamma) = 0$  for all  $\gamma \in \Gamma$ ,  $\bar{\varphi}(\kappa, \gamma)$  is strictly positive on the interior of  $\mathcal{K}$ , and  $\bar{\varphi}(\bar{\kappa}_3^*(2\gamma'), 2\gamma')$  is the unique extremum on  $\mathcal{K}$ . It follows that  $\bar{\varphi}(\bar{\kappa}_3^*(2\gamma'), 2\gamma')$  is the unique maximum on  $\mathcal{K}$  and the function  $\bar{\varphi}(\kappa, \gamma)$  is concave. Then, ultimate boundedness of the solutions and the Lyapunov-Razumikhin theorem imply that the set  $\{\text{col}(\epsilon, \tilde{\zeta}, \tilde{z}) = 0\}$  (and therefore  $\{\text{col}(\epsilon, \tilde{\zeta}, \tilde{z}) = 0\}$  as well) is a global attractor if  $\tau < \varphi(\kappa)$ ,  $\gamma > \gamma'$ ,  $\kappa > \bar{\kappa}(\gamma)$ ,  $\tau < \bar{\varphi}(\kappa, \gamma)$ , and  $(\gamma, \kappa) \in \mathcal{N}$  with  $\mathcal{N}$  the boundedness set defined in Lemma 2. ■

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