# A complex-like calculus for spherical vectorfields 

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A complex-like calculus for spherical vectorfields
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# A Complex-like Calculus for 

## Spherical Vectorfields

J. de GRAAF

Dedicated to Professor Bob Mattheij at his retirement


#### Abstract

First, $\mathbb{R}^{1+\mathfrak{d}}, \mathfrak{d} \in \mathbb{N}$, is turned into an algebra by mimicing the usual complex multiplication. Indeed the special case $\mathfrak{d}=1$ reproduces $\mathbb{C}$. For $\mathfrak{d}>1$ the considered algebra is commutative, but non-associative and even non-alternative. Next, the Dijkhuis class of mappings ('vectorfields') $\mathbb{R}^{1+\mathfrak{d}} \rightarrow \mathbb{R}^{1+\mathfrak{d}}$, suggested by C.G. Dijkhuis for $\mathfrak{d}=3, \mathfrak{d}=7$, is introduced. This special class is then fully characterized in terms of analytic functions of one complex variable. Finally, this characterization enables to show easily that the Dijkhuis-class is closed under pointwise $\mathbb{R}^{\mathfrak{d}+1}$-multiplication: It is a commutative and associative algebra of vector fields. Previously it had not been observed that the Dijkhuis-class only contains vectorfields with a 'time-dependent' spherical symmetry. Such disappointment was to be expected! The class of functions which are differentiable with respect to the algebraic structure, that we impose on $\mathbb{R}^{1+\mathfrak{d}}$, contains only linear functions if $\mathfrak{d}>1$. The Dijkhuis-class does not appear this way either! In our treatment neither quaternions nor octonions play a role.


## 1 Imitation of complex calculus in higher dimensions

On $\mathbb{R}^{1+\mathfrak{d}}$, with $\mathfrak{d} \in \mathbb{N}$, a commutative multiplication structure is introduced by

$$
\begin{equation*}
(\alpha ; \underline{a}) \cdot(\beta ; \underline{b})=\left(\alpha \beta-\underline{a}^{\top} \underline{b} ; \alpha \underline{b}+\beta \underline{a}\right), \quad \alpha, \beta \in \mathbb{R}, \underline{a}, \underline{b} \in \mathbb{R}^{\mathfrak{d}} . \tag{1.1}
\end{equation*}
$$

Note 1. This multiplication structure is non-associative (non-alternative) if $\mathfrak{d}>1$. Indeed

$$
((\alpha ; \underline{a}) \cdot(\beta ; \underline{b})) \cdot(\gamma ; \underline{c})-(\alpha ; \underline{a}) \cdot((\beta ; \underline{b}) \cdot(\gamma ; \underline{c}))=\left(0 ; \underline{b}^{\top} \underline{c a}-\underline{a}^{\top} \underline{b c}\right),
$$

which may not vanishe if for $(\lambda, \mu) \neq(0,0)$ one has $\lambda \underline{a}+\mu \underline{c} \neq \underline{0}$.
Clearly, with suitable interpretation, $\underline{b}^{\top} \underline{c a}-\underline{a}^{\top} \underline{b}=-\underline{b} \times(\underline{c} \times \underline{a})$. Note 2. If $\mathfrak{d}=3$ or $\mathfrak{d}=7$, the product $(\alpha ; \underline{a}) \cdot(\alpha ; \underline{a})$ of equal elements corresponds, respectively, to the quaternion product and the octonion product.
Note 3. Symbolically, and sometimes conveniently, (1.1) can be written

$$
(\alpha+i \underline{a}) \cdot(\beta+i \underline{b})=\left(\alpha \beta-\underline{a}^{\top} \underline{b}\right)+i(\alpha \underline{b}+\beta \underline{a}) .
$$

Note 4. If for $\mathbf{v}=v_{1}+i v_{2} \in \mathbb{C}$ and $\underline{\xi} \in \mathbb{R}^{\mathfrak{D}}$ we introduce $\mathbf{v} \underline{\xi} \in \mathbb{R}^{1+\mathfrak{o}}$ by

$$
\mathrm{v} \underline{\xi}=\left(v_{1} ; \frac{v_{2}}{|\underline{\xi}|} \underline{\xi}\right),
$$

we have the multiplication rule

$$
\mathrm{v} \underline{\xi} \cdot \mathrm{w} \underline{\xi}=(\mathrm{vw}) \underline{\xi} .
$$

Here vw is the usual product of complex numbers.
Note 5. By induction one easily shows that, with $r=|\underline{x}|$, one has for $n=1,2, \ldots$

$$
(t ; \underline{x})^{n}=\left(\operatorname{Re}(t+i r)^{n} ; \frac{\operatorname{Im}(t+i r)^{n}}{r} \underline{x}\right) .
$$

## Some calculations

- $(\alpha ; \underline{a}) \cdot(t ; \underline{x})^{n}=\left(\alpha \operatorname{Re}(t+i r)^{n}-\frac{\operatorname{Im}(t+i r)^{n}}{r} \underline{x}^{\top} \underline{a} ; \alpha \frac{\operatorname{Im}(t+i r)^{n}}{r} \underline{x}+\operatorname{Re}(t+i r)^{n} \underline{a}\right)$
- $(t ; \underline{x})^{m} \cdot\left((\alpha ; \underline{a}) \cdot(t ; \underline{x})^{n}\right)=\left((\alpha ; \underline{a}) \cdot(t ; \underline{x})^{n}\right) \cdot(t ; \underline{x})^{m}=(t ; \underline{x})^{n} \cdot\left((\alpha ; \underline{a}) \cdot(t ; \underline{x})^{m}\right)=$ $=\left(\alpha \operatorname{Re}(t+i r)^{m+n}-\frac{\operatorname{Im}(t+i r)^{m+n}}{r} \underline{x}^{\top} \underline{a} ;\right.$

$$
\left.;\left\{\alpha \frac{\operatorname{Im}(t+i r)^{m+n}}{r}-\frac{\operatorname{Im}(t+i r)^{m}}{r} \frac{\operatorname{Im}(t+i r)^{n}}{r} \underline{x}^{\top} \underline{a}\right\} \underline{x}+\left\{\operatorname{Re}(t+i r)^{m} \operatorname{Re}(t+i r)^{n}\right\} \underline{a}\right) .
$$

## Definition $1.1^{1}($ Dijkhuis: A special class of functions )

On open sets in $\mathbb{R}^{1+\mathfrak{o}}$ we introduce the class of functions

$$
\begin{equation*}
(t ; \underline{x}) \mapsto(T(t ; \underline{x}) ; \underline{X}(t ; \underline{x})) \in \mathbb{R}^{1+\mathfrak{d}} \tag{1.2}
\end{equation*}
$$

where $T$ and $\underline{X}=\operatorname{column}\left[X_{1}, \ldots, X_{\mathfrak{0}}\right]$ are supposed to satisfy

$$
\begin{align*}
\nabla T & =-\frac{\partial \underline{X}}{\partial t} \\
\nabla \times \underline{X} & =\underline{0} \\
\underline{x} \times \underline{X} & =\underline{0}  \tag{1.3}\\
(\underline{x} \cdot \nabla) \underline{X} & =\frac{\partial T}{\partial t} \underline{x} .
\end{align*}
$$

Here we denote

$$
\nabla T=\operatorname{column}\left[\partial_{1} T, \ldots, \partial_{\mathfrak{\imath}} T\right]
$$

$\underline{x} \times \underline{X}$ stands for the anti-symmetric matrix $\quad\left[\underline{x} \underline{X}^{T}-\underline{X} \underline{x}^{T}\right]_{k \ell}=\left[x_{k} X_{\ell}-x_{\ell} X_{k}\right] \in \mathbb{R}^{\mathfrak{d} \times \mathfrak{d}}$,
$\nabla \times \underline{X}$ stands for the anti-symmetric matrix $\left[(\mathcal{D} \underline{X})^{T}-\mathcal{D} \underline{X}\right]_{k \ell}=\left[\partial_{k} X_{\ell}-\partial_{\ell} X_{k}\right] \in \mathbb{R}^{\mathfrak{d} \times \mathfrak{D}}$.
If $\mathfrak{d}=3$ the identities in (1.3) correspond with the usual interpretation!
Note 6. From $\left[\underline{x X}^{\top}-\underline{X x}^{\top}\right]=[0]$ it immediately follows that $\underline{X}$ can only be a multiple of $\underline{x}$.

Theorem 1.2 Suppose (1.3). Then the function

$$
\begin{equation*}
(t ; \underline{x}) \mapsto(t ; \underline{x}) \cdot(T(t ; \underline{x}) ; \underline{X}(t ; \underline{x})) \tag{1.4}
\end{equation*}
$$

also satisfies (1.3).
Proof In index notation the conditions (1.3) read

$$
\partial_{k} T=-\partial_{0} X_{k}, \quad \partial_{i} X_{j}-\partial_{j} X_{i}=0, \quad x_{i}\left(\partial_{i} X_{k}\right)=\left(\partial_{0} T\right) x_{k}, \quad 1 \leq i, j, k \leq \mathfrak{d}
$$

The product (1.4) reads $\left(t T-\underline{x}^{\top} \underline{X} ; t \underline{X}+T \underline{x}\right)$. We list the components of all derivatives needed. Summation over repeated indices.

$$
\begin{array}{cll}
\nabla\left(t T-\underline{x}^{\top} \underline{X}\right) \quad & : & \partial_{k}\left(t T-x_{i} X_{i}\right)=t\left(\partial_{k} T\right)-\delta_{k i} X_{i}-x_{i}\left(\partial_{k} X_{i}\right)= \\
& =t\left(\partial_{k} T\right)-X_{k}-x_{i}\left(\partial_{i} X_{k}\right)+x_{i}\left(\partial_{i} X_{k}-\partial_{k} X_{i}\right) \\
\partial_{t}\left(t T-\underline{x}^{\top} \underline{X}\right) \quad: & T+t \partial_{0} T-x_{i} \partial_{0} X_{i}=T+t \partial_{0} T+x_{i} \partial_{i} T \\
\nabla \times(t \underline{X}+T \underline{x}) \quad: & \partial_{k}\left(t X_{\ell}+T x_{\ell}\right)-\partial_{\ell}\left(t X_{k}+T x_{k}\right)= \\
& =t\left(\partial_{k} X_{\ell}-\partial_{\ell} X_{k}\right)+\left(\partial_{k} T\right) x_{\ell}-\left(\partial_{\ell} T\right) x_{k}= \\
& =t\left(\partial_{k} X_{\ell}-\partial_{\ell} X_{k}\right)+\partial_{0}\left(X_{\ell} x_{k}-X_{k} x_{\ell}\right) \\
\partial_{t}(t \underline{X}+T \underline{x}) \quad: & X_{k}+t\left(\partial_{0} X_{k}\right)+\left(\partial_{0} T\right) x_{k} \\
(\underline{x} \cdot \nabla)(t \underline{X}+T \underline{x}): & x_{i} \partial_{i}\left(t X_{k}+T x_{k}\right)=t x_{i}\left(\partial_{i} X_{k}\right)+x_{i}\left(\partial_{i} T\right) x_{k}+x_{i} T \delta_{i k}
\end{array}
$$

[^0]Taking into account (1.3) leads to the desired result.

Corollary 1.3 Convergent power series with real coefficients $c_{n}$

$$
\begin{equation*}
\left(T(t ; \underline{x}) ; \underline{X}(t ; \underline{x})=\sum_{m=0}^{\infty} c_{n}(t ; \underline{x})^{m}\right. \tag{1.5}
\end{equation*}
$$

all lead to functions which satisfy (1.3).
Note 7. The 'vectorial part' of the sum of such power series is always a multiple of $\underline{x}$.
Note 8. If $\mathfrak{d}=3$ or $\mathfrak{d}=7$ these series correspond to quaternion and octonion power series, respectively. It is emphasized again that the coefficients are real!

Inspired by Note 5. we come to a full description of functions (1.2) that satisfy (1.3).
Theorem 1.4 The functions (1.2) satisfy (1.3) if and only if, locally, there exists an analytic function $t+i r \mapsto \mathrm{~F}(t+i r)=\operatorname{Re} \mathrm{F}(t, r)+i \operatorname{Im} \mathrm{~F}(t, r)$, such that

$$
(t ; \underline{x}) \mapsto(T(t ; \underline{x}) ; \underline{X}(t ; \underline{x}))=\left(\operatorname{ReF}(t, r) ; \frac{\operatorname{Im} \mathrm{F}(t, r)}{r} \underline{x}\right) .
$$

For convenience in the proof I first summarize

## Some properties of analytic functions

- A function $f: \mathbb{C} \rightarrow \mathbb{C}$ ia analytic iff $f(z)=f(x+i y)=\operatorname{Re} f(x, y)+i \operatorname{Im} f(x, y)$ satisfies the Cauchy-Riemann identities

$$
\frac{\partial}{\partial \bar{z}} f(z)=\frac{1}{2}\left(\partial_{x}+i \partial_{y}\right) f(x+i y)=\frac{1}{2}\left(\partial_{x}+i \partial_{y}\right)(\operatorname{Re} f(x, y)+i \operatorname{Im} f(x, y))=0
$$

which corresponds to

$$
\partial_{x} \operatorname{Re} f-\partial_{y} \operatorname{Im} f=0, \quad \partial_{y} \operatorname{Re} f+\partial_{x} \operatorname{Im} f=0
$$

- For the 'complex' derivative we have

$$
\begin{aligned}
\frac{\partial}{\partial z} f(z) & =f^{\prime}(z)=\frac{1}{2}\left(\partial_{x}-i \partial_{y}\right) f(x+i y)=\frac{1}{2}\left(\partial_{x}-i \partial_{y}\right)(\operatorname{Re} f(x, y)+i \operatorname{Im} f(x, y))= \\
& =\frac{1}{2}\left\{\partial_{x} \operatorname{Re} f+\partial_{y} \operatorname{Im} f\right\}+\frac{i}{2}\left\{\partial_{x} \operatorname{Im} f-\partial_{y} \operatorname{Re} f\right\}=\partial_{x} \operatorname{Re} f-i \partial_{y} \operatorname{Re} f .
\end{aligned}
$$

- Analytic functions are harmonic, indeed
$\Delta(\operatorname{Re} f(x, y)+i \operatorname{Im} f(x, y))=4 \frac{1}{2}\left(\partial_{x}-i \partial_{y}\right) \frac{1}{2}\left(\partial_{x}+i \partial_{y}\right)(\operatorname{Re} f(x, y)+i \operatorname{Im} f(x, y))=0$

$$
\begin{array}{r}
=\Delta \operatorname{Re} f(x, y)+i \Delta \operatorname{Im} f(x, y)=0 . \\
z \frac{\mathrm{~d}}{\mathrm{~d} z} f=z f^{\prime}(z)=\left(x \partial_{x}+y \partial_{y}\right) \operatorname{Re} f-i\left(x \partial_{y}-y \partial_{x}\right) \operatorname{Re} f
\end{array}
$$

- If $(x, y) \mapsto h(x, y)$ is harmonic, that means $\Delta h(x, y)=0$, then the function

$$
z=x+i y \quad \mapsto \quad \partial_{x} h(x, y)-i \partial_{y}(x, y)
$$

is analytic.
Proof of Theorem $1.4(\Leftarrow)$ If $T=\operatorname{ReF}$ and $\underline{X}=\frac{\operatorname{ImF}}{r} \underline{x}$, the 2 nd and 3rd property in (1.3) follow from the symmetry of

$$
x_{i} x_{j}=\frac{x_{i} x_{j}}{r} \mathrm{~F} \quad \text { and } \quad \partial_{i} X_{j}=\frac{x_{i} x_{j}}{r^{2}}\left(\partial_{r} \mathrm{~F}\right)+\left(\frac{\delta_{i j}}{r}-\frac{x_{i} x_{j}}{r^{3}}\right) \mathrm{F} .
$$

Substitution in the 1st condition leads to

$$
\partial_{r}(\operatorname{ReF}) \frac{1}{r} \underline{x}=-\partial_{t} \operatorname{Im} \mathrm{~F} \frac{1}{r} \underline{x},
$$

which is OK because of one of the Cauchy-Riemann properties. Substitution in the 4th condition, because of $(\underline{x} \cdot \nabla)\left(\frac{1}{r} \underline{x}\right)=\underline{0}$, leads to

$$
r\left(\partial_{r} \operatorname{Im} \mathrm{~F}\right) \frac{1}{r} \underline{x}=\left(\partial_{t} \operatorname{Re} \mathrm{~F}\right) \underline{x},
$$

which is also OK because of the other Cauchy-Riemann property.
$(\Rightarrow)$ Since $\underline{X}$ has rotation $\underline{0}$ it has a potential. Write $\underline{X}(t ; \underline{x})=-\nabla G(t, \underline{x})$. Further, from $\left[\underline{x} \underline{X}^{\top}-\underline{X x}^{\top}\right]=[0]$ it follows that $\underline{X}$ can only be a multiple of $\underline{x}$. It follows that there exists a scalar function $(t ; \underline{x}) \mapsto \alpha(t ; \underline{x})$, such that $\nabla G(t, \underline{x})=\alpha(t ; \underline{x}) \underline{x}$. We want to show that, for all fixed $t$, the function $\underline{x} \mapsto G(t, \underline{x})$ is constant on spheres $|\underline{x}|=r$. Take $\underline{a}, \underline{b}$ with $|\underline{a}|=|\underline{b}|=r$. Let $\mathscr{C}$ be an oriented curve $s \rightarrow \underline{x}(s)$ which runs from $\underline{a}$ to $\underline{b}$ and which lies entirely on the sphere $|\underline{x}|=r$. Then $G(t, \underline{b})-G(t, \underline{a})=\int_{\mathscr{C}} \nabla G(t, \underline{x}(s)) \cdot \underline{\dot{x}}(s) \mathrm{d} s$. The integrand vanishes at all points of the curve because $\nabla G$ is orthogonal to the sphere at all points of it. From now on we write $G(t, \underline{x})=G(t, r)$. Therefore $\underline{X}(t ; \underline{x})=-\left(\partial_{r} G(t, r)\right) \frac{1}{r} \underline{x}$. Put $T(t, r)=\partial_{t} G(t, r)$ and we only have to satisfy the final condition in (1.3). Substitute our $T$ and $G$. The condition reads

$$
-r\left(\partial_{r} \partial_{r} G\right) \frac{1}{r} \underline{x}=\partial_{t} \partial_{t} G \underline{x} .
$$

It follows that $G$ has to be harmonic: $\Delta G=0$. We now define

$$
\mathrm{F}(t+i r)=\partial_{t} G(t, r)-i \partial_{r} G(t, r)
$$

and we are done.

Examples The analytic functions $\mathrm{F}(t, r)=(t+\mathrm{i} r)^{m}, m \in \mathbb{N}$, represent the polynomial vectorfields $(t ; \underline{x})^{m}$.

Theorem 1.5 Endowed with pointwise multiplication the Dijkhuis class of vectorfields, defined by (1.3), is a commutative and associative algebra.

Proof For analytic F, G we only have to check the multiplication

$$
\left(\operatorname{ReF} ; \frac{\operatorname{Im} \mathrm{F}}{r} \underline{x}\right) \cdot\left(\operatorname{ReG} ; \frac{\operatorname{Im} \mathrm{G}}{r} \underline{x}\right)=\left(\operatorname{ReFG} ; \frac{\operatorname{Im} \mathrm{FG}}{r} \underline{x}\right) .
$$

Associativity follows because all vectorial parts are multiples of $\underline{x}$.

## Further Consequences

It will be clear by now that operations on the Dijkhuis class can be represented fully by operations on analytic functions. We mention some examples

- Multiplication by $(t ; \underline{x})$ corresponds to $\mathrm{F} \mapsto\{z \mapsto z \mathrm{~F}(z)\}$.
- The Kelvin transform corresponds to $\mathrm{F} \mapsto\left\{z \mapsto \mathrm{~F}\left(\frac{1}{z}\right)\right\}$.
- The harmonic conjugate corresponds to $\mathrm{F} \mapsto\{z \mapsto \mathrm{iF}(z)\}$.
- The Euler operator corresponds to $\mathrm{F} \mapsto\left\{z \mapsto z \frac{\mathrm{~d}}{\mathrm{~d} z} \mathrm{~F}(z)\right\}$.
- Meaningful derivatives are given by $\mathrm{F} \mapsto\left\{\mapsto z \frac{\mathrm{~d}^{m}}{\mathrm{~d} z^{m}} \mathrm{~F}(z)\right\}$.


## 2 Differentiability with respect to the algebra

A mapping

$$
\mathbb{R}^{1+\mathcal{o}} \rightarrow \mathbb{R}^{1+\mathcal{o}}: \quad\left[\begin{array}{c}
t  \tag{2.1}\\
\underline{x}
\end{array}\right] \mapsto\left[\begin{array}{c}
T(t ; \underline{x}) \\
\underline{X}(t ; \underline{x})
\end{array}\right]
$$

is differentiable (in the usual sense) at $\left[\begin{array}{c}t \\ \underline{x}\end{array}\right] \in \mathbb{R}^{1+\boldsymbol{d}}$, if for any $\left[\begin{array}{c}h \\ \underline{k}\end{array}\right] \in \mathbb{R}^{1+\boldsymbol{d}}$, we have

$$
\left[\begin{array}{c}
T(t+h ; \underline{x}+\underline{k})  \tag{2.2}\\
\underline{X}(t+h ; \underline{x}+\underline{k})
\end{array}\right]=\left[\begin{array}{cc}
T(t ; \underline{x}) \\
\underline{X}(t ; \underline{x})
\end{array}\right]+\left[\begin{array}{cc}
\partial_{t} T(t ; \underline{x}) & \nabla T(t ; \underline{x}) \\
\partial_{t} \underline{X}(t ; \underline{x}) & \mathcal{D} \underline{X}(t ; \underline{x})
\end{array}\right]\left[\begin{array}{l}
h \\
\underline{k}
\end{array}\right]+o\left(\sqrt{h^{2}+|\underline{k}|^{2}}\right) .
$$

Here, $\nabla T=\operatorname{row}\left(\partial_{1} T, \ldots, \partial_{\mathfrak{\jmath}} T\right)$ and $\mathcal{D} \underline{X}=\operatorname{matrix}\left[\partial_{j} X_{\ell}\right], 1 \leq j, \ell \leq \mathfrak{d}$.

For left/right differentiability with respect to the algebraic structure imposed on $\mathbb{R}^{1+0}$, it is required that the linearization term in (2.2) has the form

$$
\left[\begin{array}{cc}
\partial_{t} T(t ; \underline{x}) & \nabla T(t ; \underline{x})  \tag{2.3}\\
\partial_{t} \underline{X}(t ; \underline{x}) & \mathcal{D} \underline{X}(t ; \underline{x})
\end{array}\right]\left[\begin{array}{l}
h \\
\underline{k}
\end{array}\right]=\left[\begin{array}{cc}
\alpha(t ; \underline{x}) & -\underline{a}^{\top}(t ; \underline{x}) \\
\underline{a}(t ; \underline{x}) & \alpha(t ; \underline{x}) \mathcal{I}
\end{array}\right]\left[\begin{array}{l}
h \\
\underline{k}
\end{array}\right] .
$$

As a consequence the conditions for differentiability, with respect to the algebra, are

$$
\begin{equation*}
\partial_{j} X_{\ell}=0, \text { if } j \neq \ell, \quad \alpha=\partial_{t} T=\partial_{j} X_{j}, 1 \leq j, \ell \leq \mathfrak{d}, \quad \underline{a}=-\nabla T=\partial_{t} \underline{X} . \tag{2.4}
\end{equation*}
$$

It follows that, for $\mathfrak{d}>1$, the only differentiable functions are

$$
\begin{equation*}
T=B t-\underline{A} \cdot \underline{x}+D, \quad \underline{X}=B \underline{x}+t \underline{A}, \quad B, D \in \mathbb{R}, \underline{A} \in \mathbb{R}^{\mathfrak{D}}, \tag{2.5}
\end{equation*}
$$

which does not look very exciting.

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J. de Graaf, May 2011.

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[^0]:    ${ }^{1}$ Introduced by G.C. Dijkhuis for $\mathbb{R}^{1+3}$ and $\mathbb{R}^{1+7}$. Private communication.

