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UNFOLDING-BASED CORRECTOR ESTIMATES FOR A REACTION-DIFFUSION SYSTEM PREDICTING CONCRETE CORROSION

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ABSTRACT. We use the periodic unfolding technique to derive corrector estimates for a reaction-diffusion system describing concrete corrosion penetration in the sewer pipes. The system, defined in a periodically-perforated domain, is semi-linear, partially dissipative, and coupled via a non-linear ordinary differential equation posed on the solid-water interface at the pore level. After discussing the solvability of the pore scale model, we apply the periodic unfolding techniques (adapted to treat the presence of perforations) not only to get upscaled model equations, but also to prepare a proper framework for getting a convergence rate (corrector estimates) of the averaging procedure.

Keywords: Corrector estimates, periodic unfolding, homogenization, sulfate corrosion of concrete, reaction-diffusion systems.

1. Introduction

Concrete corrosion is a slow natural process that leads to the deterioration of concrete structures (buildings, bridges, highways, etc.) leading yearly to huge financial losses everywhere in the world. In this paper, we focus on one of the many mechanisms of chemical corrosion, namely the sulfation of concrete, and aim to describe it macroscopically by a system of averaged reaction-diffusion equations whose effective coefficients depend on the particular shape of the microstructure. The final aim of our research is to become capable to predict quantitatively the durability of a (well-understood) cement-based material under a controlled experimental setup (well-defined boundary conditions). The striking thing is that in spite of the fact that the basic physical-chemistry of this relatively easy material is known [1], we have no control on how the microstructure changes (in time and space) and to which extent these spatio-temporal changes affect the observable macroscopic behavior of the material. The research reported here goes along the line open in [11], where a formal asymptotic expansion ansatz was used to derive macroscopic equations for a corrosion model, posed in a domain with locally-periodic microstructure (see

[17] for a rigorous averaging approach of a reduced model defined in a domain with locally-periodic microstructures). A two-scale convergence approach for periodic microstructures was studied in [10], while preliminary multiscale simulations are reported in [3]. Within this paper we consider a partially dissipative reaction-diffusion system defined in a domain with periodically distributed microstructure. This system was originally proposed in [2] as a free-boundary problem. The model equations describe the corrosion of sewer pipes made of concrete when sulfate ions penetrate the non-saturated porous matrix of the concrete viewed as a "composite". The typical concrete microstructure includes solid, water and air parts, see Fig. 2.1. One could argue that the microstructure of a concrete is neither uniformly periodic nor locally periodic, and the randomness of the pores and of their distributions should be taken into account. However, periodic representations of concrete microstructures often provide good descriptions. For what the macroscopic corrosion process is concerned, the derivation of corrector estimates [for the periodic case] is crucial for the identification of convergence rates of microscopic solutions. The stochastic geometry of the concrete will be studied as future work with the hope to shed some light on eventual connections between the role played by a locally-periodic distributed microstructure vs. stationary random(-distributed) pores. In this spirit, we think that there is much to be learnt from [18].

The main novelty of the paper is twofold: on one hand, we obtain corrector estimates under optimal regularity assumptions on solutions of the microscopic model and obtain the desired convergence rate (hence, we have now a confidence measure of our averaging results); on the other hand, we apply for the first time an unfolding technique to derive corrector estimates in perforated media. The main ideas of the methodology were presented in [12, 13] and applied to linear elliptic equations with oscillating coefficients, posed in a fixed domain. Our approach strongly relies on these results. However, novel aspects of the method, related to the presence of perforations in the considered microscopic domain, are treated here for the first time; see section 3. The main advantage of using the unfolding technique to prove corrector estimates is that only H^1-regularity of solutions of microscopic equations and of unit cell problems is required, compared to standard methods (mostly based on energy-type estimates) used in the derivation of corrector estimates. As a natural consequence of this fact, the set of choices of microstructures is now much larger.

The paper is structured in the following fashion: After introducing model equations and the assumed microscopic geometry of the concrete material, the section 2 goes on with the main assumptions and basic estimates ensuring both the solvability of the microscopic problem and the convergence of microscopic solutions to a solution of the macroscopic equations, as $\varepsilon \to 0$. In section 3 we state and prove the corrector estimates for the concrete corrosion model, Theorem 3.6, determining the range of validity of the upscaled model.

Note that the technique developed in this article can be applied in a straightforward way to derive convergence rates for solutions of other classes of partial differential equations, posed in domains with periodically-distributed microstructures.

2. Problem description

2.1. **Geometry.** We assume that concrete piece consists of a system of pores periodically distributed inside the three-dimensional cube $\Omega = [a,b]^3$ with $a, b \in \mathbb{R}$ and $b > a$. Since usually the concrete in sewer pipes is not completely dry, we consider a partially saturated porous material. We assume that every pore has three distinct non-overlapping parts: a solid part, the water film which surrounds the solid part, and an air layer bounding the water film and filling the space of Y as shown in Fig. 2.1. Note that the dark (black) parts indicate the water-filled parts in the material where most of our model equations are defined. The reference pore, $Y = [0,1]^3$, has three pair-wise disjoint domains Y_0, Y_1 and Y_2 with smooth boundaries Γ_1 and Γ_2 as shown in Fig. 2.1. Moreover, $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$.

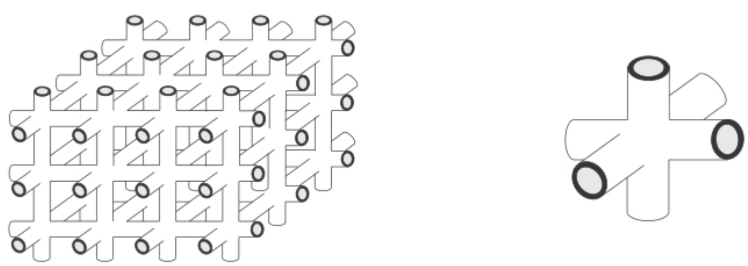

Figure 1. Left: Periodic approximation of the concrete piece. Right: Our choice of the microstructure.

Let ε be a small factor denoting the ratio between the characteristic length of the pore Y and the characteristic length of the domain Ω. Let χ_1 and χ_2 be the characteristic functions of the sets Y_1 and Y_2, respectively. The shifted set Y_1^k is defined by $Y_1^k := Y_1 + \Sigma_{j=0}^3 k_j e_j$ for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, where e_j is the j^{th} unit vector. The union of all Y_1^k multiplied by ε that are contained within Ω defines the perforated domain Ω_1^ε, namely $\Omega_1^\varepsilon := \cup_{k\in\mathbb{Z}^3}\{\varepsilon Y_1^k \mid \varepsilon Y_1^k \subset \Omega\}$.
Similarly, Ω_2^ε, Γ_1^ε, and Γ_2^ε denote the union of εY_2^k, $\varepsilon \Gamma_1^k$, and $\varepsilon \Gamma_2^k$, contained in Ω.

2.2. **Microscopic equations.** We consider a microscopic model

$$\begin{cases} \partial_t u^\varepsilon - \nabla \cdot (D_u^\varepsilon \nabla u^\varepsilon) = -f(u^\varepsilon, v^\varepsilon) & \text{in } (0,T) \times \Omega_1^\varepsilon, \\ \partial_t v^\varepsilon - \nabla \cdot (D_v^\varepsilon \nabla v^\varepsilon) = f(u^\varepsilon, v^\varepsilon) & \text{in } (0,T) \times \Omega_1^\varepsilon, \\ \partial_t w^\varepsilon - \nabla \cdot (D_w^\varepsilon \nabla w^\varepsilon) = 0 & \text{in } (0,T) \times \Omega_2^\varepsilon, \\ \partial_t r^\varepsilon = \eta(u^\varepsilon, r^\varepsilon) & \text{on } (0,T) \times \Gamma_1^\varepsilon, \end{cases} \tag{1}$$

with the initial conditions

$$\begin{cases} u^\varepsilon(0,x) = u_0(x), \quad v^\varepsilon(0,x) = v_0(x) & \text{in } \Omega_1^\varepsilon, \\ w^\varepsilon(0,x) = w_0(x) \quad \text{in } \Omega_2^\varepsilon, \quad r^\varepsilon(0,x) = r_0(x) & \text{on } \Gamma_1^\varepsilon \end{cases} \tag{2}$$

and the boundary conditions

$$u^\varepsilon = 0, v^\varepsilon = 0 \text{ on } (0,T)\times\partial\Omega\cap\partial\Omega_1^\varepsilon,\ w^\varepsilon = 0 \text{ on } (0,T)\times\partial\Omega\cap\partial\Omega_2^\varepsilon, \tag{3}$$

together with

$$\begin{cases} D_u^\varepsilon\nabla u^\varepsilon\cdot\nu = -\varepsilon\eta(u^\varepsilon, r^\varepsilon) & \text{on } (0,T)\times\Gamma_1^\varepsilon,\\ D_v^\varepsilon\nabla v^\varepsilon\cdot\nu = 0 & \text{on } (0,T)\times\Gamma_1^\varepsilon,\\ D_u^\varepsilon\nabla u^\varepsilon\cdot\nu = 0 & \text{on } (0,T)\times\Gamma_2^\varepsilon,\\ D_v^\varepsilon\nabla v^\varepsilon\cdot\nu = \varepsilon(a^\varepsilon(x)w^\varepsilon - b^\varepsilon(x)v^\varepsilon) & \text{on } (0,T)\times\Gamma_2^\varepsilon,\\ D_w^\varepsilon\nabla w^\varepsilon\cdot\nu = -\varepsilon(a^\varepsilon(x)w^\varepsilon - b^\varepsilon(x)v^\varepsilon) & \text{on } (0,T)\times\Gamma_2^\varepsilon. \end{cases} \tag{4}$$

We consider the space $H^1_{\partial\Omega}(\Omega_i^\varepsilon) = \{u\in H^1(\Omega_i^\varepsilon) : u = 0 \text{ on } \partial\Omega\cap\partial\Omega_i^\varepsilon\}$, $i=1,2$.

Assumption 2.1. (A1) *D_i, $\partial_t D_i \in L^\infty(0,T;L^\infty_{per}(Y))^{3\times 3}$, $i\in\{u,v,w\}$, $(D_i(t,x)\xi,\xi)\geq D_i^0|\xi|^2$ for $D_i^0>0$, for every $\xi\in\mathbb{R}^3$ and a.a. $(t,x)\in(0,T)\times Y$.*

(A2) *Reaction rate $k_3\in L^\infty_{per}(\Gamma_1)$ is nonnegative and $\eta(\alpha,\beta) = k_3(y)R(\alpha)Q(\beta)$, where $R:\mathbb{R}\to\mathbb{R}_+$, $Q:\mathbb{R}\to\mathbb{R}_+$ are sublinear and locally Lipschitz continuous. Furthermore, $R(\alpha)=0$ for $\alpha<0$ and $Q(\beta)=0$ for $\beta\geq\beta_{max}$, with some $\beta_{max}>0$.*

(A3) *$f\in C^1(\mathbb{R}^2)$ is sublinear and globally Lipschitz continuous in both variables, i.e. $f(\alpha,\beta)\leq C_f(1+|\alpha|+|\beta|)$, $|f(\alpha_1,\beta_1)-f(\alpha_2,\beta_2)|\leq C_L(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|)$ and $f(\alpha,\beta)=0$ for $\alpha<0$ or $\beta<0$.*

(A4) *The mass transfer functions at the boundary $a,b\in L^\infty_{per}(\Gamma_2)$, $a(y)$ and $b(y)$ are positive for a.a. $y\in\Gamma_2$ and there exists A_v, A_w, M_v, M_w such that $b(y)e^{A_v t}M_v = a(y)e^{A_w t}M_w$ for a.a. $y\in\Gamma_2$ and $t\in(0,T)$.*

(A5) *Initial data $(u_0,v_0,w_0,r_0)\in[H^2(\Omega)\cap H^1_0(\Omega)\cap L^\infty(\Omega)]^3\times L^\infty_{per}(\Gamma_1)$ and $u_0(x)\geq 0, v_0(x)\geq 0, w_0(x)\geq 0$ a.e. in Ω, $r_0(x)\geq 0$ a.e. on Γ_1.*

We define the oscillating coefficients:
$D_i^\varepsilon(t,x) := D_i\left(t,\frac{x}{\varepsilon}\right)$, $i\in\{u,v,w\}$, $a^\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$, $b^\varepsilon(x) := b\left(\frac{x}{\varepsilon}\right)$, $k^\varepsilon(x) := k\left(\frac{x}{\varepsilon}\right)$.

Definition 2.2. *We call $(u^\varepsilon,v^\varepsilon,w^\varepsilon,r^\varepsilon)$ a weak solution of* (1)–(4) *if $u^\varepsilon, v^\varepsilon\in L^2(0,T;H^1_{\partial\Omega}(\Omega_1^\varepsilon))\cap H^1(0,T;L^2(\Omega_1^\varepsilon))$, $w^\varepsilon\in L^2(0,T;H^1_{\partial\Omega}(\Omega_2^\varepsilon))\cap H^1(0,T;L^2(\Omega_2^\varepsilon))$, $r^\varepsilon\in H^1(0,T;L^2(\Gamma_1^\varepsilon))$ and* *satisfies the following equations*

$$\int_0^T\int_{\Omega_1^\varepsilon}\big(\partial_t u^\varepsilon\phi + D_u^\varepsilon\nabla u^\varepsilon\nabla\phi + f(u^\varepsilon,v^\varepsilon)\phi\big)dxdt = -\varepsilon\int_0^T\int_{\Gamma_1^\varepsilon}\eta(u^\varepsilon,r^\varepsilon)\phi d\gamma dt, \tag{5}$$

$$\int_0^T\int_{\Omega_1^\varepsilon}\big(\partial_t v^\varepsilon\phi + D_v^\varepsilon\nabla v^\varepsilon\nabla\phi - f(u^\varepsilon,v^\varepsilon)\phi\big)dxdt = \varepsilon\int_0^T\int_{\Gamma_2^\varepsilon}\big(a^\varepsilon\, w^\varepsilon - b^\varepsilon\, v^\varepsilon\big)\phi d\gamma dt, \tag{6}$$

$$\int_0^T\int_{\Omega_2^\varepsilon}\big(\partial_t w^\varepsilon\varphi + D_w^\varepsilon\nabla w^\varepsilon\nabla\varphi\big)dxdt = -\varepsilon\int_0^T\int_{\Gamma_2^\varepsilon}\big(a^\varepsilon\, w^\varepsilon - b^\varepsilon\, v^\varepsilon\big)\varphi d\gamma dt, \tag{7}$$

$$\varepsilon\int_0^T\int_{\Gamma_1^\varepsilon}\partial_t r^\varepsilon\psi d\gamma dt = \varepsilon\int_0^T\int_{\Gamma_1^\varepsilon}\eta(u^\varepsilon,r^\varepsilon)\psi d\gamma dt \tag{8}$$

for all $\phi\in L^2(0,T;H^1_{\partial\Omega}(\Omega_1^\varepsilon))$, $\varphi\in L^2(0,T;H^1_{\partial\Omega}(\Omega_2^\varepsilon))$, $\psi\in L^2((0,T)\times\Gamma_1^\varepsilon)$ *and* $u^\varepsilon(t)\to u_0$, $v^\varepsilon(t)\to v_0$ *in* $L^2(\Omega_1^\varepsilon)$, $w^\varepsilon(t)\to w_0$ *in* $L^2(\Omega_2^\varepsilon)$, $r^\varepsilon(t)\to r_0$ *in* $L^2(\Gamma_1^\varepsilon)$ *as* $t\to 0$.

Lemma 2.3. *Under the Assumption 2.1, solutions of the problem* (1)–(4) *satisfy the following* a priori *estimates:*

$$\begin{array}{lll} \|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \|\nabla u^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} & \le & C \\ \|v^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \|\nabla v^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} & \le & C, \\ \|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_2^\varepsilon))} + \|\nabla w^\varepsilon\|_{L^2((0,T)\times\Omega_2^\varepsilon)} & \le & C, \\ \varepsilon^{1/2}\|r^\varepsilon\|_{L^\infty(0,T;L^2(\Gamma_1^\varepsilon))} + \varepsilon^{1/2}\|\partial_t r^\varepsilon\|_{L^2((0,T)\times\Gamma_1^\varepsilon)} & \le & C, \end{array} \tag{9}$$

where the constant C *is independent of* ε.

Proof. First, we consider as test functions $\phi = u^\varepsilon$ in (5), $\phi = v^\varepsilon$ in (6), $\psi = w^\varepsilon$ in (7) and use Assumption 2.1, Young's inequality, and the trace inequality, i.e.

$$\varepsilon\int_0^t\int_{\Gamma_2^\varepsilon} w^\varepsilon v^\varepsilon d\gamma d\tau \le C\int_0^t\int_{\Omega_2^\varepsilon}(|w^\varepsilon|^2+\varepsilon^2|\nabla w^\varepsilon|^2)d\gamma d\tau + C\int_0^t\int_{\Omega_1^\varepsilon}(|v^\varepsilon|^2+\varepsilon^2|\nabla v^\varepsilon|^2)d\gamma d\tau.$$

Then, adding the obtained inequalities, choosing ε conveniently and applying Gronwall's inequality imply the first three estimates in Lemma.
Taking $\psi = r^\varepsilon$ as a test function in (8) and using (A2) from Assumption 2.1 and the estimates for u^ε, yield the estimate for r^ε. The test function $\psi = \partial_t r^\varepsilon$ in (8), the sub-linearity of R, the boundedness of Q and the estimates for u^ε imply the boundedness of $\varepsilon^{1/2}\|\partial_t r^\varepsilon\|_{L^2((0,T)\times\Gamma_1^\varepsilon)}$. □

Lemma 2.4. *(Positivity and boundedness) Let Assumption 2.1 be fulfilled. Then the following estimates hold:*

(i) $u^\varepsilon(t), v^\varepsilon(t)\ge 0$ *a.e. in* Ω_1^ε, $w^\varepsilon(t)\ge 0$ *a.e. in* Ω_2^ε *and* $u^\varepsilon(t), r^\varepsilon(t)\ge 0$ *a.e. on* Γ_1^ε, *for a.a.* $t\in(0,T)$.

(ii) $u^\varepsilon(t) \le M_u e^{A_u t}$, $v^\varepsilon(t) \le M_v e^{A_v t}$ *a.e. in* Ω_1^ε , $w^\varepsilon(t) \le M_w e^{A_w t}$ *a.e. in* Ω_2^ε *and* $u^\varepsilon(t) \le M_u e^{A_u t}$, $r^\varepsilon(t) \le M_r e^{A_r t}$ *a.e. on* Γ_1^ε, *for a.a.* $t \in (0,T)$.

Proof. (i) To show the positivity of a weak solution we consider $u^{\varepsilon -}$ as test function in (5), $v^{\varepsilon -}$ in (6), $w^{\varepsilon -}$ in (7) , and $r^{\varepsilon -}$ in (8), where $\phi^- = \min\{0,\phi\}$ with $\phi^+\phi^- = 0$. The integrals involving $f(u^\varepsilon, v^\varepsilon)u^{\varepsilon -}$, $f(u^\varepsilon, v^\varepsilon)v^{\varepsilon -}$and $\eta(u^\varepsilon, r^\varepsilon)u^{\varepsilon -}$ are zero, since by Assumption 2.1 $f(u,v)$ is zero for negative u or v and $\eta(u,r)$ is zero for negative u. In the integrals over Γ_2^ε we use the positivity of a and b and the estimate $v^\varepsilon w^{\varepsilon -} = (v^{\varepsilon +} + v^{\varepsilon -})w^{\varepsilon -} \le v^{\varepsilon -}w^{\varepsilon -}$. Due to the positivity of η, the right hand side in the equation for r^ε, with the test function $\psi = r^{\varepsilon -}$, is nonpositive. Adding the obtained inequalities, applying both Young's and the trace inequalities, considering ε sufficiently small, we obtain, due to positivity of the initial data and using Gronwall's inequality, that

$$\|u^{\varepsilon -}(t)\|_{L^2(\Omega_1^\varepsilon)} + \|v^{\varepsilon -}(t)\|_{L^2(\Omega_1^\varepsilon)} + \|w^{\varepsilon -}(t)\|_{L^2(\Omega_2^\varepsilon)} + \|r^{\varepsilon -}(t)\|_{L^2(\Gamma_1^\varepsilon)} \le 0,$$

for a.a. $t \in (0,T)$. Thus, negative parts of the involved concentrations are equal zero a.e. in $(0,T)\times\Omega_i^\varepsilon$, $i = 1,2$, or in $(0,T)\times\Gamma_1^\varepsilon$, respectively.

(ii) To show the boundedness of solutions, we consider $(u^\varepsilon - e^{A_u t}M_u)^+$ as a test function in (5), $(v^\varepsilon - e^{A_v t}M_v)^+$ in (6) and $(w^\varepsilon - e^{A_w t}M_w)^+$ in (7), where $(\phi - M)^+ = \max\{0, \phi - M\}$ and A_i, M_i, $i = u, v, w$ are positive numbers, such that $u_0(x) \le M_u$, $v_0(x) \le M_v$, $w_0(x) \le M_w$ a.e in Ω, and A_i, M_i for $i = v, w$ are given by $(A4)$ in Assumption 2.1. Adding the equations for u^ε, v^ε, w^ε and using Assumption 2.1 yield, with $U_M^\varepsilon = (u^\varepsilon - e^{A_u t}M_u)^+$, $V_M^\varepsilon = (v^\varepsilon - e^{A_v t}M_v)^+$, and $W_M^\varepsilon = (w^\varepsilon - e^{A_w t}M_w)^+$,

$$\int_0^\tau \Big(\int_{\Omega_1^\varepsilon} \partial_t(|U_M^\varepsilon|^2 + |V_M^\varepsilon|^2) + |\nabla U_M^\varepsilon|^2 + |\nabla V_M^\varepsilon|^2 dx + \int_{\Omega_2^\varepsilon} \partial_t |W_M^\varepsilon|^2 + |\nabla W_M^\varepsilon|^2 dx\Big) dt$$

$$\le C \int_0^\tau \Big[\int_{\Omega_1^\varepsilon} \Big(\big(C_f(e^{A_u t}M_u + e^{A_v t}M_v) - A_u e^{A_u t}M_u\big)U_M^\varepsilon + |U_M^\varepsilon|^2 + |V_M^\varepsilon|^2 + \varepsilon^2|\nabla V_M^\varepsilon|^2$$

$$+\big(C_f(e^{A_u t}M_u + e^{A_v t}M_v) - A_v e^{A_v t}M_v\big)V_M^\varepsilon\Big) dx + \int_{\Omega_2^\varepsilon} \Big(|W_M^\varepsilon|^2 + \varepsilon^2|\nabla W_M^\varepsilon|^2\Big) dx\Big] dt.$$

Choosing A_u, M_u such that $C_f e^{A_u t}M_u + C_f e^{A_v t}M_v - A_u e^{A_u t}M_u \le 0$ and $C_f e^{A_u t}M_u + C_f e^{A_v t}M_v - A_v e^{A_v t}M_v \le 0$, and ε sufficiently small, Gronwall's inequality implies the estimates for u^ε, v^ε, w^ε, stated in Lemma.

Lemma 5.1 in Appendix and H^1-estimates for u^ε in Lemma 2.3 imply $u^\varepsilon(t) \ge 0$ and $u^\varepsilon(t) \le e^{A_u t}M_u$ a.e on Γ_1^ε for a.a. $t \in (0,T)$. The assumption on η and equation (8) with the test function $(r^\varepsilon - e^{A_r t}M_r)^+$, where $r_0(x) \le M_r$ a.e. on Γ_1, yield

$$\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \Big(\frac{1}{2}\partial_t |(r^\varepsilon - e^{A_r t}M_r)^+|^2 + A_r e^{A_r t}M_r (r^\varepsilon - e^{A_r t}M_r)^+\Big) d\gamma dt =$$

$$\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)(r^\varepsilon - e^{A_r t}M_r)^+ d\gamma dt \le C_\eta(A_u, M_u)\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} (r^\varepsilon - e^{A_r t}M_r)^+ d\gamma dt.$$

This, for A_r and M_r, such that $C_\eta \leq A_r M_r e^{A_r T}$, implies the boundedness of r^ε on Γ_1^ε for a.a. $t \in (0,T)$. □

Lemma 2.5. *Under Assumption 2.1, we have the following estimates, independent of* ε*:*

$$\|\partial_t u^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|\partial_t v^\varepsilon\|_{L^2(0,T;H^1(\Omega_1^\varepsilon))} + \|\partial_t w^\varepsilon\|_{L^2(0,T;H^1(\Omega_2^\varepsilon))} \leq C.$$

Proof. We test (5) with $\phi = \partial_t u^\varepsilon$, and using the structure of η, the regularity assumptions on R and Q and the boundedness of u^ε and r^ε on Γ_1^ε, we estimate the boundary integral by

$$\varepsilon \int_0^t \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)\partial_t u^\varepsilon d\gamma d\tau = \varepsilon \int_0^t \int_{\Gamma_1^\varepsilon} k^\varepsilon \Big(\partial_t\big(\mathcal{R}(u^\varepsilon)Q(r^\varepsilon)\big) - \mathcal{R}(u^\varepsilon)Q'(r^\varepsilon)\partial_t r^\varepsilon\Big) d\gamma d\tau$$
$$\leq C\int_{\Omega_1^\varepsilon} \Big(|u^\varepsilon|^2 + \varepsilon^2|\nabla u^\varepsilon|^2 + |u_0|^2 + \varepsilon^2|\nabla u_0|^2\Big)dx + C\varepsilon \int_0^t \int_{\Gamma_1^\varepsilon} \Big(1 + |\partial_t r^\varepsilon|^2\Big)d\gamma d\tau,$$

where $\mathcal{R}(\alpha) = \int_0^\alpha R(\xi)d\xi$. Then, Assumption 2.1, estimates in Lemma 2.3 and the fact that $D_u^0/2 - \varepsilon^2 \geq 0$ for appropriate ε, imply the estimate for $\partial_t u^\varepsilon$.

In order to estimate $\partial_t v^\varepsilon$ and $\partial_t w^\varepsilon$, we differentiate the corresponding equations with respect to the time variable and then test the result with $\partial_t v^\varepsilon$ and $\partial_t w^\varepsilon$, respectively. Due to assumptions on f and using the trace inequality, we obtain

$$\int_{\Omega_1^\varepsilon} |\partial_t v^\varepsilon|^2 dx + C\int_0^t \int_{\Omega_1^\varepsilon} |\nabla\partial_t v^\varepsilon|^2 dx d\tau \leq C\int_0^t \int_{\Omega_2^\varepsilon} \big(|\partial_t w^\varepsilon|^2 + \varepsilon^2|\nabla\partial_t w^\varepsilon|^2\big)dx d\tau$$
$$+C\int_0^t \int_{\Omega_1^\varepsilon} \big(|\partial_t u^\varepsilon|^2 + |\partial_t v^\varepsilon|^2 + |\nabla v^\varepsilon|^2\big)dx d\tau + \int_{\Omega_1^\varepsilon} |\partial_t v^\varepsilon(0)|^2 dx, \tag{10}$$
$$\int_{\Omega_2^\varepsilon} |\partial_t w^\varepsilon|^2 dx + C\int_0^t \int_{\Omega_2^\varepsilon} |\nabla\partial_t w^\varepsilon|^2 dx d\tau \leq C\int_0^t \int_{\Omega_2^\varepsilon} \big(|\partial_t w^\varepsilon|^2 + |\nabla w^\varepsilon|^2\big)dx d\tau$$
$$+\int_{\Omega_2^\varepsilon} |\partial_t w^\varepsilon(0)|^2 dx + C\int_0^t \int_{\Omega_1^\varepsilon} \big(|\partial_t v^\varepsilon|^2 + \varepsilon^2|\nabla\partial_t v^\varepsilon|^2\big)dx d\tau. \tag{11}$$

The regularity assumptions imply that $||\partial_t v^\varepsilon(0)||_{L^2(\Omega_1^\varepsilon)}$ and $||\partial_t w^\varepsilon(0)||_{L^2(\Omega_2^\varepsilon)}$ can be estimated by the H^2-norm of v_0 and w_0. Adding (10) and (11), making use of estimates for $\partial_t u^\varepsilon$, ∇v^ε and ∇w^ε, and applying Gronwall's Lemma, give the desired estimates. □

Lemma 2.6. *(Existence & Uniqueness) Let Assumption 2.1 be fulfilled. Then there exists a unique global-in-time weak solution in the sense of Definition 2.2.*

Proof. The Lipschitz continuity of f, local Lipschitz continuity of η and the boundedness of u^ε and r^ε on Γ_1^ε ensure the uniqueness result. The existence of weak solutions follows by a standard Galerkin approach, [14], using the a priori estimates in Lemmata 2.3, 2.4 and 2.5. □

2.3. **Unfolded limit equations.** We define $\tilde{\Omega}^\varepsilon_{int} = Int(\cup_{k\in\mathbb{Z}^3}\{\varepsilon\overline{Y^k}, \varepsilon Y^k \subset \Omega\})$, $\tilde{\Gamma}^\varepsilon_{i,int} = \cup_{k\in\mathbb{Z}^3}\{\varepsilon\overline{\Gamma^k_i}, \varepsilon Y^k\subset\Omega\}$, $(\mathbb{R}^n)^i = \mathbb{R}^n \cap \{\varepsilon(Y_i+\xi), \xi\in\mathbb{Z}^n\}$, $\tilde{\Omega}^{\varepsilon,l}_i = \{x\in(\mathbb{R}^n)^i : \mathrm{dist}(x,\Omega^\varepsilon_i) < l\sqrt{n}\varepsilon\}$, $l=1,2$.

Definition 2.7. [4, 5, 7] *1. For any function ϕ Lebesgue-measurable on perforated domain Ω^ε_i, the unfolding operator $\mathcal{T}^\varepsilon_{Y_i} : \Omega^\varepsilon_i \to \Omega\times Y_i$, $i=1,2$, is defined by*

$$\mathcal{T}^\varepsilon_{Y_i}(\phi)(x,y) = \begin{cases} \phi(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & a.e.\ for\ \ y\in Y_i,\ x\in\tilde{\Omega}^\varepsilon_{int}, \\ 0 & a.e.\ for\ \ y\in Y_i,\ x\in\Omega\setminus\tilde{\Omega}^\varepsilon_{int}, \end{cases}$$

where $k := [\frac{x}{\varepsilon}]$ denotes the unique integer combination $\Sigma^3_{j=1}k_je_j$ of the periods such that $x - [\frac{x}{\varepsilon}]$ belongs to Y_i,
2. For any function ϕ Lebesgue-measurable on oscillating boundary Γ^ε_i, the boundary unfolding operator $\mathcal{T}^\varepsilon_{\Gamma_i} : \Gamma^\varepsilon_i \to \Omega\times\Gamma_i$, $i=1,2$ is defined by

$$\mathcal{T}^\varepsilon_{\Gamma_i}(\phi)(x,y) = \begin{cases} \phi(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & a.e.\ for\ \ y\in\Gamma_i,\ x\in\tilde{\Omega}^\varepsilon_{int}, \\ 0 & a.e.\ for\ \ y\in\Gamma_i,\ x\in\Omega\setminus\tilde{\Omega}^\varepsilon_{int}. \end{cases}$$

We note that for $w\in H^1(\Omega)$ it holds that $\mathcal{T}^\varepsilon_{Y_i}(w|_{\Omega^\varepsilon_i}) = \mathcal{T}^\varepsilon_Y(w)|_{\Omega\times Y_i}$.

Lemma 2.8. *Under the Assumption 2.1, there exist $u, v, w \in L^2(0,T; H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$, $\tilde{u}$, $\tilde{v}\in L^2((0,T)\times\Omega; H^1_{per}(Y_1))$, $\tilde{w}\in L^2((0,T)\times\Omega; H^1_{per}(Y_2))$, and $r\in H^1(0,T,L^2(\Omega\times\Gamma_1))$ such that (up to a subsequence) for $\varepsilon\to 0$*

$$\begin{array}{lll} \mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)\to u, & \mathcal{T}^\varepsilon_{Y_1}(v^\varepsilon)\to v & in\ L^2((0,T)\times\Omega; H^1(Y_1)), \\ \partial_t\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)\rightharpoonup\partial_t u, & \partial_t\mathcal{T}^\varepsilon_{Y_1}(v^\varepsilon)\rightharpoonup\partial_t v & in\ L^2((0,T)\times\Omega\times Y_1), \\ \mathcal{T}^\varepsilon_{Y_2}(w^\varepsilon)\to w, & \partial_t\mathcal{T}^\varepsilon_{Y_2}(w^\varepsilon)\rightharpoonup\partial_t w & in\ L^2((0,T)\times\Omega; H^1(Y_2)), \\ \mathcal{T}^\varepsilon_{Y_1}(\nabla u^\varepsilon)\rightharpoonup\nabla u+\nabla_y\tilde{u} & & in\ L^2((0,T)\times\Omega\times Y_1), \\ \mathcal{T}^\varepsilon_{Y_1}(\nabla v^\varepsilon)\rightharpoonup\nabla v+\nabla_y\tilde{v} & & in\ L^2((0,T)\times\Omega\times Y_1), \\ \mathcal{T}^\varepsilon_{Y_2}(\nabla w^\varepsilon)\rightharpoonup\nabla w+\nabla_y\tilde{w} & & in\ L^2((0,T)\times\Omega\times Y_2), \end{array} \tag{12}$$

and

$$\begin{array}{lll} \mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)\to r, \quad \partial_t\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)\rightharpoonup\partial_t r & & in\ L^2((0,T)\times\Omega\times\Gamma_1), \\ \mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)\to u & & in\ L^2((0,T)\times\Omega\times\Gamma_1), \\ \mathcal{T}^\varepsilon_{\Gamma_2}(v^\varepsilon)\to v, \quad \mathcal{T}^\varepsilon_{\Gamma_2}(w^\varepsilon)\to w & & in\ L^2((0,T)\times\Omega\times\Gamma_2). \end{array} \tag{13}$$

Proof. Applying estimates in Lemmata 2.3, 2.5 and Convergence Theorem [7, 8], see Theorem 5.3 in Appendix, implies the convergences for u^ε, v^ε, w^ε in (12). The strong convergence of r^ε is achieved by showing that $\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)$ is a Cauchy sequence in $L^2((0,T)\times\Omega\times\Gamma_1)$, for the proof see [10, 16]. A priori estimate for $\partial_t r^\varepsilon$ and the convergence properties of $\mathcal{T}^\varepsilon_{\Gamma_1}$, [7], imply the convergences of $\mathcal{T}^\varepsilon_{\Gamma_1}(\partial_t r^\varepsilon)$. To show the convergences (13), we make use of the trace theorem, [9], and of the strong convergence of $T^\varepsilon_{Y_1}(u^\varepsilon)$ as $\varepsilon\to 0$, i.e.
$\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon) - u\|_{L^2((0,T)\times\Omega\times\Gamma_1)} \leq C\|\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)-u\|_{L^2((0,T)\times\Omega;H^1(Y_1))}\to 0$ as $\varepsilon\to 0$. □

Theorem 2.9. *Under the Assumption 2.1, the sequences of weak solutions of the problem (1)-(4) converges as $\varepsilon \to 0$ to a weak solution (u, v, w, r) of a macroscopic model, i.e. $u, v, w \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$, $r \in H^1(0,T; L^2(\Omega \times \Gamma_1))$ and u, v, w, r satisfy the macroscopic equations*

$$\begin{aligned}
&\int_0^T \int_{\Omega \times Y_1} \partial_t u \phi_1 + D_u(t,y)\Big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j} \nabla_y \omega_u^j\Big)(\nabla \phi_1 + \nabla_y \tilde{\phi}_1) + f(u,v)\phi_1 dy dx dt \\
&= -\int_0^T \int_{\Omega \times \Gamma_1} \eta(u,r)\phi_1 d\gamma_y dx dt, \\
&\int_0^T \int_{\Omega \times Y_1} \partial_t v \phi_1 + D_v(t,y)\Big(\nabla v + \sum_{j=1}^n \frac{\partial v}{\partial x_j} \nabla_y \omega_v^j\Big)(\nabla \phi_1 + \nabla_y \tilde{\phi}_1) - f(u,v)\phi_1 dy dx dt \\
&= \int_0^T \int_{\Omega \times \Gamma_2} (a(y)w - b(y)v)\phi_1 d\gamma_y dx dt, \qquad (14)\\
&\int_0^T \int_{\Omega \times Y_2} \partial_t w \phi_2 + D_w(t,y)\Big(\nabla w + \sum_{j=1}^n \frac{\partial w}{\partial x_j} \nabla_y \omega_w^j\Big)(\nabla \phi_2 + \nabla_y \tilde{\phi}_2) dy dx dt \\
&= -\int_0^T \int_{\Omega \times \Gamma_2} (a(y)w - b(y)v)\phi_2 d\gamma_y dx dt, \\
&\int_0^T \int_{\Omega \times \Gamma_1} \partial_t r \psi d\gamma_y dx dt = \int_0^T \int_{\Omega \times \Gamma_1} \eta(u,r)\psi d\gamma_y dx dt,
\end{aligned}$$

for $\phi_1, \phi_2 \in L^2(0,T; H_0^1(\Omega))$, $\tilde{\phi}_1 \in L^2((0,T) \times \Omega; H^1_{per}(Y_1))$, $\tilde{\phi}_2 \in L^2((0,T) \times \Omega; H^1_{per}(Y_2))$ and $\psi \in L^2((0,T) \times \Omega \times \Gamma_1)$, where ω_u^j, ω_v^j and ω_w^j are solutions of the correspondent unit cell problems

$$-\nabla_y (D_\zeta(t,y) \nabla_y \omega_\zeta^j) = \sum_{k=1}^3 \partial_{y_k} D_\zeta^{kj}(t,y) \ \textit{in } Y_1, \qquad \zeta = u, v, \qquad (15)$$

$$-D_\zeta(t,y) \nabla \omega_\zeta^j \cdot \nu = \sum_{k=1}^3 D_\zeta^{kj}(t,y)\nu_k \ \textit{on } \Gamma_1 \cup \Gamma_2, \quad \omega_\zeta^j \textit{ is } Y\textit{-periodic}, \int_{Y_1} \omega_\zeta^j(y) dy = 0,$$

$$-\nabla_y (D_w(t,y) \nabla_y \omega_w^j) = \sum_{k=1}^3 \partial_{y_k} D_w^{kj}(t,y) \ \textit{in } Y_2, \qquad (16)$$

$$-D_w(t,y) \nabla \omega_w^j \cdot \nu = \sum_{k=1}^3 D_w^{kj}(t,y)\nu_k \ \textit{on } \Gamma_2, \quad \omega_w^j \textit{ is } Y\textit{-periodic}, \int_{Y_2} \omega_w^j(y) dy = 0.$$

Proof. Due to considered geometry of Ω_1^ε and Ω_2^ε we have

$$\int_0^T \int_{\Omega_i^\varepsilon} u^\varepsilon \phi dx dt = \int_0^T \int_{\Omega \times Y_i} \mathcal{T}_{Y_i}^\varepsilon(u^\varepsilon) \mathcal{T}_{Y_i}^\varepsilon(\phi) dy dx dt, \quad i = 1, 2.$$

Applying the unfolding operator to (5)-(8), using $\mathcal{T}_{Y_1}^\varepsilon D_i(t, \frac{x}{\varepsilon}) = D_i(t,y), i \in \{u, v\}$ and $\mathcal{T}_{Y_2}^\varepsilon D_w(t, \frac{x}{\varepsilon}) = D_w(t,y)$, considering the limit as $\varepsilon \to 0$ and the convergences stated in Theorem 2.8, we obtain the unfolded limit problem. Similarly as for microscopic problem, using local Lipschitz continuity of η and f and boundedness of macroscopic solutions, which follows directly from the boundedness of microscopic solutions, we can show the uniqueness of a solution of the macroscopic model. Thus the whole sequence of microscopic solutions converge to a solution of the unfolded limit problem. The functions $\tilde{u}, \tilde{v}, \tilde{w}$ are defined in terms of u, v, w and solutions $\omega_u^j, \omega_v^j, \omega_w^j$ of unit cell problems (15) and (16), see [10, 16]. □

3. Corrector estimates

First of all, we introduce the definition of local average and averaging operators. After that, we show some technical estimates needed in the following.

Definition 3.1. [12, 4] *1. For any $\phi \in L^p(\Omega_i^\varepsilon)$, $p \in [1, \infty]$ and $i = 1, 2$, we define the local average operator ("mean in the cells") $\mathcal{M}_{Y_i}^\varepsilon : L^p(\Omega_i^\varepsilon) \to L^p(\Omega)$*

$$\mathcal{M}_{Y_i}^\varepsilon(\phi)(x) = \frac{1}{|Y_i|} \int_{Y_i} \mathcal{T}_{Y_i}^\varepsilon(\phi)(x, y) dy = \frac{1}{\varepsilon^n |Y_i|} \int_{\varepsilon[\frac{x}{\varepsilon}] + \varepsilon Y_i} \phi(y) dy, \quad x \in \Omega.$$

2. The operator $Q_i^\varepsilon : L^p(\tilde{\Omega}_i^{\varepsilon,2}) \to W^{1,\infty}(\Omega)$, $i = 1, 2$ is defined as Q_1–interpolation of $M_{Y_i}^\varepsilon(\phi)$, i.e. $Q_i^\varepsilon(\phi)(\varepsilon\xi) = \mathcal{M}_{Y_i}^\varepsilon(\phi)(\varepsilon\xi)$ for $\xi \in \mathbb{Z}^n$ and

$$Q_i^\varepsilon(\phi)(x) = \sum_{k \in \{0,1\}^n} Q_i^\varepsilon(\phi)(\varepsilon\xi + \varepsilon k) \bar{x}_1^{k_1} \dots \bar{x}_n^{k_n} \quad \text{for } x \in \varepsilon(Y_i + \xi), \quad \xi \in \mathbb{Z}^n,$$

where for $x \in \varepsilon(Y_i + \xi)$ and $k = (k_1, \dots, k_n) \in \{0, 1\}^n$ points $\bar{x}_l^{k_l}$ are given by

$$\bar{x}_l^{k_l} = \begin{cases} \frac{x_l - \varepsilon\xi_l}{\varepsilon}, & \text{if } k_l = 1, \\ 1 - \frac{x_l - \varepsilon\xi_l}{\varepsilon}, & \text{if } k_l = 0. \end{cases}$$

3. The operator $Q_i^\varepsilon : W^{1,p}(\Omega_i^\varepsilon) \to W^{1,\infty}(\Omega)$ is defined by $Q_i^\varepsilon(\phi) = Q_i^\varepsilon(\mathcal{P}(\phi))|_{\Omega_i^\varepsilon}$, where Q_i^ε is given in 2. and $\mathcal{P} : W^{1,p}(\Omega_i^\varepsilon) \to W^{1,p}((\mathbb{R}^n)^i)$ is an extension operator, in the case there exists $\mathcal{P}$, such that $\|\mathcal{P}(\phi)\|_{W^{1,p}((\mathbb{R}^n)^i)} \le C\|\phi\|_{W^{1,p}(\Omega_i^\varepsilon)}$.

Note $\mathcal{T}_{Y_i}^\varepsilon \circ \mathcal{M}_{Y_i}^\varepsilon(\phi) = \mathcal{M}_{Y_i}^\varepsilon(\phi)$ for $\phi \in L^p(\Omega_i^\varepsilon)$ and $\mathcal{M}_{Y_i}^\varepsilon(\phi)(x) = \mathcal{M}_{Y_i}(\mathcal{T}_{Y_i}^\varepsilon(\phi))(x)$, additionally $\sum_{k \in \{0,1\}^n} \bar{x}_1^{k_1} \dots \bar{x}_n^{k_n} = 1$.

Definition 3.2. [7, 8] *1. For $p \in [1 + \infty]$ and $i = 1, 2$, the averaging operator $\mathcal{U}_{Y_i}^\varepsilon$: $L^p(\Omega \times Y_i) \to L^p(\Omega_i^\varepsilon)$ is defined as*

$$\mathcal{U}_{Y_i}^\varepsilon(\Phi)(x) = \begin{cases} \frac{1}{|Y|} \int\limits_Y \Phi(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y) dz & \text{for a.a. } x \in \tilde{\Omega}_{i,int}^\varepsilon, \\ 0 & \text{for a.a. } x \in \Omega_i^\varepsilon \setminus \tilde{\Omega}_{i,int}^\varepsilon. \end{cases}$$

2. $\mathcal{U}_{\Gamma_i}^\varepsilon : L^p(\Omega \times \Gamma_i) \to L^p(\Gamma_i^\varepsilon)$ is defined as

$$\mathcal{U}_{\Gamma_i}^\varepsilon(\Phi)(x) = \begin{cases} \frac{1}{|Y|} \int\limits_Y \Phi(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y) dz & \text{for a.a. } x \in \tilde{\Gamma}_{i,int}^\varepsilon, \\ 0 & \text{for a.a. } x \in \Gamma_i^\varepsilon \setminus \tilde{\Gamma}_{i,int}^\varepsilon. \end{cases}$$

For $\omega^i \in H^1_{\text{per}}(Y_i)$, due to $\nabla_y \omega^i(y) = \nabla_y \mathcal{T}_{Y_i}^\varepsilon\big(\omega^i\big(\frac{x}{\varepsilon}\big)\big) = \varepsilon \mathcal{T}_{Y_i}^\varepsilon\big(\nabla_x \omega^i\big(\frac{x}{\varepsilon}\big)\big)$ and $\mathcal{U}_{Y_i}^\varepsilon(\nabla_y \omega^i(y)) = \varepsilon \mathcal{U}_{Y_i}^\varepsilon\big(\mathcal{T}_{Y_i}^\varepsilon\big(\nabla_x \omega^i\big(\frac{x}{\varepsilon}\big)\big)\big) = \varepsilon \nabla_x \omega^i\big(\frac{x}{\varepsilon}\big) = \nabla_y \omega^i\big(\frac{x}{\varepsilon}\big)$, we have that $\mathcal{U}_{Y_i}^\varepsilon(\nabla_y \omega^i(y)) = \nabla_y \omega^i\big(\frac{x}{\varepsilon}\big)$.

3.1. **Basic estimates.** In this subsection, we prove some technical estimates, used in the derivation of corrector estimates.

Proposition 3.3. *For $\phi_1 \in L^2(0,T;H^1(\Omega))$ and $\phi_2 \in L^2(0,T;H^1(\Omega_i^\varepsilon))$ we have*

$$\begin{aligned} \|\phi_1 - \mathcal{M}^\varepsilon_{Y_i}(\phi_1)\|_{L^2((0,T)\times\Omega)} &\leq \varepsilon C \|\nabla\phi_1\|_{L^2((0,T)\times\Omega)}, \\ \|\phi_2 - \mathcal{M}^\varepsilon_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega_i^\varepsilon)} &\leq \varepsilon C \|\nabla\phi_2\|_{L^2((0,T)\times\Omega_i^\varepsilon)}. \end{aligned} \tag{17}$$

Proof. This proof is similar to [12]. For $\phi_1 \in L^2(0,T;H^1(\Omega))$ we can write

$$x \to \phi_1|_{\varepsilon(\xi+Y)}(x) - \mathcal{M}^\varepsilon_{Y_i}(\phi_1)(\varepsilon\xi) \in L^2(0,T;H^1(\varepsilon\xi+\varepsilon Y)) \text{ with } \varepsilon(\xi+Y)\subset\Omega.$$

Using $Y_i \subset Y$ and applying Poincaré inequality, we obtain

$$\int_0^T \int_{\varepsilon(\xi+Y)} |\phi_1 - \mathcal{M}^\varepsilon_{Y_i}(\phi_1)(\varepsilon\xi)|^2 dxdt = \int_0^T \int_{\xi+Y} \Big|\phi_1(\varepsilon y) - \frac{1}{|Y_i|}\int_{\xi+Y_i} \phi_1(\varepsilon z)dz\Big|^2 \varepsilon^n dydt$$

$$\leq C\varepsilon^n \int_0^T \int_{\xi+Y} |\nabla_y \phi_1(\varepsilon y)|^2 dydt = C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\nabla_x\phi_1(x)|^2 dxdt.$$

Then, we add all inequalities for $\xi \in \mathbb{Z}^n$, such that $\varepsilon(\xi+Y)\subset\Omega$, and obtain the first estimate in (17). The second estimate follows from the decomposition of Ω_i^ε into $\cup_{\xi\in\mathbb{Z}^n}\varepsilon(\xi+Y_i)$ and Poincaré's inequality as in the previous estimate. □

Lemma 3.4. *For $\phi \in L^2(0,T;H^2(\Omega))$, $\phi_2 \in L^2(0,T;H^1(\tilde\Omega_i^\varepsilon))$ and $\omega \in H^1_{per}(Y_i)$, we have the following estimates*

$$\begin{aligned} &\|\nabla\phi - \mathcal{M}^\varepsilon_{Y_i}(\nabla\phi)\|_{L^2((0,T)\times\Omega)} \leq \varepsilon C\|\phi\|_{L^2(0,T;H^2(\Omega))}, \\ &\|(\mathcal{M}^\varepsilon_{Y_i}(\partial_{x_i}\phi) - Q^\varepsilon_{Y_i}(\partial_{x_i}\phi))\nabla_y\omega\|_{L^2((0,T)\times\Omega_i^\varepsilon)} \leq \varepsilon C\|\phi\|_{L^2(0,T;H^2(\Omega))}\|\nabla\omega\|_{L^2(Y_i)}, \\ &\|Q^\varepsilon_{Y_i}(\phi_2) - \mathcal{M}^\varepsilon_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)} \leq \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^\varepsilon)}, \\ &\|Q^\varepsilon_{Y_i}(\phi) - \phi\|_{L^2((0,T)\times\Omega)} \leq \varepsilon C\|\nabla\phi\|_{L^2((0,T)\times\tilde\Omega)}, \\ &\|Q^\varepsilon_{Y_i}(\phi_2) - \phi_2\|_{L^2((0,T)\times\Omega_i^\varepsilon)} \leq \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^\varepsilon)}, \\ &\|\phi - \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)\|_{L^2((0,T)\times\Omega\times\Gamma_i)} \leq \varepsilon C\|\nabla\phi\|_{L^2((0,T)\times\Omega)} + \varepsilon C\|\nabla\phi\|_{L^2((0,T)\times\Omega_i^\varepsilon)}, \\ &\|\nabla Q^\varepsilon_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)} \leq C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^\varepsilon)}, \\ &\|Q^\varepsilon_{Y_i}(\omega(y)) - \omega(y)\|_{L^2(Y_i)} \leq C\|\nabla_y\omega\|_{L^2(Y_i)}, \\ &\|\mathcal{T}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2)) - Q^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega\times Y_i)} \leq \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^\varepsilon)}. \end{aligned} \tag{18}$$

Proof. The first inequality follows directly from the first estimate in (17) applied to $\nabla\phi$. To show the second inequality, we use the definition of the operator Q_ε, the equality $\sum_{k\in\{0,1\}^n} \bar{x}_1^{k_1}\dots\bar{x}_n^{k_n} = 1$, and obtain

$$Q^\varepsilon_{Y_i}(\phi)(x) - \mathcal{M}^\varepsilon_{Y_i}(\phi)(x) = \sum_{k\in\{0,1\}^n} \big(Q^\varepsilon_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k) - \mathcal{M}^\varepsilon_{Y_i}(\phi)(\varepsilon\xi)\big)\bar{x}_1^{k_1}\dots\bar{x}_n^{k_n}.$$

Then, it follows

$$\int_{\varepsilon(\xi+Y_i)} \left|Q^\varepsilon_{Y_i}(\phi)(x) - \mathcal{M}^\varepsilon_{Y_i}(\phi)(x)\right|^2 \left|\nabla_y \omega\Big(\frac{x}{\varepsilon}\Big)\right|^2 dx$$
$$\leq 2^n \sum_{k\in\{0,1\}^n} \left|Q^\varepsilon_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k) - Q^\varepsilon_{Y_i}(\phi)(\varepsilon\xi)\right|^2 \varepsilon^n \int_{Y_i} |\nabla_y \omega(y)|^2\, dy.$$

For any $\phi \in W^{1,p}(Int(\overline{Y_i \cup (Y_i + e_j)}))$, the following estimate holds

$$|\mathcal{M}_{Y_i+e_j}(\phi) - \mathcal{M}_{Y_i}(\phi)| = |\mathcal{M}_{Y_i}(\phi(\cdot + e_j) - \phi(\cdot))|$$
$$\leq ||(\phi(\cdot+e_j) - \phi(\cdot)||_{L^p(Y_i)} \leq C||\nabla\phi||_{L^p(Y_i\cup(Y_i+e_j))}.$$

Thus, by the definition of $Q^\varepsilon_{Y_i}(\phi)(x)$ and by a scaling argument this implies

$$|Q^\varepsilon_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k) - Q^\varepsilon_{Y_i}(\phi)(\varepsilon\xi)| \leq \varepsilon C||\nabla\phi||_{L^2(\varepsilon(\xi+Y_i)\cup\varepsilon(\xi+k+Y_i))}. \tag{19}$$

We sum over $\xi\in\mathbb{Z}^n$ with $\varepsilon(\xi+Y_i)\subset\tilde{\Omega}^\varepsilon_i$ and obtain the desired estimate. Using (19) we obtain also that

$$\int_\Omega |Q^\varepsilon_{Y_i}(\phi) - \mathcal{M}^\varepsilon_{Y_i}(\phi)|^2 dx$$
$$\leq \varepsilon^2 C \sum_{\varepsilon(\xi+Y_i)\subset\tilde{\Omega}^\varepsilon_i} \varepsilon^n \sum_{k\in\{0,1\}^n} ||\nabla\phi||^2_{L^2(\varepsilon(\xi+Y_i)\cup\varepsilon(\xi+k+Y_i))} \leq \varepsilon^2 C \int_{\tilde{\Omega}^\varepsilon_i} |\nabla\phi|^2 dx.$$

In the same way, using the estimates stated in Proposition 3.3, the fourth and fifth estimates in (18) follows from:

$$\|Q^\varepsilon_{Y_i}(\phi_2) - \phi_2\|_{L^2((0,T)\times\Omega^\varepsilon_i)} \leq \|Q^\varepsilon_{Y_i}(\phi_2) - \mathcal{M}^\varepsilon_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)}$$
$$+\|\mathcal{M}^\varepsilon_{Y_i}(\phi_2) - \phi_2\|_{L^2((0,T)\times\Omega^\varepsilon_i)} \leq \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde{\Omega}^\varepsilon_i)}.$$

For $\phi\in H^1(\Omega)$ applying the trace theorem to a function in $L^2(\Gamma_i)$ yields

$$\int\limits_{\Omega\times\Gamma_i} |\phi - \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)|^2 d\gamma dx \leq \int\limits_{\Omega\times\Gamma_i} \Big(|\phi - \mathcal{M}^\varepsilon_{Y_i}(\phi)|^2 + |\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)|^2\Big) d\gamma dx \leq$$
$$C\varepsilon^2|\Gamma_i| \int\limits_\Omega |\nabla\phi|^2 dx + C\int\limits_{\Omega\times Y_i} \Big(|\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{Y_i}(\phi)|^2 + |\nabla_y(\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{Y_i}(\phi))|^2\Big) dy dx$$
$$\leq C\varepsilon^2|\Gamma_i| \int\limits_\Omega |\nabla\phi|^2 dx + C\int\limits_{\Omega^\varepsilon_i} |\mathcal{M}^\varepsilon_{Y_i}(\phi) - \phi|^2 dx + \int\limits_{\Omega\times Y_i} |\nabla_y \mathcal{T}^\varepsilon_{Y_i}(\phi))|^2 dy dx$$
$$\leq \varepsilon^2 C\Big(\int\limits_\Omega |\nabla\phi|^2 dx + \int\limits_{\Omega^\varepsilon_i} |\nabla\phi|^2 dx\Big).$$

To obtain an estimate for the gradient of $Q^\varepsilon_{Y_i}(\phi_2)$, with $\phi_2\in L^2(0,T;H^1(\tilde{\Omega}^\varepsilon))$, we define $\tilde{k}^j = (k_1,\dots,k_{j-1},k_{j+1},\dots,k_n)$, $\tilde{k}^j_1 = (k_1,\dots,k_{j-1},1,k_{j+1},\dots,k_n)$, $\tilde{k}^j_0 = (k_1,\dots,k_{j-1},0,k_{j+1},\dots,k_n)$ and calculate

$$\frac{\partial Q^\varepsilon_{Y_i}(\phi_2)}{\partial x_j} = \sum_{\tilde{k}^j} \frac{Q^\varepsilon_{Y_i}(\phi_2)(\varepsilon\xi+\varepsilon\tilde{k}^j_1) - Q^\varepsilon_{Y_i}(\phi_2)(\varepsilon\xi+\varepsilon\tilde{k}^j_0)}{\varepsilon} \bar{x}_1^{k_1}\dots\bar{x}_{j-1}^{k_{j-1}}\dots\bar{x}_{j+1}^{k_{j+1}}\bar{x}_n^{k_n}.$$

Now, applying (19) we obtain the estimates for $\nabla Q^\varepsilon_{Y_i}(\phi_2)$ in $L^2((0,T)\times\Omega)$.

For $y \in Y_i$ we have $Q_{Y_i}(\omega(y))(y) - \omega(y) = \sum_{k\in\{0,1\}^n} (Q^\varepsilon_{Y_i}(\omega)(k) - \omega(y))\bar{y}_1^{k_1} \dots \bar{y}_n^{k_n}$, where $\bar{y}_l^{k_l} = \begin{cases} y_l - \xi_l, & \text{if } k_l = 1, \\ 1 - (y_l - \xi_l), & \text{if } k_l = 0 \end{cases}$. The Poincaré's inequality and the periodicity of ω imply the estimate for $Q^\varepsilon_{Y_i}(\omega(y)) - \omega(y)$.

To derive the last estimate, we consider

$$\|\mathcal{T}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2)) - Q^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega\times Y_i)} \leq \|\mathcal{T}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2)) - \mathcal{M}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega\times Y_i)}$$
$$+\|\mathcal{M}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2)) - Q^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega\times Y_i)} \leq C\|Q^\varepsilon_{Y_i}(\phi_2) - \mathcal{M}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega_i^\varepsilon)}$$
$$+C\|\mathcal{M}^\varepsilon_{Y_i}(Q^\varepsilon_{Y_i}(\phi_2)) - Q^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega)} \leq \varepsilon C\|\nabla Q^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega)} \leq \varepsilon C\|\nabla\phi_2\|_{L^2(\tilde{\Omega}_i^\varepsilon)}.$$

□

3.2. **Periodicity defect.** In the derivation of error estimates we use a generalization of the Theorem 3.4 proved in [12] for functions defined in a perforated domain:

Theorem 3.5. *For any $\phi \in H^1(\Omega_i^\varepsilon)$, $i = 1, 2$, there exists $\hat{\psi}^\varepsilon \in L^2(\Omega; H^1_{per}(Y_i))$:*

$$\|\hat{\psi}^\varepsilon\|_{L^2(\Omega;H^1(Y_i))} \leq C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n},$$
$$\|\mathcal{T}^\varepsilon_{Y_i}(\nabla\phi) - \nabla\phi^\varepsilon - \nabla_y\hat{\psi}^\varepsilon\|_{L^2(Y_i;H^{-1}(\Omega))} \leq C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}.$$

Here $\phi^\varepsilon = Q^\varepsilon_{Y_i}(\phi)$.

The proofs of Theorem 3.5 go the same lines as in [12, Theorem 3.4], using the estimates

$$\|\mathcal{T}^\varepsilon_{Y_i}(\phi)\|_{L^2(\Omega\times Y_i)} \leq C\|\phi\|_{L^2(\Omega_i^\varepsilon)}, \quad \|\nabla Q^\varepsilon_{Y_i}(\phi)\|_{L^2(\Omega)} \leq C\|\nabla\phi\|_{L^2(\tilde{\Omega}_i^\varepsilon)}.$$

3.3. **Error estimates.** Under additional regularity assumptions on the solution of the macroscopic problem, we obtain a set of error estimates. We emphasize here again that the most important point is that only H^1-regularity for the solutions of the microscopic model and of the cell problems is required.

Theorem 3.6. *Suppose $(u^\varepsilon, v^\varepsilon, w^\varepsilon, r^\varepsilon)$ are solutions of the microscopic problem* (1)-(4) *and $u, v, w \in L^2(0,T; H^2(\Omega)) \cap H^1((0,T)\times\Omega))$, $r \in H^1(0,T; L^2(\Omega\times\Gamma_1))$ are solutions of the macroscopic equations* (14)*. Then we have the following corrector estimates:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon - \nabla u - \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\|^2_{L^2((0,T)\times\Omega_1^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}},$$

$$\|v^\varepsilon - v\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|\nabla v^\varepsilon - \nabla v - \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j}v)\nabla_y\omega_v^j\|^2_{L^2((0,T)\times\Omega_1^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}},$$

$$\|w^\varepsilon - w\|_{L^2((0,T)\times\Omega_2^\varepsilon)} + \|\nabla w^\varepsilon - \nabla w - \sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j}w)\nabla_y\omega_w^j\|^2_{L^2((0,T)\times\Omega_2^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}},$$

$$\varepsilon^{\frac{1}{2}}\|r^\varepsilon - \mathcal{U}^\varepsilon_{\Gamma_1}(r(t,x,y))\|_{L^2((0,T)\times\Gamma_1^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}.$$

4. Proof of Theorem 3.6.

We define distance function $\rho(x) = \text{dist}(x, \partial\Omega)$, domains $\hat{\Omega}^\varepsilon_{\rho,in} = \{x \in \Omega, \rho(x) < \varepsilon\}$ and $\hat{\Omega}^\varepsilon_{i,\rho,in} = \{x \in \Omega^\varepsilon_i, \rho(x) < \varepsilon\}$, where $\rho^\varepsilon(\cdot) = \inf\{\frac{\rho(\cdot)}{\varepsilon}, 1\}$. Definition of ρ^ε yields

$$\|\nabla_x \rho^\varepsilon\|_{L^\infty(\Omega)^n} = \|\nabla_x \rho^\varepsilon\|_{L^\infty(\hat{\Omega}^\varepsilon_{\rho,in})^n} = \varepsilon^{-1}. \tag{20}$$

Then, for $\Phi \in H^2(\Omega)$ and $\omega^j \in H^1(Y_i)$, $i = 1, 2$, $j = u, v, w$, we obtain the following estimates, [12],

$$\begin{aligned}
&\|\nabla \Phi\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,in})^n} + \|Q^\varepsilon_{Y_i}(\nabla\Phi)\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,in})^n} + \|\mathcal{M}^\varepsilon_{Y_i}(\nabla\Phi)\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,in})^n} \leq C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)},\\
&\left\|\omega^j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}^\varepsilon_{i,\rho,in})} + \left\|\nabla\omega^j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}^\varepsilon_{i,\rho,in})^n} \leq C\varepsilon^{\frac{1}{2}}\|\nabla_y \omega^j\|_{L^2(Y_i)^n},\\
&\|(1-\rho_\varepsilon)\nabla_x\Phi\|_{L^2(\Omega)^n} \leq \|\nabla_x \Phi\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,in})^n} \leq C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)},\\
&\|\nabla_x(\rho_\varepsilon \partial_{x_j}\Phi)\|_{L^2(\Omega)^n} \leq C(\varepsilon^{-\frac{1}{2}} + 1)\|\Phi\|_{H^2(\Omega)},\\
&\left\|\varepsilon\partial_{x_i}\rho_\varepsilon Q^\varepsilon_{Y_i}(\partial_{x_j}\Phi)\omega^j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega^\varepsilon_i)} \leq C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)}\|\omega^j\|_{L^2(Y_i)},\\
&\left\|\varepsilon\rho_\varepsilon\partial_{x_i} Q^\varepsilon_{Y_i}(\partial_{x_j}\Phi)\omega^j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega^\varepsilon_i)} \leq C\varepsilon\|\Phi\|_{H^2(\Omega)}\|\omega^j\|_{L^2(Y_i)}.
\end{aligned} \tag{21}$$

Now, for $\phi_1 \in L^2(0, T; H^1(\Omega^\varepsilon_i))$ given by

$$\phi_1(x) = u^\varepsilon(x) - u(x) - \varepsilon\rho^\varepsilon(x)\sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j}u)(x)\omega^j_u\left(\frac{x}{\varepsilon}\right)$$

we consider an extension $\tilde{\phi}^\varepsilon_1$ from $(0, T) \times \Omega^\varepsilon_1$ into $(0, T) \times \Omega$ such that

$$\|\tilde{\phi}^\varepsilon_1\|_{L^2((0,T)\times\Omega)} \leq C\|\phi_1\|_{L^2((0,T)\times\Omega^\varepsilon_1)} \quad \text{and} \quad \|\nabla\tilde{\phi}^\varepsilon_1\|_{L^2((0,T)\times\Omega)} \leq C\|\nabla\phi_1\|_{L^2((0,T)\times\Omega^\varepsilon_1)}.$$

Due to zero boundary conditions such extension can be defined for whole Ω. Notice that $Q^\varepsilon_{Y_i}(\partial_{x_j}u)$ and ∇u are in $L^2(0, T; H^1(\Omega))$, but not in $L^2(0, T; H^1_0(\Omega))$.

We consider $\tilde{\phi}^\varepsilon_1 \in L^2(0, T; H^1_0(\Omega))$ and $\hat{\psi}^\varepsilon_1 \in L^2((0, T) \times \Omega, H^1_{\text{per}}(Y_1))$, given by Theorem 3.5, as test functions in the macroscopic equation (14) for u:

$$\begin{aligned}
&\int_0^\tau \int_{\Omega\times Y_1} \partial_t u\tilde{\phi}^\varepsilon_1 + D_u(y)\Big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y \omega^j_u\Big)\big(\nabla\tilde{\phi}^\varepsilon_1 + \nabla_y\hat{\psi}^\varepsilon_1\big)dydxdt\\
&+ \int_0^\tau \int_{\Omega\times Y_1} f(u, v)\tilde{\phi}^\varepsilon_1 dydxdt + \int_0^\tau \int_{\Omega\times\Gamma_1} \eta(u, r)\tilde{\phi}^\varepsilon_1 d\gamma dxdt = 0.
\end{aligned}$$

In the first term and in the last two integrals we replace $\tilde{\phi}^\varepsilon_1$ by $\mathcal{M}^\varepsilon_{Y_1}(\phi_1)$, $\tilde{\phi}^\varepsilon_1$ by $\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)$, and u by $\mathcal{T}^\varepsilon_{Y_1}(u)$. As next step, we introduce ρ^ε in front of ∇u and $\partial_{x_j}u$ and replace $\nabla\tilde{\phi}^\varepsilon_1$ by $\nabla Q^\varepsilon_{Y_1}(\phi_1)$. Now, using Theorem 3.5, we replace $\nabla\phi^\varepsilon_1 + \nabla_y\hat{\psi}^\varepsilon_1$, by $\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)$, where $\phi^\varepsilon_1 = Q^\varepsilon_{Y_1}(\phi_1)$ and obtain

$$\begin{aligned}
&\int_0^\tau \int_{\Omega\times Y_1} \partial_t \mathcal{T}^\varepsilon_{Y_1}(u)\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + D_u(y)\rho^\varepsilon\Big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega^j_u\Big)\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)dydxdt\\
&+ \int_0^\tau \int_{\Omega\times Y_1} \mathcal{T}^\varepsilon_{Y_1}(f(u, v))\mathcal{M}^\varepsilon_{Y_1}(\phi_1)dydxdt + \int_0^\tau \int_{\Omega\times\Gamma_1} \eta(u, r)\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)d\gamma dxdt = R^1_u,
\end{aligned}$$

where

$$
\begin{aligned}
R_u^1 \quad &= \int_0^\tau \int_{\Omega\times Y_1} \Big[\partial_t(u - \mathcal{T}_{Y_1}^\varepsilon(u))\mathcal{M}_{Y_1}^\varepsilon(\phi_1) + \partial_t u(\tilde{\phi}_1^\varepsilon - \mathcal{M}_{Y_1}^\varepsilon(\phi_1))\\
&+\rho^\varepsilon D_u\big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y \omega_u^j\big)\Big(\nabla(Q_{Y_i}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon) + (\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1) - \nabla\phi_1^\varepsilon - \nabla_y\hat{\psi}_1^\varepsilon)\Big)\\
&+(\rho^\varepsilon - 1)D_u\big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\big)(\nabla\tilde{\phi}_1^\varepsilon + \nabla_y\hat{\psi}_1^\varepsilon) + f(u,v)(\tilde{\phi}_1^\varepsilon - \mathcal{M}_{Y_1}^\varepsilon(\phi_1))\\
&+(f - \mathcal{T}_{Y_1}^\varepsilon(f))\mathcal{M}_{Y_1}^\varepsilon(\phi_1)\Big]dydxdt + \int_0^\tau\int_{\Omega\times\Gamma_1}\eta(u,r)(\mathcal{T}_{\Gamma_1}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon)d\gamma dxdt.
\end{aligned}
$$

Then we remove ρ^ε, replace ∇u by $\mathcal{M}_{Y_1}^\varepsilon(\nabla u)$, $\partial_{x_j}u$ by $\mathcal{M}_{Y_1}^\varepsilon(\partial_{x_j}u)$ and, using $\mathcal{M}_{Y_1}^\varepsilon(\phi) = \mathcal{T}_{Y_1}^\varepsilon \circ \mathcal{M}_{Y_1}^\varepsilon(\phi)$, we apply the inverse unfolding

$$
\begin{aligned}
&\int_0^\tau\int_{\Omega_1^\varepsilon}\Big(\partial_t u\mathcal{M}_{Y_1}^\varepsilon(\phi_1) + D_u^\varepsilon\Big(\mathcal{M}_{Y_1}^\varepsilon(\nabla u) + \sum_{j=1}^n \mathcal{M}_{Y_1}^\varepsilon(\partial_{x_j}u)\nabla_y\omega_u^j\Big(\frac{x}{\varepsilon}\Big)\Big)\nabla\phi_1\Big)dxdt\\
&+\int_0^\tau\int_{\Omega_1^\varepsilon} f(u,v)\mathcal{M}_{Y_1}^\varepsilon(\phi_1)dxdt + \int_0^\tau\int_{\Omega\times\Gamma_1}\eta(u,r)\mathcal{T}_{\Gamma_1}^\varepsilon(\phi_1)d\gamma dxdt = R_u^1 + R_u^2,
\end{aligned}
$$

where

$$
\begin{aligned}
R_u^2 = &\int_0^\tau\int_{\Omega\times Y_1}\Big[(1-\rho^\varepsilon)D_u(y)\Big(\nabla u + \sum_{j=1}^n \partial_{x_j}u\nabla_y\omega_u^j(y)\Big)\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1)\\
&+D_u(y)\Big(\mathcal{M}_{Y_1}^\varepsilon(\nabla u) - \nabla u + \sum_{j=1}^n\big(\mathcal{M}_{Y_1}^\varepsilon(\partial_{x_j}u) - \partial_{x_j}u\big)\nabla_y\omega_u^j(y)\Big)\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1)\Big]dydxdt.
\end{aligned}
$$

Introducing ρ^ε in front of $\mathcal{M}_{Y_i}^\varepsilon(\partial_{x_j}u)$ and replacing $\mathcal{M}_{Y_1}^\varepsilon(\phi_1)$ by ϕ_1, $\mathcal{M}_{Y_1}^\varepsilon(\nabla u)$ by ∇u, $\mathcal{M}_{Y_1}^\varepsilon(\partial_{x_j}u)$ by $Q_{Y_1}^\varepsilon(\partial_{x_j}u)$ yield

$$
\begin{aligned}
&\int_0^\tau\int_{\Omega_1^\varepsilon}\Big[\partial_t u\phi_1 + D_u^\varepsilon\Big(\nabla u + \sum_{j=1}^n\rho^\varepsilon Q_{Y_1}^\varepsilon(\partial_{x_j}u)\nabla_y\omega_u^j\big(\frac{x}{\varepsilon}\big)\Big)\nabla\phi_1 + f(u,v)\phi_1\Big]dxdt\\
&= -\int_0^\tau\int_{\Omega\times\Gamma_1}\eta(u,r)\mathcal{T}_{Y_1}^\varepsilon(\phi_1)d\gamma dxdt + R_u^1 + R_u^2 + R_u^3, \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
R_u^3 = &\int_0^\tau\int_{\Omega_1^\varepsilon}\Big[\big(\partial_t u + f\big)(\phi_1 - \mathcal{M}_{Y_1}^\varepsilon(\phi_1)) + (\rho^\varepsilon - 1)D_u^\varepsilon\sum_{j=1}^n\mathcal{M}_{Y_i}^\varepsilon(\partial_{x_j}u)\nabla_y\omega_u^j\Big(\frac{x}{\varepsilon}\Big)\nabla\phi_1\\
&+D_u^\varepsilon\Big(\nabla u - \mathcal{M}_{Y_1}^\varepsilon(\nabla u) + \sum_{j=1}^n\rho^\varepsilon\big(Q_{Y_1}^\varepsilon(\partial_{x_j}u) - \mathcal{M}_{Y_1}^\varepsilon(\partial_{x_j}u)\big)\nabla_y\omega_u^j\Big(\frac{x}{\varepsilon}\Big)\Big)\nabla\phi_1\Big]dxdt.
\end{aligned}
$$

Now, we subtract from the equation for u^ε the equation (22) and obtain for the test function $\phi_1 = u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum\limits_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} u)\omega^j_u$ the equality

$$\int\limits_0^\tau \int\limits_{\Omega_1^\varepsilon} \Big[\partial_t(u^\varepsilon - u)(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_i}(\partial_{x_j} u)\omega^j_u) +$$

$$D^\varepsilon_u\big(\nabla(u^\varepsilon - u) - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_i}\big(\partial_{x_j} u\big)\nabla_y \omega^j_u\big)\big(\nabla(u^\varepsilon - u) - \varepsilon \sum_{j=1}^n \nabla_x\big(\rho^\varepsilon Q^\varepsilon_{Y_i}\big(\partial_{x_j} u\big)\omega^j_u\big)\big)$$

$$+\big(f(u^\varepsilon, v^\varepsilon) - f(u,v)\big)\big(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_i}(\partial_{x_j} u)\omega^j_u\big)\Big] dxdt +$$

$$\int\limits_0^\tau \int\limits_{\Omega\times\Gamma_1} \big(\eta(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon r^\varepsilon) - \eta(u,r)\big)\mathcal{T}^\varepsilon_{\Gamma_1}\big(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_i}(\partial_{x_j} u)\omega^j_u\big) d\gamma dxdt = R_u,$$

where $R_u = R^1_u + R^2_u + R^3_u$.
We consider $\psi^\varepsilon = \mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r$ as a test function in the equations for $\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)$ and r and, using local Lipschitz continuity of η and boundedness of u^ε, u, r^ε, r, obtain

$$\int_0^\tau \int_{\Omega\times\Gamma_1} \partial_t |\mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r|^2 d\gamma dxdt \le C \int_0^\tau \int_{\Omega\times\Gamma_1} \big(|\mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r|^2 + |\mathcal{T}^\varepsilon_{\Gamma_1} u^\varepsilon - u|^2\big) d\gamma dxdt.$$

Applying Gronwall's inequality and considering $\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon_0)(x,y) = r_0(y)$ yield

$$\begin{aligned}
&\|\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon) - r\|^2_{L^2(\Omega\times\Gamma_1)} \le C\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon) - u\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon_0) - r_0\|^2_{L^2(\Omega\times\Gamma_1)} \\
&\le C\Big(\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon - u)\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(u) - u\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\Big).
\end{aligned}$$

Then, for the boundary integral using the estimate in Lemma 3.4 we obtain

$$\begin{aligned}
&\int_0^\tau \int_{\Omega\times\Gamma_1} (\eta(\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon), \mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)) - \eta(r,u))\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1) d\gamma dxdt \le \\
&C\big(\|\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon) - r\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon) - u\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big)\varepsilon\|\phi_1\|_{L^2((0,\tau)\times\Gamma^\varepsilon_1)} \\
&\le C\big(\|u^\varepsilon - u\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)} + \varepsilon\|\nabla(u^\varepsilon - u)\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)} + \varepsilon\|\nabla u\|_{L^2((0,\tau)\times\Omega)}\big) \times \\
&\big(\|\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)} + \varepsilon\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)}\big).
\end{aligned} \tag{23}$$

Therefore, the ellipticity assumption, the Lipschitz continuity of f and Young inequality, applied to the estimate for the boundary integral (23), imply

$$\begin{aligned}
&\int_0^\tau \int_{\Omega^\varepsilon_1} \Big(\partial_t\big|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} u)\omega^j_u\big|^2 + \big|\nabla\hat{u}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} u)\nabla_y\omega^j_u\big|^2\Big) dxdt \\
&\le C\int_0^\tau \int_{\Omega^\varepsilon_1} \Big(\big|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} u)\omega^j_u\big|^2 + \big|\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega^j_v\big|^2\Big) dxdt \\
&+\varepsilon^2 \int_0^\tau \int_\Omega |\nabla u|^2 dxdt + R_u + C^\varepsilon_u,
\end{aligned}$$

where $\hat{u}^\varepsilon = u^\varepsilon - u$, $\hat{v}^\varepsilon = v^\varepsilon - v$ and

$$C_u^\varepsilon := C\varepsilon^2 \int_0^\tau \int_{\Omega_1^\varepsilon} \sum_{j=1}^n \Big(|Q_{Y_1}^\varepsilon(\partial_t \partial_{x_j} u)\omega_u^j|^2 + (1+\varepsilon^2)|\nabla Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j|^2 + |Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j|^2$$

$$+|Q_{Y_1}^\varepsilon(\partial_{x_j} v)\omega_v^j|^2 + |Q_{Y_1}^\varepsilon(\partial_{x_j} u)\nabla_y \omega_u^j|^2\Big) dxdt + C\int_0^\tau \int_{\hat{\Omega}_{1,\rho,in}^\varepsilon} \sum_{j=1}^n \left|Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j\right|^2 dxdt$$

$$\leq C\big(\varepsilon^2 \|u\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2\|u\|^2_{H^1((0,T)\times\Omega)} + \varepsilon\|u\|^2_{L^2(0,T;H^2(\Omega))}\big)\|\omega_u\|^2_{H^1(Y_1)^n}$$

$$+C\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_v\|^2_{L^2(Y_1)^n}.$$

Here we used that

$$\varepsilon^2 \int_{\Omega_1^\varepsilon} |\nabla\big(\rho^\varepsilon Q_{Y_1}^\varepsilon(\partial_{x_j} u)\big)\omega_u^j|^2 dx \leq \varepsilon^2 \int_{\Omega_1^\varepsilon} |\nabla Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j|^2 dx + \int_{\hat{\Omega}_{1,\rho,in}^\varepsilon} |Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j|^2 dx.$$

The estimates of the error terms in the subsection 4.1 imply

$$|R_u| = |R_u^1 + R_u^2 + R_u^3| \leq \varepsilon^{1/2} C\big(1 + \|u\|_{H^1((0,T)\times\Omega)} + \|u\|_{L^2(0,T;H^2(\Omega))}$$

$$+\|v\|_{L^2(0,T;H^1(\Omega))} + \|r\|_{L^2((0,T)\times\Omega\times\Gamma_1)}\big)\|\phi_1\|_{L^2(0,T;H^1(\Omega_1^\varepsilon))}.$$

Then, applying Young's inequality, we obtain

$$\int_0^\tau \int_{\Omega_1^\varepsilon} \big(\partial_t |\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q_\varepsilon(\partial_{x_j} u)\omega_u^j|^2 + |\nabla \hat{u}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} u)\nabla_y \omega_u^j|^2\big) dxdt$$

$$\leq C\int_0^\tau \int_{\Omega_1^\varepsilon} \big(|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j|^2 + |\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_v^j|^2\big) dxdt$$

$$+C(\varepsilon+\varepsilon^2)(1 + \|u\|^2_{H^1((0,T)\times\Omega)} + \|u\|^2_{L^2(0,T;H^2(\Omega))})\big(1 + \|\omega_u\|^2_{H^1(Y_1)^n}\big)$$

$$+C\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega)}\big(1 + \|\omega_v\|^2_{H^1(Y_1)^n}\big) + \varepsilon^2\|r\|^2_{L^\infty((0,T)\times\Omega\times\Gamma_1)}.$$

Similarly, estimates for $v^\varepsilon - v - \varepsilon\sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} v)\omega_v^j$ and $w^\varepsilon - w - \varepsilon\sum_{j=1}^n Q_{Y_2}^\varepsilon(\partial_{x_j} w)\omega_w^j$ are obtained. The only difference is the boundary term. Applying the trace theorem and estimates in Lemma 3.4, the boundary term can be estimated by

$$\int_{\Omega\times\Gamma_2} \Big((a(y)w - b(y)v)\tilde{\phi}_1^\varepsilon - (a(y)\mathcal{T}_{\Gamma_2}^\varepsilon(w) - b(y)\mathcal{T}_{\Gamma_2}^\varepsilon(v))\mathcal{T}_{\Gamma_2}^\varepsilon(\phi_1)\Big) d\gamma dx$$

$$\leq C\int_{\Omega\times\Gamma_2} \big(|w - \mathcal{T}_{\Gamma_2}^\varepsilon(w)| + |v - \mathcal{T}_{\Gamma_2}^\varepsilon(v)|\big)\mathcal{T}_{\Gamma_2}^\varepsilon(\phi_1) + (w+v)|\tilde{\phi}_1^\varepsilon - \mathcal{M}_{Y_1}^\varepsilon(\phi_1)|$$

$$+(w+v)|\mathcal{M}_{Y_1}^\varepsilon(\phi_1) - \mathcal{T}_{\Gamma_2}^\varepsilon(\phi_1)|\big) d\gamma dx \leq \varepsilon C\big(||v||_{H^1(\Omega)} + ||w||_{H^1(\Omega)}\big)||\phi_1||_{H^1(\Omega_1^\varepsilon)}.$$

Thus, we obtain for $\hat{v}^\varepsilon = v^\varepsilon - v$ and $\hat{w}^\varepsilon = w^\varepsilon - w$

$$\int_0^\tau \int_{\Omega_1^\varepsilon} \big(\partial_t |\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega_v^j|^2 + |\nabla \hat{v}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\nabla_y \omega_v^j|^2\big) dxdt$$
$$\leq C \int_0^\tau \int_{\Omega_1^\varepsilon} \Big(|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} u)\omega_u^j|^2 + |\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega_v^j|^2\Big) dxdt +$$
$$C \int_0^\tau \int_{\Omega_2^\varepsilon} \big(|\hat{w}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\omega_w^j|^2 + \varepsilon^2|\nabla \hat{w}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\nabla_y \omega_w^j)|^2\big) dxdt$$
$$+C(\varepsilon+\varepsilon^2)\big(1 + \|v\|^2_{L^2(0,T;H^2(\Omega))} + \|v\|^2_{H^1((0,T)\times\Omega)}\big)\big(1 + \|\omega_v\|^2_{H^1(Y_1)^n}\big)$$
$$+C\varepsilon^2\big(\|u\|^2_{L^2(0,T;H^1(\Omega))} + \|w\|^2_{L^2(0,T;H^1(\Omega))}\big) + C_v,$$
$$\int_0^\tau \int_{\Omega_2^\varepsilon} \Big(\partial_t |\hat{w}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\omega_w^j|^2 + |\nabla \hat{w}^\varepsilon - \rho^\varepsilon \sum_{i=j}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\nabla_y \omega_w^j|^2\Big) dxdt$$
$$\leq C \int_0^\tau \int_{\Omega_1^\varepsilon} \Big(|\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega_v^j|^2 + \varepsilon^2|\nabla \hat{v}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j} v)\nabla_y \omega_v^j|^2\Big) dxdt +$$
$$C \int_0^\tau \int_{\Omega_2^\varepsilon} \Big(|\hat{w}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{i=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\omega_w^j|^2 + \varepsilon^2|\nabla \hat{w}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j} w)\nabla_y \omega_w^j|^2\big) dxdt$$
$$+C(\varepsilon+\varepsilon^2)\big(1 + \|w\|^2_{L^2(0,T;H^2(\Omega))} + \|w\|^2_{H^1((0,T)\times\Omega)}\big)\big(1 + \|\omega_v\|^2_{H^1(Y_1)^n}\big)$$
$$+C\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega)} + C_w,$$

where

$$C_v := C\varepsilon^2 \int_0^\tau \int_{\Omega_1^\varepsilon} \sum_{j=1}^n \Big(|Q^\varepsilon_{Y_1}(\partial_t\partial_{x_j} v)\omega_v^j)|^2 + (1+\varepsilon^2)|\nabla(Q_\varepsilon(\partial_{x_j} v))\omega_v^j|^2 + |Q^\varepsilon_{Y_1}(\partial_{x_j} u)\omega_u^j|^2$$
$$+|Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega_v^j|^2 + |Q^\varepsilon_{Y_1}(\partial_{x_j} v)\nabla_y\omega_v^j|^2\Big) dxdt + \int_0^\tau \int_{\hat{\Omega}^\varepsilon_{1,\rho,in}} |Q^\varepsilon_{Y_1}(\partial_{x_j} v)\omega_v^j|^2 dxdt$$
$$+C\varepsilon^2 \int_0^\tau \int_{\Omega_2^\varepsilon} \sum_{i=1}^n \Big(|Q^\varepsilon_{Y_2}(\partial_{x_j} w)\omega_w^j|^2 + |Q^\varepsilon_{Y_2}(\partial_{x_j} w)\nabla_y\omega_w^j|^2\Big) dxdt$$
$$\leq C(\varepsilon^2\|v\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2\|v\|^2_{H^1((0,T)\times\Omega)} + \varepsilon\|v\|^2_{L^2(0,T;H^2(\Omega))})\|\omega_v\|^2_{H^1(Y_1)^n}$$
$$+\varepsilon^2 C\|u\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_u\|^2_{L^2(Y_1)^n} + \varepsilon^2 C\|w\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_w\|^2_{H^1(Y_2)^n}$$

and

$$C_w := \varepsilon^2 C \int_0^\tau \int_{\Omega_2^\varepsilon} \sum_{j=1}^n \Big(\big|Q_{Y_2}^\varepsilon(\partial_t \partial_{x_i} w)\omega_w^j\big|^2 + \big|\nabla Q_{Y_2}^\varepsilon(\partial_{x_i} w))\omega_w^j\big|^2 + \big|Q_{Y_2}^\varepsilon(\partial_{x_j} w)\omega_w^j\big|^2 +$$

$$\big|Q_{Y_2}^\varepsilon(\partial_{x_i} w)\nabla_y \omega_w^j\big|^2\Big) dxdt + \varepsilon^2 C \int_0^\tau \int_{\Omega_1^\varepsilon} \sum_{j=1}^n \Big(\big|Q_{Y_1}^\varepsilon(\partial_{x_j} v)\omega_v^j\big|^2 + \big|Q_{Y_1}^\varepsilon(\partial_{x_j} v)\nabla_y \omega_v^j\big|^2\Big) dxdt$$

$$+ \int_0^\tau \int_{\hat\Omega_{2,\rho,in}^\varepsilon} \sum_{j=1}^n \big|Q_{Y_2}^\varepsilon(\partial_{x_j} w)\omega_w^j\big|^2 dxdt \leq \varepsilon^2 \|v\|^2_{L^2(0,T;H^1(\Omega))} \|\omega_v\|^2_{H^1(Y_1)^n}$$

$$+ C\big(\varepsilon \|w\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2 \|w\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2 \|w\|^2_{H^1((0,T)\times\Omega)}\big) \|\omega_w\|^2_{H^1(Y_2)^n}.$$

For sufficiently small ε, adding the all estimates, removing ρ^ε by using the estimates (21), applying Gronwall's inequality and considering that $u^\varepsilon(0) = u_0$, $v^\varepsilon(0) = v_0$, $v^\varepsilon(0) = v_0$ we obtain the estimates for u^ε, v^ε, w^ε, stated in the theorem.
To obtain the estimate for $r^\varepsilon - \mathcal{U}_{\Gamma_1}^\varepsilon(r)$, we consider the equations for $\mathcal{T}_{\Gamma_1}^\varepsilon r^\varepsilon$ and r with the test function $\mathcal{T}_{\Gamma_1}^\varepsilon r^\varepsilon - r$. Using the properties of $\mathcal{U}_{\Gamma_1}^\varepsilon$, the local Lipschitz continuity of η, and Gronwall's inequality, yields

$$\int_{\Gamma_i^\varepsilon} |r^\varepsilon - \mathcal{U}_{\Gamma_1}^\varepsilon(r)|^2 d\gamma \leq C \int_{\Omega\times\Gamma_1} |\mathcal{T}_{\Gamma_1}^\varepsilon(r^\varepsilon) - r|^2 d\gamma \leq C \int_0^t \int_{\Omega\times\Gamma_1} |\mathcal{T}_{\Gamma_1}^\varepsilon(u^\varepsilon) - u|^2 d\gamma d\tau +$$

$$\int_{\Omega\times\Gamma_1} |\mathcal{T}_{\Gamma_1}^\varepsilon(r_0) - r_0|^2 d\gamma_y dx \leq \int_0^t \int_{\Omega\times\Gamma_1} |\mathcal{T}_{\Gamma_1}^\varepsilon(u) - \mathcal{M}_{Y_1}^\varepsilon(u)|^2 + |\mathcal{M}_{Y_1}^\varepsilon(u) - u|^2 d\gamma d\tau$$

$$+ C \int_0^t \int_{\Omega_1^\varepsilon} \Big[\big|\hat u^\varepsilon - \varepsilon \sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j\big|^2 + \varepsilon^2 \big|\nabla \hat u^\varepsilon - \sum_{j=1}^n Q_{Y_1}^\varepsilon(\partial_{x_j} u)\nabla_y \omega_u^j\big|^2$$

$$+ \varepsilon^2 C \sum_{j=1}^n \Big(\big|Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j\big|^2 + \varepsilon^2 \big|\nabla Q_{Y_1}^\varepsilon(\partial_{x_j} u)\omega_u^j\big|^2 + \big|Q_{Y_1}^\varepsilon(\partial_{x_j} u)\nabla_y \omega_u^j\big|^2\Big)\Big] dxd\tau$$

$$\leq C(\varepsilon + \varepsilon^2)\big(\|u\|^2_{L^2(0,T;H^2(\Omega))} + \|u\|^2_{H^1((0,T)\times\Omega)} + \|v\|^2_{L^2(0,T;H^2(\Omega))} + \|v\|^2_{H^1((0,T)\times\Omega)}$$

$$+ \|w\|^2_{L^2(0,T;H^2(\Omega))} + \|w\|^2_{H^1((0,T)\times\Omega)} + \|r\|^2_{L^\infty((0,T)\times\Omega\times\Gamma_1)}\big).$$

4.1. **Estimates of the error terms.** Now, we proceed to estimating the error terms R_u^1, R_u^2, and R_u^3. Using the definition of ρ^ε, the extension properties of $\tilde\phi_1^\varepsilon$, Theorem 3.5, and the estimates (21) we obtain

$$\int_{\Omega\times Y_1} \Big| D_u(y)(\rho^\varepsilon - 1)\big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j} \nabla_y \omega_u^j\big)\big(\nabla \tilde\phi_1^\varepsilon + \nabla \hat\psi_1^\varepsilon\big)\Big| dydx$$

$$\leq C \|\nabla u\|_{L^2(\hat\Omega_{1,\rho,in})} \big(1 + \sum_{j=1}^n \|\nabla_y \omega_u^j\|_{L^2(Y_1)}\big)\big(\|\nabla \tilde\phi_1^\varepsilon\|_{L^2(\Omega)} + \|\nabla \hat\psi_1^\varepsilon\|_{L^2(\Omega\times Y_1)}\big)$$

$$\leq C\varepsilon^{1/2} \|u\|_{H^2(\Omega)} \big(1 + \sum_{j=1}^n \|\nabla_y \omega_u^j\|_{L^2(Y_1)}\big) \|\nabla \phi_1\|_{L^2(\Omega_1^\varepsilon)}.$$

The Theorem 3.5 and the estimates (20) and (21) imply

$$\int_0^\tau \int_{\Omega\times Y_1} \rho^\varepsilon D_u(y)\Big(\nabla u + \sum_{j=1}^n \partial_{x_j} u \nabla_y \omega_u^j\Big)\Big(\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1) - \nabla\phi_1^\varepsilon - \nabla_y \hat{\psi}_1^\varepsilon\Big) dydxdt$$
$$\leq C(\varepsilon^{1/2}+\varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\big(1+\sum_{j=1}^n \|\nabla_y\omega_u^j\|_{L^2((0,T)\times Y_1)}\big)\|\nabla\phi_1\|_{L^2((0,T)\times\Omega_1^\varepsilon)}.$$

We notice $\mathcal{M}_{Y_1}^\varepsilon(\tilde{\phi}_1^\varepsilon) = \mathcal{M}_{Y_1}^\varepsilon(\phi_1)$ and using estimates (20) and (21), Lemma 3.4, the fact that $\tilde{\phi}_1^\varepsilon$ is an extension of ϕ_1 from Ω_1^ε into Ω and $\phi_1 = \tilde{\phi}_1$ a.e in $(0,T)\times\Omega_1^\varepsilon$, implies

$$\int_0^\tau \int_{\Omega\times Y_1} \rho^\varepsilon D_u\big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\big)\nabla\big(Q_{Y_i}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon\big) dydxdt \leq$$
$$\|\nabla\big(\rho^\varepsilon D_u\big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\big)\big)\|_{L^2((0,\tau)\times\Omega\times Y_1)}\|Q_{Y_i}^\varepsilon(\tilde{\phi}_1^\varepsilon) - \tilde{\phi}_1^\varepsilon\|_{L^2((0,\tau)\times\Omega)} \leq$$
$$C\varepsilon\big(\varepsilon^{-1}\|\nabla u\|_{L^2((0,T)\times\hat{\Omega}_{1,\rho,in})} + \|\nabla^2 u\|_{L^2}\big)\big(1+\sum_{j=1}^n\|\nabla\omega_u^j\|_{L^2(Y_1)}\big)\|\nabla\tilde{\phi}_1^\varepsilon\|_{L^2((0,\tau)\times\Omega)}$$
$$\leq C(\varepsilon^{1/2}+\varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\big(1+\sum_{j=1}^n\|\nabla\omega_u^j\|_{L^2(Y_1)}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Applying the estimates in Lemma 3.4, yields

$$\int_0^\tau \int_{\Omega\times Y_1} \Big(\partial_t\big(u - \mathcal{T}_{Y_1}^\varepsilon(u)\big)\mathcal{M}_{Y_1}^\varepsilon(\phi_1) + \partial_t u\big(\tilde{\phi}_1^\varepsilon - \mathcal{M}_{Y_1}^\varepsilon(\phi_1)\big)\Big) dydxdt$$
$$\leq C\varepsilon\big(\|\partial_t\nabla u\|_{L^2((0,T)\times\Omega)}\|\phi_1\|_{L^2(\Omega_1^\varepsilon)} + \|\partial_t u\|_{L^2(\Omega)}\|\nabla\phi_1\|_{L^2(\Omega_1^\varepsilon)}\big).$$

Due to Lipschitz continuity of f, we can estimate

$$\int_0^\tau \int_{\Omega\times Y_1} \Big((f(u,v) - \mathcal{T}_{Y_1}^\varepsilon(f(u,v)))\mathcal{M}_{Y_1}^\varepsilon(\phi_1) + f(u,v)(\tilde{\phi}_1^\varepsilon - \mathcal{M}_{Y_1}^\varepsilon(\phi_1))\Big) dydxdt$$
$$\leq \varepsilon C(\|\nabla u\|_{L^2((0,T)\times\Omega)} + \|\nabla v\|_{L^2((0,T)\times\Omega)})\|\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}$$
$$+\varepsilon C(1+\|u\|_{L^2((0,T)\times\Omega)} + \|v\|_{L^2((0,T)\times\Omega)})\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

For the boundary integral we have

$$\int_0^\tau \int_{\Omega\times\Gamma_1} \eta(u,r)(\mathcal{T}_{\Gamma_1}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon)) d\gamma dxdt \leq \|\eta(u,r)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)} \times$$
$$\big(\|\mathcal{T}_{\Gamma_1}^\varepsilon(\phi_1) - \mathcal{M}_{Y_1}^\varepsilon(\phi_1)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{M}_{Y_1}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big)$$
$$\leq C\big(1+\|u\|_{L^2((0,T)\times\Omega)} + \|r\|_{L^\infty((0,T)\times\Omega\times\Gamma_1)}\big) \times$$
$$\big(\|\mathcal{T}_{Y_1}^\varepsilon(\phi_1) - \mathcal{M}_{Y_1}^\varepsilon(\phi_1)\|_{L^2((0,\tau)\times\Omega;H^1(Y_1))} + \|\mathcal{M}_{Y_1}^\varepsilon(\phi_1) - \tilde{\phi}_1^\varepsilon)\|_{L^2((0,\tau)\times\Omega)}\big)$$
$$\leq \varepsilon C\big(1+\|u\|_{L^2((0,T)\times\Omega)} + \|r\|_{L^\infty((0,T)\times\Omega\times\Gamma_1)}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Thus, collecting all estimates from above we obtain the estimate for R^1_u:

$$\begin{aligned}|R^1_u| \ \leq \ & C(\varepsilon^{1/2}+\varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\big(1+\sum_{j=1}^n \|\nabla\omega^j_u\|_{L^2(Y_1)}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_i)} \\ & +C\varepsilon\big(\|u\|_{H^1((0,T)\times\Omega)}+\|v\|_{L^2(0,T;H^1(\Omega))}\big)\|\phi_1\|_{L^2(0,\tau;H^1(\Omega^\varepsilon_1)}.\end{aligned}$$

Using the estimates (21) implies

$$\begin{aligned}&\int_0^\tau\int_{\Omega^\varepsilon_1}(1-\rho^\varepsilon)D^\varepsilon_u\sum_{j=1}^n\mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}u)\nabla_y\omega^j_u\Big(\frac{x}{\varepsilon}\Big)\nabla\phi_1 dxdt \\ &\leq \sum_{j=1}^n\|\mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}u)\|_{L^2((0,\tau)\times\hat\Omega^\varepsilon_{1,\rho,in})}\|\nabla_y\omega^j_u\Big(\frac{x}{\varepsilon}\Big)\|_{L^2(\hat\Omega^\varepsilon_{1,\rho,in})}\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)} \\ &\leq \varepsilon C\sum_{j=1}^n\|u\|_{L^2(0,T;H^2(\Omega^\varepsilon_1))}\|\nabla_y\omega^j_u\|_{L^2(Y_1)}\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)}.\end{aligned}$$

Thus, the last estimate and applying the estimates (18) and (21) yields

$$\begin{aligned}&|R^2_u|\leq\|\nabla u\|_{L^2((0,\tau)\times\hat\Omega^\varepsilon_{int,\rho,1})}(1+\|\nabla_y\omega_u\|_{L^2(Y_1)^{n\times n}})\|\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)\|_{L^2((0,\tau)\times\Omega\times Y_1)} \\ &+C\varepsilon\|u\|_{L^2(0,\tau;H^2(\Omega))}(1+\|\nabla_y\omega_u\|_{L^2(Y_1)^{n\times n}})\|\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)\|_{L^2((0,\tau)\times\Omega\times Y_1)} \\ &\leq(\varepsilon^{1/2}+\varepsilon)C\|u\|_{L^2(0,T;H^2(\Omega))}(1+\|\nabla_y\omega_u\|_{L^2(Y_1)^{n\times n}})\|\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)}.\end{aligned}$$

Due to estimates in (21) and in Lemma 3.4 we obtain also

$$\begin{aligned}|R^3_u| \ \leq \ & \varepsilon C\Big(\|\partial_t u\|_{L^2((0,T)\times\Omega^\varepsilon_1)}+\|f\|_{L^2((0,T)\times\Omega^\varepsilon_1)}+\|u\|_{L^2(0,T;H^2(\Omega^\varepsilon_1))}\|\nabla_y\omega_u\|_{L^2(Y_1)^{n\times n}} \\ & +\|\nabla^2 u\|_{L^2((0,T)\times\Omega^\varepsilon_1)}+\|\nabla^2 u\|_{L^2((0,T)\times\Omega^\varepsilon_1)}\|\nabla_y\omega_u\|_{L^2(Y_1)^{n\times n}}\Big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega^\varepsilon_1)}.\end{aligned}$$

In the similar way we show the estimates for the error terms in the equations for v and w:

$$\begin{aligned}|R_v| \ \leq \ & C\varepsilon^{\frac{1}{2}}\Big(1+\|v\|_{L^2(0,T;H^2(\Omega))}+\|v\|_{H^1((0,T)\times\Omega)}+\|u\|_{L^2(0,T;H^1(\Omega))} \\ & +\|w\|_{L^2(0,T;H^1(\Omega))}\Big)\|\phi_2\|_{L^2(0,\tau;H^1(\Omega^\varepsilon_1))}, \\ |R_w| \ \leq \ & C\varepsilon^{\frac{1}{2}}\big(1+\|w\|_{L^2(0,T;H^2(\Omega))}+\|w\|_{H^1((0,T)\times\Omega)}+\|v\|_{L^2(0,T;H^1(\Omega))}\big)\|\phi_3\|_{L^2(0,\tau;H^1(\Omega^\varepsilon_2))}.\end{aligned}$$

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5. Appendix

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $z \in H^1(\Omega) \cap L^\infty(\Omega)$, then $z \in L^\infty(\partial\Omega)$.*

Proof. Let $z \in H^1(\Omega) \cap L^\infty(\Omega)$. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, we consider a sequence of smooth functions $\{f_n\} \subset C^\infty(\overline{\Omega})$, such that $f_n \to z$ in $H^1(\Omega)$ and $\|f_n\|_{L^\infty(\Omega)} \leq \|z\|_{L^\infty(\Omega)}$. Applying the trace theorem, see [9], we obtain $f_n \to z$ in $L^2(\partial\Omega)$. Thus, there exists a subsequence $\{f_{ni}\} \subset \{f_n\}$ converging pointwise, i.e., $f_{ni}(x) \to z(x)$ a.e. $x \in \partial\Omega$, and, due to $\|f_{ni}(x)\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$, follows that $\|z\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$ a.e. $x \in \partial\Omega$. ◻

Lemma 5.2. [4, 5] *1. For $w \in L^p(\Omega_i^\varepsilon)$, $p \in [1,\infty)$, we have*

$$||\mathcal{T}_{Y_i}^\varepsilon w||_{L^p(\Omega\times Y_i)} = |Y|^{1/p}||w||_{L^2(\tilde{\Omega}_{i,int}^\varepsilon)} \leq |Y|^{1/p}||w||_{L^2(\Omega_i^\varepsilon)}.$$

2. For $u \in L^p(\Gamma_i^\varepsilon)$, $p \in [1,\infty)$, we have

$$||\mathcal{T}_{\Gamma_i}^\varepsilon u||_{L^p(\Omega\times\Gamma_i)} = \varepsilon^{1/p}|Y|^{1/p}||u||_{L^2(\tilde{\Gamma}_{i,int}^\varepsilon)} \leq \varepsilon^{1/p}|Y|^{1/p}||u||_{L^2(\Gamma_i^\varepsilon)}.$$

3. If $w \in L^p(\Omega)$, $p \in [1,\infty)$ then $\mathcal{T}_{Y_i}^\varepsilon w \to w$ strongly in $L^p(\Omega \times Y_i)$ as $\varepsilon \to 0$.
4. For $w \in W^{1,p}(\Omega_i^\varepsilon)$, $1 < p < +\infty$,

$$\|\mathcal{T}_{\Gamma_i}^\varepsilon w\|_{L^p(\Omega\times\Gamma_i)} \leq C\big(\|w\|_{L^p(\Omega_i^\varepsilon)} + \varepsilon\|\nabla w\|_{L^p(\Omega_i^\varepsilon)^n}\big).$$

5. For $w \in W^{1,p}(\Omega_i^\varepsilon)$ holds $\mathcal{T}_{Y_i}^\varepsilon(w) \in L^p(\Omega, W^{1,p}(Y_i))$ and $\nabla_y \mathcal{T}_{Y_i}^\varepsilon(w) = \varepsilon\mathcal{T}_{Y_i}^\varepsilon(\nabla w)$.
6. Let $v \in L^p_{per}(Y_i)$ and $v^\varepsilon(x) = v(\frac{x}{\varepsilon})$, then $\mathcal{T}_{Y_i}^\varepsilon(v^\varepsilon)(x,y) = v(y)$.
7. For $v, w \in L^p(\Omega_i^\varepsilon)$ and $\phi, \psi \in L^p(\Gamma_i^\varepsilon)$ holds

$$\mathcal{T}_{Y_i}^\varepsilon(v\, w) = \mathcal{T}_{Y_i}^\varepsilon(v)\mathcal{T}_{Y_i}^\varepsilon(w) \text{ and } \mathcal{T}_{\Gamma_i}^\varepsilon(\phi\,\psi) = \mathcal{T}_{\Gamma_i}^\varepsilon(\phi)\mathcal{T}_{\Gamma_i}^\varepsilon(\psi).$$

Theorem 5.3. [7, 8] *Let $p \in (1,\infty)$ and $i = 1, 2$.*
1. For $\{\phi_\varepsilon\} \subset W^{1,p}(\Omega_i^\varepsilon)$ satisfies $\|\phi_\varepsilon\|_{W^{1,p}(\Omega_i^\varepsilon)} \leq C$, there exists a subsequence of $\{\phi^\varepsilon\}$ (still denoted by ϕ_ε), and $\phi \in W^{1,p}(\Omega)$, $\hat{\phi} \in L^p(\Omega; W^{1,p}_{per}(Y_i))$, such that

$$\begin{aligned} \mathcal{T}_{Y_1}^\varepsilon \phi_\varepsilon \to \phi \quad &\text{strongly in} \quad L^p_{loc}(\Omega; W^{1,p}_{per}(Y_i)),\\ \mathcal{T}_{Y_1}^\varepsilon \phi_\varepsilon \rightharpoonup \phi \quad &\text{weakly in} \quad L^p(\Omega; W^{1,p}_{per}(Y_i)),\\ \mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_\varepsilon) \rightharpoonup \nabla\phi + \nabla_y\hat{\phi} \quad &\text{weakly in} \quad L^p(\Omega \times Y_i). \end{aligned}$$

2. For $\{\phi^\varepsilon\} \subset W_0^{1,p}(\Omega_i^\varepsilon)$ such that $\|\phi^\varepsilon\|_{W_0^{1,p}(\Omega_i^\varepsilon)} \leq C$ there exists a subsequence of $\{\phi^\varepsilon\}$ (still denoted by ϕ_ε) and $\phi \in W_0^{1,p}(\Omega)$, $\tilde{\phi} \in L^p(\Omega; W^{1,p}_{per}(Y_i))$ such that

$$\begin{aligned} \mathcal{T}_{Y_i}^\varepsilon \phi^\varepsilon \to \phi \quad &\text{strongly in} \quad L^p(\Omega; W^{1,p}(Y_i)),\\ \mathcal{T}_{Y_i}^\varepsilon(\nabla\phi^\varepsilon) \rightharpoonup \nabla\phi + \nabla_y\tilde{\phi} \quad &\text{weakly in} \quad L^p(\Omega \times Y_i). \end{aligned}$$

3. For $\{\psi^\varepsilon\} \subset L^p(\Gamma_i^\varepsilon)$ such that $\varepsilon^{1/p}\|\psi^\varepsilon\|_{L^p(\Gamma_i^\varepsilon)} \leq C$ there exists a subsequence of $\{\psi^\varepsilon\}$ and $\psi \in L^p(\Omega \times \Gamma_i)$ such that

$$\mathcal{T}_{\Gamma_i}^\varepsilon(\psi^\varepsilon) \rightharpoonup \psi \quad \text{weakly in} \quad L^p(\Omega \times \Gamma_i).$$

Proposition 5.4. [7, 8] *1. The operator $\mathcal{U}_{Y_i}^\varepsilon$ is formal adjoint and left inverse of $\mathcal{T}_{Y_i}^\varepsilon$, i.e for $\phi \in L^p(\Omega_i^\varepsilon)$*

$$\mathcal{U}_{Y_i}^\varepsilon(\mathcal{T}_{Y_i}^\varepsilon(\phi))(x) = \begin{cases} \phi(x) & a.e. \text{ for } \in \tilde{\Omega}_{i,int}^\varepsilon, \\ 0 & a.e. \text{ for } \in \Omega_i^\varepsilon \setminus \tilde{\Omega}_{i,int}^\varepsilon. \end{cases}$$

2. For $\phi \in L^p(\Omega \times Y_i)$ holds $\|\mathcal{U}_{Y_i}^\varepsilon(\phi)\|_{L^p(\Omega_i^\varepsilon)} \leq |Y|^{-1/p}\|\phi\|_{L^p(\Omega\times Y_i)}$.

Theorem 5.5. [12] *For any $\phi \in H^1(\Omega)$, there exists $\hat{\phi}_\varepsilon \in H^1_{per}(Y; L^2(\Omega))$:*

$$\begin{aligned} \|\hat{\phi}_\varepsilon\|_{H^1(Y;L^2(\Omega))} &\leq C\|\nabla\phi\|_{L^2(\Omega)^n}, \\ \|\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla\phi - \nabla_y\hat{\phi}_\varepsilon\|_{L^2(Y;H^{-1}(\Omega))^n} &\leq C\varepsilon\|\nabla\phi\|_{L^2(\Omega)^n} \end{aligned}$$

Theorem 5.6. [13] *For any $\phi \in H^1(\Omega)$ there exists $\hat{\phi}_\varepsilon \in H^1_{per}(Y; L^2(\Omega))$:*

$$\begin{aligned} \|\hat{\phi}_\varepsilon\|_{H^1(Y;L^2(\Omega))} &\leq C\|\nabla\phi\|_{L^2(\Omega)^n}, \\ \|\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla\phi - \nabla_y\hat{\phi}_\varepsilon\|_{L^2(Y;(H^1(\Omega))')^n} &\leq C\varepsilon\|\nabla\phi\|_{L^2(\Omega)^n} + C\sqrt{\varepsilon}\|\nabla\phi\|_{L^2(\hat{\Omega}^{\varepsilon,3})^n}, \end{aligned}$$

where $\hat{\Omega}^{\varepsilon,l} = \{x \in \mathbb{R}^n : dist(x, \partial\Omega) < l\sqrt{n}\varepsilon\}$.

The proofs of Theorems 5.5, 5.6 and 3.5 are based on the following fundamental results:

Theorem 5.7. [12] *For any $\phi \in H^1(Y_i, X)$ and X separable Hilbert space, there exists a unique $\hat{\phi} \in H^1_{per}(Y_i, X)$, $i = 1, 2$, such that $\phi - \hat{\phi} \in (H^1_{per}(Y_i, X))^\perp$ and*

$$\|\hat{\phi}\|_{H^1(Y_i,X)} \leq \|\phi\|_{H^1(Y_i,X)}, \qquad \|\phi - \hat{\phi}\|_{H^1(Y_i,X)} \leq C\sum_{j=1}^n \|\phi|_{e_j+Y_i^j} - \phi|_{Y_i^j}\|_{H^{1/2}(Y_i^j,X)}.$$

Theorem 5.8. [12] *For any $\Phi \in W^{1,p}(Y_i)$ and for any k, $k \in \{1, \dots, n\}$, there exists $\hat{\Phi}_k \in W_k = \{\phi \in W^{1,p}(Y_i), \phi(\cdot) = \phi(\cdot + e_j), j \in \{1, \dots, k\}\}$, such that*

$$\|\Phi - \hat{\Phi}_k\|_{W^{1,p}(Y_i)} \leq C\sum_{j=1}^k \|\Phi|_{e_j+Y_i^j} - \Phi|_{Y_i^j}\|_{W^{1-1/p}(Y_i^j)}, i = 1, 2,$$

where the constant C is independent on n, $Y_i^j = \{y \in \overline{Y}_i, \, y_j = 0\}$, $j \in \{1, \dots, n\}$.

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