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INDEX-AWARE MODEL ORDER REDUCTION FOR INDEX-2 DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract. A model order reduction (MOR) method for index-2 differential-algebraic equations (DAEs) is introduced, which is based on the intrinsic differential equations contained in the starting system and on the remaining algebraic constraints. This extends the method introduced in a previous paper for index-1 DAEs. This procedure is implemented numerically and the results show numerical evidence of its robustness over the traditional methods.

Key words. differential algebraic equations, tractability index, model order reduction, modified decomposition of DAEs

AMS subject classifications. 78M34, 65L80

1. Introduction.

Consider a linear time invariant differential-algebraic equation (DAE) in descriptor form:

$$Ex' = Ax + Bu, \quad x(0) = x_0, \tag{1.1a}$$

$$y = C^T x, \tag{1.1b}$$

with matrices $E \in \mathbb{R}^{n,n}$, $A \in \mathbb{R}^{n,n}$, $C \in \mathbb{R}^{n,\ell}$, $B \in \mathbb{R}^{n,m}$, state vector $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^\ell$ and initial value $x_0 \in \mathbb{R}^n$. We assume that the matrix pencil $(E, A) := \{A - \lambda E \,|\, \lambda \in \mathbb{C}\}$ is regular, that is, $\det(A - \lambda E) \neq 0$ for at least one value of $\lambda \in \mathbb{C}$. Moreover, the matrix E is assumed to be singular, so that (1.1a) is not an ordinary differential equation (ODE). We also assume that x_0 is a consistent initial value for the DAE and u is smooth enough.

There exist many developed MOR methods for reducing ODEs, i.e., when E is non-singular, but little has yet been done to reduce DAEs. Some of these methods can be found in [18, 2]. Most recent attempts to reduce DAEs are given in [4, 14, 16, 7]. Usually, the balanced truncation method is used. This method involves solving a Lyapunov equation, which can be computationally very expensive. In [7], the authors reduce index-2 systems from electric power grids by first converting the DAE into an ODE. In [14] the reduction approach involves solving a projected Lyapunov equation. In [16] a passivity preserving approach for circuits has been developed based on certain projected Lyapunov equations. In practice balanced truncation methods cannot be used on very large systems. Other attempts where made by using the Krylov-based subspace methods for descriptor systems such as PRIMA, SPRIM [15, 8], but these methods should not be used on DAEs with index greater than one, as discussed in Section 3.1. Furthermore, these methods do not always preserve the index of a DAE, which can lead to loss of relevant information of the original DAE.

In this paper we propose a new strategy of reducing index-2 systems. This strategy is based on the ideas of März [12] of splitting a DAE into its differential and

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algebraic parts (possibly involving differentiations of other algebraic parts), by using appropriate projectors. This approach leads to a decoupled system of dimension $(\mu+1)n$, where μ is the tractability index. The spectrum of the decoupled system consists not only of the spectrum of the matrix pencil (E, A) of the original system but also of additional infinite eigenvalues. This motivated us to do some modifications in the decomposition, using special basis vectors, which leads to a modified decoupled system of dimension n. Moreover, this decoupling preserves the spectrum of the matrix pencil (E, A) of the DAE. We can now apply MOR on both the differential and the algebraic parts. For the differential part, we use one of the existing MOR methods for ODEs, and for the algebraic parts we propose a new method, which is based on the reduction on the algebraic variables induced by the reduction on the differential variables. We call the resulting method the index-aware MOR (IMOR) method. This method leads to sparse reduced-order models and also always preserves the index of the original DAE.

This paper is organized as follows. In Section 2 we discuss the decoupling of index-2 DAEs into differential and algebraic parts. In particular, in Subsection 2.1 we give an overview of März decomposition and its limitations. In Subsection 2.2 we propose a modified decomposition which preserves the size of the original DAE. Section 3 is devoted to model order reduction methods for DAEs. In Subsection 3.1 we illustrate the limitation of using conventional methods for descriptor systems to reduce DAEs, and in Subsection 3.2 we introduce the new IMOR method for reducing index-2 systems, which we call IMOR-2. This method uses the decoupled system derived in Subsection 3 instead of the original DAE in order to obtain the reduced-order model. In Subsection 3.3 we compare the IMOR-2 method with the conventional methods based on Krylov subspaces. In Section 4 we present some numerical examples, divided into simple and industrial examples. The simple examples are used to illustrate the idea of the method, and to show that the splitting of the DAE in differential and algebraic equations is beneficial also for the numerical solution of the system. The industrial examples show the feasibility of the method for real-life applications. The paper is concluded by some final remarks, in Section 5.

2. Decomposition of index-2 systems. In this section, we discuss two ways of decoupling index-2 systems by using projectors. The first decomposition, which we call März decomposition, is achieved via canonical projectors. An extensive discussion of decomposition of DAEs via canonical projectors can be found in [12]. The second decomposition is simply a modification of the former decomposition which is suitable for numerical implementation.

2.1. März decomposition. Let us consider equation (1.1a), given by

$$Ex' = Ax + Bu. \tag{2.1}$$

Following [12, 13], it is possible to introduce the notion of tractability index of system (2.1). Roughly speaking, this index measures how many derivatives of the input u appear in the solution. In fact, if the index is μ, then at most $\mu - 1$ derivatives of u will appear in the solution of (2.1). It is possible to show that the tractability index is equivalent to the Kronecker index of the matrix pencil (E, A) for constant matrices. In [12, 13], a different construction, based on geometrical concepts, making use of appropriate projectors, was proposed. This construction has been summarized in [1]. Here we specialize it briefly for index-2 systems, and we use it to decompose the system (2.1) into a differential part and two algebraic parts.

We assume that the tractability index of (2.1) is 2. We define

$$E_0 := E, \quad A_0 := A.$$

Let Q_0 be a projector on the nullspace of E_0, that is,

$$Q_0^2 = Q_0, \quad \operatorname{im} Q_0 = \ker E_0,$$

and let $P_0 = \mathrm{I} - Q_0$. Then we define

$$E_1 := E_0 - A_0 Q_0, \quad A_1 := A_0 P_0.$$

Since we consider an index-2 system, E_1 is singular. Let Q_1 be a projector on the nullspace of E_1, that is,

$$Q_1^2 = Q_1, \quad \operatorname{im} Q_1 = \ker E_1.$$

We can choose Q_1 so that it satisfies the additional condition

$$Q_1 Q_0 = 0. \tag{2.2}$$

In fact, if $\tilde{Q}_1$ is any projector onto $\ker E_1$, then we can define

$$Q_1 = -\tilde{Q}_1 (E_1 - A_1 \tilde{Q}_1)^{-1} A_1.$$

It is possible to see that Q_1 is a new projector onto $\ker E_1$ which satisfies (2.2). Let $P_1 = \mathrm{I} - Q_1$, and introduce the matrices

$$E_2 = E_1 - A_1 Q_1, \quad A_2 = A_1 P_1.$$

The index-2 condition is equivalent to assuming E_2 non singular. It is possible to prove that the following equivalent form of equation (2.1) holds [12]:

$$E_2 (P_1 P_0 x' + Q_1 x + Q_0 x) = A_2 x + B u. \tag{2.3}$$

Since E_2 is invertible, we can left-multiply (2.3) by E_2^{-1}, and then use the projectors $P_0 P_1$, $P_0 Q_1$ and $Q_0 P_1$. Then we can introduce the variables

$$x_P = P_0 P_1 x, \quad x_{Q,1} = P_0 Q_1 x, \quad x_{Q,0} = Q_0 x,$$

which satisfy the projected equations:

$$x_P' = P_0 P_1 E_2^{-1} (A_2 x_P + B u), \tag{2.4a}$$
$$x_{Q,1} = P_0 Q_1 E_2^{-1} (A_2 x_P + B u), \tag{2.4b}$$
$$x_{Q,0} = Q_0 P_1 E_2^{-1} (A_2 x_P + B u) + Q_0 Q_1 x_{Q,1}'. \tag{2.4c}$$

Equations (2.4a)-(2.4b) can be written as

$$x_P' = A_P x_P + B_P u, \tag{2.5a}$$
$$x_{Q,1} = A_{Q,1} x_P + B_{Q,1} u, \tag{2.5b}$$
$$x_{Q,0} = A_{Q,0} x_P + B_{Q,0} u + A_{Q,01} x_{Q,1}', \tag{2.5c}$$

with

$$\begin{aligned} A_P &:= P_0P_1E_2^{-1}A_2, \quad B_P := P_0P_1E_2^{-1}B, \\ A_{Q,1} &:= P_0Q_1E_2^{-1}A_2, \quad B_{Q,1} := P_0Q_1E_2^{-1}B, \\ A_{Q,0} &:= Q_0P_1E_2^{-1}A_2, \quad B_{Q,0} := Q_0P_1E_2^{-1}B, \quad A_{Q,01} := Q_0Q_1. \end{aligned}$$

It is possible to prove the following decomposition of the identity:

$$\mathrm{I}_n = P_0P_1 + P_0Q_1 + Q_0. \tag{2.6}$$

It is simple to verify that the three terms on the right-hand side of (2.6) are mutually orthogonal projectors, provided $Q_1Q_0 = 0$. Then the desired solution x of an index-2 system (2.1) can the be computed after solving the ODE (2.5a), by using the formula,

$$x = x_P + x_{Q,1} + x_{Q,0}.$$

This shows that the initial data x_0 must be consistent with the equations. In fact, if we decompose

$$x_0 = x_{P,0} + x_{Q,1,0} + x_{Q,0,0} := P_0P_1x_0 + P_0Q_1x_0 + Q_0x_0,$$

then the component $x_{P,0}$ determines uniquely the solution x_P of the ODE (2.5a), by means of the initial condition

$$x_P(0) = x_{P,0}, \tag{2.7}$$

thus determining the other components $x_{Q,1}$, $x_{Q,0}$ by means of the algebraic constraints (2.5b) and (2.5c). It follows that for generic initial data x_0 there might be an initial boundary layer, if the constraints (2.5b), (2.5c) are not satisfied initially, for $t = 0$.

The previous decomposition is still valid if $P_0P_1 = 0$, equality which is compatible with index-2 conditions. In this case the components x_P vanishes, and the projected equations (2.5) reduce to

$$x_{Q,1} = B_{Q,1}u, \tag{2.8a}$$
$$x_{Q,0} = B_{Q,0}u + A_{Q,01}x'_{Q,1}, \tag{2.8b}$$

which is a system without differential equations. Notice that $P_0P_1 = 0$ implies $P_0 = P_0Q_1$, so that $x_{Q,1} = P_0Q_1x = P_0x$. The solution of the original system can be recovered immediately from (2.8),

$$x = x_{Q,1} + x_{Q,0} = B_{Q,1}u + B_{Q,0}u + A_{Q,01}B_{Q,1}u'. \tag{2.9}$$

This special case corresponds to a matrix pencil (E, A) with no finite eigenvalues. This statement will become clear at the end of the following section.

2.2. Modified März decomposition. In the previous section, we have seen that we can write the index-2 system (2.1) in the projected form (2.5), where x_P, $x_{Q,1}$, $x_{Q,0} \in \mathbb{R}^n$. The projected system (2.5) is a decoupled system of total dimension $3n$, while the original index-2 system (2.1) has dimension n. This may be expensive in terms of storage and memory consumption, especially when solving systems in tens of thousands of degrees of freedom, thus making it even more difficult to apply MOR

methods on such a system. In general, decoupling an index-μ system by using the projector approach leads to a decoupled system of dimension $n(1+\mu)$. In this section we come up with a strategy in order to eliminate this limitation.

We show how to represent system (2.5) in a simpler way by constructing new basis column matrices from the projectors P_0P_1, P_0Q_1 and Q_0. This procedure extends the construction presented in [1] for index-1 DAEs. We start from the projectors Q_0, P_0, Q_1, P_1 constructed in the previous section, with $Q_1Q_0 = 0$.

Let $k_0 = \dim(\ker E_0)$, $n_0 = n - k_0$, and let us consider an orthonormal basis matrix $(p_0, q_0) = (p_{0,1}, \cdots, p_{0,n_0}, q_{0,1}, \cdots, q_{0,k_0}) \in \mathbb{R}^n$ which contains k_0 independent columns $q_{0,i}$ of Q_0, which span $\operatorname{im} Q_0 = \ker E_0$, and n_0 independent columns $p_{0,i}$ of P_0, which span $\operatorname{im} P_0 = \ker Q_0$. Since (p_0, q_0) is a basis matrix, it is invertible, and let $(p_0^*, q_0^*)^T$ be its inverse, with $q_0^* \in \mathbb{R}^{n,k_0}$ and $p_0^* \in \mathbb{R}^{n,n_0}$. Then, $(p_0^*, q_0^*)^T(p_0, q_0) = \mathrm{I}_n = (p_0, q_0)(p_0^*, q_0^*)^T$, that is,

$$q_0^{*T}q_0 = \mathrm{I}_{k_0}, \quad q_0^{*T}p_0 = 0, \quad p_0^{*T}q_0 = 0, \quad p_0^{*T}p_0 = \mathrm{I}_{n_0}, \tag{2.10}$$

$$q_0q_0^{*T} + p_0p_0^{*T} = \mathrm{I}_n. \tag{2.11}$$

The previous relations imply that we can represent the projectors Q_0 and P_0 as

$$Q_0 = q_0q_0^{*T}, \quad P_0 = p_0p_0^{*T}. \tag{2.12}$$

Note that, by construction, we have

$$Q_0q_0 = q_0, \quad Q_0p_0 = 0, \quad P_0q_0 = 0, \quad P_0p_0 = p_0.$$

We are now going to find a simple representation of the projectors P_0P_1 and P_0Q_1, which appear in (2.6) and are used for the decomposition of the variable x. Recalling the identities (2.10), after multiplying (2.6) by p_0^{*T} from the left, and by p_0 from the right, we obtain

$$\mathrm{I}_{n_0} = P_{01} + Q_{01} := p_0^{*T}P_1p_0 + p_0^{*T}Q_1p_0.$$

It is immediate to see that P_{01} and Q_{01} are mutually orthogonal projectors, acting on $\mathbb{R}^{n_0}$. This leads to the following proposition.

PROPOSITION 2.1. *Let $P_{01} = p_0^{*T}P_1p_0$, $Q_{01} = p_0^{*T}Q_1p_0$, then $P_{01}, Q_{01} \in \mathbb{R}^{n_0,n_0}$ are projectors in $\mathbb{R}^{n_0}$ provided the constraint condition $Q_1Q_0 = 0$ holds. Moreover they are mutually orthogonal.*

Let $k_1 = \dim(\operatorname{im} Q_{01})$, and $n_{01} = n_0 - k_1$. Here we need to distinguish two cases: $n_{01} > 0$, or $n_{01} = 0$. The first case corresponds to the matrix pencil (E, A) having at least one finite eigenvalue, while the second case corresponds to (E, A) with no finite eigenvalues.

2.2.1. Matrix pencil (E, A) with finite eigenvalues ($n_{01} > 0$). If $n_{01} > 0$, we can proceed for Q_{01} and P_{01} as we have done for Q_0 and P_0. Let us consider a basis matrix $(p_{01}, q_{01}) \in \mathbb{R}^{n_0}$ made of n_{01} independent columns of projection matrix P_{01} and k_1 independent columns of the complementary projection matrix Q_{01}. We denote by $(p_{01}^*, q_{01}^*)^T$ the inverse of (p_{01}, q_{01}), such that

$$\begin{aligned} &p_{01}^{*T}p_{01} = \mathrm{I}_{n_{01}}, \quad p_{01}^{*T}q_{01} = 0, \quad q_{01}^{*T}p_{01} = 0, \quad q_{01}^{*T}q_{01} = \mathrm{I}_{k_1}, \\ &p_{01}p_{01}^{*T} + q_{01}q_{01}^{*T} = \mathrm{I}_{n_0}. \end{aligned} \tag{2.13}$$

Then, we can represent P_{01}, Q_{01} as

$$P_{01} = p_{01}p_{01}^{*T}, \quad Q_{01} = q_{01}q_{01}^{*T},$$

and we have

$$P_{01}p_{01} = p_{01}, \quad P_{01}q_{01} = 0, \quad Q_{01}p_{01} = 0, \quad Q_{01}q_{01} = q_{01}.$$

PROPOSITION 2.2. *Let $n_{01} > 0$, $P_{01} = p_{01}p_{01}^{*T}$, $Q_{01} = q_{01}q_{01}^{*T}$, and let the constraint $Q_1Q_0 = 0$ holds. Then the projectors P_0P_1, P_0Q_1 can be decomposed as follows:*

$$P_0P_1 = p_0p_{01}p_{01}^{*T}p_0^{*T}, \quad P_0Q_1 = p_0q_{01}q_{01}^{*T}p_0^{*T}. \tag{2.14}$$

Proof. Since $P_{01} = p_0^{*T}P_1p_0$, $Q_{01} \equiv p_0^{*T}Q_1p_0$, we have:

$$p_{01}p_{01}^{*T} = p_0^{*T}P_1p_0, \quad q_{01}q_{01}^{*T} = p_0^{*T}Q_1p_0.$$

Multiplying the above identities by p_0 from the left, and by p_0^{*T} from the right, and recalling that $p_0p_0^{*T} = P_0$, we obtain

$$p_0p_{01}p_{01}^{*T}p_0^{*T} = P_0P_1P_0, \quad p_0q_{01}q_{01}^{*T}p_0^{*T} = P_0Q_1P_0.$$

Since $Q_1Q_0 = 0$, we have $Q_1P_0 = Q_1(\mathrm{I} - Q_0) = Q_1$, and $P_0P_1P_0 = P_0^2 - P_0Q_1P_0 = P_0 - P_0Q_1 = P_0P_1$, hence the thesis. □

We can now expand x with respect to the basis $(p_0p_{01}, p_0q_{01}, q_0)$, obtaining the decomposition

$$x = p_0p_{01}\xi_p + p_0q_{01}\xi_{q,1} + q_0\xi_{q,0}, \tag{2.15}$$

where $\xi_p \in \mathbb{R}^{n_{01}}$, $\xi_{q,1} \in \mathbb{R}^{k_1}$, $\xi_{q,0} \in \mathbb{R}^{k_0}$, with inversion expressions

$$\xi_p = p_{01}^{*T}p_0^{*T}x, \quad \xi_{q,1} = q_{01}^{*T}p_0^{*T}x, \quad \xi_{q,0} = q_0^{*T}x. \tag{2.16}$$

We note that the variables ξ_p, $\xi_{q,1}$, $\xi_{q,0}$ are related to the variables x_P, $x_{Q,1}$, $x_{Q,0}$ by the relations

$$\begin{aligned} &x_P = p_0p_{01}\xi_p, \quad x_{Q,1} = p_0q_{01}\xi_{q,1}, \quad x_{Q,0} = q_0\xi_{q,0}, \\ &\xi_p = p_{01}^{*T}p_0^{*T}x_P, \quad \xi_{q,1} = q_{01}^{*T}p_0^{*T}x_{Q,1}, \quad \xi_{q,0} = q_0^{*T}x_{Q,0}. \end{aligned}$$

The projected equations (2.4) can be written as

$$\xi_p' = A_p\xi_p + B_pu, \tag{2.17a}$$
$$\xi_{q,1} = A_{q,1}\xi_p + B_{q,1}u, \tag{2.17b}$$
$$\xi_{q,0} = A_{q,0}\xi_p + B_{q,0}u + A_{q,01}\xi_{q,1}', \tag{2.17c}$$

with

$$\begin{aligned} &A_p := p_{01}^{*T}p_0^{*T}E_2^{-1}A_2p_0p_{01} \in \mathbb{R}^{n_{01},n_{01}}, \quad B_p := p_{01}^{*T}p_0^{*T}E_2^{-1}B \in \mathbb{R}^{n_{01},m}, \\ &A_{q,1} := q_{01}^{*T}p_0^{*T}E_2^{-1}A_2p_0p_{01} \in \mathbb{R}^{k_1,n_{01}}, \quad B_{q,1} := q_{01}^{*T}p_0^{*T}E_2^{-1}B \in \mathbb{R}^{k_1,m}, \\ &A_{q,0} := q_0^{*T}P_1E_2^{-1}A_2p_0p_{01} \in \mathbb{R}^{k_0,n_{01}}, \quad B_{q,0} := q_0^{*T}P_1E_2^{-1}B \in \mathbb{R}^{k_0,m}, \\ &A_{q,01} := q_0^{*T}Q_1p_0q_{01} \in \mathbb{R}^{k_0,k_1}. \end{aligned}$$

We can see that the number of differential equations is equal to n_{01} and $k_1 + k_0$ is the total number of algebraic equations, thus the total system dimension is $n_{01} + k_1 + k_0 = n_0 + k_0 = n$. This is illustrated in Example 1. We note that the rank of E is no longer

equal to the number of differential equations as for the case of index-1 systems, rather it is equal to $n_{01} + k_1 = n_0$. If we apply initial condition $\xi_p(0) = p_{01}^{*T} p_0^{*T} x_0$, where x_0 is an initial condition, we can solve the differential part (2.17a), then solve algebraic parts (2.17b) and (2.17c). If the initial data is consistent, we obtain numerically stable solutions, otherwise we see the formation of an initial boundary layer. In general, solving system (2.17) is computationally cheaper than solving system (2.5).

In order to gain some insight on the decomposed system (2.17), we write it in the descriptor form:

$$\tilde{E}\xi' = \tilde{A}\xi + \tilde{B}u, \tag{2.18}$$

where

$$\tilde{E} = \begin{bmatrix} \mathrm{I}_{n_{01}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -A_{q,01} & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_p & 0 & 0 \\ A_{q,1} & -\mathrm{I}_{k_1} & 0 \\ A_{q,0} & 0 & -\mathrm{I}_{k_0} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_p \\ B_{q,1} \\ B_{q,0} \end{bmatrix},$$

and $\xi = \begin{bmatrix} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{bmatrix}$ is the projected state space. By construction, we see that

$$\xi = V^{-1}x, \tag{2.19}$$

where

$$V := \begin{bmatrix} p_0 p_{01} & p_0 q_{01} & q_0 \end{bmatrix} = \begin{bmatrix} p_{01}^{*T} p_0^{*T} \\ q_{01}^{*T} p_0^{*T} \\ q_0^{*T} \end{bmatrix}^{-1},$$

and by comparison with the original system (2.1) we find

$$(\tilde{E}, \tilde{A}) = W(E, A)V, \quad \tilde{B} = WB, \tag{2.20}$$

where $W = M^{-1}V^{-1}E_2^{-1} = (E_2VM)^{-1}$,

$$M = \begin{bmatrix} \mathrm{I}_{n_{01}} & 0 & 0 \\ 0 & \mathrm{I}_{k_1} & 0 \\ 0 & A_{q,01} & I_{k_0} \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{n_{01}} & 0 & 0 \\ 0 & \mathrm{I}_{k_1} & 0 \\ 0 & -A_{q,01} & \mathrm{I}_{k_0} \end{bmatrix}^{-1}.$$

Since the matrices V and W are invertible, it follows that the matrix pencil $(\tilde{E}, \tilde{A})$ is equivalent to (E, A), so they have the same spectrum. It is simple to check that

$$\det(\tilde{A} - \lambda\tilde{E}) = (-1)^{k_0+k_1} \det(A_p - \lambda \mathrm{I}_{n_{01}}). \tag{2.21}$$

This identity shows that the finite eigenvalues of the matrix pencil (E, A) coincide with the (possibly complex) eigenvalues of the matrix A_p of the ordinary differential system (2.17), which are exactly n_{01}, counting their multiplicity. This also shows that the stability of the DAE (2.1) is equivalent to the stability of the ODE system (2.17a).

2.2.2. Matrix pencil (E, A) with no finite eigenvalues ($n_{01} = 0$). If $n_{01} = 0$, then $\operatorname{im} Q_{01} = \mathbb{R}^{n_0}$, thus $P_{01} = 0$. It follows that $P_0 P_1 = 0$, since by definition $0 = p_0 P_{01} p_0^{*T} = P_0 P_1 P_0 = P_0 P_1$. Then, we have also $P_0 Q_1 = P_0$, so the decomposition (2.15) reduces to

$$x = x_{Q,1} + x_{Q,0} = p_0 \xi_{q,1} + q_0 \xi_{q,0}, \tag{2.22}$$

where $\xi_{q,1} \in \mathbb{R}^{k_1}$, $\xi_{q,0} \in \mathbb{R}^{k_0}$, $k_1 = n_0$, and with inversion expressions

$$\xi_{q,1} = p_0^{*T} x_{Q,1}, \quad \xi_{q,0} = q_0^{*T} x_{Q,0}.$$

The projected system (2.8) becomes

$$\xi_{q,1} = B_{q,1} u, \tag{2.23a}$$
$$\xi_{q,0} = B_{q,0} u + A_{q,01} \xi'_{q,1}, \tag{2.23b}$$

with

$$B_{q,1} := p_0^{*T} E_2^{-1} B \in \mathbb{R}^{n_0, m}, \quad B_{q,0} := q_0^{*T} P_1 E_2^{-1} B \in \mathbb{R}^{k_0, m},$$
$$A_{q,01} := q_0^{*T} Q_1 p_0 \in \mathbb{R}^{k_0, n_0}.$$

We can see that this system does not involve differential equations, and the total number of algebraic equations is equal to $k_1 + k_0 = n_0 + k_0 = n$, which is the dimension of the DAE, as illustrated in Example 2. In order to solve (2.23), we first solve the algebraic part (2.23a), then (2.23b), and the solution is:

$$x = p_0 \xi_{q,1} + q_0 \xi_{q,0} = p_0 B_{q,1} u + q_0 B_{q,0} u + q_0 A_{q,01} B_{q,1} u'.$$

As in the previous case, this system (2.23) can also be written in the descriptor form:

$$\tilde{E} \xi' = \tilde{A} \xi + \tilde{B} u, \tag{2.24}$$

where

$$\tilde{E} = \begin{bmatrix} 0 & 0 \\ -A_{q,01} & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -\mathrm{I}_{k_1} & 0 \\ 0 & -\mathrm{I}_{k_0} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_{q,1} \\ B_{q,0} \end{bmatrix},$$

and $\xi = \begin{bmatrix} \xi_{q,1} \\ \xi_{q,0} \end{bmatrix}$ is the projected state space. By construction, we see that

$$\xi = V^{-1} x, \tag{2.25}$$

where

$$V := \begin{bmatrix} p_0 & q_0 \end{bmatrix} = \begin{bmatrix} p_0^{*T} \\ q_0^{*T} \end{bmatrix}^{-1},$$

and by comparison with the original system (2.1) we find

$$(\tilde{E}, \tilde{A}) = W(E, A)V, \quad \tilde{B} = WB, \tag{2.26}$$

with $W = M^{-1} V^{-1} E_2^{-1} = (E_2 V M)^{-1}$,

$$M = \begin{bmatrix} \mathrm{I}_{k_1} & 0 \\ A_{q,01} & \mathrm{I}_{k_0} \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{k_1} & 0 \\ -A_{q,01} & \mathrm{I}_{k_0} \end{bmatrix}^{-1}.$$

The matrix V and W are nonsingular, so the matrix pencil $(\tilde{E}, \tilde{A})$ is equivalent to (E, A), thus they have the same spectrum. It is simple to check that

$$\det(\tilde{A} - \lambda\tilde{E}) = (-1)^{k_0+k_1} \neq 0. \tag{2.27}$$

This identity shows that the matrix pencil (E, A), equivalent to $(\tilde{E}, \tilde{A})$, has no finite eigenvalues. From system (2.18) and (2.24), we observe that this form also reveals the interconnection structure of the DAE (1.1a).

2.2.3. Examples of decomposition of DAEs. In this subsection we illustrate the decomposition of a system of DAEs by means of projectors and by the modified decomposition procedure. Example 1 refers to a system with differential variables, corresponding to a matrix pencil (E, A) with finite eigenvalues. Example 2 illustrates both decompositions for a system with no differential variables, that is, corresponding to a matrix pencil with no finite eigenvalues.

Example 1. *Consider an index 2 system* (2.1)*, with:*

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since $\det(\lambda E - A) = s - 2 \neq 0$*, the matrix pencil* (E, A) *is regular, and the DAE is solvable. We can choose projectors*

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfy the condition $Q_1Q_0 = 0$*, and the corresponding complementary projectors are* $P_i = \mathrm{I} - Q_i$*,* $i = 0, 1$*. The differential and algebraic variables are*

$$x_P = P_0P_1x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}, \; x_{Q,1} = P_0Q_1x = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}, \; x_{Q,0} = Q_0x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}.$$

The last values of the matrix chains are given by:

$$E_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus system is indeed index 2*, since* E_2 *is non-singular and its inverse is given by*

$$E_2^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, we can decoupled this system into the form (2.5)*, given by:*

$$x'_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_P + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{2.28a}$$

$$x_{Q,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_P + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} u, \tag{2.28b}$$

$$x_{Q,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_P + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x'_{Q,1}. \tag{2.28c}$$

We assign zero initial data for the differential variable, $x_P(0) = 0$*, that is,* $x_3(0) = 0$*. Then, using the decoupled system* (2.28)*, we find the initial conditions for the algebraic variables,*

$$x_{Q,1}(0) = \begin{bmatrix} 0 \\ -u(0) \\ 0 \end{bmatrix}, \quad x_{Q,0}(0) = \begin{bmatrix} -u(0) - u'(0) \\ 0 \\ 0 \end{bmatrix}.$$

The initial conditions for x_P*,* $x_{Q,1}$ *and* $x_{Q,0}$ *correspond to the following consistent initial data for the original unknown* x*,*

$$x(0) = \begin{bmatrix} -u'(0) - u(0) \\ -u(0) \\ 0 \end{bmatrix}.$$

We remark that the expression of the consistent initial data cannot be derived by direct inspection of the original system (2.1)*.*

We can see that the decoupled system (2.28) *is of total dimension* 9 *while the original system has dimension* 3*.*

Next we modify the decoupled system (2.28) *into a compact form, as discussed in section 2.2. First, we need to construct a basis matrix* (p_0, q_0) *in* $\mathbb{R}^n$*, and the corresponding inverse* $(p_0^*, q_0^*)^T$*, given in this case by*

$$p_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad q_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_0^{*T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q_0^{*T} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{2.29}$$

so that $Q_0 = q_0 q_0^{*T}$*,* $P_0 = p_0 p_0^{*T}$*. Then, using Proposition 2.1, and the projectors* Q_1*,* P_1*, above introduced, we construct projectors in* $\mathbb{R}^{n_{01}}$ *given by:*

$$P_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we construct a basis matrix (p_{01}, q_{01}) *in* $\mathbb{R}^{n_{01}}$*, and the corresponding inverse* $(p_{01}^*, q_{01}^*)^T$*, given by*

$$p_{01} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad p_{01}^{*T} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad q_{01}^{*T} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{2.30}$$

so that $Q_{01} = q_{01}q_{01}^{*T}$, $P_{01} = p_{01}p_{01}^{*T}$. *The variables* ξ_p, $\xi_{q,1}$, $\xi_{q,0}$, *are given by*

$$\xi_p = p_{01}^{*T}p_0^{*T}x = x_3, \quad \xi_{q,1} = q_{01}^{*T}p_0^{*T}x = x_2, \quad \xi_{q,0} = q_0^{*T}x = x_1.$$

Using (2.29) *and* (2.30), *we can rewrite* (2.28) *into the compact form* (2.17) *given by:*

$$\xi_p' = 2\xi_p + u, \tag{2.31a}$$
$$\xi_{q,1} = -u, \tag{2.31b}$$
$$\xi_{q,0} = -u + \xi_{q,1}'. \tag{2.31c}$$

The consistent initial data $x(0)$, *corresponding to* $x_P(0) = 0$, *reduces to the initial data* $\xi_p(0) = 0$ *for the differential variable. Then, we obtain the desired solution as:*

$$x = \begin{bmatrix}0\\0\\1\end{bmatrix}\xi_p + \begin{bmatrix}0\\1\\0\end{bmatrix}\xi_{q,1} + \begin{bmatrix}1\\0\\0\end{bmatrix}\xi_{q,0}. \tag{2.32}$$

We can now see that the decoupled system (2.31) *has the same dimension as the original system.*

Both decoupled systems (2.28) *and* (2.31) *lead to same solution,*

$$x_1(t) = -u'(t) - u(t), \quad x_2(t) = -u(t), \quad x_3(t) = e^{2t}\int_0^t e^{-2\tau}u(\tau)\,\mathrm{d}\tau,$$

though (2.31) *is much simpler to solve than* (2.28). *We can also observe that* $\sigma(E, A) = \sigma(A_p) = \{2\}$ *as expected.*

Example 2. *Consider the simple RL network below:*

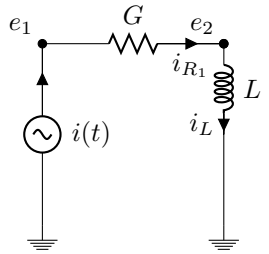


FIG. 2.1. *Simple RL network.*

Using the MNA on the above network leads to a DAE of the form (2.1), *where* $x = [e_1, e_2, i_L]^T$ *and* $u = i(t)$, *with system matrices given by*

$$E = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & L\end{bmatrix}, \quad A = \begin{bmatrix}-G & G & 0\\G & -G & -1\\0 & 1 & 0\end{bmatrix}, \quad B = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

We can see that $\det(\lambda E - A) = G > 0$, *so this system is solvable and its matrix pencil* (E, A) *has only infinite eigenvalues. Thus its decoupled system has no inherent differential equations, so it must take the form* (2.23).

We choose the projectors

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 & L \\ 0 & 0 & L \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.33}$$

such that $Q_1Q_0 = 0$ *holds. We can check that* $P_0P_1 = 0$*, so there are no differential variables. The algebraic variables are*

$$x_{Q,1} = P_0Q_1x = P_0x = \begin{bmatrix} 0 \\ 0 \\ i_L \end{bmatrix}, \quad x_{Q,0} = Q_0x = \begin{bmatrix} e_1 \\ e_2 \\ 0 \end{bmatrix}.$$

The last matrices in the matrix chain are

$$E_2 = \begin{bmatrix} G & -G & 0 \\ -G & G & 1 \\ 0 & -1 & L \end{bmatrix}, \quad A_2 = 0. \tag{2.34}$$

Since E_2 *is non-singular, this is an index* 2 *system. In the split form* (2.8)*, the system becomes:*

$$x_{Q,1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{2.35a}$$

$$x_{Q,0} = \begin{bmatrix} G^{-1} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & L \\ 0 & 0 & L \\ 0 & 0 & 0 \end{bmatrix} x'_{Q,1}. \tag{2.35b}$$

To obtain the compact decomposition we need a basic matrix (p_0, q_0) *and its inverse* (p_0^{*T}, q_0^{*T})*, with* $\operatorname{span} q_0 = \operatorname{span} Q_0$*,* $\operatorname{span} p_0 = \operatorname{span} P_0$*:*

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad p_0^{*T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad q_0^{*T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.36}$$

The variables $\xi_{q,1}$*,* $\xi_{q,0}$*, are given by*

$$\xi_{q,1} = p_0^{*T}x = i_L, \quad \xi_{q,0} = q_0^{*T}x = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Thus, using (2.36)*, the DAE can be written in the form* (2.23)*:*

$$\xi_{q,1} = u, \tag{2.37a}$$

$$\xi_{q,0} = \begin{bmatrix} G^{-1} \\ 0 \end{bmatrix} u + \begin{bmatrix} L \\ L \end{bmatrix} \xi'_{q,1}, \tag{2.37b}$$

which can be solved immediately, since it has no differential equations. The solution is

$$\xi_{q,1} = u, \quad \xi_{q,0} = \begin{bmatrix} G^{-1}u + Lu' \\ Lu' \end{bmatrix},$$

from which we find

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xi_{q,1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xi_{q,0} = \begin{bmatrix} G^{-1}u + Lu' \\ Lu' \\ u \end{bmatrix}. \tag{2.38}$$

Thus the solution of the circuit system is $e_1 = G^{-1}u + Lu'$*,* $e_2 = Lu'$*,* $i_L = u$*.*

3. Model order reduction. In this section, we discuss the model order reduction for index-2 systems. The section is divided into two subsections. In Subsection 3.1, we discuss the limitations of applying conventional methods directly on DAEs, while in Subsection 3.2, we propose a new MOR method to overcome these limitations.

3.1. Limitations of convectional MOR methods. In this subsection, we discuss how using conventional MOR methods, such as Krylov based methods, on higher index DAEs can lead to a good approximation of the transfer function but the resulting reduced-order models may be wrong or very difficult to solve.

Example 3. *Consider an index-2 dynamical system below:*

$$Ex' = Ax + Bu, \quad x(0) = x_0 \tag{3.1a}$$

$$y = C^T x, \tag{3.1b}$$

with system matrices,

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 2 & -1 & 1 & 0.5 \\ 1 & -1 & 1 & 0 & -0.5 \\ -1 & 1 & 0 & 1 & 0 \\ 1.25 & 2.25 & 0 & -4 & 1 \\ -0.5 & -0.5 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

This system is solvable since the polynomial $\det(\lambda E - A) = 2\lambda + 3$ *doesn't vanish identically and in addition, we assume that input function* u *is differentiable in the desired time interval and* $x(0)$ *is a consistent initial condition. In this example we consider two different cases of control input matrix* B *with input data* $u(t) = \cos(t)$.

(i) If $B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, *then the consistent initial condition is given by* $x(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. *We then apply the PRIMA method [15] on the DAE* (3.1). *Using* $s_0 = 0$ *as the expansion point leads to a reduced-order model below:*

$$E_r = \begin{bmatrix} 0.73684 & 0.12114 & 0.42065 \\ 0.12114 & 0.019915 & 0.069155 \\ 0.42065 & 0.069155 & 0.25289 \end{bmatrix},$$

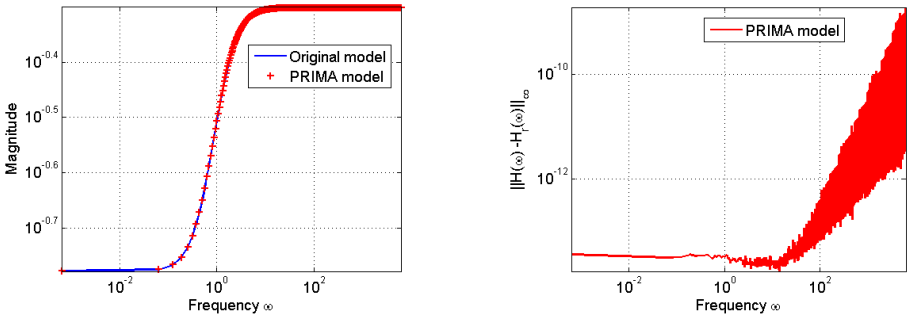
$$A_r = \begin{bmatrix} -0.94737 & -0.32015 & -1.7338 \\ -0.15575 & -0.052632 & -0.24306 \\ -0.5754 & -0.15246 & -1.3469 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0.68825 \\ 0.11315 \\ 0.41802 \end{bmatrix},$$

$$C_r = \begin{bmatrix} -0.22942 \\ 0.67888 \\ -0.88987 \end{bmatrix}, \quad x_r(0) = \begin{bmatrix} 0.11471 \\ -0.69774 \\ -1.8834e-016 \end{bmatrix}. \tag{3.2}$$

The reduced-order model is an ODE, that is, E_r *is invertible. In Figure 3.1, we compare the transfer function of the original model with that of the reduced-order model. We can see that the transfer functions coincide with a very small error as shown in Figure 3.1(b).*

The next step is to solve the reduced-order model and compare its solution with that of the original model. We solved the reduced-order model and the differential part of the original model in Matlab software, using in both cases ode15s *at* RelTol $= 10^{-6}$. *We observed that the original model requires 69 timesteps while the reduced-order model requires 75 timesteps to reach the*

desired accuracy. Figure 3.2 shows that the solution of the original model coincides with that of the reduced-order model (PRIMA model). Thus the PRIMA model (3.2) *is a good reduced-order model for the original system* (3.1) *since the reduced-order model leads to accurate solutions.*



(a) Frequency response (b) Frequency response error

FIG. 3.1. *Example 3(i). Comparison of frequency response and its error.*

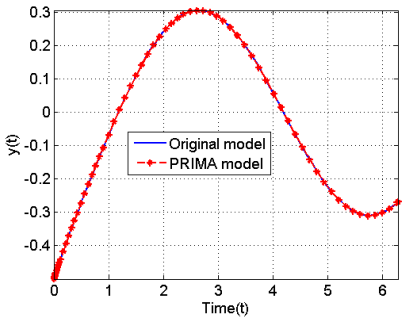


FIG. 3.2. *Example 3(i). Output solution,* RelTol $= 10^{-6}$.

(ii) If $B = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T$, *then the consistent initial condition is given by* $x(0) = \begin{bmatrix} 0.5 & -0.5 & 0.75 & 1 & 4 \end{bmatrix}^T$. *Using the same expansion point as before we obtain a reduced-order model given by:*

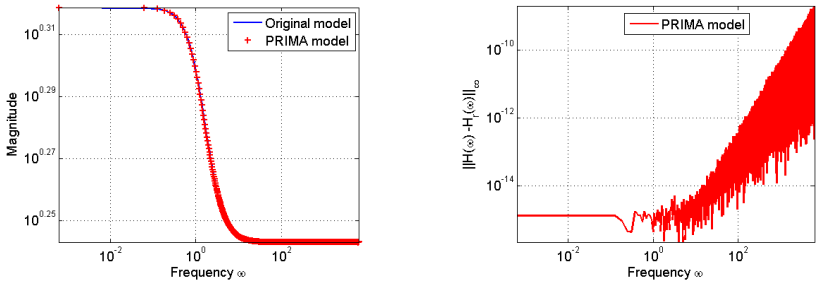
$$E_r = \begin{bmatrix} 0.043881 & 0.017375 & -0.1498 \\ 0.017375 & 0.25518 & 0.24928 \\ -0.1498 & 0.24928 & 0.89488 \end{bmatrix},$$

$$A_r = \begin{bmatrix} 0.20569 & -0.06545 & 0.052663 \\ -0.06545 & -0.15133 & -0.84866 \\ 0.052663 & 0.86786 & -3.9271 \end{bmatrix} \quad B_r = \begin{bmatrix} -0.92577 \\ 0.29458 \\ -0.23702 \end{bmatrix},$$

$$C_r = \begin{bmatrix} 0.46288 \\ 0.66642 \\ -0.97969 \end{bmatrix}, \quad x_r(0) = \begin{bmatrix} 4.1613 \\ -0.78707 \\ -0.35553 \end{bmatrix}. \tag{3.3}$$

Still the PRIMA method leads to a ODE reduced-order model and also for

this case the transfer function of the original model coincides with that of the reduced-order model (3.3) *as shown in Figure 3.3.*
We then solved the reduced-order model (3.3) *and we observed that the reduced-order model leads to a good solution, if the* RelTol ≥ 0.1 *as shown in Figure 3.4.*



(a) Frequency response (b) Frequency response error

FIG. 3.3. *Example 3(ii). Comparison of frequency response and its error.*

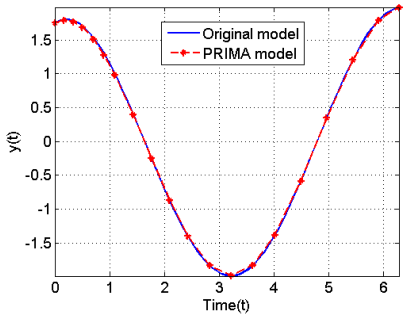


FIG. 3.4. *Example 3(ii). Output solution,* RelTol $= 0.1$.

The above example shows that solving the reduced-order model (3.3) is more difficult than solving (3.2), since we cannot achieve any better accuracy in the solution. This is due to the fact that the consistent initial condition x_0 in this example depends on u and its derivative, while in the former it only depends on u. We know that conventional methods always assume that $x(0)$ vanishes, but this assumption is not valid for DAEs, since we do not have enough freedom to choose the initial condition. We note that the conventional methods can reduce the DAE if its consistent initial condition x_0 only depends on u, otherwise the resulting reduced-order model is not acceptable.

The above discussion shows that we cannot use conventional methods to reduce higher index DAEs in general. This motivated us to develop a new technique for reducing index-2 systems which eliminates this inconvenience, as discussed in the next subsection.

3.2. Index-aware MOR for index-2 systems (IMOR-2). In Section 2.2, we saw that an index-2 system can be decoupled in two ways depending on the spectrum

of the matrix pencil (E, A). In this section, we propose a new technique of reducing index-2 systems depending on the nature of its matrix pencil. We call this technique "index-aware MOR" for index-2 systems (IMOR-2).

In short the idea is the following: attempt to reduce only the differential part, by using any MOR method; then, make explicit the limitation to the algebraic variables due to their explicit expression, obtained by the projection procedure. If there is no differential part, still the second part of the above reduction procedure can be applied.

For definiteness, and for its wide application and simplicity of implementation, we concentrate on Krylov-based MOR methods.

3.2.1. Matrix pencil (E, A) with finite eigenvalues. Assume that the DAE in descriptor form (1.1) is an index-2 dynamical system and its matrix pencil (E, A) has at least one finite eigenvalue. Then, recalling (2.15) and (2.17), equation (1.1) can be written as:

$$\xi_p' = A_p\xi_p + B_p u, \tag{3.4a}$$
$$\xi_{q,1} = A_{q,1}\xi_p + B_{q,1}u, \tag{3.4b}$$
$$\xi_{q,0} = A_{q,0}\xi_p + B_{q,0}u + A_{q,01}\xi_{q,1}', \tag{3.4c}$$
$$y = C_p^T\xi_p + C_{q,1}^T\xi_{q,1} + C_{q,0}^T\xi_{q,0}, \tag{3.4d}$$

where $C_p = p_{01}^T p_0^T C \in \mathbb{R}^{n_{01},\ell}$, $C_{q,1} = q_{01}^T p_0^T C \in \mathbb{R}^{k_1,\ell}$ and $C_{q,0} = q_0^T C \in \mathbb{R}^{k_0,\ell}$. In descriptor form, this system can be written as

$$\tilde{E}\xi' = \tilde{A}\xi + \tilde{B}u, \tag{3.5a}$$
$$y = \tilde{C}^T\xi, \tag{3.5b}$$

with $(\tilde{E}, \tilde{A}) = W(E, A)V$, $\tilde{B} = WB$, $\tilde{C} = V^T C$, and V, W defined as in (2.18).

We can rewrite (3.4) in three blocks, strictly separating the differential and algebraic parts,

$$\xi_p' = A_p\xi_p + B_p u, \tag{3.6a}$$
$$y_p = C_p^T\xi_p, \tag{3.6b}$$

$$\xi_{q,1} = A_{q,1}\xi_p + B_{q,1}u, \tag{3.7a}$$
$$y_{q,1} = C_{q,1}^T\xi_{q,1}, \tag{3.7b}$$

and

$$\xi_{q,0} = A_{q,0}\xi_p + B_{q,0}u + A_{q,01}\xi_{q,1}', \tag{3.8a}$$
$$y_{q,0} = C_{q,0}^T\xi_{q,0}. \tag{3.8b}$$

We observe that the subsystem (3.6) is an ODE, while (3.7) and (3.8) are algebraic subsystems. We can also see that using the output equations of these subsystems, we can reconstruct the output equation of (3.4) as,

$$y = y_p + y_{q,1} + y_{q,0}. \tag{3.9}$$

1. ***Reduction of the differential part*** ξ_p.
To reduce the differential part (3.6) we use a Krylov-subspace-based method, which

preserves the first r moments of the transfer function. The transfer function measures the sensitivity of the output with respect to the input, in frequency domain. Taking the Laplace transform of (3.6) and simplifying, we obtain:

$$\Xi_p(s) = (s\mathrm{I} - A_p)^{-1}B_pU(s) + (s\mathrm{I} - A_p)^{-1}\xi_p(0), \tag{3.10}$$
$$Y_p(s) = C_p^T\Xi_p(s), \tag{3.11}$$

which yields

$$Y_p(s) = C_p^T(s\mathrm{I} - A_p)^{-1}B_pU(s) + C_p^T(s\mathrm{I} - A_p)^{-1}\xi_p(0). \tag{3.12}$$

Thus, the transfer function restricted to the ODE part is given by

$$H_p(s) = C_p^T(s\mathrm{I} - A_p)^{-1}B_p. \tag{3.13}$$

We choose $s_0 \in \mathbb{C} \setminus \sigma(A_p)$, and we consider the subspace

$$\mathcal{V}_p := \mathcal{K}_r(M_p(s_0), R_p(s_0)), \quad r \leq n_{01}, \tag{3.14}$$

where

$$M_p(s_0) = (s_0 I - A_p)^{-1}, \quad \text{and} \quad R_p(s_0) = (s_0 I - A_p)^{-1}B_p,$$

and $\mathcal{K}_r(M_p, R_p)$ is the order-r Krylov subspace generated by M_p and R_p,

$$\mathcal{K}_r(M_p, R_p) = \mathrm{span}\,\{R_p, M_pR_p, \ldots, M_p^{r-1}R_p\}.$$

We denote by $V_p \in \mathbb{R}^{n_{01},rm}$ the matrix of an orthonormal basis of $\mathcal{V}_p$, so that we have $V_p^TV_p = \mathrm{I}$. We seek an approximate solution of the form $\xi_p = V_p\hat{\xi}_p$, that is, we replace equation (3.6) with

$$\hat{\xi}_p' = \hat{A}_p\hat{\xi}_p + \hat{B}_pu, \tag{3.15a}$$
$$\hat{y}_p = \hat{C}_p^T\hat{\xi}_p, \tag{3.15b}$$

where

$$\hat{A}_p = V_p^TA_pV_p, \quad \hat{B}_p = V_p^TB_p, \quad \text{and} \quad \hat{C}_p = V_p^TC_p.$$

By construction, this reduced system has a transfer function whose first r moments around s_0 coincide with the first r moments of the original transfer function $H_p(s)$.

2. ***Reduction of the algebraic part*** $\xi_{q,1}$.

The reduction of the differential part induces a reduction of the algebraic parts. First we consider the algebraic part $\xi_{q,1}$. We denote by $\xi_{q,1}^*$ the expression obtained from (3.7a) by using the approximation $\xi_p = V_p\hat{\xi}_p$, that is,

$$\xi_{q,1}^* = A_{q,1}V_p\hat{\xi}_p + B_{q,1}u.$$

This expression shows that $\xi_{q,1}^*$ belongs to the subspace

$$\mathcal{V}_{q,1} = \mathrm{span}\,\{B_{q,1}, A_{q,1}V_p\} = \mathrm{span}\,\{B_{q,1}\} + A_{q,1}\mathcal{K}_r(M_p, R_p). \tag{3.16}$$

We denote by $n_{q,1}$ the dimension of $\mathcal{V}_{q,1}$, and by $V_{q,1} \in \mathbb{R}^{k_1, n_{q,1}}$ the matrix of an orthonormal basis of $\mathcal{V}_{q,1}$, so that $V_{q,1}^T V_{q,1} = \mathrm{I}$. Then we can approximate the algebraic solution in the form $\xi_{q,1}^* = V_{q,1}\hat{\xi}_{q,1}$, that is, we replace (3.7) with

$$\hat{\xi}_{q,1} = \hat{A}_{q,1}\hat{\xi}_p + \hat{B}_{q,1}u, \tag{3.17a}$$
$$\hat{y}_{q,1} = \hat{C}_{q,1}^T\hat{\xi}_{q,1}, \tag{3.17b}$$

with

$$\hat{A}_{q,1} = V_{q,1}^T A_{q,1} V_p, \quad \hat{B}_{q,1} = V_{q,1}^T B_{q,1}, \quad \text{and} \quad \hat{C}_{q,1} = V_{q,1}^T C_{q,1}.$$

3. ***Reduction of the algebraic part*** $\xi_{q,0}$.
Finally, we consider the reduction of the algebraic part $\xi_{q,0}$, which involves differentiations of $\xi_{q,1}$. We denote by $\xi_{q,0}^*$ the expression obtained from (3.8a) by writing $\xi_p = V_p^T\hat{\xi}_p$, $\xi_{q,1} = V_{q,1}^T\hat{\xi}_{q,1}$,

$$\xi_{q,0}^* = A_{q,0}V_p\hat{\xi}_p + B_{q,0}u + A_{q,01}V_{q,1}\hat{\xi}'_{q,1}.$$

This expression shows that $\xi_{q,0}^*$ belongs to the subspace

$$\begin{aligned}\mathcal{V}_{q,0} &= \operatorname{span}\{B_{q,0}, A_{q,01}V_{q,1}, A_{q,0}V_p\},\\ &\equiv \operatorname{span}\{B_{q,0}, A_{q,01}B_{q,1}, A_{q,01}A_{q,1}V_p, A_{q,0}V_p\}.\end{aligned} \tag{3.18}$$

We denote by $n_{q,0}$ the dimension of $\mathcal{V}_{q,0}$, and by $V_{q,0} \in \mathbb{R}^{k_0, n_{q,0}}$ the matrix of an orthonormal basis of $\mathcal{V}_{q,0}$, so that $V_{q,0}^T V_{q,0} = \mathrm{I}$. Then we can approximate the algebraic solution in the form $\xi_{q,0}^* = V_{q,0}\hat{\xi}_{q,0}$, that is, we replace (3.8) with

$$\hat{\xi}_{q,0} = \hat{A}_{q,0}\hat{\xi}_p + \hat{B}_{q,0}u + \hat{A}_{q,01}\hat{\xi}'_{q,1}, \tag{3.19a}$$
$$\hat{y}_{q,0} = \hat{C}_{q,0}^T\hat{\xi}_{q,0}, \tag{3.19b}$$

where $\hat{A}_{q,0} = V_{q,0}^T A_{q,0} V_p$, $\hat{A}_{q,01} = V_{q,0}^T A_{q,01} V_{q,1}$, $\hat{B}_{q,0} = V_{q,0}^T B_{q,0}$, and $\hat{C}_{q,0} = V_{q,0}^T C_{q,0}$. Combining subsystem (3.15), (3.17) and (3.19), we obtain a reduced-order model of the DAE (2.1) given by:

$$\hat{E}\hat{\xi}' = \hat{A}\hat{\xi} + \hat{B}u, \tag{3.20a}$$
$$\hat{y} = \hat{C}^T\hat{\xi}, \tag{3.20b}$$

with

$$(\hat{A}, \hat{E}) = \hat{V}^T(\tilde{A}, \tilde{E})\hat{V}, \quad \hat{B} = \hat{V}^T\tilde{B}, \quad \hat{C} = \hat{V}^T\tilde{C},$$
$$\hat{\xi} = \begin{bmatrix}\hat{\xi}_p\\ \hat{\xi}_{q,1}\\ \hat{\xi}_{q,0}\end{bmatrix}, \quad \hat{V} = \begin{bmatrix}V_p & 0 & 0\\ 0 & V_{q,1} & 0\\ 0 & 0 & V_{q,0}\end{bmatrix}.$$

Here, the matrices $\tilde{A}$, $\tilde{E}$ are as in (3.5).

3.2.2. Matrix pencil (E, A) has no finite eigenvalues. Next we assume that the DAE (1.1) is an index-2 dynamical system and that its matrix pencil (E, A) has no finite eigenvalues. Then, recalling (2.22) and (2.23), equation (1.1) can be written as

$$\xi_{q,1} = B_{q,1}u, \tag{3.21a}$$
$$\xi_{q,0} = B_{q,0}u + A_{q,01}\xi'_{q,1}, \tag{3.21b}$$
$$y = C_{q,1}^T\xi_{q,1} + C_{q,0}^T\xi_{q,0}, \tag{3.21c}$$

where $C_{q,1} = p_0^T C \in \mathbb{R}^{n_0,\ell}$, $C_{q,0} = q_0^T C \in \mathbb{R}^{k_0,\ell}$. In compact form, this system can be written, again, as

$$\tilde{E}\xi' = \tilde{A}\xi + \tilde{B}u, \tag{3.22a}$$
$$y = \tilde{C}^T\xi, \tag{3.22b}$$

with $(\tilde{E}, \tilde{A}) = W(E, A)V$, $\tilde{B} = WB$, $\tilde{C} = V^T C$, and V, W as in (2.24).

We can rewrite (3.21) in two blocks, strictly separating the algebraic parts,

$$\xi_{q,1} = B_{q,1}u, \tag{3.23a}$$
$$y_{q,1} = C_{q,1}^T\xi_{q,1}, \tag{3.23b}$$

and

$$\xi_{q,0} = B_{q,0}u + A_{q,01}\xi'_{q,1}, \tag{3.24a}$$
$$y_{q,0} = C_{q,0}^T\xi_{q,0}. \tag{3.24b}$$

Using the output equations of these algebraic subsystems, we can reconstruct the output equation of (3.21) as,

$$y = y_{q,1} + y_{q,0}. \tag{3.25}$$

1. ***Reduction of the algebraic part*** $\xi_{q,1}$.
In this case the algebraic variable $\xi_{q,1}$ can be computed directly from equation (3.23a). This expression shows that $\xi_{q,1}$ belongs to the subspace

$$\mathcal{V}_{q,1} = \operatorname{span} B_{q,1}. \tag{3.26}$$

We denote by $n_{q,1}$ the dimension of $\mathcal{V}_{q,1}$, and by $V_{q,1} \in \mathbb{R}^{k_1,n_{q,1}}$ the matrix of an orthonormal basis of $\mathcal{V}_{q,1}$, so that $V_{q,1}^T V_{q,1} = \mathrm{I}$. Then we can approximate the algebraic solution in the form $\xi^*_{q,1} = V_{q,1}\hat{\xi}_{q,1}$, that is, we replace (3.23) with

$$\hat{\xi}_{q,1} = \hat{B}_{q,1}u, \tag{3.27a}$$
$$\hat{y}_{q,1} = \hat{C}_{q,1}^T\hat{\xi}_{q,1}, \tag{3.27b}$$

where

$$\hat{B}_{q,1} = V_{q,1}^T B_{q,1}, \quad \hat{C}_{q,1} = V_{q,1}^T C_{q,1}.$$

2. ***Reduction of the algebraic part*** $\xi_{q,0}$.
Finally, we consider the reduction of algebraic part $\xi_{q,0}$. We denote by $\xi^*_{q,0}$ the expression obtained from (3.24a) by using the approximation $\xi_{q,1} = V_{q,1}^T\hat{\xi}_{q,1}$, that is,

$$\xi^*_{q,0} = B_{q,0}u + A_{q,01}V_{q,1}\hat{\xi}'_{q,1}.$$

This expression shows that $\xi^1_{q,0}$ belongs to the subspace

$$\mathcal{V}_{q,0} = \operatorname{span}\{B_{q,0}, A_{q,01}V_{q,1}\}. \tag{3.28}$$

We denote by $n_{q,0}$ the dimension of $\mathcal{V}_{q,0}$, and by $V_{q,0} \in \mathbb{R}^{k_0,n_{q,0}}$ the matrix of an orthonormal basis of $\mathcal{V}_{q,0}$, so that $V_{q,0}^T V_{q,0} = \mathrm{I}$. Then we can approximate the algebraic solution in the form $\xi^*_{q,0} = V_{q,0}\hat{\xi}_{q,0}$, that is, we replace (3.24) with

$$\hat{\xi}_{q,0} = \hat{B}_{q,0}u + \hat{A}_{q,01}\hat{\xi}'_{q,1}, \tag{3.29a}$$

$$\hat{y}_{q,0} = \hat{C}^T_{q,0}\hat{\xi}_{q,0}, \tag{3.29b}$$

where

$$\hat{A}_{q,01} = V^T_{q,0}A_{q,01}V_{q,1},\ \hat{B}_{q,0} = V^T_{q,0}B_{q,0},\ \hat{C}_{q,0} = V^T_{q,0}C_{q,0}.$$

Also combining Equation (3.23) and (3.24), we obtain a reduced-order model in descriptor form given by:

$$\hat{E}\hat{\xi}' = \hat{A}\hat{\xi} + \hat{B}u, \tag{3.30a}$$

$$\hat{y} = \hat{C}^T\hat{\xi}, \tag{3.30b}$$

with

$$(\hat{A}, \hat{E}) = \hat{V}^T(\tilde{A}, \tilde{E})\hat{V}, \quad \hat{B} = \hat{V}^T\tilde{B}, \quad \hat{C} = \hat{V}^T\tilde{C},$$
$$\hat{\xi} = \begin{bmatrix}\hat{\xi}_{q,1}\\ \hat{\xi}_{q,0}\end{bmatrix}, \quad \hat{V} = \begin{bmatrix}V_{q,1} & 0\\ 0 & V_{q,0}\end{bmatrix}.$$

Here, the matrices $\tilde{A}$, $\tilde{E}$ are as in (3.22).

We note that the orthonormal basis matrices $V_{q,1}$ and $V_{q,0}$ can be computed numerically either using the modified Gram-Schmidt or SVD based methods. In this paper, we use the SVD method, in this way the dimension of the reduced parts can be determined by truncating the columns of either $V_{q,1}$ or $V_{q,0}$ which corresponds to the number largest singular values. The approach gives a further reduction of the order of the reduced algebraic parts, as shown in Example 4.

3.3. Comparison with traditional projection methods. In the previous subsection, we have proposed a new MOR procedure, based on the decomposition of a DAE in differential and algebraic components. Starting from the system in decoupled form, we reduce first the ODE part, and then observe that this reduction induces a reduction also on the other parts. In this section we compare the new, index-aware MOR method with traditional MOR methods. In order to make the comparison more effective, we concentrate on a specific class of MOR methods, namely, projection methods.

In traditional projection methods, the starting point is the state space system (1.1). The main idea is to find a reduction procedure which preserves the first moments of the transfer function of the system. The transfer function is defined by taking the Laplace transform of the previous expression and computing explicitly the Laplace transform $Y(s)$ of $y(t)$ as a function of the data, that is, the Laplace transform $U(s)$ of the input $u(t)$, and the initial data x_0. Explicitly, we find

$$Y(s) = C^T R(s)U(s) + C^T M(s)x_0, \tag{3.31}$$

with

$$M(s) := (sE - A)^{-1}E, \quad R(s) := (sE - A)^{-1}B.$$

Then, the transfer function $H(s)$ is the term in front of $U(s)$, which measure the dependence of the output on the input, that is,

$$H(s) := C^T R(s). \tag{3.32}$$

It is simple to see that, for any s_0 which is not in the spectrum of (E, A), we can write

$$R(s) = [\mathrm{I} + M(s_0)(s - s_0)]^{-1} R(s_0). \tag{3.33}$$

By using the Neumann expansion, we find

$$R(s) = \sum_{k=0}^{\infty} R^{(k)}(s_0)(s - s_0)^k,$$

with

$$R^{(k)}(s_0) := (-1)^k M(s_0)^k R(s_0).$$

Thus the transfer function can be expanded around $s = s_0$ as:

$$H(s) = \sum_{k=0}^{\infty} h^{(k)}(s_0)(s - s_0), \tag{3.34}$$

where the k-th moment $h^{(k)}(s_0)$ of $H(s)$ around s_0 is given by the formula

$$h^{(k)}(s_0) := C^T R^{(k)}(s_0) = (-1)^k C^T M(s_0)^k R(s_0). \tag{3.35}$$

One wishes to find a subspace such that the projection of the original system into this subspace is a reduced system which preserves the first r moments. We consider an orthonormal basis of this hypothetical subspace, which we can write in the columns of a matrix $\hat{V}$, with $\hat{V}^T\hat{V} = \mathrm{I}$. Then the reduced system is :

$$\begin{aligned} \hat{E}\hat{x} &= \hat{A}\hat{x} + \hat{B}u, \\ \hat{y} &= \hat{C}^T\hat{x}, \end{aligned}$$

with $(\hat{E}, \hat{A}) = \hat{V}^T(E, A)\hat{V}$, $\hat{B} = \hat{V}^T B$, $\hat{C} = \hat{V}^T C$. Again, we find the formal expansion of $\hat{R}(s)$ around $s = s_0$,

$$\hat{R}(s) = \sum_{k=0}^{\infty} \hat{R}^{(k)}(s_0)(s - s_0)^k, \quad \hat{R}^{(k)}(s_0) := (-1)^k \hat{M}(s_0)^k \hat{R}(s_0),$$

where

$$\hat{M}(s) := (s\hat{E} - \hat{A})^{-1}\hat{E}, \quad \hat{R}(s) := (s\hat{E} - \hat{A})^{-1}\hat{B}.$$

The transfer function of the reduced system can be written as

$$\hat{H}(s) = \hat{C}^T \hat{R}(s) = \sum_{k=0}^{\infty} \hat{h}^{(k)}(s_0)(s - s_0), \tag{3.36}$$

with

$$\hat{h}^{(k)}(s_0) := \hat{C}^T \hat{R}^{(k)}(s_0) = (-1)^k \hat{C}^T \hat{M}(s_0)^k \hat{R}(s_0). \tag{3.37}$$

It follows that the first r moments of $H(s)$ are preserved if and only if

$$\hat{C}^T \hat{R}^{(k)}(s_0) = C^T R^{(k)}(s_0), \quad k = 0, 1, \ldots, r-1. \tag{3.38}$$

It can be proved that this condition is satisfied if $\hat{V}$ is chosen so that

$$\operatorname{span}\{\hat{V}\} = \mathcal{K}_r(M(s_0), R(s_0)), \tag{3.39}$$

where $\mathcal{K}_r(M, R)$ is the order-r Krylov space generated by M, R, that is, the subspace spanned by R, MR, …, $M^{r-1}R$. This choice defines the traditional projection methods.

We wish to give some insight on the method. Since $\hat{V}\hat{V}^T$ is a projector onto $\mathcal{K}_r(M(s_0), R(s_0))$, by construction we have

$$\hat{V}\hat{V}^T M(s_0)^j R(s_0) = M(s_0)^j R(s_0), \quad j = 0, 1, \ldots, r-1, \tag{3.40}$$

which yields

$$\hat{V}\hat{V}^T R^{(j)}(s_0) = R^{(j)}(s_0), \quad j = 0, 1, \ldots, r-1. \tag{3.41}$$

It is possible to prove that condition (3.40) implies

$$\hat{M}(s_0)^j \hat{R}(s_0) = \hat{V}^T M(s_0)^j R(s_0), \quad j = 0, 1, \ldots, r-1,$$

that is,

$$\hat{R}^{(j)}(s_0) = \hat{V}^T R^{(j)}(s_0), \quad j = 0, 1, \ldots, r-1. \tag{3.42}$$

Then, for $k = 0, 1, \ldots, r-1$, we have

$$\hat{h}^{(k)}(s_0) = \hat{C}^T \hat{R}^{(k)}(s_0) = C^T \hat{V}\hat{V}^T R^{(k)}(s_0) = C^T R^{(k)}(s_0) = h^{(k)}(s_0),$$

which shows that the chosen moments are preserved.

Next, we observe that the transfer function is invariant for equivalent systems. The system

$$\begin{aligned} \tilde{E}\xi' &= \tilde{A}\xi + \tilde{B}u, \\ y &= \tilde{C}^T \xi, \end{aligned}$$

is said equivalent to system (1.1) if there exist invertible matrices V, W such that $x = V\xi$, and

$$(\tilde{E}, \tilde{A}) = W(E, A)V, \quad \tilde{B} = WB, \quad \tilde{C} = V^T C. \tag{3.43}$$

The transfer function is then

$$\begin{aligned} \tilde{H}(s) &= \tilde{C}^T \tilde{R}(s) = \tilde{C}^T (s\tilde{E} - \tilde{A})^{-1} \tilde{B} \\ &= C^T V [W(sE - A)V]^{-1} WB = C^T (sE - A)^{-1} B \\ &= C^T R(s) = H(s). \end{aligned}$$

In particular, we can use the structure of the system obtained after the decoupling, that is, we use the descriptor form (3.5) of the projected index-2 system, with transformation matrices V, W as in (2.18) :

$$\tilde{E} = \begin{bmatrix} \mathrm{I}_{n_{01}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -A_{q,01} & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_p & 0 & 0 \\ A_{q,1} & -\mathrm{I}_{k_1} & 0 \\ A_{q,0} & 0 & -\mathrm{I}_{k_0} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_p \\ B_{q,1} \\ B_{q,0} \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} C_p & C_{q,1} & C_{q,0} \end{bmatrix}^T. \tag{3.44}$$

Using these matrices we obtain,

$$(s\tilde{E} - \tilde{A})^{-1} = \begin{bmatrix} M_p(s) & 0 & 0 \\ A_{q,1}M_p(s) & \mathrm{I}_{k_1} & 0 \\ (sA_{q,01}A_{q,1} + A_{q,0})M_p(s) & 0 & \mathrm{I}_{k_0} \end{bmatrix}, \tag{3.45}$$

where $M_p(s) := (s\mathrm{I}_{n_{01}} - A_p)^{-1}$. Then we can write

$$\tilde{R}(s) = \begin{bmatrix} R_p(s) \\ R_{q,1}(s) \\ R_{q,0}(s) \end{bmatrix} := \begin{bmatrix} R_p(s) \\ A_{q,1}R_p(s) + B_{q,1} \\ A_{q,0}R_p(s) + sA_{q,01}R_{q,1}(s) + B_{q,0} \end{bmatrix}, \tag{3.46}$$

where $R_p(s) := (s\mathrm{I}_{n_{01}} - A_p)^{-1}B_p$.

Since the transfer function is the same for equivalent systems, we find

$$H(s) = \tilde{C}^T\tilde{R}(s) = H_p(s) + H_{q,1}(s) + H_{q,0}(s), \tag{3.47}$$

with

$$H_p(s) := C_p^T R_p(s),\ H_{q,1}(s) := C_{q,1}^T R_{q,1}(s),\ H_{q,0}(s) := C_{q,0}^T R_{q,0}(s). \tag{3.48}$$

The IMOR-2 method amounts to consider separately the three parts of the transfer function. We can prove that, if the first r moments of $H_p(s)$ around $s = s_0$ are preserved, then we can ensure that the first r moments of $H_{q,1}(s)$ and $H_{q,0}(s)$ around $s = s_0$ are preserved by using the projection spaces $\mathcal{V}_{q,1}$, $\mathcal{V}_{q,0}$ prescribed by the IMOR method, which leads to the reduced equation (3.20).

To see this, we expand separately the three parts of the transfer function. We recognize that $H_p(s)$ is the transfer function of the differential model (3.6). Then, reasoning as in the traditional approach, we can write

$$R_p(s) = [\mathrm{I} + M_p(s_0)(s - s_0)]^{-1}R_p(s_0) = \sum_{k=0}^{\infty} R_p^{(k)}(s_0)(s - s_0)^k, \tag{3.49}$$

with

$$R_p^{(k)}(s_0) := (-1)^k M_p(s_0)^k R_p(s_0).$$

Thus the transfer function $H_p(s)$ can be expanded around $s = s_0$ as:

$$H_p(s) = \sum_{k=0}^{\infty} h_p^{(k)}(s_0)(s - s_0), \tag{3.50}$$

where the k-th moment $h_p^{(k)}(s_0)$ of $H_p(s)$ around s_0 is given by the formula

$$h_p^{(k)}(s_0) := C_p^T R_p^{(k)}(s_0) = (-1)^k C_p^T M_p(s_0)^k R_p(s_0).$$

Since $V_p V_p^T$ is a projector onto $\mathcal{K}_r(M_p(s_0), R_p(s_0))$, we have

$$V_p V_p^T M_p(s_0)^k R_p(s_0) = M_p(s_0)^k R_p(s_0), \quad k = 0, 1, \ldots, r-1,$$

which implies

$$\hat{R}_p^{(k)}(s_0) = V_p^T R_p^{(k)}(s_0), \tag{3.51}$$

and thus

$$\hat{h}_p^{(k)}(s_0) = h_p^{(k)}(s_0). \tag{3.52}$$

Next we expand $H_{q,1}(s)$ around $s = s_0$, by using the expansion of $R_p(s)$. We find

$$R_{q,1}(s) = A_{q,1} R_p(s) + B_{q,1} = \sum_{k=0}^{\infty} R_{q,1}^{(k)}(s_0)(s - s_0)^k, \tag{3.53}$$

with

$$\begin{aligned} R_{q,1}^{(0)}(s_0) &:= A_{q,1} R_p^{(0)}(s_0) + B_{q,1} = A_{q,1} R_p(s_0) + B_{q,1}, \\ R_{q,1}^{(k)}(s_0) &:= A_{q,1} R_p^{(k)}(s_0) = (-1)^k A_{q,1} M_p(s_0)^k R_p(s_0), \quad k > 0, \end{aligned}$$

which yields

$$H_{q,1}(s) = \sum_{k=0}^{\infty} h_{q,1}^{(k)}(s_0)(s - s_0)^k, \tag{3.54}$$

with

$$h_{q,1}^{(k)}(s_0) := C_{q,1}^T R_{q,1}^{(k)}(s_0).$$

Since $V_{q,1} V_{q,1}^T$ is a projector onto $\operatorname{span}\{A_{q,1} V_p, B_{q,1}\}$, we have

$$V_{q,1} V_{q,1}^T R_{q,1}^{(k)}(s_0) = R_{q,1}^{(k)}(s_0), \quad k = 0, 1, \ldots, r-1. \tag{3.55}$$

It follows

$$\begin{aligned} \hat{R}_{q,1}^{(0)}(s_0) &= \hat{A}_{q,1} \hat{R}_p^{(0)}(s_0) + \hat{B}_{q,1} \\ &= V_{q,1}^T A_{q,1} V_p V_p^T R_p^{(0)}(s_0) + V_{q,1}^T B_{q,1} \\ &= V_{q,1}^T A_{q,1} R_p^{(0)}(s_0) + V_{q,1}^T B_{q,1} \\ &= V_{q,1}^T R_{q,1}^{(0)}(s_0), \end{aligned}$$

and similarly

$$\hat{R}_{q,1}^{(k)}(s_0) = V_{q,1}^T R_{q,1}^{(k)}(s_0), \quad k = 1, 2, \ldots, r-1.$$

Then we have

$$\hat{h}_{q,1}^{(k)}(s_0) = \hat{C}_{q,1}^T \hat{R}_{q,1}^{(k)}(s_0) = C_{q,1}^T V_{q,1} V_{q,1}^T R_{q,1}^{(k)}(s_0) = C_{q,1}^T R_{q,1}^{(k)}(s_0) = h_{q,1}^{(k)}(s_0),$$

so the first r moments of $H_{q,1}(s)$ are preserved.

Finally, we use the previous expansions of $R_p(s)$ and $R_{q,1}(s)$ to expand $H_{q,0}(s)$ around $s = s_0$,

$$R_{q,0}(s) = A_{q,0} R_p(s) + s A_{q,01} R_{q,1}(s) + B_{q,0} = \sum_{k=0}^{\infty} R_{q,0}^{(k)}(s_0)(s - s_0)^k, \tag{3.56}$$

with

$$\begin{aligned} R_{q,0}^{(0)}(s_0) &= A_{q,0} R_p^{(0)}(s_0) + s_0 A_{q,01} R_{q,1}^{(0)}(s_0) + B_{q,0}, \\ R_{q,0}^{(k)}(s_0) &= A_{q,0} R_p^{(k)}(s_0) + A_{q,01} R_{q,1}^{(k-1)}(s_0), \quad k > 0. \end{aligned}$$

Since $V_{q,0} V_{q,0}^T$ is a projector onto $\operatorname{span}\{A_{q,0} V_p, A_{q,01} V_{q,1}, B_{q,0}\}$ we have

$$V_{q,0} V_{q,0}^T R_{q,0}^{(k)}(s_0) = R_{q,0}^{(k)}(s_0), \quad k = 0, 1, \ldots, r-1. \tag{3.57}$$

Then, proceeding as before, it is possible to show that

$$\hat{R}_{q,0}^{(k)}(s_0) = V_{q,0}^T R_{q,0}^{(k)}(s_0), \quad k = 0, 1, \ldots, r-1, \tag{3.58}$$

which implies

$$\hat{h}_{q,0}^{(k)}(s_0) = h_{q,0}^{(k)}(s_0), \quad k = 0, 1, \ldots, r-1. \tag{3.59}$$

In this way, we have shown that the iMOR-2 method preserves the first r moments of the transfer function.

4. Numerical experiments. In this section, we present simple and industrial problems of index 2 DAEs to demonstrate the effectiveness of our IMOR approach. In subsection 4.2, we compare IMOR with the traditional method (PRIMA method) using industrial examples. All the results are computed under Matlab environment version 2010a on a laptop with 2.53 GHz Intel(R) Core(TM) 2 Duo CPU and 4 GB RAM.

4.1. Simple problem. We consider again Example 3, applying IMOR-2 instead of PRIMA.

Example 4. *In this example we use system matrices E, A, B, C from Example 3. We have* $\det(\lambda E - A) = 2\lambda + 3 \neq 0$, $\forall \lambda \in \mathbb{C}$, *thus this system is solvable and its matrix pencil (E, A) has one finite eigenvalue $\sigma_f(E, A) = \{-\frac{3}{2}\}$. This implies that we can use the IMOR-2 approach discussed in Section 3.2.1.*

(i) Here we consider the case when $B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. *The transfer function of the DAE is given by:*

$$H(s) = C^T (sE - A)^{-1} B = \frac{1}{2s+3} - \frac{1}{2}. \tag{4.1}$$

In order to apply the IMOR-2 we need to first decompose the DAE into differential and algebraic parts given by:

$$\xi_p' = -\frac{3}{2}\xi_p - \frac{3}{4}u, \tag{4.2a}$$

$$\xi_{q,1} = \begin{bmatrix}0\\0\end{bmatrix}\xi_p + \begin{bmatrix}0\\0\end{bmatrix}u, \tag{4.2b}$$

$$\xi_{q,0} = \begin{bmatrix}-\frac{4}{3}\\-\frac{1}{3}\end{bmatrix}\xi_p + \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}u + \begin{bmatrix}-1 & \frac{11}{30}\\ 0 & 1\end{bmatrix}\xi_{q,1}', \tag{4.2c}$$

$$y = \frac{2}{3}\xi_p + \begin{bmatrix}0 & 1\end{bmatrix}\xi_{q,1} + \begin{bmatrix}1 & 0\end{bmatrix}\xi_{q,0}. \tag{4.2d}$$

Using formula (3.47) *the transfer function* (4.1) *can be decomposed as:*

$$H(s) = \underbrace{-\frac{1}{2s+3}}_{H_p(s)} + \underbrace{0}_{H_{q,1}(s)} + \underbrace{\frac{2}{2s+3} - \frac{1}{2}}_{H_{q,0}(s)}. \tag{4.3}$$

The projected DAE (4.2) *can be written in descriptor form as,*

$$\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&-\frac{11}{30}&0&0\\0&0&-1&0&0\end{bmatrix}\xi' = \begin{bmatrix}-\frac{3}{2}&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\-\frac{4}{3}&0&0&-1&0\\-\frac{1}{3}&0&0&0&-1\end{bmatrix}\xi + \begin{bmatrix}-\frac{3}{4}\\0\\0\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}u, \tag{4.4a}$$

$$y = \begin{bmatrix}\frac{2}{3} & 0 & 1 & 1 & 0\end{bmatrix}\xi. \tag{4.4b}$$

The next step is to apply the proposed MOR technique (IMOR-2) for index 2 *system discussed in Section 3.2.1 on the projected DAE* (4.4). *Choosing* $s_0 = 0$ *as the expansion point, we can construct orthonormal bases for the differential and algebraic parts of the decoupled system* (4.2)*, given by:*

$$V_p = -1,$$
$$V_{q,1} = \begin{bmatrix}-0.72015 & -0.69382\\ 0.69382 & -0.72015\end{bmatrix},\ V_{q,0} = \begin{bmatrix}-0.90749 & -0.42008\\ -0.42008 & 0.90749\end{bmatrix}. \tag{4.5}$$

We computed $V_{q,1}$ *and* $V_{q,0}$ *using the SVD method and the corresponding singular values are given by:*

$$S_{q,1} = \begin{bmatrix}3.1\times10^{-16} & 0\\ 0 & 5.0\times10^{-17}\end{bmatrix},\ S_{q,0} = \begin{bmatrix}1.9114 & 0\\ 0 & 0.9327\end{bmatrix}.$$

The singular values can now give us the number columns of $V_{q,1}$ *and* $V_{q,0}$ *we can truncate. We observe that all columns of* $V_{q,1}$ *can be truncated since their corresponding singular values are close to zero, while* $V_{q,0}$ *remains unchanged. Thus the first algebraic part can be ignored, and the system reduces to an index-1 system. Using* (4.5) *the reduced-order model can be written as:*

$$\hat{\xi}_p' = -1.5\hat{\xi}_p - 0.75u, \tag{4.6a}$$

$$\hat{\xi}_{q,0} = \begin{bmatrix}-1.3672\\-0.14048\end{bmatrix}\hat{\xi}_p + \begin{bmatrix}0.64035\\-0.29992\end{bmatrix}u, \tag{4.6b}$$

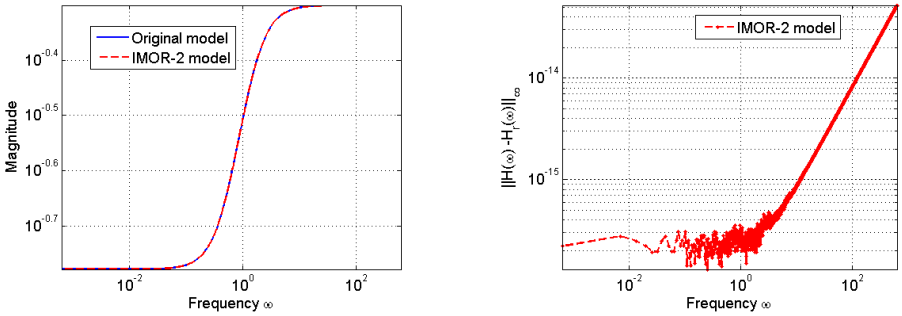
$$\hat{y} = -0.66667\hat{\xi}_p + \begin{bmatrix}-0.94027 & -0.34043\end{bmatrix}\hat{\xi}_{q,0}. \tag{4.6c}$$

In descriptor form the reduced-order model can be written as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{\xi}' = \begin{bmatrix} -1.5 & 0 & 0 \\ -1.3672 & -1 & 0 \\ -0.14048 & 0 & -1 \end{bmatrix} \hat{\xi} + \begin{bmatrix} 0.75 \\ 0.64035 \\ -0.29992 \end{bmatrix} u \tag{4.7a}$$

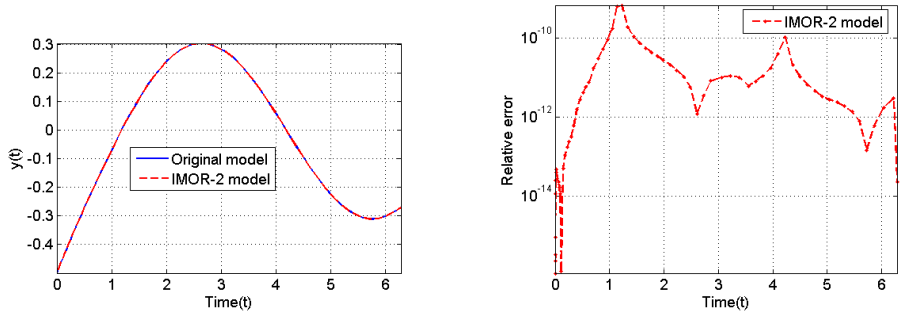
$$\hat{y} = \begin{bmatrix} -0.66667 & -0.94027 & -0.34043 \end{bmatrix} \hat{\xi} \tag{4.7b}$$

Thus the original DAE is reduced from the dimension 5 to 3 using the IMOR-2 method. In Figure 4.1, we compare the magnitude of the transfer function of the reduced-order model (IMOR-2 model) with that of the original model. We observe that their transfer functions coincide with a very small error, as shown in Figure 4.1(b). The reduced-order model also leads to very accurate solutions, as shown in Figure 4.2.



(a) Frequency response. (b) Frequency response error.

FIG. 4.1. *Example 4(i). Frequency response and its error.*



(a) Solution (b) Approximation error

FIG. 4.2. *Example 4(i). Solution and its approximation error,* $u(t) = \cos(t)$.

(ii) We now consider the case $B = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T$. *If we use the popular formula for transfer function then we have,*

$$H(s) = C^T(sE - A)^{-1}B = \frac{1}{2s+3} + \frac{7}{4}. \tag{4.8}$$

In order to apply the IMOR-2 method we need to first decompose the DAE into the differential and algebraic parts, given by:

$$\xi_p' = -\frac{3}{2}\xi_p - \frac{3}{4}u, \tag{4.9a}$$

$$\xi_{q,1} = \begin{bmatrix}0\\0\end{bmatrix}\xi_p + \begin{bmatrix}\frac{11}{30}\\1\end{bmatrix}u, \tag{4.9b}$$

$$\xi_{q,0} = \begin{bmatrix}-\frac{4}{3}\\-\frac{1}{3}\end{bmatrix}\xi_p + \begin{bmatrix}\frac{3}{4}\\4\end{bmatrix}u + \begin{bmatrix}-1 & \frac{11}{30}\\0 & 1\end{bmatrix}\xi_{q,1}', \tag{4.9c}$$

$$y = \frac{2}{3}\xi_p + \begin{bmatrix}0 & 1\end{bmatrix}\xi_{q,1} + \begin{bmatrix}1 & 0\end{bmatrix}\xi_{q,0}. \tag{4.9d}$$

Thus the transfer function (4.8) *can also be decomposed as:*

$$H(s) = \underbrace{-\frac{1}{2s+3}}_{H_p(s)} + \underbrace{1}_{H_{q,1}} + \underbrace{\frac{2}{2s+3} + \frac{3}{4}}_{H_{q,0}(s)}.$$

The projected DAE (4.9) *in descriptor form is given by:*

$$\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&-\frac{11}{30}&0&0\\0&0&-1&0&0\end{bmatrix}\xi' = \begin{bmatrix}-\frac{3}{2}&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\-\frac{4}{3}&0&0&-1&0\\-\frac{1}{3}&0&0&0&-1\end{bmatrix}\xi + \begin{bmatrix}-\frac{3}{4}\\\frac{11}{30}\\1\\\frac{3}{4}\\4\end{bmatrix}u, \tag{4.10a}$$

$$y = \begin{bmatrix}\frac{2}{3} & 0 & 1 & 1 & 0\end{bmatrix}\xi. \tag{4.10b}$$

The next step is to apply the proposed MOR technique (IMOR-2) on the projected DAE (4.10)*. Choosing* $s_0 = 0$ *as the expansion point, we can construct orthonormal bases for differential and algebraic parts of decoupled system* (4.9) *given by:*

$$V_p = -1,$$
$$V_{q,1} = \begin{bmatrix}-0.34425 & -0.93888\\-0.93888 & 0.34425\end{bmatrix},\ V_{q,0} = \begin{bmatrix}-0.21798 & -0.97595\\-0.97595 & 0.21798\end{bmatrix}. \tag{4.11}$$

The corresponding singular values for $V_{q,1}$ *and* $V_{q,0}$ *are given by*

$$S_{q,1} = \begin{bmatrix}1.0651 & 0\\0 & 2.7341\times10^{-16}\end{bmatrix},\quad S_{q,0} = \begin{bmatrix}4.2145 & 0\\0 & 1.2534\end{bmatrix}.$$

We can see that the last column of $V_{q,1}$ *can be truncated since its corresponding singular value is close to zero. Thus the block diagonal orthonormal basis matrix can be written as:*

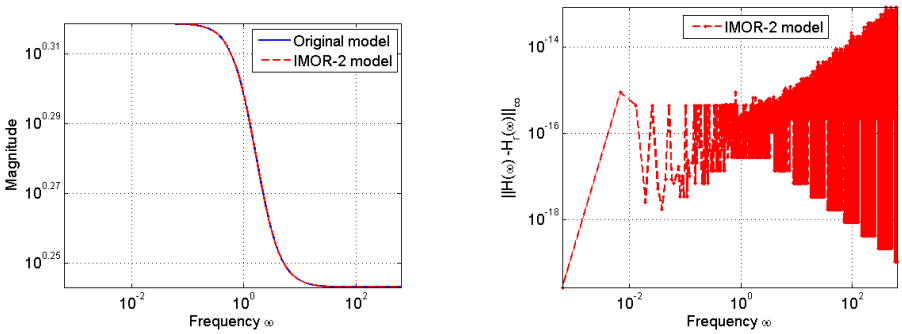
$$\hat{V} = \begin{bmatrix}-1 & 0 & 0 & 0\\0 & -0.34425 & 0 & 0\\0 & -0.93888 & 0 & 0\\0 & 0 & -0.21798 & -0.97595\\0 & 0 & -0.97595 & 0.21798\end{bmatrix}.$$

If we substitute $\xi = \hat{V}\hat{\xi}$ *into Equation* (4.10) *leads to the reduced-order model:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -0.9163 & 0 & 0 \\ 0 & 0.20466 & 0 & 0 \end{bmatrix} \hat{\xi}' = \begin{bmatrix} -1.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -0.61596 & 0 & -1 & 0 \\ -1.2286 & 0 & 0 & -1 \end{bmatrix} \hat{\xi} + \begin{bmatrix} 0.75 \\ -1.0651 \\ -4.0673 \\ 0.13996 \end{bmatrix} u \tag{4.12a}$$

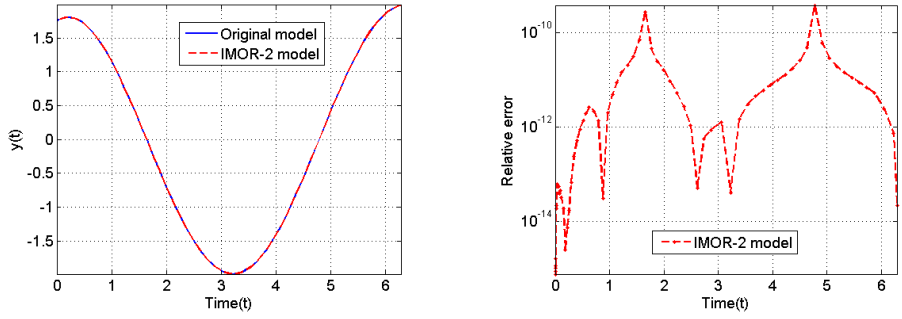
$$\hat{y} = \begin{bmatrix} -0.66667 & -0.93888 & -0.21798 & -0.97595 \end{bmatrix} \hat{\xi} \tag{4.12b}$$

We can see that the dimension of the original model is reduced to 4, and the index of the original system is preserved.
In Figure 4.3, we observe that the transfer function of the IMOR-2 model coincides with that of the original model with very small error. The reduced-order model leads to accurate solution as shown in Figure 4.4. From this example we observe that the IMOR-2 method leads to a reduced-order model which is more accurate than the reduced-order models obtained from the PRIMA method.



(a) Frequency response. (b) Frequency response error.

FIG. 4.3. *Example 4(ii). Frequency response and its error.*

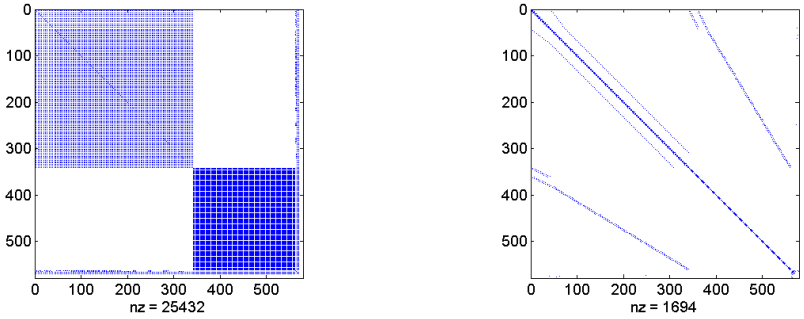


(a) Solution (b) Approximation error

FIG. 4.4. *Example 4(ii). Solution and its relative error,* $u(t) = \cos(t)$.

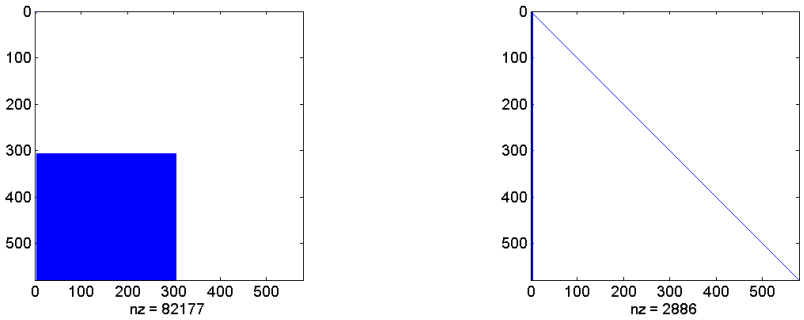
4.2. Industrial problems. In this section we test the IMOR-2 method on large scale problems.

Example 5. *This is a MNA model that originates from [5]. It is an index-2 system with dimension* 578*. The sparsity of its matrix* E *and* A *are shown in Figure 4.5. Using the procedure in Section 2.2, we decouple the system into differential and algebraic parts, as shown in the third row of Table 4.1. We then reconstruct the projected DAE in the descriptor form, and the sparsity of its matrix pencil is shown in Figure 4.6.*



(a) Sparsity of matrix E. (b) Sparsity of matrix A.

FIG. 4.5. *Example 5. Sparsity of the matrices* (E, A).



(a) Sparsity of projected matrix $\tilde{E}$. (b) Sparsity of projected matrix $\tilde{A}$.

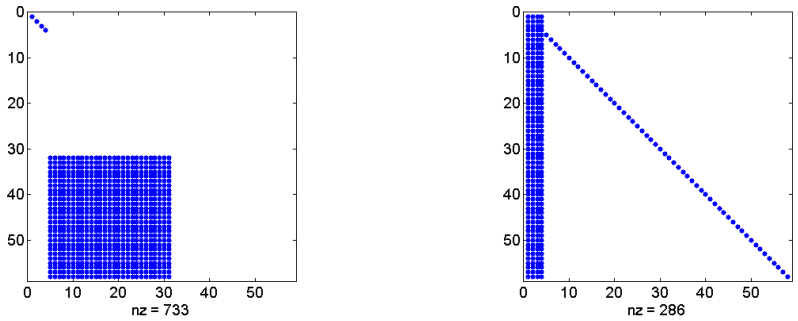
FIG. 4.6. *Example 5. Sparsity of the projected matrices* $(\tilde{E}, \tilde{A})$.

We used $s_0 = 0$ *as the expansion point and, we were able to reduce the decoupled system of dimension* 578 *to a reduced system of total dimension* 58 *as shown in the fourth row of Table 4.1.*

TABLE 4.1
Example 5. Dimension of the original and reduced-order model.

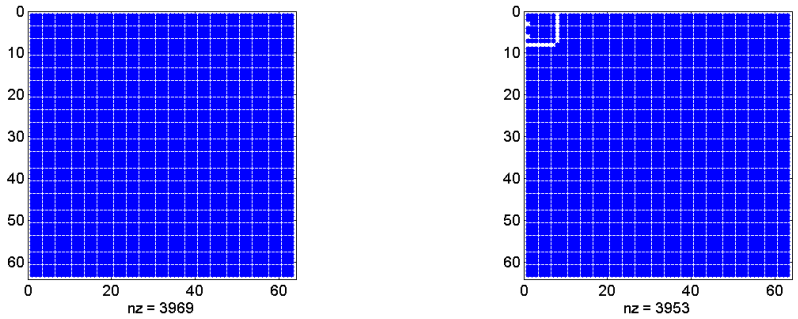
Models	Dimension		
	# differential eqns	# 1st algebraic eqns	# 2nd algebraic eqns
Original Model	4	301	273
IMOR-2 Model	4	26	28

We compared the results with that of the PRIMA method applied directly on the original DAE. Unfortunately, we could not obtain a reduced-order model of the same dimension as the one obtained with IMOR-2. Figure 4.7 shows the sparsity of the matrix pencil of the IMOR-2 model, while Figure 4.8 shows that of the PRIMA model. If we compare the two figures you can see that the IMOR-2 method leads to a sparse model, while the PRIMA method leads to a dense reduced-order model.



(a) Sparsity of $\hat{E}$. (b) Sparsity of projected matrix $\hat{A}$.

FIG. 4.7. *Example 5. Sparsity of the IMOR-2 model (n=58).*



(a) Sparsity of E_r. (b) Sparsity of projected matrix A_r.

FIG. 4.8. *Example 5. Sparsity of the PRIMA model (n=63).*

We observe that the PRIMA model is an ODE, thus it does not always preserve the index of the DAE, while the IMOR-2 model does. In Figure 4.9, we compare the transfer function of the original model and that of the reduced-order models. We can observe that the magnitude of the transfer function of the original model coincides with that of both reduced-order models. But when we solved both reduced-order models, we observe that the PRIMA model leads to wrong solutions while the IMOR-2 model leads to good solutions, as shown in Fig 4.10 .

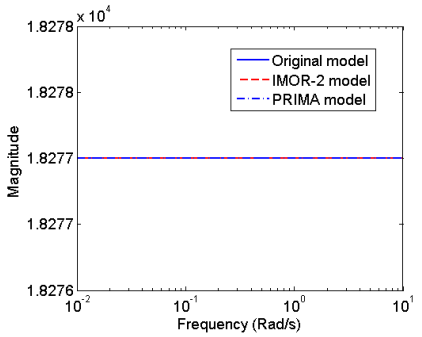
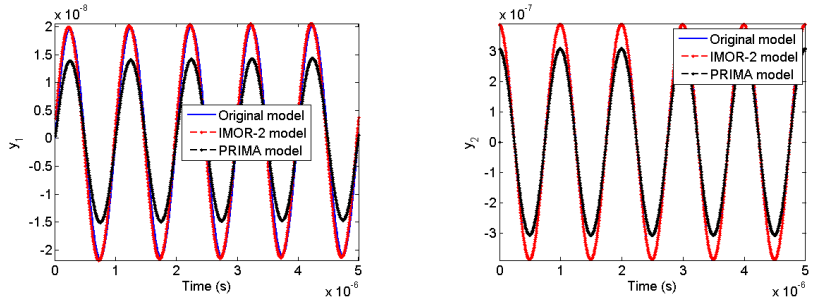
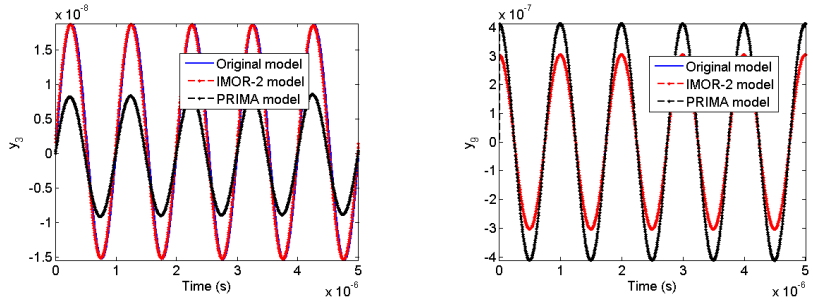


FIG. 4.9. *Example 5. Comparison of the magnitude of the transfer function.*



(a) Solution y_1 (b) Solution y_2



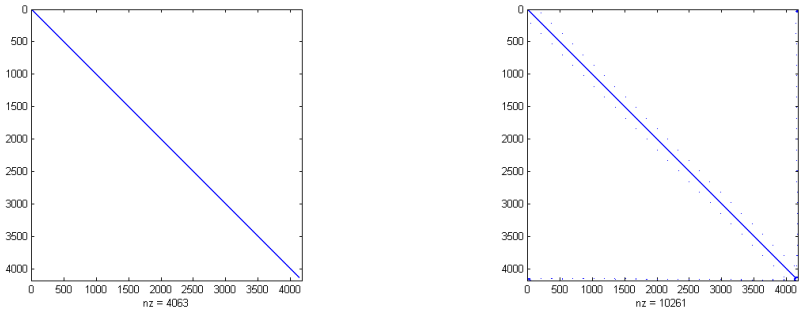
(c) Solution y_3 (d) Solution y_9

FIG. 4.10. *Example 5. Solutions of the reduced-order model,* $u(t) = \text{ones}(9,1)\sin(2\pi \times 10^6 t)$.

Example 6. *This is an electric power grid system [7] which can be found in [17]. It is a SISO system of dimension* 4182 *with the sparsity of matrix pencil (E,A) shown in Figure 4.11. We were able to decouple the system into differential and algebraic parts as shown in the third row of Table 4.2.*

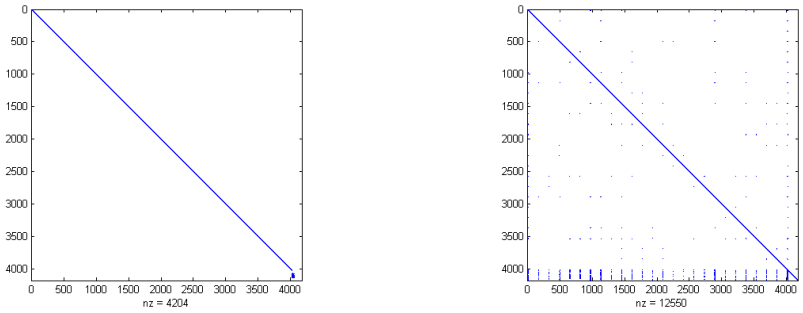
TABLE 4.2
Example 6. Dimension of the original and reduced-order model.

Models	Dimension		
	# differential eqns	# 1st algebraic eqns	# 2nd algebraic eqns
Original Model	4028	35	119
IMOR-2 Model	170	0	84



(a) Sparsity of matrix E. (b) Sparsity of matrix A.

FIG. 4.11. *Example 6. Sparsity of the matrices (E,A) of the power system.*



(a) Sparsity of projected matrix $\tilde{E}$. (b) Sparsity of projected matrix $\tilde{A}$.

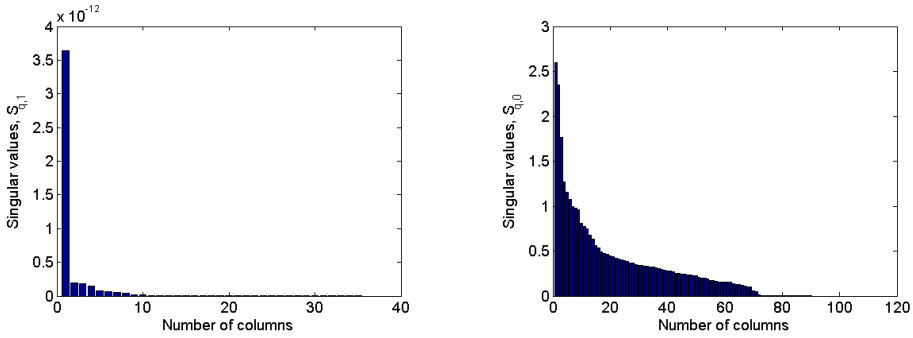
FIG. 4.12. *Example 6. Sparsity of the projected matrices* $(\tilde{E}, \tilde{A})$ *of the power system.*

Figure 4.12 shows the sparsity of the matrix pencil $(\tilde{E}, \tilde{A})$ *of the projected system. Using* $s_0 = 1$ *as the expansion point we were able to reduce the differential and algebraic parts as shown in the fourth row of Table 4.2. We reduced the dimension of the algebraic parts in the same way as in the previous example, using singular values as shown in Figure 4.13. Also in this example we can eliminate all the equations of the first algebraic part since its corresponding singular values are close to zero as shown in Figure 4.13(a). Thus the power system is reduced to a dimension of* 254 *and the sparsity of its matrix pencil is shown in Figure 4.14 which is also sparse.*

Figure 4.15 shows the sparsity of the reduced system using the PRIMA method and we can observe that it leads to a dense model. We also observe that the PRIMA method leads to an ODE while the IMOR method preserves the index of the system.

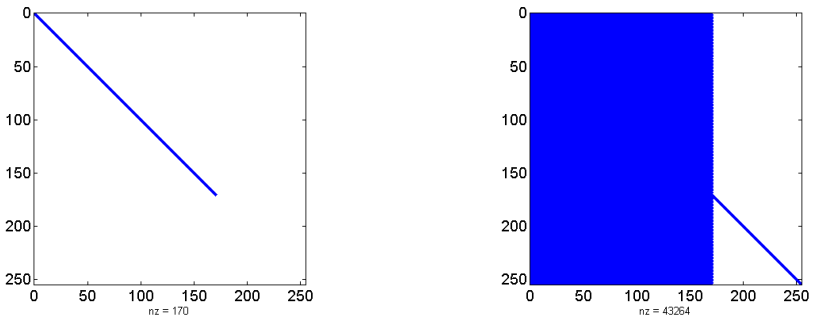
In Figure 4.16, we compare the magnitude of the transfer function of the original model with that of the reduced-order models of both methods. We observe that both reduced-order models coincide with that of the original model at low frequencies with small error as shown in Figure 4.16. In Figure 4.17, we compare the output solutions and their respective errors of the reduced-order models. We can observe that both reduced-order models lead to accurate solutions.

In Table 4.3, we compare the computational cost of solving the reduced-order models. We observe that the IMOR-2 model is easier to solve than the PRIMA model since



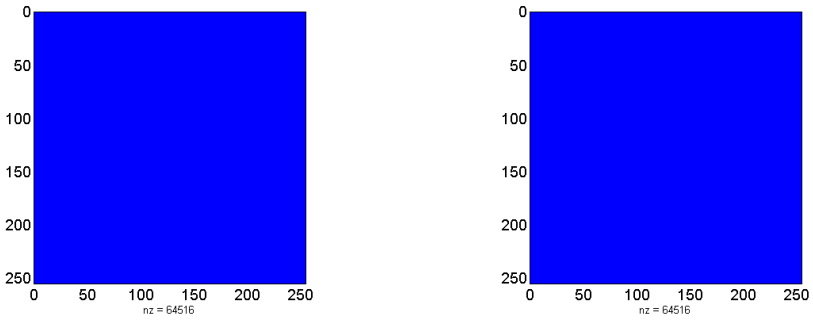
(a) Singular values of $V_{q,1}$. (b) Singular values of $V_{q,0}$.

FIG. 4.13. *Example 6. Singular values corresponding to $V_{q,1}$ and $V_{q,0}$.*



(a) Sparsity of E_r. (b) Sparsity of projected matrix A_r.

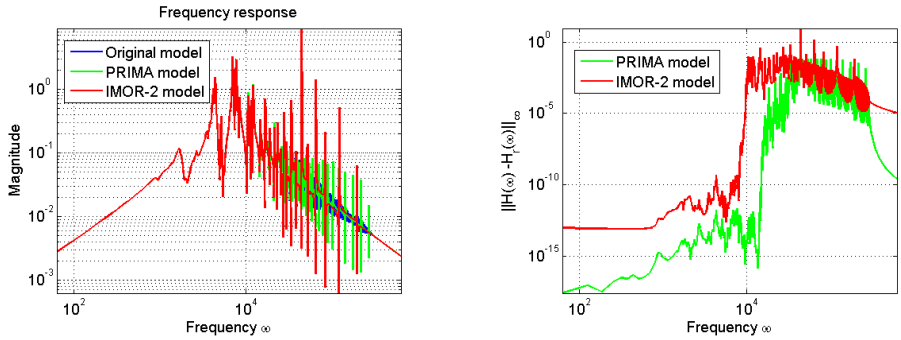
FIG. 4.14. *Example 6. Sparsity of the reduced power system using the IMOR-2 method.*



(a) Sparsity of E_r. (b) Sparsity of projected matrix A_r.

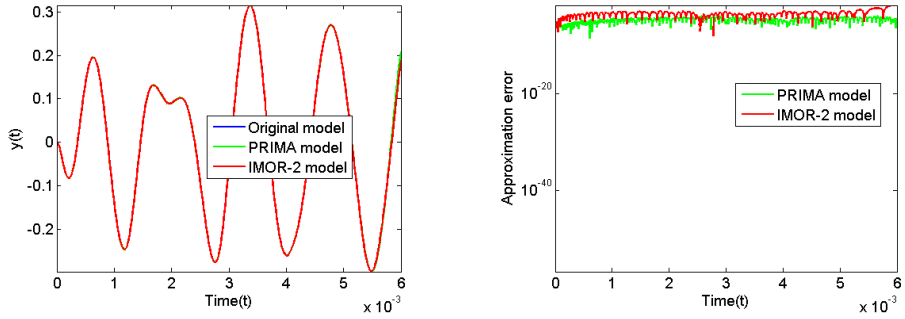
FIG. 4.15. *Example 6. Sparsity of the reduced power system using the PRIMA method.*

it requires less time, which is not a surprise. This is because IMOR-2 leads to sparse reduced-order models while PRIMA leads to very dense reduced-order models.



(a) Frequency response. (b) Frequency response error.

FIG. 4.16. *Example 6. Comparison of the frequency response and its error.*



(a) Output solution (b) Approximation error

FIG. 4.17. *Example 6.* $u = \sin(2\pi 750(t - t_1))(1 - e^{-\frac{t}{\tau}})$, $t_1 = 3\,\text{ms}, \tau = 0.1\,\text{ms}$.

TABLE 4.3
Example 6. Computation cost, RelTol $= 10^{-6}$.

	Original model	PRIMA model	IMOR-2 model
Time (s)	81.5	261.5	4.7

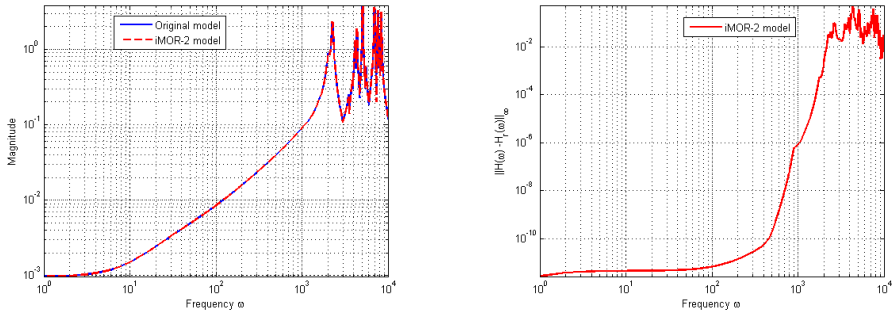
Example 7. *This example originates from the power test cases [17], which is a MIMO index-2 dynamical system with 3 inputs and 3 outputs. This is a system of dimension* 4182. *Using* $s_0 = 1$ *as the expansion point, we were able to reduce the system's dimension to* 329 *using the IMOR-2 method, as shown in the Table 4.4.*

TABLE 4.4
Example 7. Dimension of the original and reduced-order model.

Models	Dimension		
	# differential eqns	# 1st algebraic eqns	# 2nd algebraic eqns
Original Model	4028	35	119
IMOR-2 Model	270	2	57

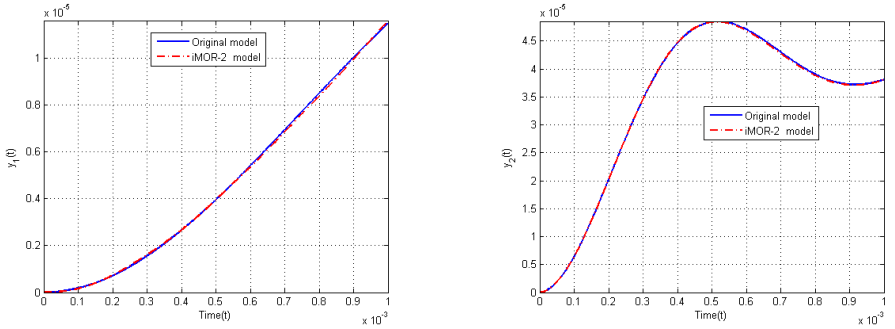
We observe that the transfer function of the IMOR-2 model coincides with that of the original model with very small error, as shown in Figure 4.18. We then solved the IMOR-2 model using $sin(t)$ *in all inputs and observed that it leads to good solutions*

with very small approximation error as shown in Figure 4.19. We also compared the computational costs and observed that the original model and the IMOR-2 model take 2.5 *and* 0.8 *seconds, respectively, at a relative tolerance* 10^{-6}.

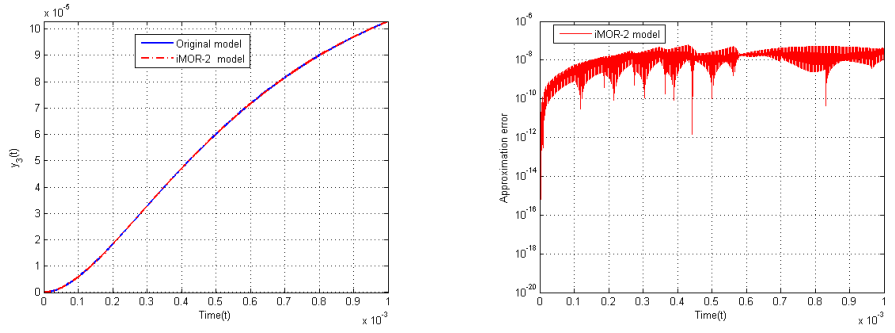


(a) Frequency response. (b) Frequency response error.

FIG. 4.18. *Example 7. Comparison of the frequency response and its error.*



(a) Solution $y_1(t)$. (b) Solution $y_2(t)$.



(c) Solution $y_3(t)$. (d) Approximation error

FIG. 4.19. *Example 7. Solutions and their approximation error.*

5. Conclusion. We extended a new MOR method developed for linear index-1 DAEs [1] with constant coefficients to linear index-2 DAEs with constant coefficients. In contrast to conventional approaches treating the DAE systems as a whole, the

presented IMOR-2 method first splits the index-2 DAE into three parts: the inherent differential equation part, the pure algebraic part and one part including differentiations of the algebraic part. Then, the PRIMA method is used to reduce the differential part whereas the algebraic and differentiation parts are treated by adapted projections.

We have discussed that conventional methods based on Krylov subspaces (like the PRIMA method) may lead to wrong reduced-order models, or the reduced-order models may be difficult to solve, if the consistent initial data depend on the derivatives of the input vector u. It is caused by the fact that - in a conventional approach - such methods are applied to the whole DAE system. Additionally, we have seen that the the conventional PRIMA method may lead to reduced-order models with dense matrices and a DAE index which differs from the one of the original system.

The IMOR-2 approach has the advantage that it leads to reduced-order models which are sparse, always solvable and index preserving. An interesting additional feature of this method is that it can also be applied to index-2 systems without inherent differential equations. For the reduction of the differential part one could also use other methods based on Krylov subspace instead of the PRIMA method. Furthermore, an extension to DAEs with an index greater than 2 is naturally given by exploiting an index-adapted decoupling approach as given in [11]. The development of IMOR methods for higher index DAEs is not straightforward, and will be the topic of a forthcoming paper [3].

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